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Evolutionary dynamics for bimatrix games: A Hamiltonian system?

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Abstract. We review some properties of the evolutionary dynamics for asymmetric conflicts, give a simplified approach to them, and present some new results on the stability and bifurcations occurring in these conservative systems. In particular, we compare their dynamics to those of Hamiltonian systems.

Key words: Evolutionary game theory – Replicator dynamics – Bimatrix games – Hamiltonian systems – Volume-preserving flows – Bipartite systems – Stability

1 Introduction

In the standard situation of evolutionary game theory, as initiated by Maynard Smith and Price, there is *one* population of players. However, there are situations, called asymmetric conflicts in [MS], where interactions or conflicts take place only between two separate populations. The resulting evolutionary games correspond to the bimatrix games of classical game theory:

Suppose the first population has a repertoire of $n + 1$ pure strategies $E_0, \ldots, E_n$, occurring with relative frequencies $x_0, \ldots, x_n$, and the second population plays strategies $F_0, \ldots, F_m$ with frequencies $y_0, \ldots, y_m$, respectively. After a contest E_i versus F_j, the payoff for the first player is a_{ij}, and for the second player b_{ji}. For such games the following evolutionary dynamics was introduced by [SS] and [SSHW], see also ([HS, Chs. 17, 27]):

$$\begin{aligned} \dot{x}_i &= x_i((Ay)_i - x \cdot Ay) \quad i = 0, \ldots, n \\ \dot{y}_j &= y_j((Bx)_j - y \cdot Bx) \quad j = 0, \ldots, m \,. \end{aligned} \tag{1.1}$$

It is the analog of the replicator equation for bimatrix games. It is a differential equation on the product $S_n \times S_m$ of two probability simplices, where

$S_n = \{x \in \mathbb{R}^{n+1}: x_i \geqq 0, \sum x_i = 1\}$. The essential assumptions for this dynamics are:

(1) A strategy not played at time 0 is not played at any other time t. (Mathematically: The boundary faces of the state space $S_n \times S_m$ are invariant. Biologically: no mutations.)

(2) The growth rates of the frequencies of two strategies are determined by the mean payoff for these strategies:

$$\frac{\dot{x}_i}{x_i} - \frac{\dot{x}_j}{x_j} = (Ay)_i - (Ay)_j \tag{1.2}$$

It is easy to see that (1.2) plus its analog for the second population is an equivalent formulation of (1.1).

Other versions of evolutionary dynamics have been suggested. In particular, Maynard Smith [MS, Appendix J] discusses a discrete time version as well as a similar looking differential equation which differs from (1.1) by the mean payoffs in the denominator. These dynamics are qualitatively completely different from (1.1), see [HS, Ch. 27], and Sect. 9 below. In fact, we will suggest a different discrete time dynamics in Sect. 8 below, which seems to behave qualitatively exactly like (1.1) and is therefore well suitable for numerical simulations. The usual ODE solvers are not recommended for studying (1.1), because they do not take care of its peculiar conservative properties.

2 An example: 2×2 games

My favorite example of such asymmetric conflicts is Dawkins' *battle of the sexes*: Here the two populations are males and females, the conflict is about the costs of raising the offspring. The two strategies for males are philandering versus being faithful, while females have the choice between fast and coy (=insisting on a long courtship period). The dynamics (1.1) for this game was derived and completely analyzed by [SS], see also [HS, Ch. 17]. It turns out that in the case of $n = m = 1$, i.e. two strategies in each population, (1.1) simplifies to

$$\begin{aligned} \dot{x} &= x(1-x)\,(a-(a+b)y)\,, \\ \dot{y} &= y(1-y)\,(-c+(c+d)x)\,. \end{aligned} \tag{2.1}$$

The orbits of this differential equation on the square can be easily obtained by separation of variables, and a constant of motion can be computed to be

$$H = c\log x + d\log(1-x) + a\log y + b\log(1-y) \tag{2.2}$$

By a change of variables, or by taking a suitable symplectic form on int $S_1 \times S_1$, (2.1) is actually a Hamiltonian system with H as its Hamilton function. Indeed, (2.1) can be rewritten as

$$\dot{x} = P(x,y)\,\frac{\partial H}{\partial y}\,, \qquad \dot{y} = -P(x,y)\,\frac{\partial H}{\partial x} \tag{2.3}$$

with $P(x, y) = x(1-x)\,y(1-y)$.

In the battle of the sexes game (where $a, b, c, d > 0$) this shows that the interior fixed point is a center, surrounded by periodic orbits. The equations (2.1) and their dynamics are similar to the classical Lotka–Volterra predator–prey equations. For other sign combinations of $a, b, c, d,$ the interior fixed point may be a saddle, and there are two stable (Nash) equilibria on the boundary. This is not in contradiction with the Hamiltonian nature of (2.1), since the symplectic form blows up on the boundary.

This raises the question whether (1.1) is a Hamiltonian system also in higher dimensions ($n, m \geqq 2$).

3 Conservative properties of the replicator equation

The conservative character of (1.1) is summarized in the following three properties (see [HS, Chs. 17, 27], a new proof will be given in Sects. 4 and 5):

(3.1) *The eigenvalues at an interior equilibrium are symmetric with respect to the imaginary axis.* Hence if λ is an eigenvalue, then also $-\lambda$. This means that the linearized equation is Hamiltonian.

(3.2) *The game dynamics* (1.1) *preserves volume.* This property was discovered by E. Akin (see the remark in [EA, p. 133]). Actually it is not the standard Euclidean volume, which is preserved, but a certain volume form. The total volume of the state space $S_n \times S_m$ is infinite. This implies that there cannot be any asymptotically stable fixed points or other attractors in the interior of the state space. This is the dynamic equivalent to a result of Selten [S1] that such games cannot have mixed ESS. The motion along an orbit of (1.1) should therefore be imagined as the motion of a particle in an incompressible fluid.

(3.3) *If* $A \approx cB^T$, *then* (1.1) *has a constant of motion* similar to (2.2). This includes zero-sum games ($c = -1$) and partnership games ($c = 1$), and their rescalings ($\approx$ means equality after addition of suitable constants to the columns of A, B, or equivalently, $\xi \cdot A\eta = c\eta \cdot B\xi$ for all vectors ξ, η whose components sum up to zero). It is even a Hamiltonian system (again not in the usual sense but only after choosing a suitable Poisson structure) on the interior of $S_n \times S_m$. This covers in particular the two dimensional case $n = m = 1$ discussed above, but only special cases for more than two strategies per population. If $c < 0$ and the game has an interior equilibrium (p, q), then the Hamiltonian function (5.3) is definite and hence the equilibrium is Ljapunov stable (in both positive and negative times). These equilibria have been characterized in purely game theoretic terms as Nash–Pareto pairs (see [HS, Ch. 27.6]). It is the only case where local stability of an interior equilibrium is known.

4 Bipartite systems

In the following we give a different and maybe simpler proof of these results, by putting them in a more general framework. The idea is to rewrite (1.1) in the form

$$\dot{u} = f(v), \quad \dot{v} = g(u) \qquad u \in \mathbb{R}^n, v \in \mathbb{R}^m \tag{4.1}$$

by a suitable change of variables: We first set $\xi_i = x_i/x_0$ and $\eta_j = y_j/y_0$. Then (1.1) transforms into

$$\dot{\xi}_i = \xi_i((Ay)_i - (Ay)_0) = \xi_i((\tilde{A}\eta)_i + \tilde{a}_{i0})y_0 = \xi_i \frac{(\tilde{A}\eta)_i + \tilde{a}_{i0}}{1 + \sum_{j=1}^m \eta_j}$$

(with $\tilde{a}_{ij} = a_{ij} - a_{0j}$) and a similar equation for η_j. Then taking $u_i = \log \xi_i$ and $v_j = \log \eta_j$, we obtain

$$\dot{u}_i = \frac{\sum_{j=1}^m \tilde{a}_{ij} e^{v_j} + \tilde{a}_{i0}}{1 + \sum_{j=1}^m e^{v_j}} \tag{4.2}$$

and similar for v_j. Hence we have found a smooth conjugacy of system (1.1) (restricted to the interior of $S_n \times S_m$) to a system of the form (4.1) on $\mathbb{R}^n \times \mathbb{R}^m$.

Systems of the form (4.1), which might be called *bipartite systems* (in analogy to bipartite graphs), occur also in other situations:

The most prominent are *Newton's equations of motion*

$$\ddot{u} = g(u), \quad u \in \mathbb{R}^n. \tag{4.3}$$

With $\dot{u} = v$ we obtain a system of form (4.1) that is reversible (see [M]) under the involution $(u, v) \to (u, -v)$: If $u(t)$ is a solution of (4.3), then also $u(-t)$. Again, these systems are in general not Hamiltonian (only when g is a gradient), and are therefore called 'nonconservative' systems in mechanics. Still, they are volume preserving and hence conservative dynamical systems.

Another example are '*conservative*' *predator–prey systems*: The Lotka–Volterra equations for a two level ecosystem with n prey (densities x_i) and m predators (densities y_j), under the assumption of no competitive or other interaction within the trophic levels, take the form

$$\begin{aligned} \dot{x}_i &= x_i(r_i - (Ay)_i) \quad i = 1, \ldots, n \\ \dot{y}_j &= y_j(-s_j + (Bx)_j) \quad j = 1, \ldots, m\,. \end{aligned} \tag{4.4}$$

In the new variables $u_i = \log x_i$ and $v_j = \log y_j$, they reduce to the form (4.1). In the special case $A = B^T$ (much studied by Volterra) there is a constant of motion, and (4.4) is again Hamiltonian, see [Pl] or (5.2). In the general case not much seems to be known.

For bipartite systems (4.1) the conservative properties (3.1) and (3.2) mentioned in the previous section are immediate:

(1) The divergence of the vector field (4.1) is 0, hence by Liouville's theorem the flow preserves volume (now it is really Euclidean volume!).

(2) Linear bipartite systems

$$\dot{u} = Av, \quad \dot{v} = Bu \qquad u \in \mathbb{R}^n, v \in \mathbb{R}^m \tag{4.5}$$

are again reversible in the sense of (4.3): the matrix $J = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$ is similar to $-J$ and hence each eigenvalue λ comes in pair with $-\lambda$. Hence linear bipartite systems have the same properties as linear reversible or linear Hamiltonian systems: Nonzero eigenvalues occur as real pairs $\pm\lambda$, imaginary pairs $\pm i\omega$, or complex quadruples $\pm\alpha \pm i\beta$.

5 Hamiltonian systems

A Hamiltonian system is a system of the form $\dot{x} = \{H, x\}$ for a certain Poisson structure $\{\ ,\ \}$ and Hamiltonian function H, or more explicitly,

$$\dot{x} = J\nabla H(x), \quad x \in \mathbb{R}^N \tag{5.1}$$

with J a skew-symmetrix matrix. Obviously, H is a constant of motion: $\dot{H} = \nabla H(x) \cdot J\nabla H(x) = 0$. In general J may depend on x, but then an additional condition (Jacobi's identity) is required to make (5.1) a Hamiltonian system. In the case of a constant Poisson structure J, these additional conditions are automatically satisfied. I recommend the first chapter of [Pe] for a concrete and concise introduction to this modern and general view of Hamiltonian systems, which applies also to odd dimensions N. For nondegenerate Poisson structures, the inverse matrix J^{-1} determines the symplectic structure, which is the classical framework for Hamiltonian systems. Otherwise the state space foliates into symplectic manifolds. In many applications, like here in game dynamics, for Lotka Volterra equations (see [Pl]), or in many problems in classical mechanics, it is the Poisson structure, and not the symplectic structure, which is given more naturally and explicitely.

Linear Hamiltonian systems, which are given by matrices that are products JA of a skew-symmetric and a symmetric matrix, have a spectrum symmetric to the imaginary axis, like (4.5). Every Hamiltonian system (5.1) with nondegenerate Poisson structure preserves volume, actually Euclidean volume in case of a constant J.

As an example consider *Hamiltonian systems with separate variables* which are both Hamiltonian and bipartite systems:

$$\begin{aligned} \dot{u} &= P\nabla_v F(v), \quad u \in \mathbb{R}^n \\ \dot{v} &= -P^T\nabla_u G(u), \quad v \in \mathbb{R}^m \end{aligned} \tag{5.2}$$

with $H(u, v) = G(u) + F(v)$ and a Poisson structure defined by the skew-symmetric $(n+m)\times(n+m)$ matrix $J = \begin{pmatrix} O & P \\ -P^T & O \end{pmatrix}$).
For bimatrix games with $A \approx cB^T$ (see (3.3)) and an interior equilibrium (p, q), the replicator equation (1.1), in the equivalent form (4.2), can be expressed in the form (5.2) with Hamiltonian

$$\begin{aligned} H &= \sum_0^n p_i \log x_i + c\sum_0^m q_j \log y_j \\ &= \sum_1^n p_i u_i - \log\left(1 + \sum_1^n e^{u_i}\right) + c\left(\sum_1^n q_i v_i - \log\left(1 + \sum_1^n e^{v_i}\right)\right) \end{aligned} \tag{5.3}$$

and Poisson structure given by $p_{ij} = \tilde{a}_{i0} - \tilde{a}_{ij} = a_{i0} + a_{0j} - a_{00} - a_{ij}$. It is not known whether this is the most general class of bimatrix games (A, B) where (1.1) is a Hamiltonian system.

The Hamiltonian character of the replicator equation for zero-sum games (within one population) was first noted in [AL], who used the more technical approach of symplectic structures.

6 Local behaviour near elliptic fixed points

Up to now we have essentially only reviewed known results from a new point of view. However, the more general and simpler form of equations (4.1) allows us now to proceed further. The most urgent question concerns the stability of the fixed points of bipartite systems, i.e. interior fixed points of (1.1).

The only fixed points for which we can expect stability are the *elliptic fixed points* where all eigenvalues of the linearization are on the imaginary axis. The problem of stability of such fixed points of (1.1) was raised in [SSHW] and [HS, Ch. 17].

The obvious way to attack this is to compute the Poincaré or Birkhoff normal form. It is well known [A2] that by a nonlinear change of coordinates, the vector field near an elliptic fixed point with n pairs of rationally independent, imaginary eigenvalues $\pm i\omega_k$ can be simplified and expressed in complex coordinates in the form

$$\dot{z}_k = z_k\left(i\omega_k + \sum_l \alpha_{kl}|z_l|^2 + h.o.t.\right). \tag{6.1}$$

Writing $|z_k|^2 = \rho_k$, $2\mathrm{Re}\alpha_{kl} = a_{kl}$, and truncating at degree 3, we obtain homogeneous Lotka–Volterra equations

$$\dot{\rho}_k = \rho_k \sum_l a_{kl}\rho_l \tag{6.2}$$

For Hamiltonian systems (5.1) (with $\det J \neq 0$) and also for reversible systems like (4.3), the coefficients a_{kl} all vanish. Even if higher order terms are included, the action part of the normal form always reads $\dot{\rho}_k = 0$, see [A1, M]. Each fixed point of the amplitude or action part of the normal form equations corresponds to an invariant torus with a linear flow on it, whose frequencies are determined by the imaginary part of (6.1). However, the actual behaviour of the Hamiltonian system near an elliptic fixed point cannot be immediately concluded from this. There are always higher order terms left after a nonlinear change of variables, which cause a small perturbation of this completely integrable behaviour. Since this is by no means structurally stable, the small perturbations will change the dynamics in general. If the vector field is analytic, one could hope to arrive at the normal form after performing infinitely many changes of coordinates. But this procedure diverges in general, because of the famous problem of small denominators. Still, the classical

Ljapunov center theorem states that – corresponding to each ρ_k-axis – there is a two-dimensional smooth manifold composed of periodic orbits with periods approximately $2\pi/\omega_k$. This subcenter manifold is tangent to the corresponding two-dimensional eigenspace. And the famous KAM theorem [A1, M] says that in general most of the n-dimensional invariant tori survive. But in general also chaotic regimes are created between these tori, due to the perturbation from the normal form.

This is the situation for Hamiltonian systems (like (1.1) for zero-sum games, whose equilibria are automatically elliptic), and reversible systems (like (4.3), or (1.1) for bimatrix games with $A = -B$, and the involution $(x, y) \to (y, x)$).

For general bipartite systems (4.1), and (1.1) for general bimatrix games, the situation is rather different. For simplicity, let us restrict to the case $n = m = 2$ in (4.1) which corresponds to 3 strategies per population in (1.1). Then the truncated normal form (6.2) turns out to be

$$\begin{aligned} \dot{\rho}_1 &= \rho_1(a\rho_1 - 2b\rho_2) \\ \dot{\rho}_2 &= \rho_2(-2a\rho_1 + b\rho_2)\,. \end{aligned} \tag{6.3}$$

This is the normal form of a general 4d volume-preserving system near a nonresonant elliptic fixed point, see Broer [B]. It can be shown, that also for bipartite systems (4.1), and (1.1) in particular, a, b can take arbitrary values. See the appendix for more details.

The system (6.3) is area preserving and hence Hamiltonian with $H = \rho_1\rho_2(b\rho_1 - a\rho_2)$. The possible phase portraits are those shown in Fig. 1, and their time-reversals. They show that *a generic elliptic fixed point of a bipartite system is not stable.* Figure 1a has to be interpreted in the following way: Orbits spiral in towards the fixed point, with angular velocity approximately ω_2, along a two dimensional invariant manifold, and spiral away along another 2d manifold with angular velocity ω_1. In Fig. 1b orbits may spiral in along both 2d manifolds, and come out of the fixed point in form of 'quasiperiodic' spirals winding around a growing 2d torus. That these pictures derived from the truncated normal form actually describe the flow near the elliptic fixed point of the original system, is again highly nontrivial, and was shown by Takens [T]. In particular, he proved (Prop. 4.17) the existence of these 2d subcenter manifolds under the assumptions $a, b \neq 0$, and of the 3d variety of the growing tori.

Numerical simulations of (1.1) confirm this behaviour, but one has to be patient to observe the movement away from the elliptic fixed point. The waiting time is inverse proportional to the distance of the initial point to the fixed point (because the normal form (6.1) starts with cubic terms only) which is exponentially longer than for a hyperbolic fixed point.

The extension of these results to more degrees of freedom is not straightforward. The above suggests that bipartite systems may have the same normal form as divergence free systems more generally. I don't see how to check this, say for an elliptic fixed point with n imaginary pairs. Given that, I could

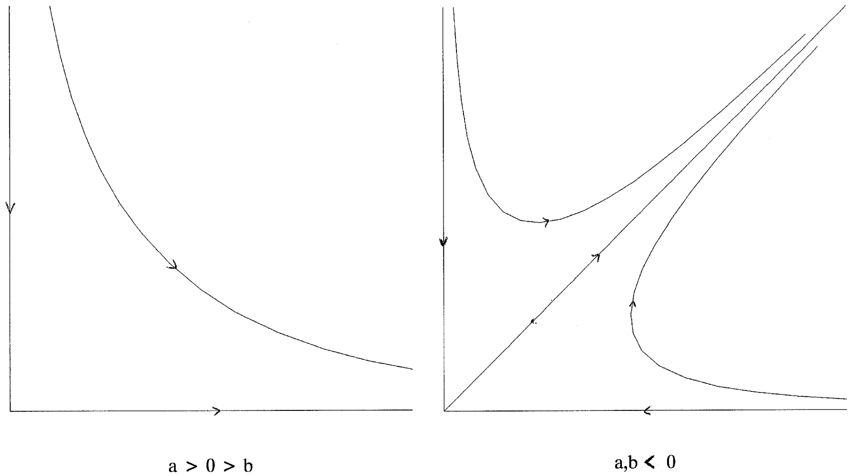

Fig. 1. Phase portraits of the truncated normal form (6.3) of a bipartite system near an elliptic fixed point

imagine to classify the possible behaviours of the normal form for 3 pairs, as this can be reduced to Zeeman's classification [Z] of the replicator (= game dynamical) equation on the two dimensional simplex. Still, the main difficulty will be to extend Takens' conjugacy result. However, it should be possible to extend the result on generic instability of elliptic fixed points.

7 Bifurcation near elliptic fixed points

The above result on the generic instability of elliptic fixed points may be somewhat disappointing. It raises the question, where orbits actually go, if they do not stay within a neighbourhood of the fixed point. Of course they may go to the boundary, either to a Nash equilibrium, to a periodic orbit, or to a heteroclinic cycle. But by studying bifurcations near elliptic fixed points, one can identify regions of stability in the interior. In such a bounded invariant region almost all orbits will be recurrent, according to Poincaré's recurrence theorem.

The simplest bifurcation occurring in our systems is when one of the parameters in (6.3), say a, changes sign. Then a kind of Hopf bifurcation happens on one 2d subcenter manifold. To study it, one has to take into account higher order terms of the normal form. Suppose, on the ρ_1-axis, the dynamics reads approximately

$$\dot{\rho}_1 = \rho_1(a\rho_1 + c\rho_1^2) .$$

The possible phase portraits are then shown in Fig. 2 ($c < 0$) and Fig. 3 ($c > 0$). The periodic orbits in the last picture correspond to a family of three

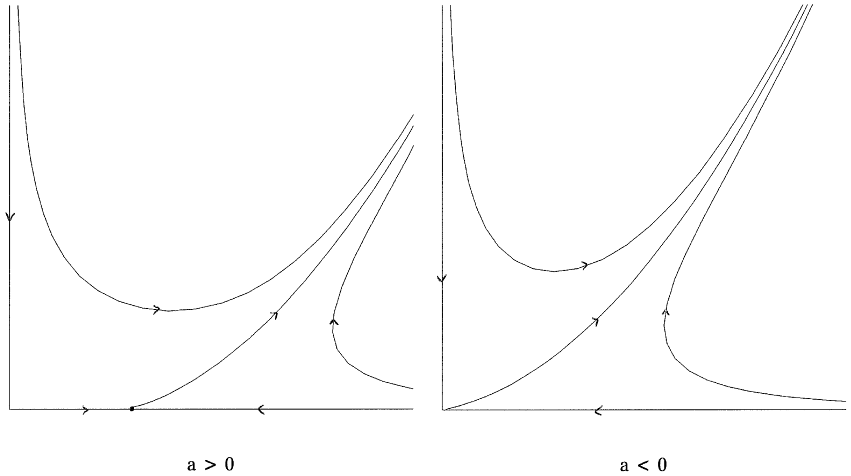

Fig. 2. Bifurcation diagram for the transition from Fig. 1a to Fig. 1b. The case $b, c < 0$

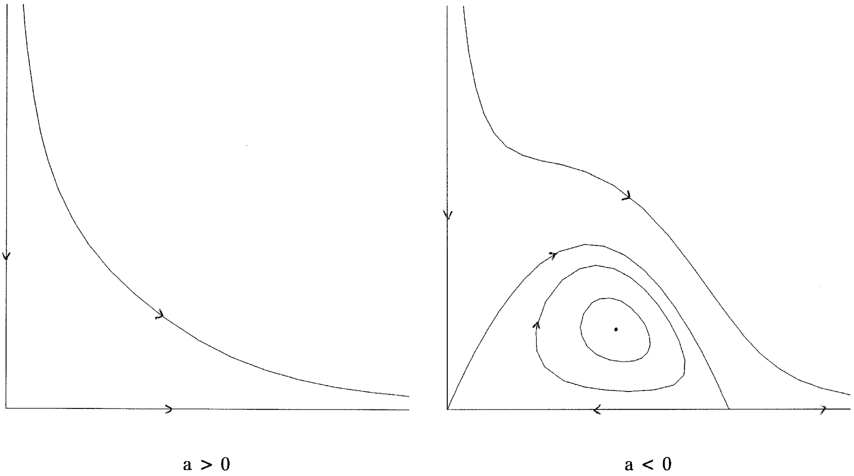

Fig. 3. The case $b < 0 < c$

dimensional tori. The actual dynamics is again a volume-preserving perturbation of this 4d integrable flow, and so by a version of KAM theory, most of these 3d tori can be expected to survive. This could be proved rigorously using the work of Broer et al. [BHT], but requires some effort. Thus a set of positive measure of recurrent orbits are confined to a neighbourhood of the elliptic fixed point, close to this bifurcation. Moreover a chaotic regime can be expected due to a transverse heteroclinic connection from the elliptic fixed point to the periodic orbit corresponding to the fixed point on the ρ_1-axis.

8 Discrete time

We propose the following 'canonical' discrete time analog for (1.1) :

$$\begin{aligned} x_i' &= x_i \frac{(Ay)_i}{x \cdot Ay} \quad i = 0, \ldots, n \\ y_j' &= y_j \frac{(Bx')_j}{y \cdot Bx'} \quad j = 0, \ldots, m\,. \end{aligned} \tag{8.1}$$

Here we require that the entries of the payoff matrices are positive. If this is not the case one has to replace a_{ij} by $a_{ij} + C$ (and similar for the b_{ji}), with C a large constant, representing the 'background fitness'.

Note the small but essential difference to the discrete time dynamics in [MS, Appendix J], and [HS, p. 273]: the x' instead of the x in the equation for y'. Hence, while the new x depends on the old x and the old y, the new y depends on the old y but already on the new x. The (at first sight) asymmetry in the roles of x and y disappeares if we think of the first population to readjust its frequencies at even times 0, 2, 4 , . . . , but the second population at odd times 1, 3, 5 , . . . : *The two players alternate their moves*. In mechanics, this kind of time staggering has been used for a long time in actual computations. Applying the same change of variables as in (5), (8.1) reduces to the form

$$u' = u + hf(v), \qquad u' = v + hg(u')\,, \tag{8.2}$$

with h a step size of order $1/C$. In the case of a Hamiltonian system, this is the canonical discrete analog, a classical symplectic integration scheme. It can be shown, that also in our more general situation of game dynamics and bipartite dynamics, this time staggering preserves the essential structure of (4.1): The map (8.2) preserves volume, and its linearization is time reversible, so its eigenvalues are those of a symplectic map. See [H] for details.

In the simplest case $n = m = 1$, or two strategies per population, we obtain an area preserving twist map (near elliptic fixed points). Numerical simulations of the discrete version of the 'battle of the sexes' game suggest that this twist map is integrable: Orbits seem to lie on smooth invariant curves. This is somewhat surprising, as generically, twist maps have wild dynamics, with stable island chains around elliptic periodic points within a chaotic sea [A1, M]. That this does not occur for (8.1), is maybe another indication that (8.1) is well suitable for numerical simulations of the continuous game dynamics (1.1).

9 Dissipative perturbations

We conclude with some remarks and consequences of our results on more general versions of evolutionary dynamics. One such version, already mentioned, is due to Maynard Smith, and takes the general form

$$\begin{aligned}\dot{x}_i &= x_i((Ay)_i - x \cdot Ay)m_A(x, y) \quad i = 0, \ldots, n \\ \dot{y}_j &= y_j((Bx)_j - y \cdot Bx)m_B(x, y) \quad j = 0, \ldots, m\,.\end{aligned} \tag{9.1}$$

with positive functions $m_A(x, y), m_B(x, y)$. [MS, Appendix J] assumed $m_A(x, y) = (x \cdot Ay + C)^{-1}, m_B(x, y) = (y \cdot Bx + C)^{-1}$ with some background fitness C again. This modified or multiplied replicator equation (9.1) was called aggregate monotone dynamics in [SZ], where an abstract characterization of this type of dynamics is given. It is a special case of even more general *monotone selection dynamics* which are characterized by

$$\frac{\dot{x}_i}{x_i} < \frac{\dot{x}_j}{x_j} \Leftrightarrow (Ay)_i < (Ay)_j \tag{9.2}$$

and similar for the second population. Hence the two expressions in (1.2) which are equal for the replicator dynamics, are here required only to have the same sign.

Most of the conservative properties of (1.1) are no longer shared by these extensions: They are in general not volume preserving, and in particular not Hamiltonian for zero-sum games. Indeed, the Maynard Smith dynamics contracts volume, and equilibria of constant sum games are globally asymptotically stable, see [HS, Ch. 27]. For other choices of the multipliers in (9.1), instability of interior equilibria and convergence of almost all orbits to the boundary can be shown.

However, the symmetry property of the eigenvalues still holds for these more general dynamics. Actually the spectrum of the linearization of (9.1) and even (9.2) at an interior equilibrium is always a multiple of the spectrum of the replicator equation (1.1). (For (9.2) there may be the degenerate situation that the Jacobian vanishes, so all eigenvalues are 0.) Hence one could define the 'spectrum of an equilibrium', rather independently of the dynamics. In particular, elliptic fixed points of (1.1) are still elliptic fixed points of (9.1), and generically even for (9.2). But the normal form is different:

$$\begin{aligned}\dot{\rho}_1 &= \rho_1(a_{11}\rho_1 + a_{12}\rho_2) \\ \dot{\rho}_2 &= \rho_2(a_{21}\rho_1 + a_{22}\rho_2)\,,\end{aligned} \tag{9.3}$$

now with arbitrary parameters a_{ij}. The leads to 4 (resp. 7, if time reversals are taken into account) further robust possibilities for the local dynamics near an elliptic fixed point, besides the 2 (resp. 3) shown in Fig. 1, see [T]. The results in Sect. 6 imply that the Maynard Smith dynamics cannot be asymptotically stable for all elliptic points, at least for large background fitness, although it is for zero sum games. In the example in Fig. 3b, an attracting 2 torus has to be expected for this dynamics. On the other hand, chaotic regimes of (1.1), caused by transverse homoclinic points, persist to these dissipative perturbations.

Adding mutation or anticipatory effects as in Selten [S2] generally shift the eigenvalues to the left and hence would stabilize an elliptic fixed point.

Still, the above results suggest that its basin of attraction would only be small, due to the instabilities caused by the nonlinear terms.

10 Conclusion

We have shown that the standard evolutionary dynamics (replicator equation) for bimatrix games (1.1) can be written as a *bipartite system* (4.1). These bipartite systems are an interesting class of conservative dynamical systems. They share some properties with Hamiltonian systems: Their linearization is actually Hamiltonian and reversible, and they are volume preserving. However, generically (at least in dimension 4), they do not satisfy analogs of the Lyapunov center theorem or KAM theorem. This is shown by computing the normal form of such systems. In particular, for $n = m = 2$, fixed points are generically not stable, even if they are elliptic (all eigenvalues on the imaginary axis).

However, for the subclass of (rescaled) zero-sum games and partnership games (and hence for all 2×2 games) the dynamics (1.1) is actually Hamiltonian. It remains an open question whether stable isolated, interior equilibria for (1.1) occur only in rescaled zero-sum games.

11 Appendix: computation of the normal form

Consider a system of differential equations of the form

$$\dot{z}_i = \lambda_i z_i + \sum_{j,k} a_i^{jk} z_j z_k + \cdots \tag{A1}$$

with a fixed point at $z = 0$ and linear part in diagonal form. The method of normal forms (see [A2]) consists in removing as many of the nonlinear terms in the Taylor expansions (A1) as possible by performing a nonlinear change of variables

$$z_i = w_i + \sum_{j,k} A_i^{jk} w_j w_k + \cdots \tag{A2}$$

A simple calculation shows, that the transformed differential equations $\dot{w}_i$ do not contain quadratic terms $w_j w_k$ if we choose

$$A_i^{jk} = \frac{a_i^{jk}}{\lambda_j + \lambda_k - \lambda_i} . \tag{A3}$$

This choice is possible whenever none of the denominators in (A3) vanishes, i.e. if there are no resonances of degree 2 between the eigenvalues λ_i. At the next level, not all cubic terms can be eliminated, since $\lambda_i = \lambda_i + \lambda_j + \overline{\lambda}_j$ is a resonance of degree 3, whenever there is an eigenvalue λ_j with zero real part. If there are no further resonances of degree 3, this leads to a normal form (6.1) at an elliptic fixed point. The task is to express the coefficients α_{kl} in (6.1) in terms of the coefficients in (A1). For this one has to take into account the consequences of the quadratic change of coordinates (A2) to the cubic terms. Consider now a four dimensional system, with two imaginary pairs of eigenvalues $\pm i\omega_1$, $\pm i\omega_2$, which we write as

$$\begin{aligned}\dot{z} &= \lambda z + a_1 z^2 + a_2 z\bar{z} + a_3 \bar{z}^2 + b_1 zw + b_2 z\bar{w} + b_3 \bar{z}w + b_4 \bar{z}\bar{w}\\ &\quad + c_1 w^2 + c_2 w\bar{w} + c_3 \bar{w}^2\\ \dot{w} &= \mu w + d_1 z^2 + d_2 z\bar{z} + d_3 \bar{z}^2 + e_1 zw + e_2 z\bar{w} + e_3 \bar{z}w + e_4 \bar{z}\bar{w}\\ &\quad + f_1 w^2 + f_2 w\bar{w} + f_3 \bar{w}^2\end{aligned} \tag{A4}$$

Then a lengthy calculation (for which you better reserve some quiet hours) leads to the following explicit expressions for the coefficients in (6.3) or (9.3).

$$\begin{aligned} a &= a_{11}\\ &= \mathrm{Re}\left(-\frac{a_1 a_2}{\lambda} - \frac{b_1 d_2}{\mu} + \frac{b_2 \bar{d}_2}{\mu} + \frac{b_3 d_1}{2\lambda - \mu} + \frac{b_4 \bar{d}_3}{2\lambda + \mu} + \text{coeff of } z^2\bar{z} \right) \end{aligned} \tag{A5}$$

$$\begin{aligned} a_{12} = \mathrm{Re}\bigg(&-\frac{2a_1 c_2}{\lambda} + \frac{a_2 \bar{c}_2}{\lambda} - \frac{b_1 f_2}{\mu} + \frac{b_2 \bar{f}_2}{\mu} + \frac{2c_1 e_2}{\lambda - 2\mu} + c_2 \frac{e_1 + \bar{e}_3}{\lambda}\\ &+ \frac{2c_3 \bar{e}_4}{\lambda + 2\mu} + \text{coeff of } zw\bar{w} \bigg) \end{aligned}$$

The formulae for a_{21} and a_{22} follow from symmetry.

Now consider a bipartite system (4.1) with $n = m = 2$. At an elliptic fixed point, after a linear change of coordinates we get

$$\begin{aligned} \dot{x}_k &= -\omega_k y_k + f_k(y)\\ \dot{y}_k &= \omega_k x_k + g_k(x) \end{aligned} \qquad (k = 1, 2) \tag{A6}$$

In complex coordinates $z_k = x_k + iy_k$, this leads to a system (A4) that satisfies

$$a_1 = a_3 = -\bar{a}_2/2, \qquad b_1 = b_4 = -\bar{b}_2 = -\bar{b}_3 \tag{A7}$$

and analogous relations between the c, d, f and e's, respectively. This simplifies (A5) considerably to

$$a = \mathrm{Re}\left(\frac{\bar{b}_1 d_1}{\mu - 2\lambda} + \frac{b_1 \bar{d}_1}{\mu + 2\lambda} \right) = \frac{\omega_1}{4(4\omega_1^2 - \omega_2^2)} \left(g_1^{12} f_2^{11} - f_1^{12} g_2^{11} \right) \tag{A8}$$

with f_i^{jk} denoting the coefficient of $y_j y_k$ in $\dot{x}_i$ in (A6), and similar for b in (6.3). This shows that in (6.3) the coefficients do not vanish for generic bipartite systems (with 2 degrees of freedom) and hence these can neither be Hamiltonian nor reversible.

The final step is to apply (A8) to game dynamics (1.1). It turns out that also for generic choice of bimatrices (A, B), the coefficients in (6.3) are nonzero, and hence elliptic fixed points are unstable, with a local behaviour as shown in Fig. 1. Almost the same computation applies to the class of conservative predator prey systems (4.4) and shows the generic instability in the case $n = m = 2$.

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