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AN INTEGRATION OF EQUILIBRIUM THEORY AND TURNPIKE THEORY*

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A turnpike theorem is proved for a general equilibrium model with finitely many immortal consumers.

1. Introduction

This paper indicates one way to link equilibrium theory with capital theory and especially with turnpike theory. I consider a model with finitely many, infinitely lived consumers. Their utility functions are additively separable with respect to time and they discount future utility. There are finitely many, infinitely lived firms. Primary resources are necessary for production and their supply is constant over time. Technology and utility functions do not change over time either. The model is simply an ordinary general equilibrium model with infinitely many commodities. The infinity arises because the horizon is infinite and commodities are distinguished according to date.

I use results from a previous paper of my own (1972) in order to prove that the model has an equilibrium. I also prove that the initial resources may be chosen so that there exists a stationary equilibrium. I prove the following analogue of Scheinkman's turnpike theorem (1976). Suppose that all consumers discount future utility at the same rate. Then, the equilibrium allocation converges provided this discount rate is sufficiently small.

Finally, I prove that if consumers do not all have equal rates of time preference, then the less patient consumers eventually consume nothing in equilibrium. The less patient consumers are those whose rates of time preference exceed the smallest rate among all consumers. In an equilibrium, the less patient mortgage all their future income beyond a certain date in order to consume more, earlier.

It is easy to relate the above results to capital theory. In capital theory, authors tend to use a reduced form, aggregate model, in which a single utility

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function is defined directly on a single intertemporal production possibility set. The following is typical of the maximization problems studied in capital theory:

$$\max\left\{\sum_{t=0}^{\infty} \delta^t u(k_t, k_{t+1}) \middle| (k_t, k_{t+1}) \in D \quad \text{for all } t, \quad k_0 = \bar{k}_0\right\}, \tag{1.1}$$

where $\bar{k}_0$ is given. In this problem, t is the index for time, δ is the discount factor applied to future utility where $0<\delta<1$, k_t is the vector of capital stocks at time t, u is the utility function, and D is the intertemporal production possibility set.

If one makes appropriate assumptions, it is not hard to prove that (1.1) has a solution. A solution corresponds to an equilibrium in my model. Sutherland (1970) and Peleg and Ryder (1974) proved that one may choose $\bar{k}_0$ so that (1.1) has a stationary solution. (A solution is stationary if $k_t = \bar{k}_0$ for all t.) Such a stationary optimum is known as the modified golden rule. It corresponds to a stationary equilibrium in my case. Scheinkman (1976) proved that, under appropriate conditions, any solution to (1.1) converges to a unique stationary optimum, provided that δ is sufficiently close to one.

There are subtle differences between capital theory and the theory I develop. In the first place, equilibria are not necessarily unique, whereas the optima of capital theory normally are unique. Also, the turnpike theorem of capital theory asserts that optimum paths converge to a stationary optimum which is independent of the initial conditions. In my case, the limit of an equilibrium is not necessarily a stationary equilibrium and the limit depends on the initial conditions. The initial conditions affect the limit of an equilibrium because they affect the relative wealths of the consumers. Also, because conditions change over time, some consumers may borrow or lend early in time. For this reason, they may be paying or earning interest when in the asymptotic state approached as the equilibrium converges. Thus, the limit is strictly speaking not a stationary equilibrium, but a stationary equilibrium with transfer payments.

It is easy to see why the turnpike theorem applies to equilibrium. I assume that all utility functions are concave. Hence, equilibrium maximizes a weighted sum of consumers' utility functions, the weights being the inverses of the marginal utilities of expenditure. Since I assume that all consumers discount future utility at the same rate, the maximand may be written as

$$\sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1} u_i(x_i^t). \tag{1.2}$$

In this expression, i is the index for consumers, Λ_i is the marginal utility of expenditure for consumer i, u_i is his utility function, and x_i^t is his consumption vector at time t. δ is the discount factor applied to future utility.

(1.2) looks much like the objective function in (1.1). Hence, a version of Scheinkman's theorem should imply that equilibrium converges.

In fact, I do not apply Scheinkman's theorem or any of the recent generalizations of it. Instead, I provide a direct proof of the convergence result. I do so for three reasons: (1) I do not want to make unnecessary assumptions; (2) I obtain exponential convergence, which is stronger than that of corresponding theorems in the literature; and (3) my method of proof seems to improve on existing methods.

My proof is in many ways simply a modification of existing proofs. I use the value loss method. My main innovation is to use a one-sided value loss rather than a two-sided value loss. This value loss is easy to interpret and leads to many simplifications.

Nevertheless, my proof is very long and complicated. The complications arise largely because I use a full general equilibrium model rather than the reduced form, aggregate model of capital theory.

If one assumes that there are one consumer and one firm, then my turnpike theorem becomes a turnpike theorem in the sense of capital theory and can be compared with theorems in the literature. In this case, my result is neither more general nor more special than existing ones. I elaborate in section 6.

I emphasize that my goal is only to link two distinct branches of economic theory. I do not claim that my model is realistic or that it justifies capital theory. The assumption of immortality is certainly not realistic. Also, prices in my model can be interpreted only as Arrow–Debreu prices of contracts for future delivery. Such prices seem especially unrealistic when there is an infinite horizon.

I emphasize that I cannot avoid interpreting prices as prices for forward contracts. I cannot interpret prices as spot prices and say that agents have perfect foresight. This last point of view is the one sometimes taken in capital theory. In my model, consumers may borrow and lend, which means that there must be forward markets. In capital theory, there is only one consumer, and he owns the firm or firms. Hence, it is impossible for the consumer to borrow or lend.

By linking equilibrium theory and capital theory, I do give some insight into the nature of the assumptions that must be made in order to obtain the turnpike property. Dealing with a general equilibrium model obliges one to state assumptions only in terms of individual utility functions, endowments and production possibility sets. Assumptions in capital theory do not always have a concrete interpretation, since the models are aggregated.

I make strong assumptions. I assume that utility functions are strictly concave and that production possibility sets are strictly convex. The assumption about production possibility sets is especially strong, for it excludes constant returns to scale. However, if one allows constant returns to scale, then there can exist optimal programs which oscillate forever, even if future utility is not discounted. Therefore, the turnpike theorem is not valid if one insists on convergence to a point.

In order to exploit strict convexity, I must assure that price ratios exactly equal marginal rates of transformation in production. (The prices referred to are for the limit stationary equilibrium with transfer payments.) I assure equality by assuming that firms can use inputs and produce outputs efficiently in any ratios they like. This assumption excludes the fixed coefficients, linear production model.

2. Definitions, notation and the model

2.1. Commodities

There are L types of commodities, $L_c \subset \{1,\dots,L\}$ denotes the set of consumption goods. $L_o \subset \{1,\dots,L\}$ denotes the set of primary commodities, such as land, labor and raw materials. $L_p = \{k = 1,\dots,L | k \notin L_o]$ denotes the set of producible goods. Goods not in either L_c or L_o should be thought of as intermediate goods or goods in process.

2.2. Vector space notation

R^L denotes L-dimensional Euclidean space. A *standard subspace* of R^L is one of the form

$$R^{L'} = \{x \in R^L | x_k = 0 \quad \text{if} \quad k \notin L'\} \quad \text{where} \quad L' \subset \{1,\dots,L\}.$$

$R^{L'}$ is said to be the subspace corresponding to L'. R^{L_c}, R^{L_o}, and R^{L_p} are the subspaces corresponding to L_c, L_o, and L_p, respectively. It is important to keep in mind that vectors in R^{L_c}, R^{L_o}, and R^{L_p} are thought of as belonging to R^L.

Infinite-dimensional vectors are always written in bold face.

2.3. Consumers

There are I consumers, where I is a positive integer. The utility function of consumer i for consumption in one period is $u_i: R_+^{L_c} \to (-\infty, \infty)$. Utility is additively separable with respect to time and consumer i discounts future utility by a factor δ_i, where $0 < \delta_i < 1$. That is, if consumer i consumes the bundle $x_t \in R_+^{L_c}$ in period t, for $t = 0, 1, 2, \dots$, then his total utility from the point of view of period zero is $\sum_{t=0}^{\infty} \delta_i^t u_i(x^t)$.

The endowment of each consumer in each period is $\omega_i \in R_+^{L_o}$. Notice that each consumer is endowed only with primary goods. This assumption is not necessary. It is made only for convenience.

2.4. Firms

There are J firms, where J is a positive integer. A firm transforms inputs $y_0 \in R^L_-$ in one period into outputs $y_1 \in R^{L_p}_+$ in the succeeding period. Inputs carry a negative sign and outputs a positive sign. The production possibility set of firm j is $Y_j \subset R^L_- \times R^{L_p}_+$. $y=(y_0, y_1)$ denotes a typical vector in Y_j, where $y_0 \in R^L_-$ and $y_1 \in R^{L_p}_+$.

Firms have an endowment of producible goods, available at time zero. These goods should be thought of as having been produced from inputs in period -1. The vector of goods available to firm j is denoted by $y_{j1}^{-1} \in R^{L_p}_+$. $\sum_{j=1}^{J} y_{j1}^{-1}$ is the initial capital stock of the economy.

Firms are owned by consumers. Consumer i owns a proportion θ_{ij} of firm j, where $i=1,\ldots,I$ and $j=1,\ldots,J$. $0 \leqq \theta_{ij} \leqq 1$ for all i and j, and $\sum_i \theta_{ij}=1$ for all j.

2.5. The economy

The economy is described by the list

$$\mathscr{E}=\{(u_i, \delta_i, \omega_i), (Y_j, y_{j1}^{-1}), \theta_{ij}\colon i=1,\ldots,I \text{ and } 1,\ldots,J\}.$$

2.6. Allocations

A *consumption program* for a particular consumer is of the form $\boldsymbol{x}=(x^0, x^1, \ldots)$, where $x^t \in R^{L_c}_+$ for all t, and $\sup_{t,k} x^t_k < \infty$. x^t is the consumption vector at time t. A consumption program is said to be *stationary* if $x^t = x^0$ for all t.

A production program is of the form $\boldsymbol{y}=(y^0, y^1, \ldots)$, where $y^t=(y^t_0, y^t_1) \in R^L_- \times R^{L_p}_+$ for all $t \geqq 0$, and $\sup_{t,k} y^t_k < \infty$. The program is *feasible* for firm j if $y^t \in Y_t$ for all $t \geqq 0$. A production program is said to be *stationary* if $y^t = y^0$ for all t.

An *allocation* for the economy is of the form $((\boldsymbol{x}_i), (\boldsymbol{y}_j))$, where each $\boldsymbol{x}_i=(x_i^0, x_i^1, \ldots)$ is a consumption program and each $\boldsymbol{y}_j=(y_j^0, y_j^1, \ldots)$ is a production program feasible for firm j. The allocation $((\boldsymbol{x}_i), (\boldsymbol{y}_j))$ is *feasible* if

$$\sum_i x_i^t \leqq \sum_i \omega_i + \sum_i (y^t_{j0} + y^{t-1}_{j1}) \quad \text{for all} \quad t \geqq 0.$$

Notice that the feasibility of an allocation depends on the endowments y_{j1}^{-1} of the firms. Also, the definition of feasibility implies free disposability.

$((\boldsymbol{x}_i), (\boldsymbol{y}_j))$ is said to be *stationary* if each of the programs $\boldsymbol{x}_i$ and $\boldsymbol{y}_j$ are stationary, and if in addition $y^t_{j1} = y^{-1}_{j1}$ for all t.

The vector $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ should not be confused with $((x_i^t),(y_j^t))$, which is the vector of allocations at time t.

2.7. Pareto optimality

A feasible allocation $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ is said to be Pareto optimal if there exists no feasible allocation $((\bar{\boldsymbol{x}}_i),(\bar{\boldsymbol{y}}_j))$ such that

$$\sum_{t=0}^{\infty} \delta_i^t u_i(\bar{x}_i^t) \geqq \sum_{t=0}^{\infty} \delta_i^t u_i(x_i^t) \quad \text{for all} \quad i,$$

with strict inequality for some i.

2.8. Prices

A *price system* is simply a non-zero vector $\boldsymbol{p}$ of the form $(p^0, p^1, \ldots)$, where $p^t \in R_+^L$ and $\sum_{t=0}^{\infty} p_k^t < \infty$ for $k=1,\ldots,L$. p_k^t is the price of commodity k in period t. $\boldsymbol{p}$ is said to be *stationary* if $p^t = \delta^t p^0$ for all t, where $0<\delta<1$.

If $\boldsymbol{p}$ is a price vector and $\boldsymbol{x}$ is a consumption program, then

$$\boldsymbol{p}\cdot\boldsymbol{x} = \sum_{t=0}^{\infty} p^t \cdot x^t.$$

2.9. Profit maximization

Given the price system $\boldsymbol{p}$, each firm chooses a program so as to maximize its profit. That is, firm j solves the problem

$$\max\left\{\sum_{t=0}^{\infty}(p^t\cdot y_0^t + p^{t+1}\cdot y_1^t) \,\middle|\, \boldsymbol{y} \text{ is a production program feasible for firm } j\right\}.$$

$\eta_j(\boldsymbol{p})$ denotes the set of solutions to this problem, and $\pi_j(\boldsymbol{p})$ denotes the maximum profit plus the value of the firm's initial endowment. That is,

$$\pi_j(\boldsymbol{p}) = p^0\cdot y_{j1}^{-1} + \sum_{t=0}^{\infty}(p^t\cdot y_0^t + p^{t+1}\cdot y_1^t),$$

where $\boldsymbol{y}\in\eta_j(\boldsymbol{p})$. $\eta_j(\boldsymbol{p})$ and $\pi_j(\boldsymbol{p})$ may be empty.

2.10. Utility maximization

Given the price system $\boldsymbol{p}$, consumer i's budget set is

$$\beta_i(\boldsymbol{p})=\left\{\boldsymbol{x} \,\middle|\, \boldsymbol{x} \text{ is a consumption program, and } \boldsymbol{p}\cdot\boldsymbol{x} \leqq \sum_{t=0}^{\infty} p^t\cdot\omega_i+\sum_{j=1}^{J}\theta_{ij}\pi_j(\boldsymbol{p})\right\}.$$

He solves the problem

$$\max\left\{\sum_{t=0}^{\infty}\delta_i^t u_i(x^t) \,\middle|\, \boldsymbol{x}\in\beta_i(\boldsymbol{p})\right\}. \tag{2.1}$$

$\xi_i(\boldsymbol{p})$ denotes the set of solutions to this problem. $\xi_i(\boldsymbol{p})$ may be empty.

2.11. Equilibrium

An *equilibrium* consists of $((\boldsymbol{x}_i),(\boldsymbol{y}_j),\boldsymbol{p})$ such that

$$((\boldsymbol{x}_i),(\boldsymbol{y}_j)) \quad \text{is a feasible allocation,} \tag{2.2}$$

$\boldsymbol{p}$ is a price system, and for all t and k,

$$p_k^t=0 \quad \text{if} \quad \sum_{i=1}^{I} x_k^t<\sum_{i=1}^{I}\omega_{ik}+\sum_{j=1}^{J}(y_{j0k}^t+y_{j1k}^{t-1}). \tag{2.3}$$

$$\boldsymbol{y}_j\in\eta_j(\boldsymbol{p}) \quad \text{for all } j, \tag{2.4}$$

$$\boldsymbol{x}_i\in\xi_i(\boldsymbol{p}) \quad \text{for all } i. \tag{2.5}$$

An equilibrium with transfer payments consists of $((\boldsymbol{x}_i),(\boldsymbol{y}_j),\boldsymbol{p})$ which satisfies conditions (2.2)–(2.4) and, for each i, $\boldsymbol{x}_i$ solves the problem

$$\max\left\{\sum_{t=0}^{\infty}\delta_i^t u_i(x^t) \,\middle|\, \boldsymbol{x}\in l_{\infty,L_c}^{+} \text{and } \boldsymbol{p}\cdot\boldsymbol{x}\leqq\boldsymbol{p}\cdot\boldsymbol{x}_i\right\}. \tag{2.6}$$

The transfer payment made by consumer i is

$$\sum_{t=0}^{\infty} p^t\cdot\omega_i+\sum_{j=1}^{J}\theta_{ij}\pi_j(\boldsymbol{p})-\boldsymbol{p}\cdot\boldsymbol{x}_i.$$

An equilibrium is said to be *stationary* if the allocation $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ and the price system $\boldsymbol{p}$ are all stationary.

2.12. Convergence of allocations

Let $((\bar{\boldsymbol{x}}_i),(\bar{\boldsymbol{y}}_j))$ be a stationary allocation. An allocation $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ is said to converge to $((\bar{\boldsymbol{x}}_i),(\bar{\boldsymbol{y}}_j))$ if

$$\lim_{t\to\infty} |((x_i^t),(y_j^t))-((\bar{x}_i),(\bar{y}_j))|=0,$$

where $\bar{x}_i$ and $\bar{y}_j$ are defined by $\bar{\boldsymbol{x}}_i=(\bar{x}_i,\bar{x}_i,\ldots)$ and $\bar{\boldsymbol{y}}_j=(\bar{y}_j,\bar{y}_j,\ldots)$.

The convergence is said to exponential if there is a such that

$$0<a<1 \quad \text{and} \quad \|((x_i^t),(y_j^t))-((\bar{x}_i),(\bar{y}_j))\|<a^t,$$

for all sufficiently large t.

2.13. Marginal utilities of expenditure

Corresponding to any equilibrium $((\boldsymbol{x}_i),(\boldsymbol{y}_i),\boldsymbol{p})$, there is a vector of marginal utilities of expenditure, $\Lambda=(\Lambda_1,\ldots,\Lambda_I)$. Each Λ_i is simply the Lagrange multiplier corresponding to the budget constraint in consumer i's utility maximization problem (2.1).

3. Assumptions

I list below the assumptions I use. Some have already been mentioned.

3.1. Consumers

$$\omega_i \in R_+^{L_0} \quad \text{for all} \quad i. \tag{3.1}$$

$$u_i: R_+^{L_c}\to(-\infty,\infty) \quad \text{is twice continuously differentiable.} \tag{3.2}$$

Df and D^2f denote the first and second derivatives, respectively, of the function f.

Strict monotonicity. For all i, $Du_i(x)\gg 0$, for all $x\in R_+^{L_c}$. (3.3)

Strict concavity. For all i, $D^2u_i(x)$ is negative definite, for all $x\in R_+^{L_c}$. (3.4)

3.2. Firms

$$y_{j1}^{-1} \in R_{+}^{L_p} \quad \text{for all } j. \tag{3.5}$$

I represent production in the following way:

For all j, $Y_j = \{y \in M_{j0}^{-} \times M_{j1}^{+} \mid g_j(y) \leqq 0\}$,
where M_{j0} and M_{j1} are subspaces of R^L and R^{L_p}, respectively,
where $M_{j0}^{-} = R_{-}^{L} \cap M_{j0}$ and $M_{j1}^{+} = R_{+}^{L_p} \cap M_{j1}$,
and where $g_j: M_{j0}^{-} \times M_{j1}^{+} \to R$. (3.6)

For all j, M_{j0} and M_{j1} are standard subspaces of R^L. (3.7)

For all j, g_j is twice continuously differentiable. (3.8)

For all j, $Dg_j(y) \gg 0$ for all $y \in M_{j0}^{-} \times M_{j1}^{+}$. (3.9)

For all j and for all $y \in M_{j0}^{-} \times M_{j1}^{+}$, $D^2 g_j(y)$ is positive definite on the subspace of $M_{j0} \times M_{j1}$ orthogonal to $Dg_j(y)$. (3.10)

This assumption says that production possibility frontiers have positive curvature. In other words, production possibility sets are differentiably strictly convex.

Possibility of zero production. $g_j(0) = 0$ for all j. (3.11)

Necessity of primary inputs. The following is true, for all j:
Let $y = (y_0, y_1) \in M_{j0} \times M_{j1}$. If $y_1 > 0$ and $y_{0k} = 0$, for all $k \in L_o$, then $g_j(y) > 0$. (3.12)

3.3. Adequacy

The final assumptions guarantee that no consumer would have a zero income in equilibrium.

For each i, $\omega_{ik} > 0$ for some $k \in L_c \cap L_o$. (3.13)

That is, every consumer is endowed with some primary good, such as labor, which is also a consumption good.

$$\sum_{i=1}^{I} \omega_{ik} > 0 \quad \text{for all} \quad k \in L_o. \tag{3.14}$$

That is, there is a positive endowment of every primary good.

$$\text{There are} \quad \bar{\omega} \in R_{+}^{L_o} \quad \text{and} \quad (y_{j0}, y_{j1}) \in Y_j \quad \text{for} \quad j = 1, \ldots, J,$$

$$\text{such that} \quad \bar{\omega} + \sum_{j=1}^{J} (y_{j0} + y_{j1}) \gg 0. \tag{3.15}$$

That is, it is possible to produce some of every good in every period while using only primary inputs from outside the production system.

4. Theorems

I assume that assumptions (3.1)–(3.15) apply.

Theorem. *Suppose that* $\sum_{j=1}^{J} y_{j1k}^{-1} > 0$ *for all* $k \in L_p$. *Then there exists an equilbrium.* (4.1)

Theorem. *Suppose that* $\delta_i = \delta$ *for all* i. *If* δ *is sufficiently close to one, then the vector* (y_{j1}^{-1}) *may be chosen so that an equilibrium exists which is stationary. The equilibrium price vector is of the form* $\boldsymbol{p} = (p, \delta p, \delta^2 p, \ldots)$. (4.2)

Theorem. *Any equilibrium allocation is Pareto optimal.* (4.3)

Theorem. *Let* $((\boldsymbol{x}_i), (\boldsymbol{y}_j), \boldsymbol{p})$ *be a competitive equilibrium. If* n *is such that* $\delta_n < \max_i \delta_i$, *then* $x_n^t = 0$ *for* t *sufficiently large.* (4.4)

For the turnpike theorem, I need the following assumption:

Interiority assumption. There exists $\zeta > 0$ and $\underline{\delta}$ such that $0 < \underline{\delta} < 1$ and the following are true. If $\delta_i = \delta$ for all i, where $\underline{\delta} < \delta < 1$ and if $((\boldsymbol{x}_i), (\boldsymbol{y}_j), \boldsymbol{p})$ is a stationary equilibrium with transfer payments, then

$$\sum_{j} y_{j1k}^{-1} > \zeta \quad \text{for all} \quad k \in L_p.$$

Theorem. The turnpike property. Suppose that the interiority assumption is satisfied. Suppose also that $\sum_j y_{j1k}^{-1} > 0$ *for all* $k \in L_p$, *and that* $\delta_i = \delta$ *for all i. If* δ *is sufficiently close to one, then the following is true. If* $((x_i),$ $(y_j), p)$ *is a competitive equilibrium, then* $((x_i), (y_j))$ *converges exponentially to a stationary allocation* $((\bar{x}_i), (\bar{y}))$. (4.5)

5. Discussion of assumptions

All my assumptions are more or less standard in equilibrium theory, except for assumption (3.7) — no fixed coefficients in production, assumption (3.10) — strict convexity in production, and the interiority assumption. I now discuss what is wrong with these assumptions.

Assumption (3.10) excludes constant returns to scale. Constant returns to scale is a very natural assumption to make. Production possibility sets really describe production processes, not firms. There is no compelling reason to keep the number of firms fixed. In fact, one imagines that firms can be replicated. This possibility is one justification for assuming constant returns to scale. All these considerations are especially persuasive in the context of growth theory, where one thinks in terms of a very long run.

The interiority assumption is traditional in turnpike theory. It would be better to replace this assumption by assumptions about preferences and technology. It is, no doubt, possible to do so, but I have not found a convincing set of assumptions which do not lead to an excessively complicated proof.

Assumption (3.7) is especially awkward. It makes it impossible to represent the use of capital equipment in production. The conventional representation is as follows. One labels equipment according to age. A production process using a machine transforms the machine and other inputs into an older machine and other outputs. The process transforms one younger machine into one older machine. A fixed coefficient of one is unavoidable.

6. Relation to the literature

This paper links Arrow–Debreu equilibrium theory with capital theory. The equilibrium theory I use is that for an economy with infinitely many commodities. The extension of equilibrium theory to such economies was made by Debreu (1954), Peleg and Yaari (1970), myself (1972) and Stigum (1972, 1973). Debreu proved that equilibria in such economies are Pareto optimal and that Pareto optima may be realized as equilibria with transfer payments. Peleg and Yaari, Stigum and I proved that equilibria exist.

Capital theory has a long history. McKenzie (1979) has written an excellent up to date survey of turnpike theory.

The turnpike theory existing in the literature, deals with models which have only one utility function and one production possibility set. I simply introduce many firms, many consumers and budget constraints for consumers. The turnpike theorem I prove is an analogue of that of Scheinkman (1976, theorem 3, p. 28).

In my proof, I use a variation of the value loss method. I believe that this method traces back to the work of Radner (1961). It has since been improved by Atsumi (1965), Brock (1970), McKenzie (1974, 1976), Cass and Shell (1976), Rockafellar (1976), Brock and Scheinkman (1976), and Magill (1977).

The most recent turnpike theorem of the type I prove is contained in McKenzie (1979, theorem 10′). His proof builds on that of Scheinkman and uses methods developed in the list of papers just cited.

It is hard to compare McKenzie's theorem with my own, since our models are so different. In order to clarify the connection between his work and my own, I show how to derive from my model the reduced form used by McKenzie. This reduced form model is the one typically used in turnpike theory.

Suppose that in my model there are one consumer and one firm. The utility function of the consumer is $u: R_+^{L_c} \to (-\infty, \infty)$. His initial endowment is ω. The production possibility set of the firm is $Y \subset R_-^L \times R_+^{L_p}$. Let

$$D = \{(K^0, K^1) \in R_+^{L_p} \times R_+^{L_p} \mid \text{there exists } x \in R_+^{L_c} \text{ such that } (x - \omega - K^0, K^1) \in Y\}.$$

Let $v: D \to (-\infty, \infty)$ be defined by

$$v(K^0, K^1) = \max\{u(x) \mid x \in R_+^{L_c} \quad \text{and} \quad (x - \omega - K^0, K^1) \in Y\}.$$

McKenzie's economy is defined by D and v.

The key concavity assumption of McKenzie is stated in terms of the concavity of v. But the concavity of v depends on the properties of both u and Y in a complicated way. Some of the long arguments in my proof of the turnpike theorem may be interpreted as proving that v is concave. Benhabib and Nishimura (1981, section 3.3, remark a) have already pointed out that the concavity of v requires very strong assumptions about underlying production relations. They work with a continuous time model.

I now return to the comparison of McKenzie's theorem with my own. My theorem is more general in that I prove exponential convergence and he does not. McKenzie's theorem is more general than mine in that he makes only a local strict concavity assumption. I assume strict concavity or convexity everywhere. McKenzie's assumption is that the Hessian of v is negative definite at $(\bar{K}, \bar{K})$, where $(\bar{K}, \bar{K})$ satisfies

$$v(\bar{K}, \bar{K}) = \max\{v(K, K) \mid (K, K) \in D\}.$$

($\bar{K}$ is the vector of golden rule capital stocks.) The other differences between McKenzie's theorem and my own are of no great interest.

Araujo and Scheinkman (1977) prove a turnpike theorem with exponential convergence [theorem (3.2)]. They assume that a certain infinite-dimensional matrix has the dominant diagonal property. I do not see that this condition necessarily applies in my case. For this reason, I did not use their result.

Remark. Yano (1980) has generalized the turnpike theorem of this paper. He assumed constant returns to scale while assuming that the von Neumann facet containing the golden rule input–output vector is a single ray.

I now turn to my result that there exists a stationary equilibrium [theorem (4.2)]. When there is only one firm and one consumer, stationary equilibrium becomes what is known as the modified golden rule. Therefore, the proof of theorem (4.2) provides a way to prove the existence of a modified golden rule. Brock has already pointed out that the modified golden rule is a competitive equilibrium with distortions, in section 2.3 of this paper of 1973.

Sutherland (1970) and Peleg and Ryder (1974) proved that a modified golden rule exists. They used fixed point arguments. I do so as well, but my argument is simply a modification of the usual argument which proves that a general equilibrium model has an equilibrium. Thus, I clarify the tie between general equilibrium theory and the work of Sutherland, Peleg and Ryder. My proof is a variation of one given in a previous paper (1979) of my own. In that paper, there is uncertainty and there is no discounting of future utility.

The idea expressed by my theorem (4.4) has already appeared in the literature. This theorem asserts that less patient consumers eventually consume nothing. Ramsey (1928, pp. 558–559) pointed out that in long-run or stationary equilibrium, those consumers with the highest rate of time preference would live at a subsistence level. Rader (1971, chapter I) makes an argument similar to my own. A related idea appears in the last section of one of his recent paper [Rader (1979)]. Finally, Becker (1980) proved an assertion similar to Ramsey's.

Becker studies capital theory using a disaggregated model, just as I do. He proves the existence of an equilibrium [Becker (1978, chapter 4)] and the existence of a stationary equilibrium [Becker (1980)]. Becker's work differs from mine in that his model is one of temporary equilibrium, not of general equilibrium. Consumers can sell or accumulate capital but they cannot borrow.

7. Some lemmas

The boundedness of feasible allocations is expressed by the following two lemmas.

Lemma. *Let (y_{j1}^{-1}) be given. There is $B>0$ such that if $((x_i), (y_j))$ is a feasible allocation with initial resources $\sum_j y_{j1}^{-1}$, then all its components are bounded in absolute value by B.* (7.1)

Lemma. *Let (y_{j1}^{-1}) be variable. There is $B>0$ such that if $((x_i),(y_j))$ is a feasible stationary allocation, then all its components are bounded in absolute value by B.* (7.2)

The proofs of these lemmas are completely routine and are not given. They make use of the facts that production is impossible without primary goods and that production sets are closed and convex.

In proving theorem (4.1), it is easier to deal with an economy in which production possibility sets are cones. For this reason, I now modify the economy in order to obtain an equivalent economy $\mathscr{E}^*$ in which all production possibility sets are cones. $\mathscr{E}^*$ will also be such that free disposability is incorporated in the production process rather than in the definition of feasibility.

I introduce one new factor of production for each firm. The jth such factor can be used only by firm j. This factor may be thought of as the entrepreneurial factor. McKenzie (1959) has suggested introducing such a factor in just the way I do.

More explicitly, I introduce J new commodities, so that the list of commodities in $\mathscr{E}^*$ is $L\cup\{1,\dots,J\}$. The commodity space of $\mathscr{E}^*$ is $R^L\times R^J$. An input vector for a firm may be written as (y_0, y_0^*), where $y_0\in R^L_-$ and $y_0^*\in R^J_-$. Let $e_j=(0,\dots,0,1,0,\dots,0)$ be the jth standard basis vector of R^J. The production possibility set of the jth firm in $\mathscr{E}^*$ is

$$Y_j^*=\{t(y_0,-e_j,y_1)\,|\,(y_0,y_1)\in Y_j,\ t\geqq 0,\ \text{and every component of } (y_0,y_1) \text{ is bounded in absolute value by } B\},$$

where B is as in lemma (7.1).

I let the endowment of the ith consumer be

$$\omega_i^*=(\omega_i,0)+\sum_{j=1}^{J}\theta_{ij}(0,e_j)\in R^L_+\times R^J_+.$$

Notice that I have introduced one unit of factor j into the economy, for each j. The consumption set of every consumer is still $R^{L_c}_+$ and the utility function of consumer i is still $u_i\colon R^{L_c}_+\to(-\infty,\infty)$.

Finally, I introduce an extra firm, firm $J+1$, which disposes of goods. The production possibility set of firm $J+1$ is

$$Y_{J+1}^{*}=R_{-}^{L}\times R_{-}^{J}\times\{0\}\subset R^{L+J}\times R^{L_{p}}.$$

Let $\theta_{i,J+1}=I^{-1}$, for all i. The initial endowment of firm $J+1$ is $y_{J+1,1}^{-1}=0$.

In summary, the economy $\mathscr{E}^{*}$ is

$$\{(u_i,\delta_i,\omega_i^{*}),(y_{j1}^{-1},Y_j^{*}),\theta_{ij}:\ i=1,\dots,I,\ j=1,\dots,J+1\}.$$

An equilibrium for $\mathscr{E}^{*}$ is defined in the obvious way. It is easy to see the following.

Lemma. There is a one to one correspondence between equilibria for $\mathscr{E}$ and for $\mathscr{E}^{*}$. (7.3)

8. Proof of theorem (4.1)

I prove this theorem by applying results from a previous paper (1972) on the existence of equilibrium when there are infinitely many commodities. The appropriate economy with infinitely many commodities is $\mathscr{E}^{**}$, defined as follows. The commodity space is

$$l_{\infty}=\left\{x=(x^{0},x^{1},\dots)\,\middle|\,x^{t}\in R^{L+J}\quad\text{and}\quad\sup_{t,k}|x_k^t|<\infty\right\}.$$

The consumption set of each consumer is

$$X=\{x\in l_{\infty}\,|\,x_k^t\geqq 0,\ \text{all } t \text{ and } k,\ \text{and } x_k^t=0 \text{ if } k\notin L_c\}.$$

The utility function of the ith consumer is

$$U_i(x)=\sum_{t=0}^{\infty}\delta_i^t u_i(x^t).$$

The initial endowment of the ith consumer, $\omega_i^{**}=(\omega_i^{**0},\omega_i^{**1},\dots)$, is defined by

$$\omega_i^{**0}=\omega_i^{*}+\sum_{j=1}^{J}\theta_{ij}(y_{j1}^{-1},0)\in R^{L}\times R^{J}\quad\text{and}\quad\omega_i^{**t}=\omega_i^{*}\quad\text{for}\quad t>0.$$

Notice that the firms' initial endowments have been transferred to the consumers.

The production possibility set of firm j is

$$Y_j^{**}=\{y=(y^{0},y^{1},\dots)\in l_{\infty}\,|\,\text{there exist } (z_0^t,z_1^t)\in Y_j^{*},\ \text{for } t=0,1,\dots,\ \text{such that } y^{0}=z_0^{0} \text{ and } y^{t}=z_0^{t}+z_1^{t-1} \text{ for } t>0\}.$$

In summary, the economy $\mathscr{E}^{**}$ is

$$\{(X, U_i, \omega_i^{**}),\ Y_j^{**},\ \theta_{ij} \mid i=1,\dots,I,\ j=1,\dots,J+1\}.$$

Price systems for $\mathscr{E}^{*}$ are non-zero vectors in

$$l_1^+ = \left\{ \boldsymbol{x}=(x^0, x^1,\dots) \mid x^t \in R_+^{L+J} \text{ and } \sum_{t=0}^{\infty} x_k^t < \infty, \text{ for all } k \right\}.$$

$\boldsymbol{p} \in l_1^+$ is written as $\boldsymbol{p}=(p^0, p^1,\dots)$, where $p^t \in R_+^L \times R_+^J$ for all t.

An equilibrium for $\mathscr{E}^{**}$ is defined in the obvious way. It should be clear that an equilibrium for $\mathscr{E}^{**}$ may be interpreted as an equilibrium for $\mathscr{E}^{*}$. Hence by lemma (7.3), it is sufficient to prove that $\mathscr{E}^{**}$ has an equilibrium. That $\mathscr{E}^{**}$ has an equilibrium follows from theorems 1 and 3 of my previous paper (1972). Some routine arguments are needed to prove that $\mathscr{E}^{**}$ satisfies the assumptions of that paper. In order to save space, these arguments are not given here [they do appear in my paper of (1980)]. Q.E.D.

9. Proof of theorem (4.2)

I prove the existence of what I call a δ-equilibrium for a two-period economy, $\mathscr{E}^0$, where δ is the discount factor applied to future utility. It will be easy to see that a δ-equilibrium for $\mathscr{E}^0$ corresponds to a stationary equilibrium for $\mathscr{E}$. In $\mathscr{E}^0$, consumption takes place in the first period. Firms $j=1,\dots,J$ use inputs in the first period in order to produce outputs in the second period. An artificial firm, firm zero, transfers goods from the second period back to the first. Firms j, for $j=1,\dots,J$, are subject to a sales tax of $1-\delta$ times the value of their output. This tax is paid to consumers according to the shares θ_{ij}. Firm zero pays no tax. The tax embodies the distortion caused by discounting future utility.

I now define $\mathscr{E}^0$ precisely. The commodity space of $\mathscr{E}^0$ is $R^L \times R^{L_p}$. The production set of firm zero is

$$Y_0 = \{(y_0, y_1) \in R^L \times R^{L_p} \mid y_0 = -y_1\}.$$

The production set of firm j, for $j=1,\dots,J$, is simply Y_j. The consumption set of each consumer is

$$X = R_+^{L_c} \times \{0\} \subset R^L \times R^{L_p}.$$

The utility function of consumer i is

$$u_i^0(x, 0) = u_i(x).$$

His initial endowment is

$$\omega_i^0 = (\omega_i, 0) \in R^L \times R^{L_p}.$$

The profit shares, θ_{ij}, are as before, for $j = 1, \ldots, J$. $\theta_{i0} = I^{-1}$ for all i. Formally,

$$\mathscr{E}^0 = \{(X, u_i^0, \omega_i^0), Y_j, \theta_{ij} : i = 1, \ldots, I,\ j = 0, 1, \ldots, J\}.$$

An allocation for $\mathscr{E}^0$ will be written as $((x_i^0), (y_j))$, where $x_i^0 = (x_i, 0) \in X$ for all i, and $y_j = (y_{j0}, y_{j1}) \in Y_j$ for $j = 0, 1, \ldots, J$. $((x_i^0), (y_j))$ is feasible if

$$\sum_i x_i^0 \leqq \sum_i \omega_i^0 + \sum_{j=0}^{J} y_j.$$

Price systems for $\mathscr{E}^0$ belong to

$$\Delta = \left\{ p = (p_0, p_1) \in R_+^L \times R_+^{L_p} \,\middle|\, \sum_{k \in L} p_{0k} + \sum_{k \in L_p} p_{ik} = 1 \right\}.$$

If $p \in \Delta$, then $p_0 \in R^L$ and $p_1 \in R^{L_p}$ always denote the component vectors of p.

Given $p \in \Delta$, the maximization problem of firm zero is simply

$$\max \{ p_0 \cdot y_0 + p_1 \cdot y_1 \,|\, (y_0, y_1) \in Y_0 \}.$$

$\eta_0^0(p)$ denotes the set of solutions of this problem. For $j = 1, \ldots, J$, the maximization problem of firm j is

$$\max \{ p_0 \cdot y_0 + \delta p_1 \cdot y_1 \,|\, (y_0, y_1) \in Y_j \}.$$

$\eta_j^0(p)$ denotes the set of solutions of this problem. Notice that for $j \geqq 1$, firm j maximizes his after tax profits, the tax being $(1 - \delta) p_1 \cdot y_1$.

If $j \geqq 1$, the tax paid by firm j to consumer i is $\theta_{ij}(1 - \delta) p_1 \cdot y_1$, where $(y_0, y_1) \in \eta_j^0(p)$. Hence, the income of consumer i, given $p \in \Delta$, is

$$w_i(p) = p_0 \cdot \omega_i + \sum_{j=0}^{J} \theta_{ij}(p_0 \cdot y_{j0} + p_1 \cdot y_{j1}),$$

where $(y_{j0}, y_{j1}) \in \eta_j^0(p_1)$ for $j = 0, 1, \ldots, J$. $w_i(p)$ is well-defined provided that $p_0 \cdot y_{j0} + p_1 \cdot y_{j1}$ is independent of $(y_{j0}, y_{j1}) \in \eta_j^0(p)$ for $j = 1, \ldots, J$.

The maximization problem of consumer i, given $p \in \Delta$, is

$$\max \{ u_i^0(x^0) \,|\, x^0 \in X \quad \text{and} \quad p \cdot x^0 \leqq w_i(p) \}.$$

$\xi_i^0(p)$ denotes the set of solutions to this problem, for $i = 1, \ldots, I$.

A δ-equilibrium for $\mathscr{E}^0$ is of the form $((x_i^0), (y_j), p)$, where $((x_i^0), (y_j))$ is a feasible allocation for $\mathscr{E}^0$;

$$p \in \Delta, \quad p_{0k}=0 \quad \text{if} \quad \sum_i x_{ik} < \sum_i \omega_{ik} + \sum_{j=0}^{J} y_{j0k},$$

and

$$p_{1k}=0 \quad \text{if} \quad 0 < \sum_{j=0}^{J} y_{j1k};$$

$y_j \in \eta_j^0(p)$ for $j=0,1,\ldots,J$, and $x_i^0 \in \xi_i^0(p)$ for all i.

A stationary equilibrium for $\mathscr{E}$ corresponds to every δ-equilibrium for $\mathscr{E}^0$. Let $((x_i^0), (y_j), p)$ be a δ-equilibrium for $\mathscr{E}^0$, where $x_i^0=(x_i, 0)$ and $y_j=(y_{j0}, y_{j1})$ for all i and j. Let $\boldsymbol{x}_i=(x_i, x_i, \ldots.)$ and let $\boldsymbol{y}_j=(y_j, y_j, \ldots)$ for all i and j. Finally, let $\boldsymbol{p}=(p_0, \delta p_0, \delta^2 p_0, \ldots)$. It is not hard to show that $((\boldsymbol{x}_i)_{i=1}^I, (\boldsymbol{y}_j)_{j=1}^J, \boldsymbol{p})$ is a stationary equilibrium for $\mathscr{E}$. Hence, theorem 4.2 is true provided $\mathscr{E}^0$ has an equilibrium. The proof that $\mathscr{E}^0$ does have an equilibrium is standard and may be done by imitating the arguments Debreu (1959, chapter 5). Q.E.D.

10. Proof of theorem (4.3)

I do not give a detailed proof of this theorem, since the proof is completely routine. One approach is as follows. To any equilibrium for $\mathscr{E}$, there corresponds an equilibrium for the economy $\mathscr{E}^{**}$ defined in section 8. A theorem of Debreu (1954, theorem 1, p. 589) implies that the $\mathscr{E}^{**}$-equilibrium allocation is Pareto optimal among feasible allocations for $\mathscr{E}^{**}$. It follows at once that the corresponding equilibrium allocation for $\mathscr{E}$ is Pareto optimal.

11. Proof of theorem (4.4)

Let $((\boldsymbol{x}_i), (\boldsymbol{y}_j), \boldsymbol{p})$ be an equilibrium and let $\Lambda_i > 0$ be the marginal utility of expenditure for consumer i in the equilibrium. Suppose that p is so normalized that $\sum_i \Lambda_i = 1$. Let B be as in lemma (7.1). Then, $|x_i^t| \leqq B$ for all i and t. By assumptions (3.2) and (3.3), $Du_i(x)$ is a continuous function of x with positive components. Therefore, there exist positive numbers a and b such that $a \leqq \partial u_i(x)/\partial x_k \leqq b$ for all i and k, if $|x| \leqq B$. Let $\delta = \max_i \delta_i$.

Let i be such that $\delta_i = \delta$. Then, for any t and k,

$$\delta^{-t} p_k^t \geqq \Lambda_i \delta^{-t} p_k^t \geqq \partial u_i(x_i^t)/\partial x_k \geqq a.$$

Now suppose that i is such that $\delta_i < \delta$. If $x_{ik}^t > 0$, then

$$b \geqq \partial u_i(x_i^t)/\partial x_k = \Lambda_i \delta_i^{-t} p_k^t \geqq \Lambda_i (\delta_i^{-1} \delta)^t a.$$

Let T be such that $b < \Lambda_i(\delta_i^{-1}\delta)^T a$. Then if $t \geqq T$, it must be that $x_i^t = 0$. Q.E.D.

12. Proof of theorem (4.5)

Let $((\hat{x}_i), (\hat{y}_j), \hat{p})$ be a competitive equilibrium for $\mathscr{E}$. I first define the allocation to which $((\hat{x}_i), (\hat{y}_j))$ converges. For each i, let $\Lambda_i > 0$ be the marginal utility of expenditure for consumer i in the equilibrium $((\hat{x}_i), (\hat{y}_j), \hat{p})$. This marginal utility was defined in section 2. I assume that $\hat{p}$ is so normalized that $\sum_i \Lambda_i = 1$. Let $U: R_+^{L_c} \to (-\infty, \infty)$ be defined by

$$U(x) = \max\left\{\sum_i \Lambda_i^{-1} u_i(x_i) \,\middle|\, x_i \in R_+^{L_c} \text{ for all } i, \text{ and } \sum_i x_i = x\right\}.$$

Let $\mathscr{E}'$ be the economy obtained from $\mathscr{E}$ by replacing all the consumers with a single consumer whose utility function is U and whose initial endowment is $\omega = \sum_i \omega_i$. By a slight modification of theorem (4.2), $\mathscr{E}'$ has a stationary equilibrium $(\bar{x}, (\bar{y}_j), \bar{p})$. I will assume that $\bar{p}$ is so normalized tha the marginal utility of expenditure of the single consumer is one. Let $(\bar{x}_i)$ be such that

$$\bar{x} = \sum \bar{x}_i \quad \text{and} \quad U(\bar{x}) = \sum \Lambda_i^{-1} u(\bar{x}_i).$$

It is easy to see that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is a stationary equilibrium for $\mathscr{E}$ with transfer payments. In this equilibrium, the marginal utility of expenditure for each consumer i is Λ_i. $((\bar{x}_i), (\bar{y}_j))$ is the stationary allocation to which $((\hat{x}_i), (\hat{y}_j))$ converges.

The proof that $((\hat{x}_i), (\hat{y}_j))$ converges to $((\bar{x}_i), (\bar{y}_j))$ uses the fact that these allocations solve related maximization problems. The set of feasible allocations for $\mathscr{E}$ depends on the initial holdings, $\sum_j y_{j1}^{-1}$, of produced goods in the economy. Think of these initial stocks as a variable. This variable is denoted by K, where $K \in R_+^{L_p}$. For each value of K, let $\mathscr{F}(K)$ be the set of feasible allocations $((x_i), (y_j))$ for $\mathscr{E}$ such that

$$\sum_i x_i^0 \leqq \sum_i \omega_i + \sum_j y_{j0}^0 + K.$$

The relevant maximization problem is the following:

$$\max\left\{\sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1} u_i(x_i^t) \,\middle|\, ((x_i), (y_j)) \in \mathscr{F}(K)\right\}. \tag{12.1}$$

The stationary allocation $((\bar{x}_i, (\bar{y}_j))$ solves this problem with initial resources $\bar{K}$

$=\sum_j \bar{y}_{j1}$. The allocation $((\hat{x}_i),(\hat{y}_j))$ solves this problem with initial resources $\hat{K}^0=\sum_j y_{j1}^{-1}$. These assertions may be proved as follows. Because $((\bar{x}_i),(\bar{y}_j),\bar{p})$ and $((\hat{x}_i),(\hat{y}_j),\hat{p})$ are both equilibria with marginal utilities of expenditure $\Lambda_1,\ldots,\Lambda_I$, they solve the first-order conditions for solutions of (12.1). Because the constraints are convex and the objective function is concave, any solution of the first-order conditions is an optimum.

Problem (12.1) is a variant of the maximization problem traditional in growth theory. I now simply adapt the well-known proofs of the turnpike theorem to the situation here.

First, I define an appropriate Liapunov function. For $K\in R_+^{L_P}$, let

$$V_\delta(K)=\sum_{t=0}^{\infty}\delta^t\sum_i \Lambda_i^{-1}[u_i(x_i^t)-u_i(\bar{x}_i)],$$

where $((x_i),(y_j))$ is a solution to problem (12.1) with initial stocks K. [Of course, $V_\delta(K)$ exists only if (12.1) has a solution with initial stocks K.] Recall that $\bar{p}$ is of the form $\bar{p}=(\bar{p},\delta\bar{p},\delta^2\bar{p},\ldots)$. Let $F_\delta(K)=\bar{p}\cdot(K-\bar{K})-V_\delta(K)$. F_δ is the Liapunov function I will use. The diagram presented in fig. 1 may help one visualize F_δ.

I now turn to a few technical matters. A series of lemmas then follow which establish properties of F_δ. The actual proof of convergence is contained in the last few paragraphs.

Lemma. There exist $\lambda>0$ *such that* $\Lambda_i>\lambda$ *for all* i, *no matter what the value of* δ. (12.2)

Proof. As in the proof of theorem (4.4), there exist numbers a and b such that $a\leqq\partial u_i(x_i^t)/\partial x_k\leqq b$ for all i, t and k.

By the definition of Λ_i, $\delta^t\partial u_i(x_i^t)/\partial x_k\leqq\Lambda_i p_k^t$, with equality if $x_{ik}^t>0$ for all i, k and t. Since $\sum_i\Lambda_i=1$, there is i such that $\Lambda_i\geqq I^{-1}$. At the end of the proof of theorem (4.1), I noted that the income of every consumer is positive in equilibrium. Therefore, $x_{ik}^t>0$ for some t and k, so that

$$b\geqq\partial u_i(x_i^t)/\partial x_k=\Lambda_i\delta^{-t}p_k^t\geqq I^{-1}\delta^{-t}p_k^t.$$

That is, $\delta^{-t}p_k^t\leqq bI$. It now follows that for the same value of t and for any $n=1,\ldots,I$,

$$a\leqq\partial u_n(x_n^t)/\partial x_k\leqq\Lambda_n\delta^{-t}p_k^t\leqq\Lambda_n bI.$$

In conclusion, $\Lambda_n\geqq ab^{-1}I^{-1}$. Q.E.D.

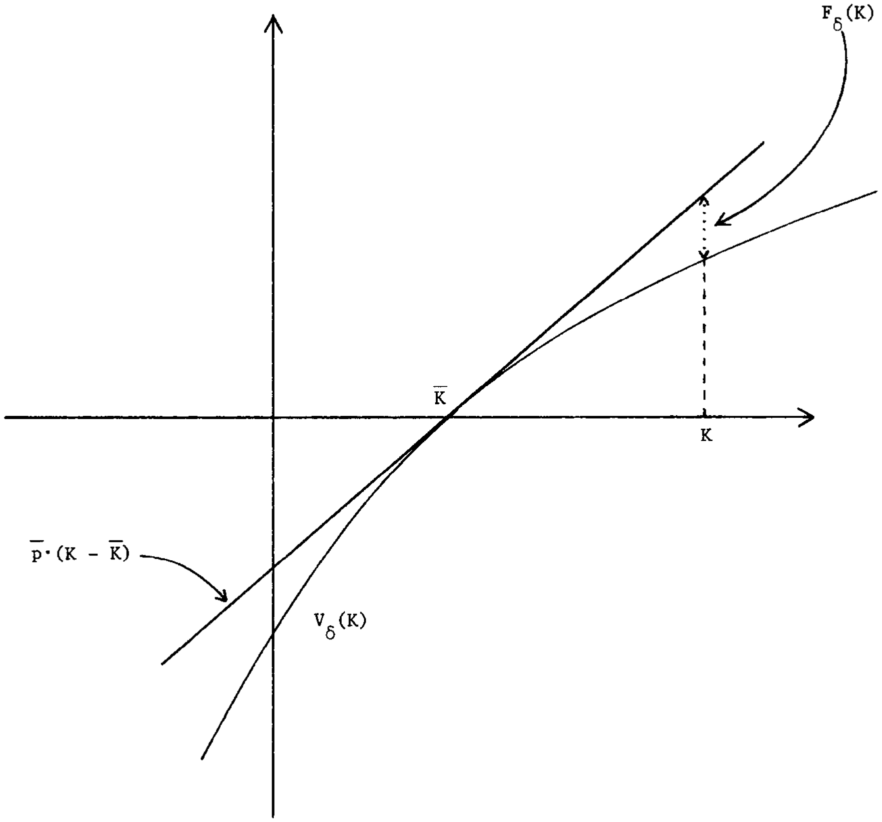

Fig. 1

I next show that

$$\text{if } ((\boldsymbol{x}_i), (\boldsymbol{y}_j), \boldsymbol{p}) \text{ is any equilibrium for } \mathscr{E}, \text{ then}$$

$$p^t \gg 0 \quad \text{and} \quad \sum_i x_i^t = \sum_i \omega_i + \sum_j (y_{j0}^t + y_{j1}^t) \quad \text{for all } t. \tag{12.3}$$

By the definition of an equilibrium, it is enough to prove that $p^t \gg 0$ for all t. By the monotonicity of preferences [assumption (3.3)] all consumption goods have positive price. Define a good to be *productive* if either it is a consumption good or may be used directly or indirectly to produce consumption goods. Assumptions (3.7) and (3.8) imply that this definition makes sense. Since all consumption goods have positive price, all productive goods have positive price. The interiority assumption implies that all goods are productive. This proves (12.3).

I now prove the following:

There exist numbers $\underline{q}$ and $\bar{q}$, *such that* $0<\underline{q}\leqq\bar{p}_k\leqq\bar{q}$ for all k, if $\delta\geqq\underline{\delta}$ where $\underline{\delta}$ is as in the interiority assumption. (12.4)

If (12.4) were not true, then a compactness argument would imply that there is a stationary equilibrium with some price equal to zero, which is impossible by (12.3). The compactness argument makes use of lemma (7.2), which asserts that stationary allocations are uniformly bounded.

I now turn to the properties of the Liapunov function F_δ. The next lemma says that fig. 1 is correct. It says that $\bar{p}$ is a subgradient of V at $\bar{K}$.

Lemma. *If $F_\delta(K)$ is well-defined, then* $F_\delta(K)\geqq 0=F_\delta(\bar{K})$. (12.5)

Proof. It is obvious that $F_\delta(\bar{K})=0$.

Let $((x_i),(y_j))\in\mathscr{F}(K)$. I must show that

$$\sum_{t=0}^{\infty}\delta^t\sum_i \Lambda_i^{-1}(u_i(x_i^t)-u_i(\bar{x}_i))\leqq\bar{p}(K-\bar{K}). \tag{12.6}$$

Choose (y_{j1}^{-1}) arbitrarily so that $\sum_j y_{j1}^{-1}=K$. In order to see that the equation below is true, cancel terms and use the fact that

$$\sum_i \omega_i=\sum_i \bar{x}_i-\sum_j(\bar{y}_{j0}+\bar{y}_{j1}).$$

$$\bar{p}\cdot(K-\bar{K})-\sum_{t=0}^{\infty}\delta^t\sum_i \Lambda_i^{-1}(u_i(x_i^t)-u_i(\bar{x}_i))$$

$$=\sum_{t=0}^{\infty}\delta^t\sum_i[(\Lambda_i^{-1}u_i(\bar{x}_i)-\bar{p}\cdot\bar{x}_i)-(\Lambda_i^{-1}u_i(x_i^t)-\bar{p}\cdot x_i^t)]$$

$$+\sum_{t=0}^{\infty}\delta^t\bar{p}\cdot\left[\sum_j(y_{j0}^t+y_{j1}^{t-1})+\sum_i(\omega_i-x_i^t)\right]$$

$$+\sum_{t=0}^{\infty}\delta^t\sum_j[(\bar{p}\cdot\bar{y}_{j0}+\delta\bar{p}\cdot\bar{y}_{j1})-(\bar{p}\cdot y_{j0}^t+\delta\bar{p}\cdot y_{j1}^t)]. \tag{12.7}$$

It is convenient to write the right-hand side of the above as $S_1+S_2+S_3$, where S_i is the ith infinite sum.

Clearly, $\bar{x}_i$ maximizes the function $\Lambda_i^{-1}u_i(x)-\bar{p}\cdot x$, so that $S_1 \geqq 0$. Since $((x_i), (y_j))$ is feasible, $S_2 \geqq 0$. By profit maximization in the equilibrium $((\bar{x}_i), (\bar{y}_j), \bar{p})$, $S_3 \geqq 0$. Q.E.D.

Recall that $\hat{K}^0$ is the vector of initial resources associated with the equilibrium $((\hat{x}_i), (y_j), \hat{p})$. For $t>0$, let $\hat{K}^t = \sum_j \hat{y}_{j1}^{t-1}$. By (12.3)

$$\hat{K}^t = \sum_i (\hat{x}_i - \omega_i) - \sum_j \hat{y}_{j0}^t.$$

Lemma. $\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t)$

$$= \sum_i [(\Lambda_i^{-1}u_i(\hat{x}_i^t) - \bar{p}\cdot\hat{x}_i^t) - (\Lambda_i^{-1}u_i(\bar{x}_i) - \bar{p}\cdot\bar{x}_i)]$$

$$+ \sum_j [(\bar{p}\cdot\hat{y}_{j0}^t + \delta\bar{p}\cdot\hat{y}_{j1}^t) - (\bar{p}\cdot\bar{y}_{j0} + \delta\bar{p}\cdot\bar{y}_{j1})], \tag{12.8}$$

for all $t \geqq 0$.

Proof. By the definition of F_δ,

$$\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^{t+1})$$

$$= \delta p\cdot(\hat{K}^{t+1} - \bar{K}) - p\cdot(\hat{K}^t - \bar{K}) + V_\delta(\hat{K}^t) - \delta V_\delta(\hat{K}^{t+1}). \tag{12.9}$$

Clearly,

$$V_\delta(\hat{K}^t) = \sum_{n=t}^{\infty} \delta^{n-t} \sum_i \Lambda_i^{-1}(u_i(\hat{x}_i^n) - u_i(\bar{x}_i)).$$

Substituting this into (12.9) and rearranging, I obtain

$$\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t)$$

$$= (\delta\bar{p}\cdot\hat{K}^{t+1} - \bar{p}\cdot\hat{K}^t) - (\delta\bar{p}\cdot\bar{K} - \bar{p}\cdot\bar{K}) + \sum_i \Lambda_i^{-1}(u_i(\hat{x}_i^t) - u_i(\bar{x}_i)).$$

If one substitutes $\sum_i(\hat{x}_i^t - \omega_i) - \sum_j \hat{y}_{j0}^t$ for $\hat{K}^t$, $\sum_j \hat{y}_{j1}^t$ for $\hat{K}^{t+1}$, $\sum_j \bar{y}_{j1}$ for the first $\bar{K}$, and $\sum_i(\bar{x}_i - \omega_i) - \sum_j \bar{y}_{j0}$ for the second $\bar{K}$, and rearranges terms, one obtains the lemma. Q.E.D.

In what follows, $|\cdot|$ will denote the maximum norm. That is, if v is a vector, then $|v| = \max_k |v_k|$.

Lemma. *There exist* $\alpha > 0$ *and* $\varepsilon > 0$, *such that*

$$\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t) \leqq -\alpha \min[\varepsilon^2, |((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2],$$

provided $\delta \geqq \underline{\delta}$. (12.10)

This lemma follows in a routine way from the previous one and from the differential concavity of utility functions and production possibility sets [assumptions (3.4) and (3.10)] and from the boundedness of feasible stationary allocations [lemma (7.2)].

Lemmas (12.5) and (12.10) imply the following:

$$F_\delta(\hat{K}^t) \geqq \alpha \min(\varepsilon^2, |((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2). \tag{12.11}$$

Hence, in order to demonstrate that $((\hat{x}_i), (\hat{y}_j))$ converges to $((\bar{x}_i), (\bar{y}_j))$ exponentially, it is sufficient to prove that $F_\delta(\hat{K}^t)$ converges to zero exponentially.

The next lemma is simply a corollary of the previous one:

Lemma. *There exist* $\alpha > 0$ *and* $\varepsilon > 0$, *such that*

$$F_\delta(\hat{K}^{t+1}) - \delta^{-1} F_\delta(\hat{K}^t) \leqq -2\alpha \min(\varepsilon^2, |\hat{K}^t - \bar{K}|^2),$$

provided $\delta \geqq \underline{\delta}$. (12.12)

Proof. It is enough to observe that

$$|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))| \geqq (I+J)^{-1} |\hat{K}^t - \bar{K}|. \quad \text{Q.E.D.}$$

The next lemma puts an upper bound on $F_\delta(\hat{K}^0)$:

Lemma. *There exist* $C > 0$, *such that*

$F_\delta(\hat{K}^0) \leqq C$ *for all* $\delta > 0$. (12.13)

I prove this lemma by adapting an argument of Gale (1967, proof of theorem 6, p. 12). It is at this point that I use the hypothesis that

$$\sum_j y_{j1k}^{-1} \equiv \hat{K}_k^0 > 0 \quad \text{for all} \quad k \in L_p.$$

Proof. By assumptions (3.14) and (3.15), there exist $y_j \in Y_j$ for $j = 1, \ldots, J$, such that

$$\sum_i \omega_i + \sum_j (y_{j0} + y_{j1}) \gg 0.$$

I may assume that $\sum_j y_{j1} \leqq \sum_j y_{j1}^{-1}$, for I may multiply the y_j by an arbitrarily small positive constant. Hence, I may assume that

$$\sum_i \omega_i + \sum_j (y_{j0} + y_{j1}^{-1}) \gg 0.$$

Choose α such that $0<\alpha<1$ and α is so close to one that

$$(1-\alpha)\sum_i \bar{x}_i \leqq \sum_i \omega_i + (1-\alpha)\sum_j \bar{y}_{j0} + \alpha \sum_j y_{j0} + \sum_j y_{j1}^{-1}. \tag{12.14}$$

Let

$$x_i^t = (1-\alpha^{t+1})\bar{x}_i \quad \text{and} \quad y_j^t = (1-\alpha^{t+1})\bar{y}_j + \alpha^{t+1} y_j.$$

Clearly, $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ is an allocation. One may prove that it is feasible by imitating the argument used by Gale.

Observe that $\lim_{t\to 0} x_i^t = \bar{x}_i$ exponentially. Since u_i is differentiable, it follows that there is $c>0$ such that

$$\sum_{t=0}^{\infty} (u_i(x_i^t) - u_i(\bar{x}_i)) \geqq -c \quad \text{for all } i.$$

Let λ be as in lemma (12.2). Then,

$$V_\delta(\hat{K}^0) \geqq \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1}(u_i(x_i^t) - u_i(\bar{x}_i)) \geqq -I\lambda^{-1}c.$$

Hence,

$$F_\delta(\hat{K}^0) = \bar{p}\cdot(\hat{K}^0 - \bar{K}) - V_\delta(\hat{K}^0)$$

$$\leqq \bar{p}\cdot(\hat{K}^0 - \bar{K}) + I\lambda^{-1}c \leqq \underline{q}\left(\max_k \hat{K}_k^0 + B\right) + I\lambda^{-1}c,$$

where $\bar{q}$ is as in (12.4) and B is as in lemma (7.2). Q.E.D.

The proof of the next lemma is the most difficult step in the proof of the theorem:

Lemma. *There exist* $\varepsilon>0$ *and* $A>0$, *such that if* $|K-\bar{K}|<\varepsilon$ *and* $\delta \geqq \underline{\delta}$, *then* $F_\delta(K)$ *is well-defined and* $F_\delta(K) \leqq A|K-\bar{K}|^2$. (12.15)

This lemma implies, of course, that the value function, V_δ, is differentiable at $\bar{K}$ and that $\bar{p}=DV_\delta(\bar{K})$. V_δ is probably differentiable everywhere. See Benveniste and Scheinkman (1982) and Araujo and Scheinkman (1977).
(1977).

Proof. It is sufficient to prove the following:

$$\begin{aligned}&\text{There exist}\quad \varepsilon>0 \quad\text{and}\quad A>0,\quad \text{such that if}\quad |K-\bar{K}|<\varepsilon,\\ &\text{then there exists } ((x_i),(y_j))\varepsilon\mathscr{F}(K), \text{ such that} \qquad (12.16)\\ &\bar{p}\cdot(K-\bar{K})-\sum_{t=0}^{\infty}\delta^t\sum_i A_i^{-1}(u_i(x_i^t)-u_i(\bar{x}_i))\leqq A|K-\bar{K}|^2.\end{aligned}$$

If (12.16) is true, then a Cantor diagonalization argument proves that $F_\delta(K)$ exists. [Such an argument is given in Brock (1970, proof of lemma 5, pp. 277–278).]

I start by defining a feasible allocation $(({}^0x_i), ({}^0y_j))$ which converges to $((\bar{x}_i),(\bar{y}_j))$ exponentially and from below. I do so by using the construction of Gale, which I have already used in proving lemma (12.13). Clearly,

$$\tfrac{1}{2}\sum_i \bar{x}_i \leqq \sum_i \omega_i + \tfrac{3}{4}\sum_j \bar{y}_{j0} + \sum_j \bar{y}_{j1}.$$

This is simply (12.14), with $\alpha=\frac{1}{2}$, $y_{j0}=\frac{1}{2}\bar{y}_{j0}$, and $y_{j1}^{-1}=\bar{y}_{j1}$. Let

$${}^0x_i^t=(1-\tfrac{1}{2}^{t+1})\bar{x}_i \quad\text{and}\quad {}^0y_j^t=(1-\tfrac{1}{2}^{t+2})\bar{y}_j.$$

Then, $(({}^0\boldsymbol{x}_i),({}^0\boldsymbol{y}_j))$ is a feasible allocation.

I now modify $(({}^0\boldsymbol{x}_i),({}^0\boldsymbol{y}_j))$ so as to obtain an allocation $(({}^1\boldsymbol{x}_i),({}^1\boldsymbol{y}_j))$ such that $g_j({}^1y_j^t)=0$ for all j and t. For all j and t, let

$$({}^1y_{j0}^t,{}^1y_{j1}^t)=({}^0y_{j0}^t),a_j^{t0}y_{j1}^t) \quad\text{where}\quad a_j^t=\max\{a\geqq 1\,|\,({}^0y_{j0}^t,a^0y_{j1}^t)\in Y_j\}.$$

This defines $(({}^1\boldsymbol{x}_i),({}^1\boldsymbol{y}_j))$. Since ${}^1y_{j1}^t\geqq {}^0y_{j1}^t$ for all j and t, $(({}^1\boldsymbol{x}_i),({}^1\boldsymbol{y}_j))$ is feasible.

The ε of (12.16) is defined by the formula

$$\varepsilon=(1+4\zeta^{-1}B)^{-1}D^{-1}, \qquad (12.17)$$

where B is a bound on stationary allocations [which exists by lemma (7.2)], where $\zeta>0$ is as in the interiority assumption, and where D is a constant defined just after (12.18) below. It will be seen that D could be defined before ε, so that the argument is not circular. $D>1$, and I assume that $B>1$ and $\zeta<1$, so that $\varepsilon<\zeta/4$.

I next show that the part of $((^1\boldsymbol{x}_i),(^1\boldsymbol{y}_j))$ from some time t on is feasible, provided that the vector of initial capital stocks, K, satisfies $(|K-\bar{K}|<\varepsilon$.

Suppose that $0<|K-\bar{K}|=\varepsilon$ and let T be the largest non-negative integer such that

$$\tfrac{1}{2}^{T+2}\bar{K}_k \geqq |K-\bar{K}| \quad \text{for all } k.$$

Such a T exists since $\varepsilon<\zeta/4$ and $\bar{K}_k \geqq \zeta$ for all k. I claim that the part of $((^1\boldsymbol{x}_i),(^1\boldsymbol{y}_j))$ from T on belongs to $\mathscr{F}(K)$. For all $k\in L_{\mathrm{p}}$.

$$K_k \geqq \bar{K}_k - |K-\bar{K}| \geqq (1-\tfrac{1}{2}^{T+2})\bar{K}_k.$$

Therefore,

$$\begin{aligned}\sum_i {}^1x_{ik}^T - \sum_j {}^1y_{j0k}^T &= (1-\tfrac{1}{2}^{T+1})\sum_i \bar{x}_{ik} - (1-\tfrac{1}{2}^{T+2})\sum_j \bar{y}_{j0k}\\ &\leqq (1-\tfrac{1}{2}^{T+2})\left(\sum_i \bar{x}_{ik} - \sum_j \bar{y}_{j0k}\right)\\ &= (1-\tfrac{1}{2}^{T+2})\bar{K}_k \leqq K_k \quad \text{for all} \quad k\in L_{\mathrm{p}}.\end{aligned}$$

This proves the claim.

Let $((^2\boldsymbol{x}_i),(^2\boldsymbol{y}_j))$ be defined by $((^2x_i^t),(^2y_j^t))=((^1x_i^{T+t}),(^1y_j^{T+1}))$. I have shown that $((^2\boldsymbol{x}_i)$, $(^2\boldsymbol{y}_j))$ belongs to $\mathscr{T}(K)$, that it converges exponentially to $((\bar{x}_i)$, $(\bar{y}_j))$ and that it satisfies $g_j(^2y_j^t)=0$ for all t and j. I need an additional condition, which is that the feasibility condition be satisfied with equality in every period. I define surplus vectors $z^t\in R_+^L$ as follows:

$$z^0 = K + \sum_i \omega_i - \left(\sum_i {}^2x_i^0 - \sum_j {}^2y_{j0}^0\right)$$

and

$$z^t = \sum_j {}^2y_{j1}^{t-1} + \sum_i \omega_i - \left(\sum_i {}^2x_i^t - \sum_j {}^2y_0^t\right) \quad \text{if} \quad t>0.$$

I next define a process which I call distributing surpluses. This process will transform the given allocation into one which has no surpluses and still satisfies the conditions listed below.

Suppose that we start with an allocation $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ which has the property that a positive quantity of every good is either consumed or used directly or indirectly to produce some consumption good. By the interiority assumption, the stationary allocation $((\bar{\boldsymbol{x}}_i)$, $(\bar{\boldsymbol{y}}_j))$ has this property, and it follows that $((^2\boldsymbol{x}_i)$, $(^2\boldsymbol{y}_j))$ does so as well. I show how to distribute the surplus vector $z\in R_+^L$ available at time t. I describe how to distribute the first component of z.

Each of the other components are distributed in succession in the same manner. The distribution of the first component results in a new allocation $((\tilde{x}_i),(\tilde{y}_j))$ which is defined as follows. If the first good is consumed by some consumer, say by consumer i, then let $\tilde{x}_{i1}^t = x_{i1}^t + z_1$ and let all other components of the allocation $((\tilde{x}_i),(\tilde{y}_j))$ be the same as the corresponding components of $((x_i),(y_j))$. If the first good is not consumed by anyone, then there exists a sequence of the form $k_0 j_0 k_1, \ldots, j_N k_N i$, where $k_0 = 1$, and with the following interpretation. In the allocation $((x_i),(y_j))$, firm j_n uses good k_n to produce good k_{n+1}, for $n = 0, 1, \ldots, N-1$, and a positive quantity of good k_N is consumed by consumer i. Clearly, the sequence may be chosen so that $N < L$. For $n = 0, 1, \ldots, N$, let e_n be the k_nth standard basis vector of R_+^L. Define $a_0, a_1, \ldots, a_N$ as follows. Let $a_0 = z_1$. Given a_{n-1}, let

$$a_n = \max\{a \geq 0 \mid (y_0^{t+n-1} - a_{n-1}e_{n-1},\ y_{j1}^{t+n-1} + ae_n) \in Y_j\} \quad \text{for} \quad j = j_{n-1}.$$

By assumptions (3.7)–(3.9), a_n is well-defined and $a_n > 0$ if $a_{n-1} > 0$. If $n = 0, 1, \ldots, N-1$, let

$$\tilde{y}_j^{t+n} = (y_{j0}^{t+n} - a_n e_n,\ y_{j1}^{t+n} + a_{n+1}e_{n+1}) \quad \text{for} \quad j = j_n.$$

Let $\tilde{x}_i = x_i + a_N e_N$. Let all the other components of the allocation $((\tilde{x}_i),(\tilde{y}_j))$ be the same as the corresponding components of $((x_i),(y_j))$.

The allocation $((^3x_i),(^3y_j))$ is defined to be an allocation obtained from $((^2x_i),(^2y_j))$ by distributing the surpluses z^t for $t = 0, 1, \ldots$. It should be clear that the new allocation belongs to $\mathscr{F}(K)$ and that $g_j(^3y_j^t) = 0$ for all j and t. I now show that $((^3x_i),(^3y_j))$ converges exponentially to $((\bar{x}_i),(\bar{y}_j))$.

First of all, let $a_0, a_1, \ldots, a_N$ be as in the construction just described. If the starting allocation $((x_i),(y_j))$ equalled the stationary allocation $((\bar{x}_i),(\bar{y}_j))$, then it would follow from profit maximization and from inequality (12.4) that

$$a_n \leqq \delta^{-n} \underline{q}^{-1} \bar{q} z_1 \leqq \delta^{-L} \underline{q}^{-1} \bar{q} z_1.$$

Because the production possibility sets are differentiably strictly convex [assumption (3.10)], it follows there is $C \geqq 1$ such that if $|((x_i^t),(y_j^t)) - ((\bar{x}_i),(\bar{y}_j)| \leqq 2$ for all t, then

$$a_n \leqq \zeta \underline{\delta}^{-L} \underline{q}^{-1} \bar{q} z_1 \quad \text{for all } n.$$

If one takes into account the fact that a component of $((^2x_i),\ (^2y_j))$ may be affected by at most L^2 distributed surpluses, one obtains that

$$|((^3x_i^t),(^3y_j^t)) - ((^2x_i^t),(^2y_j^t))| \leqq L^2 \zeta \underline{\delta}^{-L} \underline{q}^{-1} \bar{q} \max |z^s| \quad \text{for all} \quad t,$$

if the allocation to which surpluses are distributed is always at most distance 2 from $((x_i), (y_j))$. In summary,

$$|((^2x_i^t),(^2y_j^t))-((^3x_i^t),(^3y_j^t))| \leqq D \max_{s \leq t} |z^s| \quad \text{for all} \quad t,$$

$$\text{provided} \quad |((^2x_j^t),(^2y_j^t))-((\bar{x}_i),\bar{y}_j))| \leqq 1$$

$$\text{and} \quad |z^t| \leqq D^{-1} \quad \text{for all} \quad t, \tag{12.18}$$

where

$$D = L^2 \zeta \underline{\delta}^L \underline{q}^{-1} \bar{q}.$$

I next show that the conditions in (12.18) apply. It is easy to see that

$$|((^2x_i^t),(^2y_j^t))-((\bar{x}_i),(\bar{y}_j))| \leqq \tfrac{1}{2}^{T+t+1} B \quad \text{for all} \quad t, \tag{12.19}$$

where B is the bound on stationary allocations which appears in the definition of ε, (12.17). Recall that T is the largest non-negative integer such that $|K-\bar{K}| \leqq \frac{1}{2}^{T+2} \bar{K}_k$ for all k. Hence, for some k,

$$|K-\bar{K}| \geqq \tfrac{1}{2}^{T+3} \bar{K}_k \geqq \tfrac{1}{2}^{T+3} \zeta,$$

where ζ is as in the interiority assumption and in (12.7). It follows that

$$\tfrac{1}{2}^{T+1} \leqq 4\zeta^{-1} |K-\bar{K}| \leqq 4\zeta^{-1} \varepsilon \leqq B^{-1}. \tag{12.20}$$

The last inequality follows from the definition of ε, (12.17). Hence by inequality (12.19),

$$|((^2x_i^t)(^2y_j^t))-((\bar{x}_i),(\bar{y}_j))| \leqq 1 \quad \text{for all} \quad t,$$

and the first condition of (12.18) is satisfied.

I next show that

$$|z^t| \leqq (4\zeta^{-1} B + 1) \tfrac{1}{2}^t |K-\bar{K}| \quad \text{for all} \quad t. \tag{12.21}$$

Recall that

$$z^t = \sum_j {}^1y_{j1}^{T+t+1} + \sum_i \omega_i - \left(\sum_i {}^1x_i^{T+t} - \sum_j {}^1y_{j0}^{T+t} \right) \quad \text{if} \quad t > 0.$$

It should be clear that ${}^1y^t_{j1} \leqq \bar{y}_{j1}$ for all t and j. Therefore, for all $k \in L_p$,

$$(1-\tfrac{1}{2}^{t+1})\bar{K}_k = (1-\tfrac{1}{2}^{t+1})\left[\sum_i \bar{x}_{ik} - \sum_j \bar{y}_{j0k}\right]$$

$$\leqq (1-\tfrac{1}{2}^{t+1})\sum_i \bar{x}_{ik} - (1-\tfrac{1}{2}^{t+2})\sum_j \bar{y}_{j0k}$$

$$= \sum_i {}^1x^t_{ik} - \sum_j {}^1y^t_{j0k} \leqq \sum_j {}^1y^{t-1}_{j1k} \leqq \bar{K}_k.$$

Hence,

$$0 \leqq z^t_k \leqq \tfrac{1}{2}^{T+t}|\bar{K}| \leqq \tfrac{1}{2}^{T+t+1} B \quad \text{for all} \quad k \in L_p.$$

If $k \in L_o$, then

$$\sum_i \omega_{ik} - \left(\sum_i {}^1x^t_{ik} - \sum_j {}^1y^t_{j0k}\right)$$

$$= \sum_i \bar{x}_{ik} - \sum_j \bar{y}_{j0k} - \left[(1-\tfrac{1}{2}^{t+1})\sum_i \bar{x}_{ik} - (1-\tfrac{1}{2}^{t+2})\sum_j \bar{y}_{j0k}\right]$$

$$= \tfrac{1}{2}^{t+1}\sum_i \bar{x}_{ik} - \tfrac{1}{2}^{t+2}\sum_j \bar{y}_{j0k} \leqq \tfrac{1}{2}^{t+1}\left(\sum_i \bar{x}_{ik} - \sum_j \bar{y}_{j0k}\right)$$

$$= \tfrac{1}{2}^{t+1}\sum_i \omega_{ik} = \tfrac{1}{2}^{t+1} B.$$

Hence,

$$z^t_k \leqq \tfrac{1}{2}^{T+t+1} B.$$

Using (12.20), it now follows that

$$|z^t| \leqq 4\zeta^{-1} B \tfrac{1}{2}^t |K - \bar{K}| \quad \text{if} \quad t > 0.$$

Since

$$z^0 = K + \sum_i \omega_i - \left(\sum_i {}^1x^T_i - \sum_j {}^1y^T_{j0}\right),$$

it follows from what has been proved that

$$|z^0| \leqq |K - \bar{K}| + \left|\bar{K} + \sum_i \omega_i - \left(\sum_i {}^1x^T_i - \sum_j {}^1y^T_0\right)\right|$$

$$\leqq |K - \bar{K}| + 4\zeta^{-1}|K - \bar{K}|.$$

This completes the proof of (12.21).

Since $|K-\bar{K}|\leqq\varepsilon$, (12.21) implies that

$$|z^t|\leqq D^{-1}\quad\text{for all}\quad t,$$

and the second condition of (12.18) is satisfied.

Combining (12.18)–(12.21) and using the triangle inequality, I obtain that

$$|((^3x_i^t),(^3y_j^t))-((\bar{x}_i),(\bar{y}_j))|\leqq\tfrac{1}{2}{}^tE|K-\bar{K}|\quad\text{for all}\quad t,\tag{12.22}$$

where

$$E=4\zeta^{-1}B(D+1)+D.$$

Let $((^3\boldsymbol{x}_i),(^3\boldsymbol{y}_j))=((\boldsymbol{x}_i),(\boldsymbol{y}_j))$. I now show that $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ satisfies (12.16) for a suitable choice of A and for the ε defined in (12.17). Let B be as in the definition of ε. By assumptions (3.2) and (3.4), there exists $b_c>0$ such that for all i,

$$u_i(x)+Du_i(x)\cdot(x'-x)-u_i(x')\leqq b_c|x-x'|^2,$$

provided that $|x|\leqq B$ and $|x-x'|\leqq 1$. Let λ be as in lemma (12.2). Then for

$$\Lambda_i^{-1}u_i(\bar{x}_i)-\bar{p}\cdot\bar{x}_i-(\Lambda_i^{-1}u_i(x)-\bar{p}\cdot x)\leqq\lambda^{-1}b_c|x-\bar{x}_i|^2,$$

provided that $x_k=0$ whenever $\bar{x}_{ik}=0$ for $k=1,\dots,L$.

By assumptions (3.8) and (3.10), there exists $b_p>0$ such that for all j,

$$Dg_j(y^1)\cdot(y^1-y)\leqq b_p|y^1-y|^2,$$

provided $|y^1|\leqq B$, $|y-y^1|\leqq 1$, and $g_j(y)=g_j(y^1)=0$. Let

$$\bar{\rho}=\bar{q}\left[\min\left\{\left.\frac{\partial g_j(y)}{\partial y_{ik}}\right|\,|y|\leqq B,\ i=0,1,\ k=1,\dots,L,\ j=1,\dots,J\right\}\right]^{-1},$$

where $\bar{q}$ is as in (12.4). By assumptions (3.8) and (3.9), $\bar{\rho}$ exists. Then,

$$(\bar{p}\cdot\bar{y}_{j0}+\delta\bar{p}\cdot\bar{y}_{j1})-(\bar{p}\cdot y_0+\delta\bar{p}\cdot y_1)\leqq\bar{\rho}b_p|y-\bar{y}_j|^2,$$

provided $y_{0k}=0$ whenever $\bar{y}_{0k}=0$, and $y_{1k}=0$ whenever $\bar{y}_{1k}=0$, for $k=1,\dots,L$, and provided $g_j(y)=0$.

I have been careful to choose $((\boldsymbol{x}_i),(\boldsymbol{y}_j))$ so that the following are true: $x_{ik}^t=0$ whenever $\bar{x}_{ik}=0$ for all t, i, and k, $y_{j0k}^t=0$ whenever $\bar{y}_{j0k}=0$, and $y_{j1k}^t=0$

whenever $\bar{y}_{j1k}=0$ for all t, j, and k, and $g_j(y_j^t)=0$ for all t and j. It follows that

$$\sum_i [(\Lambda_i^{-1}u_i(\bar{x}_i)-\bar{p}\cdot\bar{x}_i)-(\Lambda_i^{-1}u_i(x_i^t)-p\cdot x_i^t)]$$

$$+\sum_j [\bar{p}\cdot\bar{y}_{j0}+\delta\bar{p}\cdot\bar{y}_{j1})-(\bar{p}\cdot y_{j0}^t+\delta\bar{p}\cdot y_{j1}^t)]$$

$$\leqq b(I+J)|((x_i^t),(y_j^t))-((\bar{x}_i),(\bar{y}_j))|^2, \quad \text{for all} \quad t, \tag{12.23}$$

where

$$b=\max(\lambda^{-1}b_c,\bar{p}b_p).$$

Let $A=\frac{4}{3}Eb(I+J)$. I now show that (12.16) is true for ε, A, and $((x_i),(y_j))$ as defined above. By (12.7),

$$\bar{p}\cdot(K-\bar{K})-\sum_{t=0}^{\infty}\delta^t\sum_i \Lambda_i^{-1}(u_i(x_i^t)-u_i(\bar{x}_i))\leqq S_1+S_2+S_3,$$

where S_1, S_2 and S_3 are defined just after (12.7). Since

$$\sum_j (y_{j0}^t+y_{j1}^{t-1})+\sum_i(\omega_i-x_i^t)=0 \quad \text{for all} \quad t,$$

it follows that $S_2=0$. By (12.22) and (12.23),

$$S_1+S_3\leqq b(I+J)\sum_{t=0}^{\infty}\delta^t|((x_i^t),(y_j^t))-((\bar{x}_i),(\bar{y}_j))|^2\leqq A|K-\bar{K}|^2.$$

This proves (12.16) and hence proves the lemma. Q.E.D.

I now may prove that $\lim_{t\to\infty}F_\delta(\hat{K}^t)=0$ exponentially. By (12.11) this proof will complete the proof of the theorem.

Choose a small positive number no larger than the ε of lemma (12.12) and the ε of lemma (12.15). Call this number ε again. Let α be as in lemma (12.12) and let A be as in lemma (12.15). Clearly, I may assume that $\alpha<A$. Finally, let C be as in lemma (12.13). Then, I know that $F_\delta(\hat{K}^0)\leqq C$.

If $1>\delta\geqq\underline{\delta}$ and $|\hat{K}^t-\bar{K}|\leqq\varepsilon$, then

$$F_\delta(\hat{K}^t)\leqq A|\hat{K}^t-\bar{K}|^2.$$

Also,

$$F_\delta(\hat{K}^{t+1})-\delta^{-1}F_\delta(\hat{K}^t)\leqq -2\alpha\min(\varepsilon^2,|\hat{K}^t-\bar{K}|^2) \quad \text{for all} \quad t.$$

Let

$$\bar{\delta}=\max(\underline{\delta}, A/(A+\alpha), C/(C+\alpha\varepsilon^2)).$$

From now on, I assume that $\delta \geqq \bar{\delta}$. I claim that

$$\hat{F}_\delta(K^{t+1}) \leqq F_\delta(\hat{K}^t)-\alpha\min(\varepsilon^2, |\hat{K}^t-K|^2). \tag{12.24}$$

The argument involves induction on t. Recall that $F_\delta(\hat{K}^0)\leqq C$. Assume by induction that $F_\delta(\hat{K}^t)\leqq C$. If $|\hat{K}^t-\bar{K}|\leqq\varepsilon$, then

$$\begin{aligned}F_\delta(\hat{K}^{t+1}) &\leqq F_\delta(\hat{K}^t)+(\delta^{-1}-1)F_\delta(\hat{K}^t)-2\alpha|\hat{K}^t-\bar{K}|^2\\ &\leqq F_\delta(\hat{K}^t)+[(\delta^{-1}-1)A-2\alpha]|\hat{K}^t-\bar{K}|^2\\ &\leqq F_\delta(\hat{K}^t)-\alpha|\hat{K}^t-\bar{K}|^2.\end{aligned}$$

The last inequality follows from the choice of $\bar{\delta}$. If $|\hat{K}^t-\bar{K}|\geqq\varepsilon$, then

$$\begin{aligned}F_\delta(\hat{K}^{t+1}) &\leqq F_\delta(\hat{K}^t)+(\delta^{-1}-1)F_\delta(\hat{K}^t)-2\alpha\varepsilon^2\\ &\leqq F_\delta(\hat{K}^t)+(\delta^{-1}-1)C-2\alpha\varepsilon^2\\ &\leqq F_\delta(\hat{K}^t)-\alpha\varepsilon^2.\end{aligned}$$

The last inequality again follows from the choice of $\bar{\delta}$. It now follows that $F_\delta(\hat{K}^{t+1})\leqq C$. Hence, I may continue the above argument inductively. This proves (12.24).

I now prove that

$$F_\delta(\hat{K}^{t+1})\leqq\max[(1-\alpha A^{-1})F_\delta(\hat{K}^t), F_\delta(\hat{K}^t)-\alpha\varepsilon^2]. \tag{12.25}$$

By (12.24),

$$F_\delta(\hat{K}^{t+1})\leqq F_\delta(\hat{K}^t)-\alpha\varepsilon^2 \quad \text{if} \quad |\hat{K}^t-\bar{K}|\geqq\varepsilon.$$

Also

$$F_\delta(\hat{K}^{t+1})\leqq F_\delta(\hat{K}^t)-\alpha|K^t-\bar{K}|^2\leqq F_\delta(\hat{K}^t)-\alpha A^{-1}F_\delta(\hat{K}^t)$$
$$\text{if} \quad |\hat{K}^t-\bar{K}|+\varepsilon.$$

This proves (12.25).

I complete the proof by showing the following:

There exists a positive integer T, such that

$$F_\delta(\hat{K}^t) \leqq C - t\alpha\varepsilon^2 \quad \text{if} \quad t \leqq T, \quad \text{and}$$

$$F_\delta(\hat{K}^t) \leqq (1-\alpha A^{-1})^{t-T} A\varepsilon^2 \quad \text{if} \quad t \geqq T. \tag{12.26}$$

If $F_\delta(\hat{K}^t) \geqq A\varepsilon^2$, then $\alpha A^{-1} F_\delta(\hat{K}^t) \geqq \alpha\varepsilon^2$, so that by (12.25), $F_\delta(\hat{K}^{t+1}) \leqq F_\delta(\hat{K}^t) - \alpha\varepsilon^2$. Similarly, if $F_\delta(\hat{K}^t) \leqq A\varepsilon^2$, then $\alpha A^{-1} F_\delta(\hat{K}^t) \leqq \alpha\varepsilon^2$, so that $F_\delta(\hat{K}^{t+1}) \leqq (1-\alpha A^{-1}) F_\delta(\hat{K}^t)$. Let T be the smallest positive integer such that $A\varepsilon^2 \geqq C - \alpha\varepsilon^2 T$. This completes the proof of assertion (12.26). Q.E.D.

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