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Adjustment Dynamics and Rational Play in Games*

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When a given strategic situation arises repeatedly, the possibility arises that equilibrium predictions can be justified by a dynamic adjustment process. We examine myopic adjustment dynamics, a class that includes replicator dynamics from evolutionary game theory, simple models of imitation, models of experimentation and adjustment, and some simple learning dynamics. We present a series of theorems showing conditions under which behavior that is asymptotically stable under some such dynamic is strategically stable in the sense of Kohlberg and Mertens. This behavior is thus *as if* the agents in the economy satisfied the extremely stringent assumptions that game theory traditionally makes about rationality and beliefs. *Journal of Economic Literature* Classification Number: C72.

I. INTRODUCTION

There is growing skepticism as to whether sophisticated strategic behavior—satisfying, for example, sequential equilibrium or forward induction—is the natural end product of introspection by economic agents. Why, and under what circumstances, should we then believe in equilibrium and equilibrium refinements?

Many strategic situations of interest arise repeatedly. In some cases, fixed players repeatedly find themselves in the same situation, as for

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example competing firms. In other cases, a given strategic situation arises repeatedly among sets of anonymous players drawn at random from a large population. The interaction of drivers on the road seems a good example. Other cases fall between. While a lawyer preparing for a trial may never have faced quite the same legal situation before, an extensive record of similar past trials is available. Individuals preparing to negotiate the purchase of a new car have friends, various consumer publications, and their experience in other bargaining situations to guide their behavior. In all these situations, one might think of a process in which behavior adjusts over time based on the experience of participants.

In this paper, we try to understand when these processes have implications for *as-if-rational* play. We find conditions under which asymptotic stability under this sort of dynamic implies behavior that is as if the agents in the economy satisfied the stringent assumptions that game theory normally makes about rationality and congruence of beliefs.

I.1. *The Literature*

Models of adjusting play have been extensively studied in both the learning and the evolutionary game theory literatures. Some of these models are explicitly dynamic. Others, while at least partly based on intuitions about dynamic adjustment, are formulated in a static way.

The literature on static formulations had its genesis in the application of evolutionary ideas to game theory by Maynard Smith (1974, 1982) and Maynard Smith and Price (1973). They argued that many interactions in the natural world could be interpreted as strategic situations, and that mutation and natural selection would tend to push organisms toward optimal play. For some economic questions—in explaining altruistic behavior or tastes, for example—a literal interpretation of these ideas from evolutionary biology may make sense. However, much more generally, both "mutation" and "natural selection" have close analogues in economic environments. In many situations there will be a general movement over time to strategies that perform well in their environment, whether by imitation, by the growth or bankruptcy of firms following superior or inferior strategies, or by a learning process.

An evolutionarily stable strategy profile is one such that members of any small group of entrants to a population who are playing a strategy different from the status quo fare worse against the post entry population than do individuals using the original strategy. It attempts to capture in a static way the notion of stability of behavior in a population when a small mutation is followed by natural selection.[1]

[1] Aside from connections to stability under any particular dynamic process, evolutionary stability (and its offspring) can also be interpreted as strong forms of the familiar "no profitable entry" condition. This may help explain the considerable appeal of evolutionary stability to economists despite its unsatisfactory dynamic foundations.

The static approach implies a remarkable amount of as-if-rationality. In particular, van Damme (1991) shows that an evolutionarily stable strategy is proper (Myerson, 1978). Van Damme (1984) also shows that a proper equilibrium is sequential (Kreps and Wilson, 1982) in associated extensive forms. Thus, an evolutionarily stable strategy profile corresponds to a sequential equilibrium in associated extensive forms.

Swinkels (1992a, b) extends this result, working with a much weaker static notion and deriving stronger implications. In that work, the entrants against which the status quo is tested are restricted to those that are best responses to the post entry environment. Solutions are allowed to be set-valued. Such a set is called *equilibrium evolutionarily stable* (*EES*). EES sets are robust to the iterative removal of weakly dominated strategies, satisfy the never-a-weak-best-response property (Kohlberg and Mertens, 1986), and depend only on the reduced normal form. Under some additional conditions (which are always satisfied for EES sets with a single element, and are generically true for EES sets for two-person extensive form games) an EES set contains a stable component in the sense of both Kohlberg and Mertens (1986) and Hillas (1990).[2] For generic extensive form games, EES sets correspond to a single outcome, with different elements reflecting different out-of-equilibrium behaviors. Because Hillas stable sets contain a proper element, this outcome is sequential.[3]

Of course, our intuitions about evolution and learning are largely about dynamics. And, indeed, these ideas have been intensively explored in explicitly dynamic frameworks (see, for example, Taylor and Jonker (1978), Foster and Young (1990), Friedman (1991), Kandori *et al.* (1992), and Nöldeke and Samuelson (1992)). In these models, there is typically a large population (or populations) from which sets of players, one for each player position, are randomly and repeatedly drawn to play the game. Players change their behavior over time based on their experiences. The state variable is typically the proportion of the population playing each pure strategy, i.e., the *population strategy profile*. This evolves in either a deterministic or a stochastic fashion.

An important example is the replicator dynamic. For this dynamic, the proportionate rate of growth of the proportion of the population playing each pure strategy is linear in the difference between the payoffs to that pure strategy and the current average payoff within the population. The replicator dynamic arises naturally in biological games, where one inter-

[2] To avoid terminological confusion, stability in the sense of Kohlberg and Mertens (1986) will henceforth be referred to as KM stability.

[3] For symmetric games in which players for the various player positions are modeled as being drawn from a single population, the appropriate version of strategic stability itself has a symmetry condition. In either case, the set will contain a proper element. See Swinkels (1992b) for details.

prets payoffs as the number of offspring. It also has some intuitive appeal as a model of imitation in an economic environment. Friedman (1991) introduces *weak compatible dynamics*, which can be thought of as generalized replicator dynamics. For these dynamics, if all strategies currently in use by a given population are performing equally well, then the play of that population is at rest. Otherwise, play adjusts among those strategies currently in use in a direction that would be payoff-increasing were the play of the opponents fixed.

Given the intuition on which evolutionary stability is based, one might hope that evolutionarily stable strategies would correspond to asymptotically stable points under replicator dynamics or its generalizations. Evolutionary stability would then capture the idea that small (one-time) mutations are driven from the population. However, as Taylor and Jonker (1978) show, evolutionary stability is sufficient, *but not necessary*, for asymptotic stability under replicator dynamics. Friedman (1991) shows that evolutionary stability is neither necessary nor sufficient for asymptotic stability under weak compatible dynamics. While Matsui (1992) finds some connection between EES sets and a variant of best response dynamics, the overall relationship between the set-valued notions discussed above and asymptotic stability under adjustment dynamics is little understood.

This casts doubt on the import of the as-if-rationality results mentioned above, and brings us to our major question: Is as-if-rational behavior truly an implication of stability under this sort of adjustment process, or is it an artifact of the (perhaps overstrong) static conditions?

A paper with goals similar to this one is Nöldeke and Samuelson (1992). They examine a model in which a finite number of agents are distinguished by a behavior strategy and a conjecture about other's play in an extensive form game. With small probability in each period, each agent learns the current play of her opponents and adjusts her play accordingly. Finally, there is a small probability of "mutation." Nöldeke and Samuelson examine the limiting support of the process as the probability of mutation goes to 0. They find in a class of simple games that the subgame perfect outcome is contained in the limiting set, but that for "interesting" extensive forms other outcomes are also included. They find somewhat more support for a notion of forward induction.

1.2. *Description of Results*

In this paper, we examine *myopic adjustment dynamics*. In common with weak compatible dynamics, these are deterministic dynamics with state space the set of population strategy profiles, such that at each instant the direction of movement in each population's strategy is (at least weakly) payoff increasing given the current behavior of the opposing populations.

They generalize weak compatible dynamics in several ways. First, the population is only required to be at rest at Nash equilibria, rather than whenever all strategies *currently* in the population are performing equally well. Nor do we restrict the adjustment to those strategies currently present in positive measure. Thus the analysis includes dynamics in which players can occasionally switch to profitable strategies not currently used by the population. Such dynamics have a much greater chance to perform well in extensive form games. It also seems quite restrictive in economic environments to rule out this sort of innovation a priori.

In ecological environments, the growth of a trait is limited by the existing population possessing that trait. In an economic environment, this need not always be so. We weaken the regularity conditions imposed by Friedman to allow for dynamics exhibiting more general behavior at boundaries of the system, including dynamics in which a particular strategy has a zero growth rate when it is not present, but spreads quickly once introduced. This also allows dynamics in which strategies can disappear in finite time.

We begin by noting that if a strategy profile is asymptotically stable under a myopic adjustment dynamic, then it is hyperstable (Kohlberg and Mertens, 1986). Thus, the as-if-rationality implications of the static notion of evolutionary stability are also implications of asymptotic stability under a myopic adjustment dynamic.

Consider an extensive form game, and any Nash equilibrium that does not reach every information set. Generically, there will be a range of behavior at out-of-equilibrium information sets that is consistent with the optimality of the equilibrium path. The Nash equilibrium will thus belong to a nontrivial component of Nash equilibria. The dynamics we have discussed so far all have the property that they are at rest on each Nash equilibrium. Since an asymptotically stable strategy profile must be isolated in the space of rest points, this implies that this equilibrium could not be asymptotically stable. The static notion of evolutionary stability similarly requires isolation in the set of Nash equilibria. Even for dynamics that need not stop on every Nash equilibrium (some of which we discuss later), it will often be the case that the dynamic system does not select among various out-of-equilibrium behaviors. Thus, strategy profiles satisfying one or the other of these conditions will fail to exist precisely when concepts such as sequential equilibrium have power.

Motivated by this, we consider asymptotic stability for *sets* of strategy profiles.[4] Set-valued notions are hard to interpret in standard rationality based game theory—what does it mean to say that rational players play

[4] This was also the motivation for the set-valued notions introduced in Swinkels (1992a).

a set? In dynamic environments, a set-valued solution makes perfect sense: we predict that play, once in such a set, will remain in the set, without making any particular prediction about which element of the set will be used at any instant in time.

As for sets satisfying rationality-based solution concepts, asymptotically stable sets are most attractive when they correspond to a single outcome in an extensive form. We show that if a given outcome in a two-person extensive form game is asymptotically stable under such a dynamic (so that different limiting behaviors differ only at out-of-equilibrium information sets) then that outcome is hyperstable.

More generally, for games with any finite number of players, if a set of strategies is asymptotically stable under some such dynamic, then it contains a hyperstable subset if an additional topological condition is satisfied.

Hyperstable sets satisfy many (but not all) of the commonly accepted rationality-based desiderata for a solution concept (see Kohlberg and Mertens, 1986, on these desiderata). They are invariant to "irrelevant" changes in the game, and satisfy a form of forward induction (the never-a-weak-best-response property). Hyperstable sets also contain proper equilibria and so satisfy a strong form of backward induction. In particular, when a hyperstable set corresponds to a single outcome in an extensive form, then that outcome must be a sequential equilibrium.

Hyperstable sets can contain strategy profiles using weakly dominated strategies. In general, this violates a major rationality notion. We view the results as most relevant when an asymptotically stable set corresponds to a single outcome in the extensive form. Under those circumstances, the weak dominance issue becomes less important, because play will be observationally equivalent to play not involving weakly dominated strategies. We conclude that behavior that is asymptotically stable under this type of dynamic satisfies a very strong notion of as-if-rational play.

We view the assumption that every Nash equilibrium is a rest point of the system as very strong, and so examine the importance of this assumption to the results. When Nash equilibria need not be rest points, the results continue to hold for KM stability (but not for hyperstability) if the derivative field of the dynamic is Lipschitz continuous. KM stable sets need not contain proper elements, and so need not imply backward induction (on the other hand, they do not involve weakly dominated strategies). Thus, rather surprisingly, the assumption that Nash equilibria are rest points is key for both the backward induction implication and the ability to extend the results to non-Lipschitz dynamics. If the dynamic fails to stop on some Nash equilibria solely because it eliminates particular weakly dominated strategies, then the results hold for KM stability without the extra continuity condition, but once again we are unable to get hyperstability and so backward induction.[5]

We next turn to some generalizations. We begin with dynamics in which the direction of movement can depend on more than just the current population strategy profile. The analysis generalizes almost immediately when the state space is the cross product of the space of population strategy profiles with a compact convex subset of a Banach space. A key restriction to this analysis is that the added dimensions are allowed to affect which myopically improving direction is chosen, but not myopic improvement itself.

The condition of compactness in the last paragraph rules out time as a dimension of the state space. If a time-varying dynamic has the property that time affects the speed of movement, but not its direction, or if the system admits a Lyapunov function, then the results go through.

An important instance of a time-varying dynamic is provided by fictitious play models, and more generally by models in which players respond not to actual play by their opponents, but rather to some perception of play that is formed from past play by the opponents. In some instances, our analysis can be made to apply to perceived play even though actual play meets few of the conditions of our analysis.

Finally, we consider the extent to which the results can be recast in a discrete time framework. If next period's population strategy profile is continuous in this period's, then we can proceed without too much difficulty. The topological condition does need to be considerably strengthened, and the continuity condition is especially strong for a discrete time model.

I.3. *Intuition*

That strategic stability, which is motivated by deep considerations involving idealized rational individuals, should have any relation to asymptotic stability under these simplistic adjustment rules is surprising. There are two main links to the connection. First, while the desiderata put forward by Kohlberg and Mertens (1986) are based on notions of rationality, the actual definition of strategic stability involves robustness of sets of equilibria to perturbations in the underlying game. A set of Nash equilibria is strategically stable if it is "structurally stable," in the sense that close-by games have Nash equilibria close to this set. Second, asymptotic stability is itself a structurally stable property: the key implications of asymptotic stability survive small changes in the dynamic. The analysis hinges on relating these changes in the dynamic to the perturbations used

[5] It remains possible that one may be able to show backward induction in one or the other of these cases by some method not involving hyperstability.

in Kohlberg and Mertens' analysis. Starting from a myopic adjustment dynamic and an asymptotically stable set Θ, and given a small perturbation to the game, we create a new dynamic that retains enough of the structure of the original dynamic to guarantee rest points near Θ, but that has as rest points only Nash equilibria of the perturbed game. Thus Θ has the stability required by Kohlberg and Mertens.

I.4. *Interpretation of Results*

If a set satisfying certain conditions is asymptotically stable under a myopic adjustment dynamic, then it contains a strategically stable subset. To what extent does this support a conclusion of as-if-rational play in games?

We begin with the words "if" and "contains." The results are clearly of more interest for games in which asymptotically stable sets satisfying the conditions of the analysis exist and are in some sense "small" than for games in which they either do not exist or also contain a wide range of behavior other than that in the strategically stable subset. Since this paper has little to say about either the size or the convergence question, it provides only one step along the way to a full understanding of when as-if-rational play can be expected to arise.

In combination with results of the sort proved in this paper, results about when "small" sets of strategy profiles are asymptotically stable could form the basis for predictions that are much richer (and perhaps empirically more successful) than those of rationality-based game theory. We may find that for signaling games of certain structure, and for a wide variety of adjustment processes, convergence of the sort required by this paper occurs, while for signaling games with another structure, it does not. We would then have a basis for predicting that forward induction will be satisfied by one type of signaling game, but not necessarily by another. Similarly, we may find that adjustment dynamics for games involving chains of backward induction show very poor convergence properties, or convergence to very large sets of behavior (recall the results of Nöldeke and Samuelson, 1992). We might then be less compelled than formerly by backward induction.

Since the relevant dynamic may depend not only on the game itself, but also on the setting of the game, the results of this paper provide a first step toward a richer theory of the type of game, *and the type of setting,* in which various equilibrium notions should or should not apply. This stands in marked contrast to standard game theory, where if in a particular game, the predictions of a particular rationality-based concept are paradoxical, or in strong opposition to observed play, then the entire concept is called into question.

We do consider the results of this paper to be most attractive when the asymptotically stable set corresopnds to a single outcome in an underlying extensive form. In that case, issues of size disappear since all limiting behaviors are observationally equivalent.

A second point involves the condition of asymptotic stability. Marimon and McGrattan (1992, Example 13) provide an example of a game and dynamic in which play converges toward a strategy profile that is not strategically stable. Thus, in considering points (or sets) that are asymptotically stable, rather than those that are merely the limit points of some convergent adjustment path, we are making a substantive restriction. Some intuition for why the extra force of asymptotic stability matters comes from realizing that while asymptotic stability is a structurally stable property, convergence to a particular point is not. In Marimon and McGrattan's example, there are arbitrarily close by KM perturbations to the game for which the original strategy profile is not the limit point (or near the limit point) of any myopic adjustment path.

Taken together, the results imply a certain fragility of the backward induction implication to the specification of rest points. Consider dynamics that are at rest whenever every strategy present in positive measure for each player position is performing equally well (as under weak compatibility). Then, if any Nash strategy profile σ for an extensive form game is a part of an asymptotically stable set, so is every strategy profile generating the same outcome as σ, including, in general, non-Nash profiles. For most plausible dynamics, this implies that the asymptotically stable set must also include strategy profiles generating other outcomes.[6] So, for dynamics with this sort of rest point, the backward induction implication fails because, while asymptotically stable sets do contain hyperstable subsets,

[6] Consider a non-Nash point in the asymptotically stable set, and small perturbations from this point. For the outcome to be asymptotically stable, it must be that as these perturbations become vanishingly small, the adjustment paths beginning from these perturbations involve only vanishingly small changes in behavior along the original equilibrium path of play in the game. But, as behavior along the equilibrium path gets arbitrarily close to the original, the incentive to make changes in behavior that only involve information sets off the original path is also vanishingly small. In contrast, as long as play remains anywhere near the original non-Nash point, there is a positive incentive to change behavior along the equilibrium path. So, for the outcome to be asymptotically stable, it must be that along the adjustment path behavior is changing a discrete amount in directions for which the rewards are vanishingly small, but a vanishingly small amount in directions for which the rewards are large. While there are myopic adjustment dynamics with this property (see Fig. 3), such dynamics do not seem very plausible in most environments. The poor behavior of this sort of dynamic is closely related to Nöldeke and Samuelson's finding that in their setting, limit sets are very large in games where backward induction is important. Behavior in their system can freely "drift" among various strategy profiles corresponding to a particular outcome until it is well outside the set of Nash equilibria. At that point, a perturbation initiates a path that leaves that outcome.

the asymptotically stable sets are typically too large in any situation where the backward induction implication of hyperstability would be interesting.

When we look at dynamics that can pass through some Nash equilibria, we have the possibility of a single outcome being asymptotically stable, but all we are able to prove is KM stability, and so we again have no backward induction implication.

Only when we look at dynamics which stop on Nash equilibria, but are in motion otherwise, do we have both the possibility that an asymptotically stable set can correspond to a single outcome, and the implication of hyperstability, and so a strong backward induction implication.

Thus, these results *do not* provide a justification for the uncritical application of rationality based game theoretic concepts. Rather, they suggest that there are conditions under which some of these concepts are appropriate, and provide one key step in understanding these conditions.

Section II covers basic definitions and strategic stability. Section III discusses dynamics. Section IV establishes the basic relationship between asymptotic stability under myopic adjustment dynamics and strategic stability. Section V discusses alternative specifications of when the dynamic is at rest. Section VI discusses extensions. Section VII concludes. Proofs are in the appendix unless noted otherwise.

II. Preliminaries

Basic Definitions

A game (S, π) consists of players $i \in N \equiv \{1, \ldots, n\}$, finite pure strategy sets S_i with $S \equiv \Pi_{i \in N} S_i$, and payoff functions $\pi = (\pi_1, \ldots, \pi_n)$. The space of mixed strategies is $\Phi = \Pi_{i \in N} \Delta(S_i)$. The vector of weights given by the mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Phi$ to $s = (s_1, \ldots, s_n) \in S$ is $\sigma(s) = (\sigma_1(s_1), \ldots, \sigma_n(s_n))$. The strategy profile obtained from σ by replacing σ_i with γ_i is denoted $\sigma \backslash \gamma_i$. We occasionally use s_i where we properly mean the mixed strategy for i that puts probability 1 on s_i. We extend π to Φ by the expected utility calculation. The set of Nash equilibria of (S, π) is $N(S, \pi)$. The set of player i's best responses to $\sigma \in \Phi$ is $BR_i(\sigma) \subseteq \Phi_i$.

$D(\mu, \nu)$ is the Euclidean distance between $\mu, \nu \in \mathbb{R}^m$. For $X \subseteq \mathbb{R}^m$ and $\mu \in \mathbb{R}^m$, $D(\mu, X) \equiv \inf_{\nu \in X} D(\mu, \nu)$. For $X \subseteq \mathbb{R}^m$ and $\varepsilon > 0$, define $B_\varepsilon(X) = \{\gamma \mid D(\gamma, X) \leq \varepsilon\}$. Note that $B_\varepsilon(X)$ is closed. Y is a neighborhood of X if there is an open set containing X but contained in Y. Thus, $B_\varepsilon(X)$ is a (closed) neighborhood of X. $\text{Int}(X)$ is the interior of X. $\text{Cl}(X)$ is its closure. For x and y functions on S_i, define $x \cdot y$ as $\Sigma_{s_i \in S_i} x(s_i)y(s_i)$. If x, y are functions on S, $x \cdot y$ is similarly defined as $\Sigma_{i \in N} \Sigma_{s_i \in S_i} x(s_i)y(s_i)$. $\mathbb{R}_+$ denotes the nonnegative real numbers.

Strategic Stability

The idea of stability (Kohlberg and Mertens, 1986) is to examine the robustness of a set of equilibria to perturbations in the underlying game. A class p of perturbed games and a metric m is established. A set $\Theta \subseteq N(S, \pi)$ is (m, p)-stable if it is a minimal closed set such that every game in p that is close to (S, π) under m has a Nash equilibrium close to Θ (in Euclidean distance).

For KM stability, a perturbed game is generated by a completely mixed strategy profile $\gamma \in \Phi$, and a vector $\delta \in [0, 1]^n$. The payoff to each pure strategy profile s in the perturbed game is the payoff in the original game when each player plays $(1 - \delta_i)s_i + \delta_i\gamma_i$. The distance from the perturbed game to the original game is $\max_{i \in N}\delta_i$.

For hyperstability, a perturbed game is obtained from the original game by first adding a finite number of redundant pure strategies, and then perturbing the payoffs to the pure strategies in the new game by a small amount. Every hyperstable set contains a KM stable subset.

III. Dynamics

We begin our exposition with deterministic dynamics that have state space Φ. The standard interpretation of $\sigma \in \Phi$ will be that according to whatever matching technology is being used, and given the behavior of individuals, the total probability of drawing an n-tuple who (after any individual randomizations) play $s \in S$ is $\sigma(s)$ (see Section VI.3 for another interpretation). We refer to σ as the *population strategy profile*.

For our purposes, it is convenient to summarize such a dynamic by a map F: $\Phi \times \mathbb{R}_+ \to \Phi$, where for $\sigma \in \Phi$, and $t \in \mathbb{R}_+$, if the population strategy profile is σ at time $t' \geq 0$, then it will be $F(\sigma, t)$ at time $t' + t$. For $i \in N$, and $s_i \in S_i$, $F(\sigma, t)(s_i)$ is the weight given to s_i by $F(\sigma, t)$. A dynamic F is *admissible* if it is continuous and right differentiable with respect to time; that is, if for all $\sigma \in \Phi$,

$$f(\sigma) \equiv \lim_{t \downarrow 0} \frac{F(\sigma, t) - \sigma}{t}$$

is a well-defined real vector.

In many cases, the right derivative f will be the primitive. If f is Lipschitz continuous, then it will have a unique and continuous solution F. Note that right differentiability implies that $F(\cdot, 0)$ is the identity map.

For $\sigma \in \Phi$, $i \in N$, and $s_i \in S_i$, $f(\sigma)(s_i)$ is the time rate of change of the proportion of s_i in the population strategy for player position i. By $f_i(\sigma)$ (resp., $F_i(\sigma, t)$), we mean the restriction of $f(\sigma)$ (resp., $F(\sigma, t)$) to S_i.

We formally analyze only the case in which the populations associated with each position of the game evolve independently. The analysis can be extended, along the lines of Swinkels (1992a, b), to cases in which several positions are symmetric and filled by players from a single population. This corresponds to restricting the state space to a subspace of Φ in which equality restrictions hold for some dimensions. Our results hold in these cases if the definition of strategic stability is correspondingly weakened to consider only perturbations satisfying the same equality restrictions.

Replicator Dynamics and Myopic Adjustment Dynamics

The replicator dynamic for a game (S, π) is given by

$$f(\gamma)(s_i) = \gamma_i(s_i)\,[\pi_i(\gamma\backslash s_i) - \pi_i(\gamma)]$$

for $i \in N$, $s_i \in S_i$, and $\gamma \in \Phi$.[7]

Thus, among the nonextinct strategies, strategies that are currently doing well are growing relative to those that are not. Only the broad qualitative features of the replicator dynamic are needed for our results. We say that an admissible dynamic F is a *myopic adjustment dynamic* if $\forall \sigma \in \Phi$,

$$f_i(\sigma) \cdot \pi_i(\sigma\backslash .) \geq 0, \text{ for all } i \in N; \tag{1.1}$$

$$\text{if } \sigma \text{ is Nash, then } f(\sigma) = 0. \tag{1.2}$$

By $f_i(\sigma) \cdot \pi_i(\sigma\backslash .)$ we mean $\sum_{s_i \in S_i} f(\sigma)(s_i)\pi_i(\sigma\backslash s_i)$. Thus, condition (1.1) states that at any moment, the direction of movement for each player population is such that *holding the strategies of the other player positions constant*, payoffs are (at least weakly) increasing.[8] If the inequality in

[7] An alternative specification has the right-hand side of the previous expression divided through by $\pi_i(\gamma)$ (where one imposes the condition that $\pi_i(\gamma) > 0$ for all γ). For symmetric games in which players are drawn from a single population, this changes the speed but not the direction of the dynamic at each point. The solution curves are thus invariant. When populations corresponding to different player positions evolve independently, the difference between the two specifications clearly matters. Our analysis covers either case.

[8] Note that $f_i(\sigma) \cdot \pi_i(\sigma\backslash .) = \frac{\partial}{\partial t}(\pi_i(\sigma\backslash F_i(\sigma, t)))|_{t=0^+}$.

(1.1) is strict whenever $\sigma_i \notin BR_i(\sigma)$, then F is a *strict myopic adjustment dynamic*. It is easily verified that the replicator dynamic (and in fact any weak compatible dynamic) is a myopic adjustment dynamic.

While these dynamics have their foundation in evolutionary biology, our results are relevant in any social or economic situation in which play adjusts in directions that are myopically improving, and in which play is stationary when all play is optimal. There is also some reason to hope that the type of analysis introduced by this paper will be useful in understanding social or economic dynamics not included in the current analysis.

IV. Myopic Adjustment Dynamics and Strategic Stability

Taylor and Jonker (1978) show that for a strategy profile σ to be evolutionarily stable is sufficient but not necessary for asymptotic stability of σ under replicator dynamics.[9] Friedman (1991) shows that asymptotic stability under a weak compatible dynamic need be neither necessary nor sufficient for evolutionary stability. Myopic adjustment dynamics generalize weak compatible dynamics. Thus, despite the strong results relating the static notion of evolutionary stability to as-if-rationality, we cannot conclude that there is any systematic relationship between asymptotic stability under myopic adjustment dynamics and as-if-rational play.

It is a special case of Theorem 1 (below) that if a strategy profile σ is asymptotically stable under a myopic adjustment dynamic, then $\{\sigma\}$ is hyperstable. Thus the implications for rational play derived as implications of evolutionary stability and its point valued generalizations hold in the dynamic case as well. Unfortunately, the weakness of those results reappears as well: as explained in the Introduction, asymptotically stable strategy profiles fail to exist precisely where the result would be most interesting.

We thus consider a set-valued notion of asymptotic stability. A set $Y \subseteq \Phi$ is *asymptotically stable* under the dynamic F if it is closed and there is a neighborhood Z of Y such that:

$$\text{for every neighborhood } W \text{ of } Y \text{ with } W \subseteq Z, \text{ there is a neighborhood } V \text{ of } Y \text{ with } F(V, t) \subseteq W \text{ for all } t \geq 0; \tag{2.1}$$

$$\text{for each } \gamma \in Z, \lim_{t \to \infty} D(F(\gamma, t), Y) = 0. \tag{2.2}$$

[9] They also show that in the "regular" case, the converse holds. While regularity is generically satisfied for normal form games, it generically fails in the extensive form when there are out-of-equilibrium information sets.

If Y has a single element, then this corresponds to the standard notion of asymptotic stability. Note that $\lim_{t\to\infty} F(\gamma, t)$ is not required to exist. This allows for convergence to, for example, limit cycles or a region on which behavior is chaotic. Minimality is not imposed: the only effect of doing so would be to reduce the generality of the results.

Introducing the set-valued notion helps matters considerably. Say that an outcome (distribution over terminal nodes) in an extensive form game is asymptotically stable under a dynamic F if there is a set of strategy profiles generating this outcome that is asymptotically stable. Then, it is again a corollary to Theorem 1 that, for two-person extensive form games, if an outcome is asymptotically stable under a strict myopic adjustment dynamic then it is hyperstable.

The proof of this hinges on three facts. First, if a set of strategy profiles corresponding to a particular outcome is asymptotically stable under a strict myopic adjustment dynamic, then that set must exactly correspond to the set of Nash equilibria supporting that outcome. Second, for two-person games, the set of Nash equilibria supporting any particular outcome is convex (Swinkels, 1992a, Lemma 8). Third, for convex asymptotically stable sets, the topological condition in Theorem 1 is trivially satisfied.

To see the first claim, consider any σ that is not Nash. Then, as F is strict, there is some player i who is moving in a strictly payoff increasing direction from σ. This must involve a change in the outcome. Thus, the asymptotically stable set cannot contain non-Nash elements. But then by (1.2), the asymptotically stable set must be precisely the component of Nash equilibria supporting the outcome.

The second fact fails for games with more than two players. An important open question is whether there is an interesting characterization of extensive form games for which sets of Nash equilibria supporting a particular outcome are sufficiently regular to guarantee the necessary topological condition of Theorem 1.

To see why some strengthening of (1.1) was needed for this result, consider the two-person extensive form game and associated normal form illustrated in Fig. 1. Figure 2 illustrates a copy of Φ for this game, and displays the gradient field for a particular dynamic. This is a myopic adjustment dynamic under which the set $T \times \Phi_2$ is asymptotically stable. It is not strict because it includes elements of $T \times \{\sigma_2 \mid \sigma_2(R) < 1/2\}$ as rest points.

The Main Theorem

We turn to our main result.

THEOREM 1. *Let (S, π) be a game, let $\Theta \subseteq \Phi$ be asymptotically stable under a myopic adjustment dynamic F, and assume there is a neighbor-*

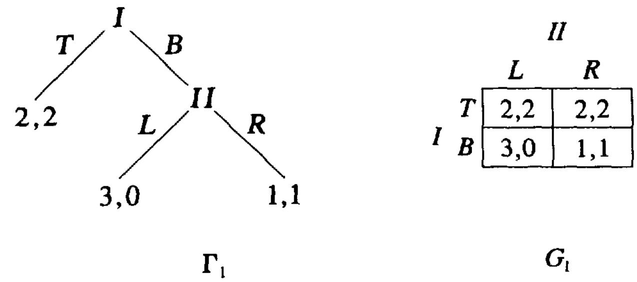

FIG. 1. An extensive form game Γ_1 and its normal form G_1.

hood U of Θ contained in the basin of attraction of Θ which is homeomorphic to Φ. Then Θ contains a hyperstable subset.

Proof. We outline a proof here. An asterisk in the proof (*) indicates that additional details for that step are contained in the appendix. We begin by defining, for any game of the form (S, ρ), where ρ may or may not equal π, a dynamic that has as its rest points precisely $N(S, \rho)$. The *canonical dynamic* for (S, ρ) has gradient field given by

$$c_\rho(\sigma)(s_i) = \max[\rho_i(\sigma\backslash s_i) - \rho_i(\sigma), 0] - \sigma_i(s_i) \sum_{t_i \in S_i} \max[\rho_i(\sigma\backslash t_i) - \rho_i(\sigma), 0]$$

for $s_i \in S_i$, $i \in N$, and $\sigma \in \Phi$. Note that c_ρ is Lipschitz and so has a unique and continuous solution C_ρ. Also $\Sigma_{s_i \in S_i} c_\rho(\sigma)(s_i) = 0$, and whenever $\sigma_i(s_i) = 0$, $c_\rho(\sigma)(s_i) \geq 0$. Thus, C_ρ maps Φ to Φ.

The first term of c_ρ increases weight on strategies for each population that would perform better than the current average in that population. The second term reduces weight on all strategies proportionately to keep

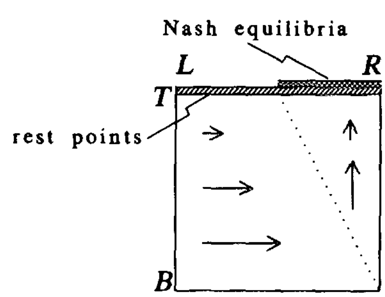

FIG. 2. A myopic adjustment dynamic for G_1 under which $T \times \Phi_2$ is asymptotically stable.

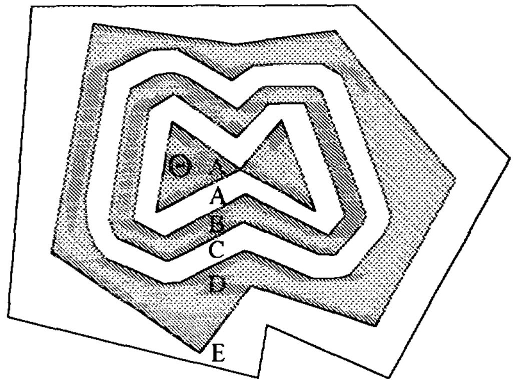

FIG. 3. The regions used in constructing the composite map. U is the entire set. V is $A \cup B \cup C \cup D$.

the system within Φ. Thus, $c_\rho(\sigma) \cdot \rho(\sigma\backslash.) \geq 0$, with equality if and only if $\sigma \in N(S, \rho)$.

We show that as $\rho \to \pi$, (S, ρ) has Nash equilibria converging to Θ. We do this by showing that for ρ close enough to π we can splice C_ρ with F in such a way that (1) the spliced map inherits enough of the structure of F to guarantee a fixed point on U, and (2) any such fixed point is near Θ and also a fixed point of C_ρ. In the appendix, we show how to extend this argument to cover the addition of redundant strategies to S.

Begin by choosing V, a closed neighborhood of Θ with $V \subseteq \text{Int}(U)$ and such that $F(V, t) \subseteq U$ for all $t \geq 0$ (*). Choose $T \geq 0$ such that $F(U, t) \subseteq U$ for all $t \geq T$ (*). Choose $\varepsilon > 0$ such that $B_{3\varepsilon}(\Theta) \subseteq \text{Int}(V)$. We will be interested in the following five subsets of U:

A	$B_\varepsilon(\Theta)$
B	$\{\gamma \mid \varepsilon \leq D(\Theta, \gamma) \leq 2\varepsilon\}$
C	$\{\gamma \mid 2\varepsilon \leq D(\Theta, \gamma)\} \leq 3\varepsilon\}$
D	$\text{Cl}(V \backslash B_{3\varepsilon}(\Theta))$
E	$\text{Cl}(U \backslash V)$.

The various regions are displayed in Fig. 3.

A and C are disjoint closed sets. Thus by Urysohn's Lemma, there is α: $U \to [0, 1]$ with $\alpha(C) = 1$, $\alpha(A) = 0$, and α continuous. Similarly, C and E are disjoint and closed, so there is β: $U \to [0, 1]$ with $\beta(E) = 1$, $\beta(C) = 0$, and β continuous.

We now define the spliced map.[10] For given ρ, consider G_ρ: $U \times \mathbb{R}_+ \to \Phi$ given by

$$G_\rho(\sigma, t) = \left\{ \begin{array}{ll} C_\rho(\sigma, t) & \text{on } A \\ \alpha(\sigma)F(\sigma, t) + [1 - \alpha(\sigma)]C_\rho(\sigma, t) & \text{on } B \\ F(\sigma, t) & \text{on } C \\ F(\sigma, \beta(\sigma)T + [1 - \beta(\sigma)]t) & \text{on } D \\ F(\sigma, T) & \text{on } E \end{array} \right\}.$$

Note that the splice on B takes place in the range, while the splice on D takes place in the domain.

For small $t > 0$, $G_\rho(., t)$ is a continuous map from U to U (*). Since U is homeomorphic to Φ, $G_\rho(., t)$ has a fixed point σ^t for each such t by Brouwer's fixed point theorem.[11] If $\gamma \in C \cup D \cup E$ is a fixed point of $G_\rho(., t)$, then there is $t' > 0$ such that $F(\gamma, t') = \gamma$. This is impossible as $C \cup D \cup E \subseteq U \backslash \Theta$ and U is in the basin of attraction of Θ under F. Thus $\sigma^t \in A \cup B$.

Let σ_ρ be an accumulation point of $\{\sigma^t\}_{\downarrow 0}$. Then, σ_ρ is a rest point of G_ρ; i.e., $G_\rho(\sigma_\rho, t') = \sigma_\rho \ \forall t' \geq 0$ (*). Let σ be a cluster point of $\{\sigma^\rho\}_{\rho \to \pi}$. Then, σ is a rest point of G_π (*). Also, if $\alpha(\sigma) = 1$, then $\sigma \in B$, and $G_\pi(\sigma,.) = F(\sigma,.)$. Since F has no rest points on B, $\alpha(\sigma) < 1$. So consider

$$\begin{aligned} g_\pi(\sigma) &\equiv \lim_{t \downarrow 0} \frac{G_\pi(\sigma, t) - \sigma}{t} \\ &= \alpha(\sigma)f(\sigma) + [1 - \alpha(\sigma)]c_\pi(\sigma). \end{aligned}$$

Since σ is a rest point of G, $g_\pi(\sigma) = 0$, and thus $g_\pi(\sigma) \cdot \pi(\sigma\backslash.) = 0$. By (1.1), $f(\sigma) \cdot \pi(\sigma\backslash.) \geq 0$. Thus, since $\alpha(\sigma) < 1$, it must be that $c_\pi(\sigma) \cdot \pi(\sigma\backslash.) = 0$. But, then $\sigma \in N(S, \pi)$. By (1.2), $N(S, \pi) \cap (A \cup B) \subseteq \Theta$ and thus, $\sigma \in \Theta$. So, as $\rho \to \pi$, $\sigma^\rho \to \Theta$. Since $G = C_\rho$ on A, $\sigma^\rho \in N(S, \rho)$ for ρ close to π, and so we are done. ∎

If Θ is convex, then for ε sufficiently small $U = B_\varepsilon(\Theta)$ satisfies the

[10] The technique used here, of cutting out part of a dynamic system and replacing it by another, is sometimes referred to as surgery.

[11] The homeomorphism assumption on U and the assumption that F is a continuous function are for the sake of Brouwer's fixed point theorem. By virtue of the Eilenberg–Montgomery fixed point theorem this could be weakened to U being an acyclic absolute normal retract and F an upper hemicontinuous acyclic correspondence (see Border, 1985, p. 73). It seems unlikely that the relaxation of the condition on U is of practical significance. Extending the results to set valued dynamics would be valuable. However, as we discuss in the working paper, acyclicity is very strong in this context.

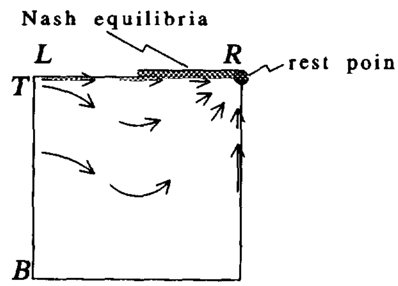

FIG. 4. A dynamic for G_1 under which (T, R) is asymptotically stable.

conditions of Theorem 1.[12] This establishes that asymptotically stable sets and asymptotically stable outcomes for strict myopic adjustment dynamics on two-person games are hyperstable. A better understanding of which games have asymptotically stable sets admitting an appropriate U would be very desirable.

Note that (1.1) was only needed on a set playing the role of B. This observation could be a first step toward results about dynamics that are only approximately myopic in the sense that players whose play is very close to optimal may move in nonimproving directions. Since B is a closed set containing no Nash equilibria, there is a strictly positive lower bound on B for the amount by which some player is short of an optimum. If (1.1) holds when σ_i is suboptimal by at least this amount, then the analysis goes through. Note also that the continuity of F was only necessary on U. In some interesting examples a given dynamic will be discontinuous only on the boundaries of the basin of attraction. See the working paper (Swinkels, 1992c) for an example and discussion.

V. ALTERNATIVE SPECIFICATIONS OF REST POINTS

In general, a set Θ that is asymptotically stable under an admissible dynamic satisfying (1.1) but not (1.2) need not contain a hyperstable subset. Consider G_1 (Fig. 1) along with the dynamic of Fig. 4. While (T, R) is asymptotically stable under this dynamic, and the dynamic satisfies (1.1), G_1 has as its unique hyperstable set $\{\sigma \mid \sigma_1(T) = 1, \sigma_2(L) \geq \frac{1}{2}\}$. It is easy to see where the proof of Theorem 1 fails: For small perturbations in which $\rho_2(T, L) > \rho_2(T, R)$ the canonical dynamic must travel from right to left on $T \times \Phi_2$, while the dynamic of Fig. 4 travels from left to right.

[12] $B_\varepsilon(\Theta)$ is a full dimensional, closed convex subset of Φ, and so homeomorphic to Φ. For ε small, $B_\varepsilon(\Theta)$ is a subset of the basin of attraction of Θ.

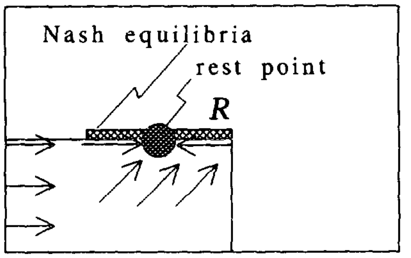

FIG. 5. A dynamic for G_1 illustrating the need for continuity of f in Theorem 2. Condition (1.1) is satisfied, but ζ is an asymptotically stable point that is not KM stable. Note that f is discontinuous as $\sigma(T) \rightarrow 1$.

These cancel each other somewhere on $T \times \Phi_2$ near (T, R), while the only Nash equilibria of the perturbed game has $\sigma_2(L) \approx \frac{1}{2}$.

The strategy profile (T, R) is KM stable. This holds in general if f is Lipschitz continuous.

THEOREM 2. *Let (S, π) be a game, let $\Theta \subseteq \Phi$ be asymptotically stable under an admissible dynamic F satisfying Eq.* (1.1) *and such that the associated f is Lipschitz continuous, and assume there is a neighborhood U of Θ contained in the basin of attraction of Θ which is homeomorphic to Φ. Then, Θ contains a KM stable subset.*

Assume that a particular outcome ζ in a two-person game is asymptotically stable under a dynamic that satisfies (1.1) strictly whenever $\sigma_i \notin \mathrm{BR}_i(\sigma)$. If the basin of attraction is large enough to include a neighborhood of the set of Nash equilibria supporting ζ, then ζ is KM stable. As before, the idea in proving this is to appeal to the convexity of sets of Nash equilibria corresponding to a particular outcome in two-person extensive form games. Since the dynamic is strict, the asymptotically stable set must be a subset of the Nash equilibria supporting that outcome. Thus, if the basin of attraction includes a neighborhood of the set Δ of Nash equilibria supporting the outcome, then, for $\varepsilon > 0$ sufficiently small, $B_\varepsilon(\Delta)$ will be the necessary U.

The dynamic for G_1 illustrated in Fig. 5 illustrates the need for the continuity of f. F is continuous in initial conditions and time and satisfies (1.1). Since ζ involves a weakly dominated strategy, it is not KM stable. The construction used in proving Theorem 2 fails for this example: For any perturbed dynamic of the sort used in that proof, (T, R) becomes the unique asymptotically stable set.

The dynamic of Fig. 4 violated (1.2) only in that it eliminated a weakly dominated strategy. For dynamics of this sort, we can reclaim the KM stability implication without the extra condition of Lipschitz continuity on f.

THEOREM 3. *Let (S, π) be a game. For each $i \in N$, let $R_i \in S_i$ be a set of weakly dominated strategies, and assume F is such that $f(\sigma)(r_i) \leq 0$ for all $\sigma \in \Phi$, $r_i \in R_i$. Assume that f satisfies* (1.1), *and satisfies* (1.2) *for any Nash equilibrium not involving $\cup_{i \in N} R_i$. If Θ is asymptotically stable under F, and there is U, a neighborhood of Θ contained in the basin of attraction of Θ and homeomorphic to Φ, then Θ contains a subset that is KM stable.*

VI. Extensions

VI.1 *Richer State Spaces*

The analysis generalizes almost immediately to a state space of the form $\Phi \times \Psi$, where Ψ is a compact convex subset of a Banach space. Note that compactness of Ψ rules out time as a dimension of the state space. We discuss time-varying dynamics in the next section.

A *generalized state space dynamic* is a map $F: \Phi \times \Psi \times \mathbb{R}_+ \rightarrow \Phi \times \Psi$. Define $P_\Phi: \Phi \times \Psi \rightarrow \Phi$ as the projection map onto Φ. A generalized state space dynamic F is *admissible* if F is continuous, and $P_\Phi(F)$ is right differentiable with respect to time. That is,

$$f(\sigma, \psi) \equiv \lim_{t \downarrow 0} \frac{P_\Phi(F(\sigma, \psi, t)) - \sigma}{t}$$

is well defined for all $(\sigma, \psi) \in \Phi \times \Psi$.

An admissible generalized state space dynamic F is a *myopic adjustment dynamic* if for all $(\sigma, \psi) \in \Phi \times \Psi$, $f_i(\sigma, \psi)$ satisfies (1.1) and (1.2). Thus, we allow the other dimensions of the state space to determine which myopically improving direction is chosen at each point, but not to affect myopic improvement.

Theorem 1 goes through with little change in this framework.

THEOREM 4. *Let (S, π) be a game, and let Ψ be a compact convex subset of a Banach space. Let $\Theta \subseteq \Phi \times \Psi$ be asymptotically stable under a generalized state space myopic adjustment dynamic F, and assume there is a neighborhood U of Θ contained in the basin of attraction of Θ which is homeomorphic to $\Phi \times \Psi$. Then, $P_\Phi(\Theta)$ contains a hyperstable subset.*

The proof is much like that of Theorem 1, except that the appropriate fixed point theorem is Schauder's rather than Brouwer's. As for Theorem 1, Theorem 4 has a much simpler form for asymptotically stable points. In particular, if $\sigma \times \psi \in \Phi \times \Psi$ is asymptotically stable under a generalized state space myopic adjustment dynamic, then $\{\sigma\}$ is hyperstable.

Similarly, consider dynamics such that $f_i(\sigma, \psi)$ satisfies (1.1) strictly whenever $\sigma_i \notin BR_i(\sigma)$, and such that whenever σ is Nash, (σ, ψ) is a rest point (this is stronger than applying (1.2) to $f_i(\sigma, \psi)$, since ψ could be in motion even though σ is not). Then, for two-person extensive form games, asymptotically stable outcomes are hyperstable. Under these assumptions, $P_\Phi(\Theta)$ exactly equals the set of Nash equilibria supporting the outcome, and Θ has the form $P_\Phi(\Theta) \times \Psi$, and so is convex.

As an example of an application of this analysis, consider a game in which each population is made up of several factions who follow different myopic adjustment rules. In addition, there is a deterministic and continuous process by which some players switch from one faction to another, based (for example) on the average performance of the various factions over the last T units of time. The state space of this process is $\Phi \times \Psi$, where Ψ is made up of a copy of Φ_i and a compact interval of the real line (representing the range of possible average payoffs) for each faction of population i, and so meets the conditions of our analysis.

VI.2. *Time Varying Dynamics*

Consider dynamics of the form $F\colon \Phi \times \mathbb{R}_+ \times \mathbb{R}_+ \to \Phi$, with the interpretation that if the system is at $\sigma \in \Phi$ at time t, then at time $t + t'$, it is at $F(\sigma, t, t')$. For such a dynamic, it is quite consistent that $F(\sigma, 0, t') = \sigma$ for some $t' > 0$ and $\sigma \in U\backslash\Theta$, but that nonetheless $\lim_{t\to\infty} D(F(\sigma, 0, t), \Theta) = 0$. This prevents us from applying the analysis of Theorem 1: If we define G in the natural way as

$$G(\sigma, t) = \left\{ \begin{array}{ll} C_p(\sigma, t) & \text{on } A \\ \alpha(\sigma)F(\sigma, 0, t) + [1 - \alpha(\sigma)]C_p(\sigma, t) & \text{on } B \\ F(\sigma, 0, t) & \text{on } C \\ F(\sigma, 0, \beta(\sigma)T + [1 - \beta(\sigma)]t) & \text{on } D \\ F(\sigma, 0, T) & \text{on } E \end{array} \right\},$$

then $G(., t)$ could have fixed points on $C \cup D \cup E$ and so need not have fixed points on $A \cup B$.

To recapture Theorem 1, we must somehow rule out $F(\sigma, 0, t') = \sigma$ for $t' > 0$ and $\sigma \in U\backslash\Theta$. For one special case, this is easily done. Given a dynamic F, assume one can find $\hat{F}$, a time invariant dynamic, and

$k: \mathbb{R}_+ \to \mathbb{R}_+$, a strictly increasing bijection, such that $F(\sigma, t, t') = \hat{F}(\sigma, k(t') - k(t))$. Thus, if ever $F(\sigma, 0, t) = \sigma$ for $t > 0$ and $\sigma \in \Phi$, then for every $T > 0$ there is $t' > T$ such that $F(\sigma, 0, t') = \sigma$. For $\sigma \in U \backslash \Theta$, this contradicts asymptotic stability, and so cannot happen.

Another situation in which one can conclude that such cycles cannot occur is when the system admits a Lyapunov function.[13] Whether there are more general conditions implying acyclicity of $F(.)$ on $U \backslash \Theta$ is an open question.

VI.3. *A Continuous Time Version of Fictitious Play*

Consider a model in which players react not to actual play, but rather to a prediction of play. Those players who are called upon to play at any instant choose a best response to this prediction (assume that players use some rule that is a function only of their current prediction to select among multiple best responses). Finally, the equation of motion of the prediction held about population i is of the form

$$\frac{\partial \sigma^p(t)}{\partial t} = \nu(t) \times (\sigma(t) - \sigma^p(t)),$$

where $\sigma^p(t)$ is predicted play at time t, $\sigma(t)$ is actual play at time t, and $\nu(t)$ is a positive scalar for all t. Setting $\nu(t) = 1/t$ generates a continuous time version of a fictitious play model (take $t \in [1, \infty)$ to avoid definitional difficulties).

Actual play is almost certainly discontinous, in both time and initial conditions, and in general is not myopically improving relative to current actual play. Perceived play is considerably better behaved. Population i plays best responses to the perception of population $-i$'s play. Population $-i$'s perception of population i's play moves in the direction of i's actual play. Thus, $-i$'s perception of i's play moves in a direction that is myopically improving relative to i's perception of $-i$'s play, i.e., perceived play satisfies (1.1). These perceptions are continuous in time, and may well in particular examples be continuous in initial conditions, at least on a region of some asymptotically stable set. Finally, if $\nu(t)$ does not go to zero too quickly then the analysis of time-varying dynamics in the last section will apply. In such examples, we can thus conclude that if perceived play is asymptotically stable (and satisfies the other conditions of Theorem 1 or

[13] Say that Λ is a Lyapunov function for the set $\Theta \subseteq \Phi$ relative to the dynamic system F if the set of minimizers of Λ is precisely Θ, and $\dot{\Lambda}(\sigma) \equiv \partial \Lambda(F(\sigma, t', t)/\partial t|_{t=t'} < 0$ for all $\{\sigma, t'\} \in U \backslash \Theta \times \mathbb{R}_+$, where the derivative is interpreted as a right-hand derivative if needed. If such a Λ exists, then Θ is asymptotically stable under F and F has no recurrent points in $U \backslash \Theta$.

2), then the asymptotically stable set of perceived plays will contain a strategically stable subset. If actual play converges to some convex region, then perceived play will as well.[14]

VI.4. *Discrete Time Dynamics*

We will consider simple discrete time dynamics with state space Φ. Such a dynamic can be represented by a map $F: \Phi \rightarrow \Phi$, with the interpretation that $\sigma \in \Phi$ is carried to $F^t(\sigma)$ in t periods. We assume F is continuous. Paralleling the continuous time case, a discrete time dynamic F is a *myopic adjustment dynamic* if for all $\sigma \in \Phi$,

$$\pi_i(\sigma \backslash F_i(\sigma)) \geq \pi_i(\sigma) \quad \text{for all } i \in N; \tag{3.1}$$

$$\text{if } \sigma \in N(S, \pi) \text{ then } F(\sigma) = \sigma. \tag{3.2}$$

Say that X is forward invariant under F if $F^t(X) \subseteq X \ \forall t \geq 0$. We then have:

THEOREM 5. *Let (S, π) be a game, let $\Theta \subseteq \Phi$ be asymptotically stable under a discrete time myopic adjustment dynamic F, and assume there is U a compact convex neighborhood of Θ such that U is in the basin of attraction of Θ under F, and U is forward invariant under F. Then Θ contains a hyperstable subset.*

The proof is similar in structure to that of Theorem 1. The major task is in constructing an analog to the canonical dynamic for the discrete time environment. By strengthening the restriction on U we are able to ignore any sort of splice from F^t to F, and so only need regions corresponding to regions A, B, and C in the proof of Theorem 1. To see why we had to do this, consider attempting to mimic the construction of Theorem 1 without the condition that U be forward invariant. It is easily shown that there is T such that $F^t(U) \subseteq U \ \forall t \geq T$. Could one then splice F^T and F together in some manner to create an aggregate map that is onto U? Since F^t is only defined for positive integers, splicing continuously in the domain is impossible. The obvious alternative is to splice in the range. That is, one might consider a splice of the form

$$\beta(\sigma)F^T(\sigma) + [1 - \beta(\sigma)]F(\sigma),$$

replacing $F(\sigma, \beta(\sigma)T + [1 - \beta(\sigma)]t)$ in the definition of G in Theorem

[14] We phrase this in terms of convergence of actual play because it is possible for perceived play to converge even though actual play does not. We find the results much less relevant in that case.

1. The difficulty is that there does not seem to be any natural condition ruling out $\sigma = \beta(\sigma)F^T(\sigma) + [1 - \beta(\sigma)]F(\sigma)$. Unlike when we convexified in the range, we can no longer conclude that such a fixed point of the map would correspond to a cycle of F.

The assumption that F is continuous is particularly strong in this context. Consider a model in which in each period one of a finite set of players switches pure strategies based on the current population strategy profile. The associated dynamic will be discontinuous on any boundary between regions where different strategies are chosen.

VII. Conclusion

We have shown that there are conditions under which asymptotic stability of behavior under a dynamic adjustment process can imply behavior that is as if the members of the economy satisfied the rationality and commonality of beliefs assumptions that underlie traditional game theory. The results in this paper, while hardly complete, do cover a wide class of situations.

It seems likely that the analysis could be adapted to dynamics that have other state spaces or do not satisfy the myopic improvement condition (1.1). Given a dynamic F on some state space, the key is the construction of another dynamic for nearby games that (a) stops only on states that correspond in some way to Nash equilibria, and (b) does not cancel F on some region playing the role of B in the proof of Theorem 1. The combination of the canonical dynamic and the myopic adjustment condition is one way of doing this. It seems likely that there are others.

The existing treatment of discrete dynamics is unsatisfactory. It seems possible that a different framework might prove productive. Finally, a treatment of stochastic dynamics would be valuable.

Appendix

Details of the proof of theorem 1. *Existence of* V: By (2.1), there is some neighborhood V' of Θ with $F(V', t) \subseteq U\ \forall t \geq 0$. Since Θ is closed and thus compact, we can choose $\varepsilon > 0$ such that $B_\varepsilon(\Theta) \subseteq V'$. Since $B_\varepsilon(\Theta)$ is a closed neighborhood of Θ, it is the necessary V.

Existence of T: For each $\gamma \in U$, define $T(\gamma) = \inf\{t \geq 0 \mid F(\gamma, t) \in V\}$. If $T(.)$ is bounded on U we are finished since $F(V, t) \subseteq U\ \forall t$. Assume $T(.)$ is not bounded on U. Since U is compact, there is $\{\gamma^k\}_{k \in \mathbb{N}} \to \gamma$, with $\gamma^k, \gamma \in U$ and such that $\lim_{k \to \infty} T(\gamma^k) = \infty$. Since V is a neighborhood of Θ, there is T' such that $F(\gamma, T') \in \mathrm{Int}(V)$. But by the continuity of F, $F(\gamma^k, T') \in V$ for k sufficiently large, contradicting $\lim_{k \to \infty} T(\gamma^k) = \infty$.

Properties of G_ρ: From the continuity of C_ρ, α, F, and β, and from the agreement of the appropriate functions on $A \cap B$, $B \cap C$, $C \cap D$, and $D \cap E$, G_ρ is continuous. Also,

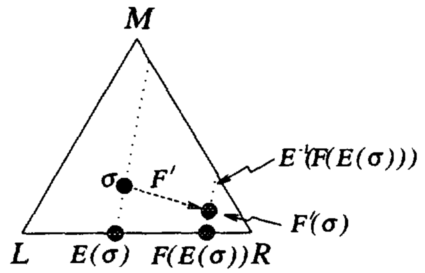

FIG. 6. The derivation of F'. Φ' is the full simplex. Φ is the simplex with pure elements L and R. In this case, M is equivalent to $2/3L + 1/3R$. Dashed lines are E-equivalence classes of Φ'.

$B_\varepsilon(A \cup B) \subseteq U$, and $G_\rho(.,0)$ is the identity map. Thus, $G_\rho(A \cup B, t) \subseteq U$ for t sufficiently small. Since $C \cup D \subseteq V$, $G_\rho(C \cup D, t) \subseteq U$ for all t. Finally, $G_\rho(E, t) \subseteq U$ by choice of T.

Rest point property of σ_ρ: Fix $t' > 0$. For any $t \in \mathbb{R}_+$, define $r(t) = \min\{t^* \mid t^* \geq 0, t^* = t' - kt \text{ for } k \in \mathbb{N}\}$. By definition of σ', $G_\rho(\sigma', t') = G_\rho(\sigma', r(t))$. Since $\lim_{t\downarrow 0} r(t) = 0$, and since G_ρ is continuous, we have

$$G_\rho(\sigma_\rho, t') = \lim_{t\downarrow 0} G_\rho(\sigma', t') = \lim_{t\downarrow 0} G_\rho(\sigma', r(t)) = G_\rho(\sigma_\rho, 0) = \sigma_\rho.$$

Since $t' > 0$ was arbitrary, we are done.

Rest point property of σ: As $\rho \to \pi$, $\| c_\rho - c_\pi \| \to 0$. Thus, as c_ρ is Lipschitz continuous, $C_\rho \to C_\pi$ pointwise (cf. Coddington and Levinson, 1955, p. 8, Theorem 2.1, or Eq. (†) below). But, then $G_\rho \to G_\pi$, and so σ is a rest point of G_π.

Hyperstability: Consider adding a redundant strategy to (S, π). That is for some i, augment S_i with a pure strategy r_i, where r_i is equivalent to some mixture $\gamma_i \in \Phi_i$. Let $S' = S_{-i} \times S_i \cup r$, let π' be the appropriately augmented payoff function, and let Φ' be the space of mixed strategy profiles for S'. For $\sigma \in \Phi'$, let $E(\sigma)$ be the strategy profile in Φ which is equivalent to σ. That is, define E: $\Phi' \to \Phi$ by $E(\sigma) = \sigma \backslash \phi_i$ where

$$\phi_i(s_i) = \sigma_i(s_i) + \sigma_i(r_i)\gamma_i(s_i) \quad \text{for } s_i \in S_i.$$

Let $\Theta \in \Phi$ be asymptotically stable under a myopic adjustment dynamic F for (S, π). To extend the result to hyperstability, we will show that $E^{-1}(\Theta)$ satisfies the conditions of the preceding analysis, and so is strategically stable relative to payoff perturbations in (S', π'). Since the argument can be repeated a finite number of times, this implies that Θ is hyperstable in (S, π).

So, for $\sigma \in \Phi'$, define

$$F'(\sigma, t) = \underset{\eta \in E^{-1}(F(E(\sigma), t))}{\operatorname{argmin}} D(\eta, \sigma).$$

That is, for any σ in Φ', first project back into Φ, then translate the projection by F, and finally return to Φ' by taking the point in Φ' that is closest to σ subject to being equivalent to the translated projection. Figure 6 may help.

Then, $E(F'(\sigma, t)) = F(E(\sigma), t)$, so that F' operates on E-equivalence classes of Φ' in the same way as F operates on Φ. From this, we conclude that $E^{-1}(\Theta)$ is asymptotically stable under F'. Continuity of F' is obvious as $E^{-1}(F(E(\sigma), t)$ is a continuous correspondence in σ and t, while $D(.,.)$ is strictly concave. Seeing that f' is well defined is a little involved. However, note that

$$\lim_{t \downarrow 0} \left[\frac{F'(\sigma, t) - \sigma}{t} \right] \cdot \pi'(\sigma \backslash .)$$

is well defined and equal to $f(E(\sigma)) \cdot \pi(E(\sigma)\backslash .) \geq 0$ even if f' is not well defined. As the only use of right differentiability and (1.1) in the previous analysis was to ensure that this product is well defined and nonnegative, we can avoid the direct proof. Also note that if σ is a Nash quilibrium of (S', π'), then $E(\sigma)$ is a Nash equilibrium of (S, π). Because F satisfies (1.2), $F(E(\sigma), t) = E(\sigma)$ for all $t \geq 0$, from which we conclude that $F'(\sigma, t) = \sigma$ for all $t \geq 0$, so that (1.2) is satisfied by F'. Finally, $E^{-1}(U)$ is homeomorphic to Φ' (this is rather intuitive, but tedious to prove: see the appendix in Swinkels, 1992c). Taken together, this implies that the previous analysis carries through for $E^{-1}(\Theta)$ and F'.

Proof of Theorem 2. Let ρ by an arbitrary KM perturbed payoff function. Then, there is $\delta = (\delta_1, \ldots, \delta_n)$, with $\delta_i \in (0, 1)$ $\forall i \in N$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i \in \text{Int}(\Delta_i)$ $\forall i \in N$, such that for each $\sigma \in \Phi$,

$$\rho(\sigma) = \pi((1 - \delta)\sigma + \delta\gamma),$$

where $(1 - \delta)\sigma + \delta\gamma$ is a convenient shorthand for

$$((1 - \delta_1)\sigma_1 + \delta_1\gamma_1, \ldots, (1 - \delta_n)\sigma_n + \delta_n\gamma_n).$$

For $\sigma \in \Phi$, define $f_\rho(\sigma)$ by

$$f_{\rho i}(\sigma) = f_i((1 - \delta)\sigma + \delta\gamma\backslash\sigma_i), i \in N. \qquad (*)$$

Then, f_ρ inherits Lipschitz continuity from f. If $\sigma_i(s_i) = 0$, then $((1 - \delta)\sigma + \delta\gamma\backslash\sigma_i)_i(s_i) = 0$, and so

$$f_{\rho i}(\sigma)(s_i) = f_i((1 - \delta)\sigma + \delta\gamma\backslash\sigma_i)(s_i) \geq 0$$

since F is onto Φ. Further, $\Sigma_{s_i \in S_i} f_{\rho i}(\sigma)(s_i) = 0$. Thus f_ρ has a unique and continuous solution F_ρ: $\Phi \times \mathbb{R}_+ \rightarrow \Phi$.

Finally,

$$\begin{aligned} f_{\rho i}(\sigma) \cdot \rho_i(\sigma) &= f_i((1 - \delta)\sigma + \delta\gamma\backslash\sigma_i) \cdot \pi_i((1 - \delta)\sigma + \delta\gamma\backslash .) \\ &= f_i((1 - \delta)\sigma + \delta\gamma\backslash\sigma_i) \cdot \pi_i(((1 - \delta)\sigma + \delta\gamma\backslash\sigma_i)\backslash .) \\ &\geq 0 \end{aligned}$$

since f satisfies (1.1). So, f_ρ satisfies (1.1) relative to ρ.

Let $\rho \rightarrow \pi$. From (*) and continuity of f, $f_\rho \rightarrow f$ pointwise. Since Φ is compact, and each f_ρ and f is continuous, $\|f_\rho - f\| \rightarrow 0$.

Choose V a neighborhood of Θ such that $F(V, t) \subseteq U \; \forall t \geq 0$. Choose any $\lambda > 0$ such that $B_\lambda(\Theta) \subseteq \text{Int}(V)$. We will show that sufficiently close by KM perturbed games (S, ρ) have Nash equilibria in $B_{2\lambda}(\Theta)$. Now, for each ρ, F_ρ is an approximate solution to F. Thus, (see, for example, Coddington and Levinson, 1955, page 8, Theorem 2.1) for all $t \geq 0$, and for all $\sigma \in \Phi$,

$$|F(\sigma, t) - F_\rho(\sigma, t)| \leq \frac{\|f - f_\rho\|}{b}[e^{bt} - 1], \qquad (\dagger)$$

where b is the Lipschitz coefficient for f. Choose $T > 0$ such that $F(U, t) \subseteq B_{\lambda/2}(\Theta) \; \forall t \geq T$. Then in particular, if f_ρ is sufficiently close to f, then $\forall t \leq 2T$, and $\forall \gamma \in$ U,

$$\|F(\gamma, t) - F_\rho(\gamma, t)\| \leq \lambda/2.$$

Choose any such ρ. Consider any $t \geq T$ and $\gamma \in U$. Now, $F(\gamma, T) \in B_{\lambda/2}(\Theta)$ and $\|F(\gamma, T) - F_\rho(\gamma, T)\| \leq \lambda/2$ by construction. Thus, $F_\rho(\gamma, T) \in B_\lambda(\Theta)$. Since $B_\lambda(\Theta) \subseteq V$, we can apply the same argument repeatedly to conclude $F_\rho(\gamma, kT) \in B_\lambda(\Theta)$, where k is the integer such that $2T > t\text{-}kT \geq T$. Now, since $t\text{-}kt \geq T$, $F[F_\rho(\gamma, kT), t\text{-}kT] \in B_{\lambda/2}(\Theta)$, and since $t\text{-}kT < 2T$,

$$\|F_\rho[F_\rho(\gamma, kT), t\text{-}kT] - F[F_\rho(\gamma, kT), t\text{-}kT]\| \leq \lambda/2.$$

Finally, note that $F_\rho[F_\rho(\gamma, kT), t\text{-}kT] = F_\rho(\gamma, t)$. Thus, $F_\rho(U, t) \subseteq V \; \forall t \geq T$. Also, F_ρ has no rest points on $U \backslash B_\lambda(\Theta)$.

Choose any such ρ. Subdivide U and define G_ρ as in the proof of Theorem 1, with $B_\lambda(\Theta)$ playing the role of Θ and F_ρ the role of F, and with T taken from the above construction. By choosing ε small enough in this construction, this can be done such that $A \cup B \subseteq B_{2\lambda}(\Theta)$. By the same analysis as in Theorem 1, there is $\sigma \in A \cup B$, σ a rest point of G_ρ. Then $g_\rho(\sigma) = 0$ and so

$$(1 - \alpha(\sigma))c_\rho(\sigma) \cdot \rho(\sigma\backslash.) + \alpha(\sigma)f_\rho(\sigma) \cdot \rho(\sigma\backslash.) = g_\rho(\sigma) \cdot \rho(\sigma\backslash.) = 0.$$

The second term of the LHS is nonnegative since f_ρ satisfies (1.1) relative to ρ. Thus, the first term of the LHS must be 0. As in Theorem 1, $\alpha(\sigma) < 1$ must hold, since $\alpha(\sigma) = 0$ would imply a rest point of F_ρ on $B \subseteq U \backslash B_\lambda(\Theta)$. Thus $c_\rho(\sigma) \cdot \rho(\sigma\backslash.) = 0$ and so $\sigma \in N(S, \rho)$. Since $A \cup B \subseteq B_{2\lambda}(\Theta)$, we are done. ∎

Proof of Theorem 3. Let (S, π_z) be obtained from (S, π) by subtracting a positive constant z from $\pi_i(\sigma\backslash r_i)$ for $i \in N$, $\sigma \in \Phi$, and $r_i \in R_i$. In (S, π_z), elements of R_i are strictly dominated, and so there are no Nash equilibria of this game that put positive weight on $\cup_{i \in N} R_i$. Thus, F satisfies (1.2) for (S, π_z). As $f(\sigma)(r_i) \leq 0$ for $r_i \in R_i$, F also satisfies (1.1). Thus by Theorem 1, Θ contains a hyperstable subset for (S, π_z). Consider any small KM perturbation of the original game. Note that weakly dominated strategy remains weakly dominated in KM perturbations. Subtract z as before to create a small perturbation of (S, π_z) that thus has a Nash equilibrium γ near Θ. γ does not use elements of $\cup_{i \in N} R_i$. Add back z. Since the elements of R_i remain weakly dominated, γ remains a Nash equilibrium. ∎

Proof of Theorem 4. Define Q_ρ: $\Phi \times \Psi \times \mathbb{R}_+ \to \Phi \times \Psi$ by $Q_\rho(\sigma, \psi, t) = (C_\rho(\sigma, t), \psi)$. Choose $\lambda > 0$. We will show that for ρ sufficiently close to π, $N(S, \rho) \cap B_\lambda(P_\Phi(\Theta)) \neq$

$\varnothing$. Choose $V \subseteq \Phi \times \Psi$, subdivide U, and choose α and β as before. By compactness of Ψ, we can choose ε such that $B_{3\varepsilon}(\Theta) \subseteq \text{Int}(V) \cap B_\lambda(\Theta)$. Define G_ρ as before, substituting Q_ρ for C_ρ. As before, for small t, $G(., t)$ is a continuous map from U to U and so has a fixed point σ^t for each t sufficiently small. Since there is no assumption that Ψ is finite dimensional, the relevant fixed point theorem is Schauder's (see for example Deimling, 1985, p. 60) rather than Brouwer's.[15] As before, a fixed point of $G(., t)$ on $C \cup D \cup E$ is a positive length cycle of F, contradicting asymptotic stability, and so there is a fixed point σ^t of $G(., t)$ in $A \cup B$ for t sufficiently small and so (again using compactness of Ψ) a rest point σ of G.

It remains to show $\sigma \notin B$. So, for any $\gamma \in B$ consider

$$\begin{aligned} g(\gamma) &\equiv \lim_{t \downarrow 0} \frac{P_\Phi(G(\gamma, t) - \gamma)}{t} \\ &= \alpha(\gamma) f(\gamma) + [1 - \alpha(\gamma)] c_\rho(\gamma). \end{aligned}$$

This expression is identical to the corresponding expression in Theorem 1. Thus, $\sigma \in A$. But then, by definition of Q_ρ, $P_\Phi(\sigma) \in N(S, \rho)$. Finally, since $\sigma \in A \subseteq B_\lambda(\Theta)$, $P_\Phi(\sigma) \in B_\lambda(P_\Phi(\Theta))$. ■

Proof of Theorem 5. For $\sigma \in \Phi$ define $\lambda(\sigma)$ by $\lambda(\sigma) = \max\{\lambda \mid 0 \le \lambda \le 1, \sigma + \lambda c_\rho(\sigma) \in \Phi\}$. For $\Lambda \ge 0$, the *discrete canonical dynamic with scaling factor* Λ for (S, ρ) is given by

$$J_\rho(\sigma) = \sigma + \left[\frac{\lambda(\sigma)}{\Lambda}\right] c_\rho(\sigma).$$

The role of Λ will be clear shortly. By construction, $J_\rho(\sigma)$ maps Φ to Φ. Continuity of J_ρ follows from the continuity of λ and c_ρ. Since c_ρ defines a continuous dynamic remaining within the simplex, $\lambda(\sigma) > 0 \ \forall\sigma$. Thus

$$\rho_i(\sigma \backslash J_{\rho i}(\sigma)) - \rho_i(\sigma) = \left[\frac{\lambda(\sigma)}{\Lambda}\right] c_{\rho i}(\sigma) \cdot \rho(\sigma\backslash .) \ge 0$$

with equality if and only if $\sigma \in N(S, \rho)$. As before, consider an arbitrary neighborhood M of Θ. Let $\varepsilon > 0$ be such that $B_{3\varepsilon}(\Theta) \subseteq M \cap U$. Subdivide U as follows:

A	$B_\varepsilon(\Theta)$
B	$\{\gamma \mid \varepsilon \le D(\Theta, \gamma) \le 2\varepsilon\}$
C	$U \cap \{\gamma \mid 2\varepsilon \le D(\Theta, \gamma)\}$.

Choose $\alpha: U \to [0, 1]$ with $\alpha(C) = 1$, $\alpha(A) = 0$, and α continuous. Then, for any given Λ, consider the following spliced map:

[15] Schauder's theorem only requires Ψ to be a closed and bounded subset of a Banach space. We use the (stronger) compactness condition elsewhere in the proof. Schauder's theorem also requires a compact (rather than merely continuous) map. Since Ψ is a compact space, continuous maps on $\Phi \times \Psi$ are compact maps.

$$G(\sigma) = \begin{cases} J_\rho(\sigma) & \text{on } A \\ (\sigma)F(\sigma) + [1 - \alpha(\sigma)]J_\rho(\sigma) & \text{on } B \\ F(\sigma) & \text{on } C \end{cases}.$$

Continuity of G is again clear from the continuity of J_ρ, α, F, and β, and from the agreement of the appropriate functions on $A \cap B$, $B \cap C$, $C \cap D$, and $D \cap E$.

Now, c_ρ is bounded, and therefore

$$D(\sigma, J_\rho(.)) = D(0, \lambda(\sigma)c_\rho(\sigma)/\Lambda)$$

is also bounded for any fixed Λ. Choosing Λ large enough, we can conclude that $J_\rho(A \cup B) \subseteq$ U. From the convexity of U, we can conclude $G(A \cup B) \subseteq U$ and so $G(U) \subseteq U$.

We can now argue as before that $G(.)$ has a rest point in $A \cup B$, and that for ρ close enough to π, this rest point must actually be in A, and so a Nash equilibrium of (S, ρ). ∎

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