JOHN GEANAKOPLOS
Yale University

MICHAEL MAGILL
University of Southern California

MARTINE QUINZII
University of California, Davis

Demography and the Long-Run Predictability of the Stock Market

THE SECULAR MOVEMENT OF the U.S. stock market in the postwar period has been characterized by three distinct twenty-year episodes of sustained increases or decreases in real stock prices: the bull market of 1945–66, the subsequent bear market of the 1970s and early 1980s, and the bull market of the middle and late 1980s and the 1990s. Explanations of the most recent and spectacular bull market have typically been based on several factors:[1] the advent of a "new economy" in which innovations create a permanently higher rate of economic growth and an accompanying increase in the intangible capital of the corporate sector;[2] the substantial increase in participation in the market; and the apparent decrease in risk aversion of the baby-boom generation.[3] Similar arguments, based on the "new economy" created by the technical innovations of the immediate

This paper was begun during a visit at the Cowles Foundation in the fall of 2000 and revised during a visit in the fall of 2002. Michael Magill and Martine Quinzii are grateful for the stimulating environment and the research support provided by the Cowles Foundation. We are also grateful to Robert Shiller for helpful discussions and to participants at the Cowles Conference on Incomplete Markets at Yale University, the SITE Workshop at Stanford University, the Incomplete Markets Workshop at the State University of New York, Stony Brook, during the summer of 2001, the Southwest Economic Conference at the University of California, Los Angeles, and the Conference for the Advancement of Economic Theory at Rhodes, Greece, in 2003 for helpful comments. The authors claim sole responsibility for the remaining weaknesses.

1. These explanations, although couched in the language of analytical models, are essentially the same as those given by Irving Fisher (1929) for the stock market boom of the 1920s.
2. McGrattan and Prescott (2000); Hall (2000).
3. Heaton and Lucas (2000).

postwar period and increased participation in the stock market, have also been used to justify the bull market of the 1950s.[4] The period of declining stock prices from 1966 to 1982 has spawned fewer rationales, as documented by the well-known paper by Franco Modigliani and Richard Cohn.[5] They argued that real earnings and interest rates could not account for the 50 percent decline in the real Standard and Poor's (S&P) index between 1966 and 1978, and they found themselves forced to conclude that the only explanation for the sustained decrease in stock prices was that investors, at least in the presence of unaccustomed and fluctuating inflation, are unable to free themselves from certain forms of money illusion and therefore look to the nominal rather than the real rate of interest when valuing equity. Although these explanations probably capture important elements underlying the behavior of stock prices in each of the three episodes, they cannot readily be pieced together to form a coherent explanation of the stock market over the whole sixty-year period.

The idea motivating this paper is that demography is a common thread that might provide a single explanation for the alternating bull and bear markets over the whole postwar period. Since the turn of the twentieth century, live births in the United States have also gone through alternating twenty-year periods of boom and bust: for example, the low birth rate during the Great Depression and the war years was followed by the baby boom of the 1950s and early 1960s and the baby bust of the 1970s. These birth waves have resulted in systematic changes in the age composition of the population over the postwar period, roughly corresponding to the twenty-year periods of boom and bust in the stock market.

People have distinct financial needs at different periods of their life, typically borrowing when young, investing for retirement when middle-aged, and disinvesting during retirement. Stocks (along with other assets such as real estate and bonds) are a vehicle for the savings of those preparing for their retirement. It seems plausible that a large middle-aged cohort seeking to save for retirement will push up the prices of these securities, and that prices will be depressed in periods when the middle-aged cohort is small. We find that this is indeed the case in the model we develop in this paper, regardless of whether economic agents are myopic or fully aware of demography and its implications. James

4. See Malkiel (1990) and Shiller (2000).
5. Modigliani and Cohn (1979).

Poterba has argued that, if agents were rational, they would anticipate any demography-induced rise in stock prices twenty years before it happened, bidding up prices at that time and thereby negating much of the effect of demographics on stock prices.[6] We show that, in our model, if agents are myopic, blindly plowing savings into stocks when middle-aged, stock prices will be proportional to the size of the middle-aged cohort. But we also show that, when agents fully anticipate demographic trends, their rational response actually reinforces the effect on stock prices, making prices rise more than proportionally to the growth of the middle-aged cohort.

To test how much of the variation in security prices can be explained by the combination of life-cycle behavior and changing demographic structure, we study the equilibria of a cyclical, stochastic, overlapping-generations exchange economy, calibrated to the stylized facts of agents' lifetime income patterns, the payoffs of securities, and the demographic structure in the United States during the postwar period. We derive three predictions from our model, which we then compare with historical data on stock and bond returns. The first prediction is that price-earnings (PE) ratios should be proportional to the ratio of middle-aged to young adults (the MY ratio). The second is that real rates of return on equity and bonds should be an increasing function of the *change* in the MY ratio. Lastly, we show in our model that the equity premium should covary with the YM ratio (the reciprocal of the MY ratio), even though the young are more risk-tolerant than the middle-aged.

The fact that the most recent stock market boom coincided with the period in which the generation of post–World War II baby-boomers reached middle age has led Wall Street participants and the financial press to attribute part of the rise in prices to the investment behavior of baby-boomers preparing for their retirement. Professional economists, on the other hand, have been skeptical of the connection between demography and stock prices. Although Gurdip Bakshi and Zhiwu Chen documented a striking relationship between the average age of the U.S. population over twenty and the movement of the real S&P index since 1945,[7] a systematic literature studying the relationship between demography and prices of financial assets has emerged only recently. On the empirical side, Diane

6. Poterba (2001).
7. Bakshi and Chen (1994).

Macunovich found a relationship between the (smoothed) rate of change of the real Dow Jones index and the rate of change of cohort sizes,[8] and Poterba tested the relationship between various indicators of demography and prices of and returns on equity, concluding that the retiring of the baby-boom generation would have only a small effect on asset prices.[9] On the theoretical side, Robin Brooks and Andrew Abel pioneered the use of equilibrium models to study the effect of demography.[10] Both used a Diamond model with random birth rates.[11] Brooks found that demography had a small effect on real rates of return and that the equity premium shrinks when the population is relatively young. Abel's model was not calibrated, but a calibrated version of it was studied by Monika Bütler and Philipp Harms, who concluded that the variation of the labor supply could smooth out some of the effects of a demographic shock such as a baby boom.[12] Bakshi and Chen had used an infinite-horizon, representative-agent pricing model to account for the behavior of security prices, in which the age of the representative agent was the population average. A key assumption was that the relative risk aversion of the representative agent is an increasing function of the average age.[13]

Our approach and our conclusions differ from those of earlier researchers in several respects. First, we study a model in which large cohorts are deterministically followed by small cohorts in a recurring cycle, as has been the case for the past century in the United States, rather than a stochastic birth model in which a large cohort might be followed by an even larger cohort. Second, we assume preferences for which saving is relatively insensitive to interest rates. Third, we take as our reference point a model in which a fixed quantity of land produces a fixed output per period, and then move to models with endogenous capital and adjustment costs. Taking this approach, we find that the demographic effect on PE ratios is larger than our predecessors have suggested. Finally, in contrast to Brooks and Bakshi and Chen, we find that the equity premium is smaller when the population of savers is older, thus reinforcing the demographic effect, as has been the case historically.

8. Macunovich (1997, 2002).
9. Poterba (2001).
10. Brooks (1998, 2002); Abel (2001, 2003).
11. Diamond (1965).
12. Bütler and Harms (2001).
13. The existing literature has been admirably summarized by Young (2002).

The first section of the paper studies the equilibria of a simple deterministic model in which generations are alternately large and small, periods last for twenty years, and equity in a fixed asset ("land" or "trees") yields a constant stream of dividends each period. The sizes of the generations, and the dividends and wages received by the young and the middle-aged, are chosen in accordance with historical averages for the United States. This certainty model gives the order of magnitude of the change in security prices that can be attributed to demographic change: even when cohort sizes fluctuate by 50 percent, output increases by only 7 percent when the large generation is in its peak earning years—yet PE ratios increase by 130 percent. We show that the lower the intertemporal elasticity of substitution in preferences, the greater the fluctuation in equity prices.

In the second section we show that the qualitative behavior of the equilibrium is not significantly changed when the model is enriched to accommodate more realistic features such as children, Social Security, or bequests. Children and Social Security both reinforce the demographic effect on asset prices, whereas bequests attenuate it, but when all are taken together at levels calibrated to fit the U.S. data, there is not much difference. The equilibria of our model can also be related to the equilibria of the standard Diamond model with endogenous capital. By introducing adjustment costs for capital, we obtain a parameterized family of models, which includes at one extreme the Diamond model, with zero adjustment costs, and at the other extreme models with progressively higher adjustment costs whose equilibria converge to the equilibrium of the land economy. The possibility that savings can go into new capital instead of pushing up the price of existing capital reduces the demographic variation in rates of return and in equity prices. However, since there is a lag between physical investment and increased output, the variation in price-dividend ratios due to demographics can be as high in the Diamond model as in the exchange model with fixed land.

In the paper's third section we show how shortening the time periods reveals the relationship between demographic structure and security prices in its most striking form: in the stationary equilibrium, equity prices are precisely in phase with the demographic structure, attaining a maximum when the number of middle-aged agents is at a maximum and the number of young agents is at a minimum, and attaining a minimum when the cohort numbers are interchanged. Rates of return, on the other hand, are not in phase with the demographic cycle. The maximum for the rate of return occurs in the middle of the ascending phase of equity prices, when the increase in the MY ratio is at its maximum, inducing a large capital gain; the minimum rate of return occurs in the middle of the descending phase of equity prices, when the decrease in the MY ratio and the capital loss are the greatest. Thus, in the absence of shocks to the economy, a cyclical birth process translates into a cyclical behavior of equity prices and interest rates, with short-term interest rates leading equity prices by half a phase, because equity prices move with the MY ratio whereas short-term interest rates move with the change in this ratio.

In the fourth section we add uncertainty in wages and dividends to the model. In the postwar period in the United States, equity prices in bull markets have had peak-to-trough ratios of the order of 5 or 6, whereas the pure demographic model delivers increments of the order of 2 or 3. Thus "other forces" must contribute a factor of order 2.5 to 3 to the changes in stock prices. The periods in which middle-aged agents were numerous relative to the young (the 1950s and early 1960s, and the late 1980s and the 1990s) were also periods in which the economy was subject to positive shocks, whereas the period of the 1970s, when the baby-boomers were young, was marked by negative shocks (oil shortages and inflation). Thus we add business cycle shocks to incomes and dividends and calculate the stationary Markov equilibrium of the resulting economy by a method similar to that recently used by George Constantinides, John B. Donaldson, and Rajnish Mehra.[14] With these shocks, our model can deliver variations in PE ratios of the order of 5 or 6.

The equity premium (the excess return stocks earn over the riskless interest rate) is the new variable of interest in the stochastic economy. Previous work has suggested that the equity premium observed historically is difficult to reconcile with a rational expectations model, on two counts. First, the historical equity premium is too large to be rationalized by reasonable levels of risk aversion.[15] Second, and more important for us, the observation, exploited by Bakshi and Chen, that young people are more risk-tolerant than old people suggests that the equity premium should be smallest when the proportion of young people is highest, but this is exactly contrary to the historical record.[16]

14. Constantinides, Donaldson, and Mehra (2002).
15. Mehra and Prescott (1985).
16. Bakshi and Chen (1994).

Our stochastic model sheds some light on the second problem. If there is a strong demographic effect, then the numerous young (and the few contemporaneous middle-aged) should rationally anticipate that investment returns will be relatively high. Since wages and dividends do not vary as dramatically with demographic shifts as do financial returns, they should anticipate that a relatively large fraction of their future wealth will come from holding risky equity capital. Although their average risk tolerance is higher, their average exposure to risk is also higher, and so we find that in our model the equity premium is larger when stock prices are low, which is consistent with the historical record.

As for the problem that the historically observed equity premium in the United States is above the ex ante equity premium generated by standard models, we have little new to contribute. We impose limited participation in equity markets (confining such participation to 50 percent of the population, a proportion consistent with recent history), and we find that the equity premium rises in our model, while preserving the demographic effect on equity prices. As is now standard, we attribute the larger historical ex post equity premium to chance.[17]

In the paper's fifth section we compare the results of the model with the stylized facts on the bond and equity markets for the period 1910–2002. The variables that most closely fit the predictions of the model are the PE ratio and the rate of return on equity. Since 1945 the PE ratio has strikingly followed the cyclical pattern of the MY ratio in the population, whereas the rate of return on equity has a significant relationship with the *changes in* the MY ratio, as predicted by the model. The behavior of real interest rates departs much more from the predictions of the model, and only after 1965 does the real interest rate have a significant relationship with the change in the MY ratio. Moreover, interest rate variations have been smaller than in the calibrated model, with the result that the level and variability of the equity premium are greater in the data than in the model. This section of the paper also briefly presents some evidence on equity markets and demography for Germany, France, the United Kingdom, and Japan. The paper concludes with some cautionary remarks on the use of the model for predicting the future course of prices in an era of globalization of equity markets.

17. See, for example, Brown, Goetzmann, and Ross (1995).

A Simple Model with Demographic Fluctuation

Consider an overlapping-generations exchange economy with a single good (income), in which the economic life of an agent lasts for three periods: young adulthood, middle age, and retirement. All agents have the same preferences and endowments and differ only by the date at which they enter the economic scene. Their preferences over lifetime consumption streams are represented by a standard discounted sum of expected utilities:

$$(1) \qquad U(c) = E\left[u(c^y) + \delta u(c^m) + \delta^2 u(c^r)\right], \quad \delta > 0,$$

where $c = (c^y, c^m, c^r)$ denotes the random consumption stream of an agent when young, middle-aged, and retired. For the calibration, u will be taken to be a power utility function

$$u(x) = \frac{1}{1-\alpha} x^{1-\alpha}, \quad \alpha > 0,$$

where α is the coefficient of relative risk aversion (and $1/\alpha$ the intertemporal elasticity of substitution). Since a "period" in the model represents twenty years in the lifetime of an agent, we take the discount factor to be $\delta = 0.5$ (corresponding to an annual discount factor of 0.97).

In this section we outline the basic features of the model and explain how we choose average values for the calibration: these average values can be taken as the characteristics of a deterministic exchange economy whose equilibrium is easy to compute, and this provides a first approximation for the effect of demographic fluctuations on the stock market.

Each agent has an endowment $w = (w^y, w^m, 0)$, which can be interpreted as the agent's labor income in the three periods (income in retirement being zero). There are two financial instruments—a riskless bond and an equity contract—which agents can trade to redistribute their income over time (and, in the stochastic version of the model, to alter their exposure to risk). The (real) bond pays one unit of income (for sure) next period and is in zero net supply; the equity contract is an infinite-lived security in positive supply (normalized to 1), which pays a dividend each period. Agents own the financial instruments only by virtue of having bought them in the past: they are not initially in any agent's endowment. In this section the dividends and wage income are nonstochastic, so that the bond and equity are perfect substitutes; later, where we introduce random shocks to both dividends and wages, bonds and equity cease to be perfect substitutes.

Since we want to study the effect of the fluctuations in the age composition of the population on capital market prices rather than the effect of a general growth of the population, we assume that the model has been "detrended" so that the systematic sources of growth in dividends and wages arising from population growth, capital accumulation, and technical progress are factored out. The sole source of variation in total output comes from the cyclical change in the demographic structure, to which we now turn, and from the random business cycle shocks to be introduced later.

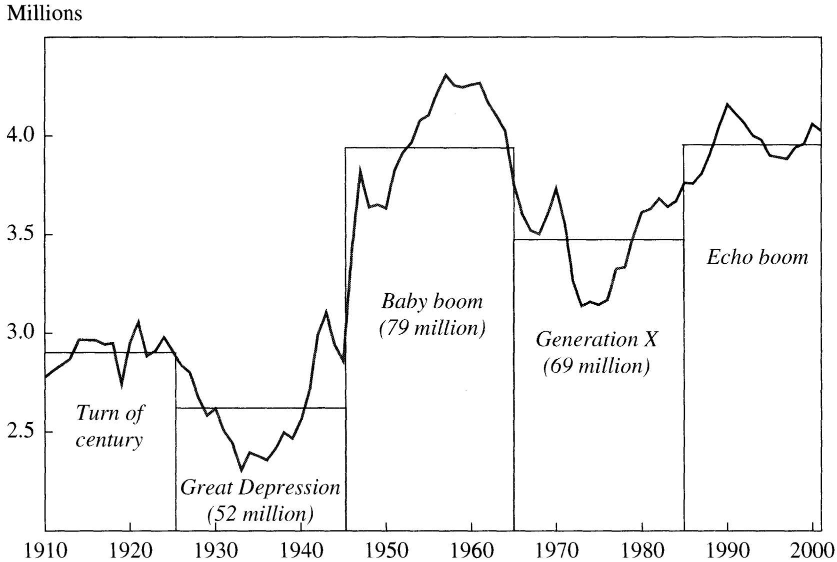
Demographic Structure

Live births in the population induce the subsequent age structure of the population: figure 1 shows annual live births for the United States during the twentieth century. If all live births over twenty consecutive years are grouped into a cohort, then the number of births can be approximated by five twenty-year periods, which create alternately large and small cohorts, as shown in figure 1.

We seek the simplest way of modeling this alternating sequence of generation sizes: time is divided into a sequence of twenty-year periods. To be commensurate with this, an individual's "biological life" is divided into four periods: from age 0 to age 19 the agent is a *child,* from 20 to 39 the agent is *young,* from 40 to 59 the agent is *middle-aged,* and from 60 to 79 the agent is *retired*. The "economic life" during which the agent earns income and trades on the financial markets consists of the last three periods. We assume that in each odd period a large cohort (N) enters the economic scene as young, and that in each even period a small cohort (n) enters. Thus there are N young, n middle-aged, and N old in every odd period (pyramid Δ_1), and n young, N middle-aged, and n old in every even period (pyramid Δ_2).

Because the typical lifetime income of an individual is low in youth, high in middle age, and low or nonexistent in retirement, agents typically seek to borrow in their youth, invest in equity and bonds in middle age, and live off this middle-age investment in their retirement. As we shall see, this life-cycle portfolio behavior implies that the relative sizes of the

Figure 1. Live Births and the Five Major Cohorts, 1910–2001[a]



Source: *Historical Statistics of the United States,* Series B1, and Bureau of the Census data.
a. Numbers in parentheses are total births, and bar heights average number of births per year, during the indicated period.

middle-aged and young cohorts play an important role in determining the behavior of equilibrium bond and equity prices. For the alternating cohort structure just described, the *medium-to-young cohort ratio* (MY ratio, for short) alternates between $n/N < 1$ in odd periods and $N/n > 1$ in even periods.

The demographic structure shown in figure 1 is not perfectly stationary. There were 52 million live births in the Great Depression generation born from 1925 to 1944, and 79 million in the baby boom from 1945 to 1964; these two generations traded with each other as middle-aged and young in the period 1965–84. Between 1965 and 1984 births fell, but only to 69 million. We refer to this baby-bust generation as "generation X" or "the Xers" for short; the baby-boom and Xer generations have traded with each other from 1985 to the present. The "echo boom generation" born since 1985 seems headed for the same order of magnitude as the baby-boom generation; this generation and the Xer generation will trade with each other from 2005 through 2024.

In order to mimic actual history with a stationary economy, we are thus led to study two cases: in the first, $n = 52$ and $N = 79$, the relative sizes of the Great Depression and baby-boom generations. This is the case for which the demographic effect is the strongest and whose equilibrium is studied in the main text of the paper. We also compute the equilibrium for a second case in which N is kept at 79 and the smaller cohort size is $n = 69$. The equilibrium values for this case are given in appendix C.

Calibrating the age pyramid using the number of live births neglects immigration, which plays an important role in the demography of the United States. We show in appendix A, however, that taking immigrants into account essentially leaves the MY ratio unchanged for the periods 1965–84 and 1985–2004, which we have taken as reference values for the calibration.

Wage Income

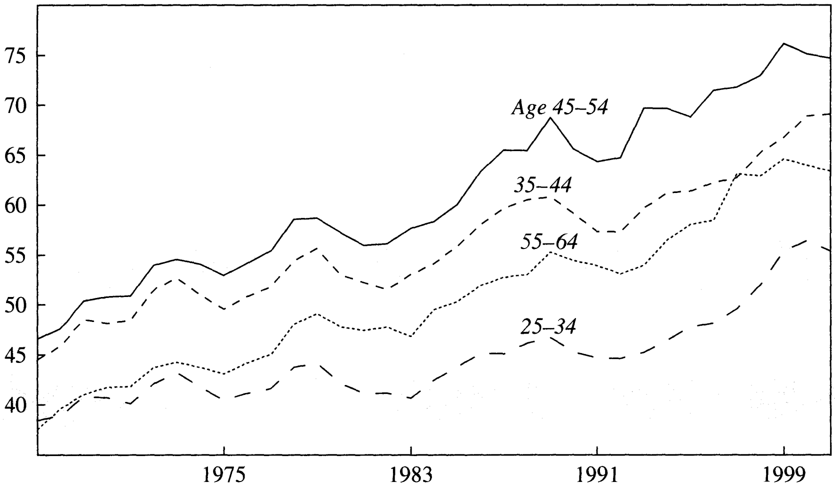
The exchange economy is viewed as an economy with fixed production plans. Equity in land or trees yields a steady stream of dividends D each period, and each young and middle-aged worker produces output w^y and w^m, respectively. To calibrate the relative shares of wage income going to young and middle-aged agents, we draw on data from the Bureau of the Census shown in figure 2: the maximum ratio of the average annual real incomes of agents in the age groups 45–54 and 25–34 is 1.54: we round this to 1.5 and calibrate the model on the basis of a wage income of $w^y = 2$ for each young agent and $w^m = 3$ for each middle-aged agent. Since the agents have homothetic (constant elasticity of substitution) preferences, the absolute levels of endowments and dividends do not influence the relative prices or relative consumption levels, which will be the primary focus of the study.

Since the wage income of middle-aged agents is greater than that of the young, the total wage is greater in even periods, when the middle-aged generation is large, than in odd periods, when the young generation is large. Since the active population is constant, this increase in wages has to be interpreted as coming from an increase in the average productivity of labor: implicitly the model presumes that the middle-aged are more experienced and productive than the young, since they are paid higher wages.

When $(N, n) = (79, 52)$, total wages alternate between $341 = (79 \times 3) + (52 \times 2)$ and $314 = (79 \times 2) + (52 \times 3)$. When the demographic structure is

Figure 2. Average Household Income, by Age of Head of Household, 1967–2001

Thousands of 2001 dollars



Source: U.S. Bureau of the Census, "Historical Income Tables—Households," table H-10a.

less skewed, as in the economy with $(N, n) = (79, 69)$, total wage income alternates between $375 = (79 \times 3) + (69 \times 2)$ and $365 = (79 \times 2) + (69 \times 3)$.

Dividends

Land produces output, which is distributed as dividends to the equity holders. We take the ratio of dividends to wages to be of the same order of magnitude as the ratio of (generalized) dividends to (generalized) wages in the National Income and Product Accounts (NIPA).[18] More precisely, we define as generalized wages the sum of the NIPA categories "compensation to employees," "supplements to wages and salaries," and half of "proprietors' income." The rationale for this is that "proprietors' income" includes the net income of unincorporated businesses (farmers, doctors, lawyers, partners, small business proprietors), which is really wage income from the perspective of our model.[19] We define generalized divi-

18. The reference data set is the annual National Income by Type of Income from 1959 to 1999, Bureau of Economic Analysis.

19. McGrattan and Prescott (2000) attribute 80 percent of proprietors' income to wages and 20 percent to dividends.

dends as the sum of "rental income," "dividends paid by corporations," "net interest," and the other half of "proprietors' income." These are the payments to capital, which are priced in long-lived securities. We postulate that the retained earnings of corporations are used to finance growth, and, since our model does not have growth and investment, we do not take them into account. On average the ratio of generalized dividends to generalized wages is 0.19. Thus, in the economy in which $(N, n) = (79, 52)$, we take $D = 0.19[(341 + 314)/2] \approx 62$, and when $(N, n) = (79, 69)$ we take $D = 0.19[(375 + 365)/2] \approx 70$.

For the demographic structure $(N, n) = (79, 52)$, in which there is a large variation in the cohort ratios between odd and even periods, total income (wages plus dividends) is on average 7.2 percent higher in even than in odd periods. For the case $(N, n) = (79, 69)$, with its smaller variation in the cohort ratio, the output difference is 2.3 percent.

Pure Demographic Equilibrium

When the only source of change in the economy comes from fluctuations in the demographic structure, it is straightforward to describe and solve for the stationary equilibrium. Let q_t^b be the price of the bond at time t, that is, the amount of the good required in period t to buy one unit of the good in the next period; then $q_t^b = 1/(1 + r_t)$, where r_t is the interest rate from period t to period $t + 1$. It is easy to show that an equilibrium exists in which $q_t^b = q_1$ whenever t is odd, and $q_t^b = q_2$ whenever t is even. Since agents can use the bond or the equity contract to transfer income across the different periods of their life, they can equalize the present value of their consumption to the present value of their income. Agents in the large cohorts, who are young in odd periods, choose a consumption stream (C^y, C^m, C^r) so as to maximize the utility function (equation 1) subject to the budget constraint

$$(2) \qquad C^y + q_1 C^m + q_1 q_2 C^r = w^y + q_1 w^m + q_1 q_2 w^r = 2 + q_1 3,$$

whereas agents in the small cohorts, who are young in even periods, choose (c^y, c^m, c^r) so as to maximize the utility function in equation 1 under the budget constraint

$$(3) \qquad c^y + q_2 c^m + q_1 q_2 c^r = w^y + q_2 w^m + q_1 q_2 w^r = 2 + q_2 3.$$

In equilibrium we must have

$$\begin{aligned}(4)\qquad NC^y + nc^m + NC^r &= (N \times 2) + (n \times 3) + D\\ nc^y + NC^m + nc^r &= (n \times 2) + (N \times 3) + D.\end{aligned}$$

Since there is no uncertainty, the bond and the equity contract must be perfect substitutes in each period. From the no-arbitrage property of equilibrium, the rate of return on the equity market and on the bond market must be the same. Thus, if bond prices alternate between q_1 and q_2, the price of equity must alternate between q_1^e and q_2^e, where

$$\frac{D + q_2^e}{q_1^e} = \frac{1}{q_1} = 1 + r_1, \quad \frac{D + q_1^e}{q_2^e} = \frac{1}{q_2} = 1 + r_2.$$

If $q_1 < q_2$, or equivalently if $r_1 > r_2$, then it must be that $q_1^e < q_2^e$: thus interest rates are high when equity prices are rising and low when equity prices are falling. Solving the rate-of-return equations yields the following relationship between bond and equity prices:

$$\begin{aligned}(5)\qquad q_1^e / D &= (q_1 q_2 + q_1)/(1 - q_1 q_2) \quad \text{and}\\ q_2^e / D &= (q_1 q_2 + q_2)/(1 - q_1 q_2).\end{aligned}$$

Note that the same result could have been obtained by expressing the price of equity as the discounted value of its dividends:

$$\begin{aligned}q_1^e &= Dq_1 + Dq_1q_2 + Dq_1q_2q_1 + Dq_1q_2q_1q_2 + \cdots\\ q_2^e &= Dq_2 + Dq_2q_1 + Dq_2q_1q_2 + Dq_2q_1q_2q_1 + \cdots.\end{aligned}$$

A convenient way of assessing the level of equity prices is to compute the price-dividend (PD) ratio, defined by $PD(i) = q_i^e/(D/20)$, $i = 1, 2$, where dividends are expressed on a yearly basis. To compare the results of the model with the well-publicized price-earnings ratios used in valuing corporate equity, a good rule of thumb is to divide by 2, since on average corporate firms distribute half their earnings as dividends.[20] We will often refer to PE ≡ PD/2 as the "price-earnings ratio." In the same way, rather than report the interest rate for a twenty-year period, we report the annualized interest rate r_i^{an} defined by $(1 + r_i^{an})^{20} = 1 + r_i$, for $i = 1, 2$.

20. For 1951–2000 the average payout ratio for firms in the S&P 500 index was 0.51.

Properties of Equilibrium

If the bond prices were to coincide with the consumer discount factor, $q_1 = q_2 = 0.5$, then individuals would attempt to completely smooth their consumption, demanding the stream $(c^y, c^m, c^r) = (2, 2, 2)$. But then, in the case where the population structure is $(N, n) = (79, 52)$, in odd periods the aggregate excess demand for consumption would be $79(2 - 2) + 52(2 - 3) + 79(2 - 0) - 62 = 44$, and in even periods it would be $52(0) + 79(-1) + 52(2) - 62 = -37$. Thus in odd periods there is excess demand for consumption, as retired agents consume beyond their income more than the middle-aged save for their retirement, whereas in even periods, when the middle-aged cohort is large, there is excess demand for saving as those households seek to invest for their retirement. To clear markets, interest rates must adjust, discouraging consumption (stimulating saving) in odd periods, and discouraging saving (stimulating consumption) in even periods: as a result, equilibrium bond prices must be below 0.5 in odd periods and above 0.5 in even periods. By no arbitrage, land prices must be higher in even than in odd periods. How far interest rates and land prices must adjust depends on how big a price change is required to move consumers away from equal consumption in each period of their lives, which in turn is connected to the relative strength of income and substitution effects, as will be shown below.

Here is another way of understanding how the demographic effect on equity prices can be so large, and how it is reinforced by rational optimization. Suppose agents myopically consume 2 when young (thus saving nothing), consume 2 again when middle-aged (investing all their savings in land), and finally sell all their land in old age to finance their retirement consumption. The price of land would then be 79 in even periods, with their large middle-aged population, and 52 in odd periods, with their small middle-aged population. Myopic behavior in which the middle-aged do all the saving explains a roughly 50 percent (79/52 – 1) variation in equity values, even though total output varies by only 7 percent.

Rationality boosts the effect: rational agents would perceive that, in following the myopic strategy, the large generations would end up with consumption of approximately (2, 2, 1.3) in youth, middle age, and old age, respectively, whereas small generations would end up with consumption approximately equal to (2, 2, 2.5). Anticipating this drop in old-age consumption twenty years ahead, and assuming sufficient aversion to drastic jumps in consumption, the large generations would save more in middle age, and the small generations less, reinforcing the demographic effect.

If agents foresaw the demographics forty years ahead (which is possible, since the size of the current child cohort gives a good idea of the middle-aged cohort forty years hence), the large generations would also tend to save more when young, buying, say, 30 percent of the land with the purpose of holding it until old age. If they did not use the land to increase their middle-aged consumption, this would still further reinforce the demographic effect: 30 percent of the land would be removed from the market in both periods, and their middle-aged savings would rise by 30 percent of land dividends.

The only damper on the demographic effect on equity prices is that rational agents will anticipate that the return on land between odd and even periods will be greater than the return between even and odd periods, rendering middle-aged consumption relatively cheap for the big generations and relatively expensive for the small generations. If their preferences have a large substitution effect, middle-aged consumption for the large generation will increase, thus partially reducing their middle-aged savings and mitigating the demographic effect. When the risk aversion parameter in the utility function is $\alpha = 4$, the intertemporal elasticity of substitution of consumption is 1/4, and the substitution effect is small.

Since in our model agents are always saving (for their retirement years) when middle-aged, the high returns to land in odd periods and the low returns in even periods favor agents born in small cohorts (who are middle-aged when returns are high) relative to those born in large cohorts. We call this the *favored cohort effect*. This income effect just offsets the substitution effect when $\alpha = 4$: large and small cohorts have the same middle-age consumption.

Calculating the stationary equilibrium for the economy with $(N, n) = (79, 52)$ and utility function parameter $\alpha = 4$ gives the equity prices, annual interest rates, and PE ratios

$$(q_1^e, q_2^e, r_1^{an}, r_2^{an}, PE_1, PE_2) = (52, 120, 6.4\%, -0.3\%, 8.4, 19.4),$$

and the consumption streams $C = (C^y, C^m, C^r)$, $c = (c^y, c^m, c^r)$ and utilities (U, u) for large and small generations

$$(C, c, U, u) = \big((1.8, 2, 1.7), (2.4, 2, 2.3), (-0.1, -0.05)\big).$$

As expected, when the large cohort is young and the small cohort middle-aged, the equity price is low, with a PE ratio around 8; when the large cohort moves into middle age and seeks to save for retirement, the equity price is more than twice as high ($q_2^e/q_1^e = 2.3$), and the PE ratio increases to 19. The variation in equity prices (or equivalently, the variation in PE ratios) is roughly equal to the variation in the MY ratio, namely, $2.3 \approx (79/52)/(52/79)$. When the equity price is low and is anticipated to increase, the annual real interest rate is high (6.4 percent); it falls to –0.3 percent when the equity price is high and going to decrease. As predicted by the favored cohort effect, the smaller generation is better off ($-0.05 > -0.1$).

When the demographic structure (N, n) is less skewed, the disequilibrium implied when the bond prices are equal ($q_1 = q_2$) is less pronounced, so that bond and equity prices do not need to fluctuate as much to establish equilibrium. With $(N, n) = (79, 69)$, equity prices are again roughly proportional to MY ratios: $89/67 = 12.6/9.5 \approx (79/69)/(69/79)$. For a given demographic structure, if the aversion to consumption variability is lower (that is, if the intertemporal elasticity of substitution is higher), the variation in prices needed to establish an equilibrium is also lower. Table 1 shows the effect on equilibrium prices of decreasing the difference in cohort sizes and of varying the coefficient α, which determines the intertemporal elasticity of substitution of consumption (equal to $1/\alpha$). The rule that equity prices are proportional to MY ratios holds very closely when $\alpha = 4$, but only approximately for $\alpha \neq 4$.

Robustness of Pure Demographic Equilibrium

Family, Bequests, and Social Security

The model of the previous section can be viewed as the simplest model for studying the consequences for the stock market of fluctuations in demographic structure. However, it abstracts from a number of important features that alter agents' needs to redistribute income over time. In particular, the presence of bequests, Social Security payments in retirement, or the fact that young agents have to provide for their children alters the need for intertemporal savings. In this section we study how the predictions of the basic model are modified by the introduction of these factors.

Table 1. Effects of Different Intertemporal Elasticity Values and Cohort Sizes on Equilibrium Prices[a]

Coeff. of relative risk aversion	*Large intercohort difference* (N = *79*, n = *52*)						*Small intercohort difference* (N = *79*, n = *69*)					
	Pyramid Δ_1			*Pyramid* Δ_2			*Pyramid* Δ_3			*Pyramid* Δ_4		
	Equity price (q_e)	*Interest rate* (r^{an})[b]	*PE ratio*	*Equity price* (q_e)	*Interest rate* (r^{an})	*PE ratio*	*Equity price* (q_e)	*Interest rate* (r^{an})	*PE ratio*	*Equity price* (q_e)	*Interest rate* (r^{an})	*PE ratio*
$\alpha = 2$	55	5.3	8.8	91	1.3	14.6	68	4	9.8	81	2.7	11.6
$\alpha = 4$	52	6.4	8.4	120	–0.25	19.4	67	4.4	9.5	89	2.2	12.6
$\alpha = 6$	51	7.3	8.2	147	–1.3	23.8	66	4.7	9.4	94	1.8	13.5

Source: Authors' calculations of the equilibrium values of the calibrated model described in the text.

a. Cohort sizes (N, n) are chosen to approximate the sizes of the Great Depression (52 million live births), baby-boom (79 million), and X (69 million) generations. Pyramid Δ_1 applies when the young and retired generations are larger (N, n, N), and pyramid Δ_2 when the middle-aged generation is larger (n, N, n).

b. Real rate in percent a year.

Implicit in the model is that parents from a large cohort have, on average, small families—each agent of a large cohort has $\nu_1 = n/N$ children—whereas parents from a small cohort have, on average, large families, $\nu_2 = N/n$ children. The Easterlin hypothesis provides an explanation for such fluctuations in the fertility ratio,[21] which can be rephrased in the setting of our model as follows. The young of any generation form their material aspirations as children in the households of their parents: in deciding their family size, they compare the material prospects they can offer to their children with the aspirations they have formed as children in their parents' household. Since the young in a small cohort have greater lifetime income than their parents, who come from a large cohort, they feel that they can offer to their children material conditions that exceed their aspirations, and are led to choose a large family size. Conversely, the young of a large cohort facing difficult conditions but having formed high aspirations choose a small family size. This suggests a simple, albeit highly stylized, way of linking the choice of family size (fertility) to the underlying economic conditions.

Let us now take into account the fact that parents provide for the consumption of their children. If ν denotes the number of children, then the utility of a young parent is $\nu\lambda u(c^k) + u(c^p)$, where c^k denotes the consumption of a child and c^p the parent's consumption, and λ is the weight given by the parent to a child's utility.[22] Assume that agents give bequests to their children, and let b denote the bequest transferred by retired parents to their middle-aged children. We take the utility in the retired period to be $u(c^r, b) = (c^r)^{1-\beta}b^\beta$, $0 < \beta < 1$. In practice, individuals end up with wealth at the time of their death, both because they hold precautionary balances against the uncertain time of death and because they derive direct utility from the bequests they leave to their children.[23] We model the combination of these two motives by assuming that the utility is a function of the total bequest and not of the bequest per child. The utility function of the representative agent, which replaces equation 1, is given by

$$(6) \qquad U(c, b) = \frac{1}{1-\alpha}\left[\nu\lambda(c^k)^{1-\alpha} + (c^p)^{1-\alpha} + \delta(c^m)^{1-\alpha} + \delta^2\left((c^r)^{1-\beta}b^\beta\right)^{1-\alpha}\right].$$

21. Easterlin (1987).

22. This is the specification used by Brooks (2002).

23. See Modigliani (1986, 1988) for a discussion and estimation of the proportion of wealth transferred through bequests.

To complete the model we add the transfers to an agent's lifetime income arising from a pay-as-you-go Social Security system. We assume that each retired agent, regardless of cohort size, receives a transfer $\theta \geq 0$ and that the labor income received in pyramids Δ_1 and Δ_2 is taxed at rates τ_1 and τ_2, respectively, where τ_1 and τ_2 are chosen so that the Social Security budget is balanced.

The lifetime budget constraint of an agent who is young in pyramid Δ_i, $i = 1, 2$ can then be written as

$$(7) \qquad \nu c_i^k + c_i^p + q_i c_i^m + q_i q_{i+1}(c_i^r + b_i) = w^y(1-\tau_i) + q_i\left(w^m(1-\tau_{i+1}) + \frac{b_{i+1}}{\nu_{i+1}}\right) + q_i q_{i+1}\theta, \; i = 1, 2,$$

where $i + 1$ is taken modulo 2 ($1 + 1 = 2$, $2 + 1 = 1$). In a stationary equilibrium with children, bequests, and Social Security, young agents in pyramid Δ_i maximize equation 6 subject to equation 7, the market-clearing equations 4 hold, and the Social Security tax satisfies the balanced-budget equations

$$(Nw^y + nw^m)\tau_1 = N\theta \quad (nw^y + Nw^m)\tau_2 = n\theta.$$

The equilibrium equity prices are then given in terms of (q_1, q_2) and D by equation 5.

Since the first-order conditions imply that $\lambda(c^k)^{-\alpha} = (c^p)^{-\alpha}$, the weight $\lambda^{1/\alpha}$ determines the ratio of the consumption of a child to the consumption of the parent (which in the literature is called the child-equivalent consumption). Since we can find estimates for this ratio in the empirical literature, it is convenient to parameterize the model by the child-equivalent consumption η and to choose $\lambda = \eta^\alpha$. The equilibrium depends on three new coefficients (η, β, θ), which parameterize the child-equivalent consumption, the strength of the bequest motive, and the magnitude of the Social Security transfer. By setting two of these coefficients equal to zero, we can study how each parameter affects the equilibrium; by choosing a representative value for each parameter, we can examine their combined effect on the equilibrium. We take the consumption of a child to be half the consumption of an adult parent ($\eta = 0.5$);[24] we take $\beta = 0.3$ to generate

24. Following Deaton (1996).

a ratio of bequests to aggregate income between 15.5 percent and 18.5 percent, which is the consensus estimate reported by Modigliani.[25] At the end of the 1990s the ratio of Social Security transfers and Medicare benefits to national income was of the order of 8 percent: by choosing $\theta = 0.5$ as the Social Security transfer per capita, we obtain a ratio of Social Security transfers to total income of 10.5 percent in pyramid Δ_1 and 6.45 percent in pyramid Δ_2. Table 2 shows the separate and combined effects of the three parameters on the equilibrium. The preference coefficient is set to $\alpha = 4$, and the demographic parameters are $(N, n) = (79, 52)$.

Poterba has argued that the presence of bequests will attenuate, if not cancel, the decrease in security prices that is expected when the baby-boomers go into retirement, since they will not attempt to sell all their securities.[26] However, if all generations transferred the same fraction of their wealth as bequests, it still implies that a large generation will need to sell the share of its wealth that it needs as retirement income to a smaller generation of middle-aged savers. Abel has shown that, in his model with production and two-period-lived agents, the presence of bequests does not change the equilibrium.[27] In our model adding a bequest motive does lower the ratio of equity prices, but it does not cancel the effect: the main effect is to lower the interest rate, since agents in both cohorts have more income in middle age by virtue of the bequests from their parents, and thus save more for retirement. The smaller ratio of equity prices comes partly from the fact that the small generation, when middle-aged, receives a larger bequest per capita ($0.7/\nu_1 = 1.06$) than the large generation ($1/\nu_2 = 0.66$), the higher income tending to compensate for the smaller size of the cohort in the aggregate saving function.

The other parameters, the child-equivalent consumption η and the Social Security benefit θ, have the reverse effect, increasing the ratio of equity prices and increasing interest rates. The need to provide for children tends to increase the demand for borrowing or, equivalently, to decrease the saving rate in each pyramid. Since small generations have more children, their savings drop more, thereby increasing the demographic effect on equity prices. Introducing Social Security benefits decreases the income of agents when they are working and increases their

25. Modigliani (1988).
26. Poterba (2001).
27. Abel (2001).

Table 2. Effects of Bequest Motive, Family Size, and Social Security on Equilibrium Prices[a]

	$\beta = 0, \eta = 0, \theta = 0$		$\beta = 0.3, \eta = 0, \theta = 0$		$\beta = 0, \eta = 0.5, \theta = 0$		$\beta = 0, \eta = 0, \theta = 0.5$		$\beta = 0.3, \eta = 0.5, \theta = 0.5$	
Variable	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
Equity price (q^e)	52	120	97	172	19	55	26	73	19.8	45
PE ratio	8.5	19	15.5	28	3	9	4.2	12	3.2	7.3
Interest rate (r^{an}; percent a year)	6.4	–0.3	4.5	–0.4	9.4	1.9	8.6	0.9	8.8	3
Bequest (b)	0	0	0.7	1	0	0	0	0	0.8	1
Tax rate (τ; percent)[b]	0	0	0	0	0	0	12.6	7.6	12.6	7.6
Equity price ratio (q_2^e/q_1^e)[c]	2.3		1.8		2.9		2.8		2.3	

Source: Authors' calculations of the equilibrium values of the calibrated model described in the text.

a. Variables β, η, and θ parameterize, respectively, the strength of the bequest motive, the child-equivalent consumption, and the Social Security transfer.

b. Tax rate necessary to preserve balance in Social Security.

c. Ratio of the equity price in pyramid 1 to that in pyramid 2.

income when retired, thus also decreasing the saving rate. When the three effects are combined, the forces causing interest rates to be high prevail, lowering the PE ratios. But the ratio of equity prices is of the same order of magnitude as in the simple model. If this more detailed institutional model were chosen as the reference model, we would need to increase the discount factor to obtain more realistic interest rates and PE ratios. For example, with $\delta = (0.99)^{20} = 0.82$ and the same parameters as in the last two columns of table 2, the equilibrium is

$$(q_1^e, q_2^e, r_1^{an}, r_2^{an}, PE_1, PE_2) = (45, 81, 6\%, 1.4\%, 7.2, 13).$$

The relatively low discount factor $\delta = (0.97)^{20} = 0.5$ used in the simple exchange model can then be viewed as a convenient proxy for these more realistic institutional features that are left out of the model and which lower the saving rate.

Comparing Equilibria of Exchange and Production Economies

In this section we study the effect of replacing the assumption that the single asset is in fixed supply with the assumption that the asset is producible "capital." Variations in savings can now be channeled into changes in the capital stock, reducing the demographic variation in interest rates and equity values. However, since there is a lag between the moment when saving occurs and the time when output and dividends are generated, the price-dividend ratio is as sensitive to demography as it was before.[28] Finally, we show that in the presence of adjustment costs—which permit equity prices to differ from the capital stock—the equilibria of the production economy become similar to those of the exchange economy and essentially coincide when the adjustment costs are sufficiently high.

Consider an economy with the same consumer side as in the exchange economy, but in which wages and dividends are endogenous. Each agent is endowed with one unit of labor when young and middle-aged and supplies labor inelastically. The efficiency of a unit of young labor is two-

28. In this section we assume that the representative firm has no debt and finances its investment from retained earnings. For an arbitrary financial policy, the ratio that we compute is the ratio of the market value of the firm to its "net" dividend, that is, the sum of what is paid to shareholders and bondholders minus new borrowing from bondholders or new shareholders.

thirds the efficiency of a unit of middle-aged labor. The effective labor supply in pyramids Δ_1 and Δ_2 is thus

$$(8) \qquad L_1 = 2/3N + n, \quad L_2 = 2/3n + N.$$

There is a single (representative, infinitely lived) firm, which uses capital and labor to produce the single output with the production function $F(K, L) = AK^aL^{1-a}$. At the beginning of period t the firm has K_t units of capital, inherited from period $t - 1$. It hires L_t units of (effective) labor paid at the wage rate w_t, and after producing $F(K_t, L_t)$ units of output is left with $(1 - \mu)K_t$ units of capital, where μ, with $0 \leq \mu \leq 1$, is the depreciation rate. The firm then decides to spend I_t on investment, where investment is subject to convex adjustment costs:[29]

$$(9) \qquad K_{t+1} = (1-\mu)K_t + I_t - \gamma(K_{t+1} - K_t)^2,$$

with $\gamma \geq 0$. The cost of replacing the depreciated units μK_t of capital is equal to μK_t, but if the firm wants to change its capital stock, then an adjustment cost, which is convex in the change $|K_{t+1} - K_t|$, has to be incurred. If $\gamma = 0$, there is no adjustment cost, and the model is the standard Diamond model.

After paying for wages and investment, the firm distributes the rest of its output as dividends:

$$(10) \qquad D_t = F(K_t, L_t) - w_tL_t - I_t.$$

The stock market opens, and agents buy and sell shares of the firm at price q_t^e. For simplicity we assume that the bond is not used,[30] and we define the rate of interest as the rate of return on equity:

29. We introduce a cost to modifying the level of capital to capture the fact that altering firm size by introducing new plant or introducing more capital-intensive technology involves a cost over and above the cost of the materials involved, whereas the maintenance of depreciated capital involves no additional cost. We make the cost symmetric in increases or decreases of capital, since it is typically costly to uninstall used capital that is not worth maintaining. Equation 9 differs from the equation for the evolution of capital $K_{t+1} = G(K_t, I_t)$ introduced by Basu (1987) and adopted by Abel (2003), where G is a Cobb-Douglas function. The latter equation expresses decreasing returns to investment but does not necessarily involve a cost for changing the level of capital.

30. Introducing borrowing and lending on the bond merely induces indeterminacy in portfolios and does not change the market value (equity plus debt) of the firm.

$$(11) \qquad 1 + r_t = \frac{D_{t+1} + q^e_{t+1}}{q^e_t}.$$

Let (w_i, q^e_i, r_i) denote the wage, equity price, and interest rate in pyramid Δ_i, $i = 1, 2$; similarly, let $(c_i, z_{e,i}) = (c^y_i, c^m_i, c^r_i, z^y_{e,i}, z^m_{e,i})$ denote the consumption stream and equity holdings of an agent who is young in pyramid Δ_i, and let (K_i, I_i, D_i) denote the capital inherited by the firm, the investment undertaken to form the capital next period, and the dividend distributed in pyramid Δ_i, $i = 1, 2$. $(c_i, z_{e,i})$ maximizes the utility function in equation 1 subject to the sequence of budget constraints

$$(12) \qquad \begin{aligned} c^y_i &= 2/3w_i - q^e_i z^y_{e,i} \\ c^m_i &= w_{i+1} + (D_{i+1} + q^e_{i+1})z^y_{e,i} - q^e_{i+1} z^m_{e,i} \\ c^r_i &= (D_i + q^e_i) z^m_{e,i}, \end{aligned}$$

where $i + 1$ is taken modulo 2 $(1 + 1 = 2, 2 + 1 = 1)$. Note that these sequential budget constraints are equivalent to the single lifetime budget constraint

$$c^y_i + q_i c^m_i + q_i q_{i+1} c^r_i = 2/3w_i + q_i w_{i+1},$$

with present-value prices $q_i = 1/(1 + r_i)$, which can be taken as equilibrating variables.

The firm is assumed to maximize its market value—the present value of its dividends—with perfect foresight of future prices. Thus at each date t the choice of labor L_t must maximize $F(K_t, L_t) - w_t L_t$ given K_t, and the choice of capital K_{t+1} must maximize

$$-I_t(K_{t+1}, K_t) + \frac{1}{1 + r_t}\big(F(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - I(K_{t+2}, K_{t+1})\big),$$

given K_t, L_{t+1}, and K_{t+2}, where $I_t(K_{t+1}, K_t)$ is given by equation 9. This leads to the first-order conditions that define the optimal production plan of the firm in the stationary equilibrium: for $i = 1, 2$,

$$\begin{aligned} F'_L(K_i, L_i) &= w_i \\ F'_L(K_{i+1}, L_{i+1}) &= r_i + \beta + 2\gamma(2 + r_i)(K_{i+1} - K_i), \end{aligned}$$

where we use the fact that $K_{i+2} = K_i$, and where L_1 and L_2 are given by equation 8, so that the labor market clears. The market-clearing conditions for the consumption good market are

$$Nc_1^y + nc_2^m + Nc_1^r + I_1 = F(K_1, L_1)$$

$$nc_2^y + Nc_1^m + nc_2^r + I_2 = F(K_2, L_2),$$

where $I_i = I(K_{i+1}, K_i)$. The simplest approach is to find the equilibrium $(\overline{c}_i, \overline{K}_i, \overline{r}_i, i = 1, 2)$, with the interest rates or, equivalently, the present-value prices $(\overline{q}_1, \overline{q}_2)$ as equilibrating variables, and to deduce the financial variables $(\overline{D}_i, \overline{q}_i^e, \overline{z}_i)$ using equations 10, 11, and 12. As in the exchange economy, the equity price is the present value of the dividends, which are now endogenous and vary between pyramids Δ_1 and Δ_2:

$$\text{(13)} \quad \overline{q}_i^e = \overline{D}_{i+1}\overline{q}_i\left(1 + \overline{q}_1\overline{q}_2 + (\overline{q}_1\overline{q}_2)^2 + \cdots\right) + \overline{D}_i\overline{q}_1\overline{q}_2\left(1 + \overline{q}_1\overline{q}_2 + (\overline{q}_1\overline{q}_2)^2 + \cdots\right)$$

$$= \frac{\overline{D}_{i+1}\overline{q}_i + \overline{D}_i\overline{q}_1\overline{q}_2}{1 - \overline{q}_1\overline{q}_2}.$$

By varying the adjustment cost parameter γ in the above model, we can now compare the equilibrium outcomes of a family of models of the stock market (table 3), starting with the Diamond equilibrium $\gamma = 0$ and ending with $\gamma = 0.1$, for which the equilibrium is close to that of the simple exchange economy analyzed above, which is shown in the last two columns. In this family of models the coefficient of relative risk aversion is fixed at $\alpha = 4$ and the demography parameters at $(N, n) = (79, 52)$, and the production parameters A, a, and μ are chosen so that the depreciation parameter is $\mu = 0.5$ (yielding depreciation of the order of 3 percent a year), and so that the Diamond equilibrium generates wages, dividends, and output close to those of the exchange economy: this leads to the choice $A = 4.2$, $a = 0.24$.

To compare the equilibria, let $s_t^y = q_t^e z_{e,t}^y$ and $s_t^m = q_t^e z_{e,t-1}^m$ denote the saving of the representative young and the representative middle-aged agent, respectively, trading at date t. Since the total demand for equity must be equal to the one unit that exists, in equilibrium the total saving of the active agents in the economy must be equal to the price of the equity, which itself is equal to the present value of the dividends. The steady-state equilibrium on the saving market in the two pyramids can then be written as

Table 3. Effects of Different Adjustment Costs and Comparison with Exchange Economy[a]

	Production economy							
	Diamond equilibrium		*Adjustment cost equilibrium*				*Exchange economy*	
	$\gamma = 0$		$\gamma = 0.01$		$\gamma = 0.1$			
Variable	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
Capital (K)	74	53	76	65	86	84	. . .	. . .
Output (Y)	405	397	406	417	419	444	376	403
Investment (I)	16	48	28	44	42	44	. . .	. . .
Dividend (D)	81	47	69	55	59	62	62	62
Wage (w)	2.9	2.7	3	2.8	3	3	3	3
Interest rate (r^{an})	4.3	3	5.4	1.3	6.1	0	6.5	–0.3
Equity price (q^e)	53	74	52	94	55	117	52	120
PD ratio	13	31	15	34	18	37	17	39
q_2^e/q_1^e	. . .	1.4	. . .	1.8	. . .	2.1	. . .	2.3
Consumption profile (C)[b]								
Young	1.84	1.9	1.79	2.12	1.79	2.37	1.77	2.38
Middle-aged	1.9	1.86	1.96	1.90	2.02	1.98	2.03	1.98
Retired	1.86	1.93	1.76	2.08	1.69	2.24	1.69	2.28

Source: Authors' calculations of the equilibrium values of the calibrated model described in the text.

a. The coefficient of relative risk aversion is fixed at $\alpha = 4$, the demography parameters at $(N, n) = (79, 52)$, and the production parameters A (technology coefficient), a (capital share), and μ (depreciation) are set at 4.2, 0.24, and 0.5, respectively.

b. Each row presents the consumption profile for the indicated age of an agent who is young in pyramid Δ_i.

$$(14)\qquad Ns^{y}(\bar{r}_1,\bar{r}_2)+ns^{m}(\bar{r}_1,\bar{r}_2)=q^{e}(\bar{r}_1,\bar{r}_2)=\frac{\bar{D}_2\bar{q}_1+\bar{D}_1\bar{q}_1\bar{q}_2}{1-\bar{q}_1\bar{q}_2}$$

$$(15)\qquad ns^{y}(\bar{r}_2,\bar{r}_1)+Ns^{m}(\bar{r}_2,\bar{r}_1)=q^{e}(\bar{r}_2,\bar{r}_1)=\frac{\bar{D}_1\bar{q}_2+\bar{D}_2\bar{q}_1\bar{q}_2}{1-\bar{q}_1\bar{q}_2}.$$

In pyramid Δ_1 (equation 14), the young and middle-aged agents receive the current interest rate r_1 on their savings; r_2 affects the young because it is the rate of return that they will obtain on their future savings in middle age, and r_2 affects the current middle-aged because it is the rate of return that they have obtained on the (possibly negative) saving that they have done when young. The same holds for equation 15 with the roles of r_1 and r_2 reversed.

In the exchange equilibrium with fixed land, greater savings can be accommodated only by adjustments in the interest rate and the accompanying changes in the value of land. In the Diamond model, new saving can instead be channeled into new capital, reducing the variation in interest rates, as seen in table 3. Furthermore, because the investment appears as capital one generation later, the large middle-aged cohort will earn lower wages, because it will work with the smaller capital stock bequeathed by the previous small generation. This reduces their savings in middle age, and we see that the variation in equity values falls from 130 percent in the exchange economy to 40 percent in the pure Diamond model.

On the other hand, the dividends D_1 and D_2 differ in a way that reinforces the effect of the difference in rates of return on PD ratios: a lower r_2 induces a higher investment I_2: the savings of the large middle-aged cohort result in high investment for building the capital stock of the following period. The high capital stock of pyramid Δ_1 leads to a large dividend D_1, both because the economy is productive and because I_1 is low. The PD ratio is thus affected even more by demographics in the Diamond model than it is in the land model.

Introducing a convex adjustment cost tends to reduce the difference between K_1 and K_2 and to limit investment to the replacement of the depreciated capital. Dividends are then almost equal in the two pyramids, and the rates of return must vary more widely, as in the exchange economy, to establish equilibrium.

Finally, the less variable the rate of return, the less marked the favored cohort effect. The large cohort is first young and later retired in states where the effective labor supply is lower: in the Diamond equilibrium there is more capital in these states, and the output is the same as in the states where there is a large, productive middle-aged cohort. There is still an adverse effect of numbers, but it is much less marked than in the equilibrium where capital is constant and output varies.

Equilibrium with Shorter Time Periods

An objection commonly presented to the idea that the increase in equity prices during the 1990s was partly due to the saving behavior of the baby-boomers reaching middle age is that interest rates in the 1990s were not historically low.[31] The argument is that if the increase in prices resulted from a higher-than-usual propensity to save due to the presence of a large generation in its saving years, then this high propensity to save should have forced interest rates down. The model that we have studied so far (with three-period-lived agents) supports this argument, since the equity price alternates between high values (when the large cohort is middle-aged) and low values (when the small cohort is middle-aged), with the result that the rate of return—and hence the interest rate—alternates between low and high values. High equity prices coincide with low interest rates and conversely.

However, the joint dynamics of interest rates and equity prices in a model with shorter time periods is, as we shall now show, more subtle. We study how security prices behave when the three active twenty-year periods of an agent's life are each divided into five periods of four years, so that the economic life of an agent now lasts for fifteen periods. Adopting a four-year period as the basic unit of time keeps the calculation manageable and suffices to show how a more detailed statement of the changing sizes of age cohorts over time carves itself precisely into a cyclical pattern for equity prices and interest rates, with a phase shift in the path of interest rates relative to equity prices.

31. See Poterba (2001).

We continue to assume that the population cycle repeats itself every forty years, or ten periods; that is, the number of agents entering the economy in period $t + 10$ is the same as the number of agents entering in period t. Since the age composition in period t is the same as in period $t + 10$, there are now ten different pyramids $\Delta_1, \ldots, \Delta_{10}$, which keep repeating themselves. For $i = 1, \ldots, 10$, let n_i denote the size of the cohort beginning its economic life in pyramid Δ_i. The sequence $n_1, \ldots, n_{10}$ can better approximate the progressive increase and decrease in live births shown in figure 1. The choice of $n_1, \ldots, n_{10}$, which approximates the Great Depression and the baby-boom generations, is shown in figure 3: during the first five periods (twenty years) the small cohort enters, with $n_1 + n_2 + n_3 + n_4 + n_5 = 52$, and in the next five periods the large cohort enters, with $n_6 + n_7 + n_8 + n_9 + n_{10} = 79$. The cycle then repeats itself.

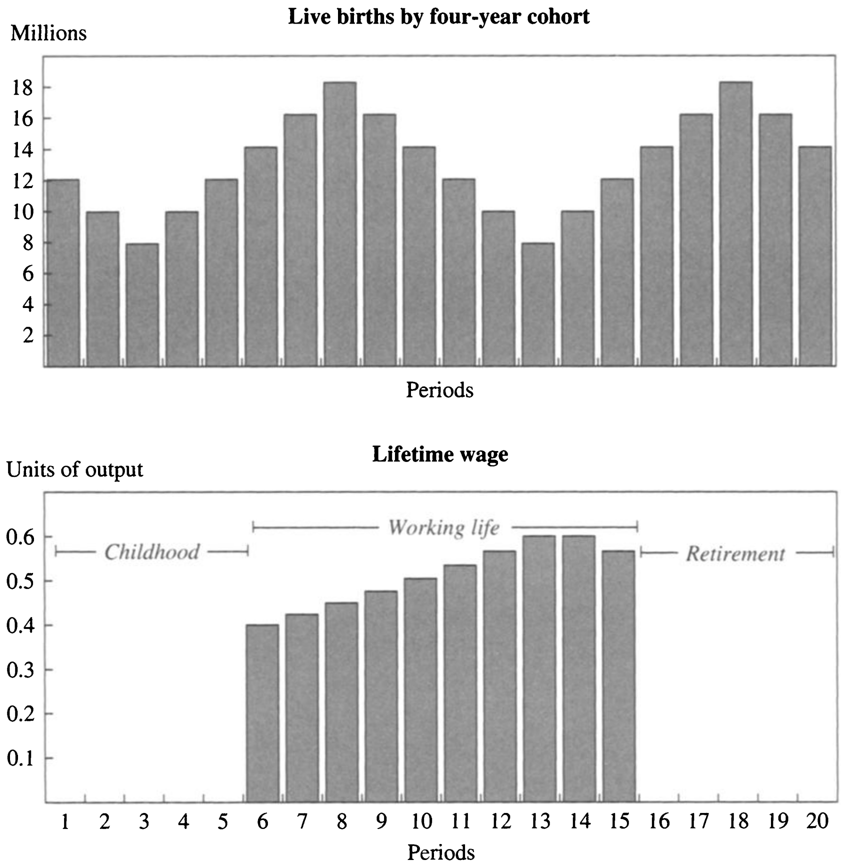
To keep the structure of the economy comparable and consistent with the previous calibration, we assume that the wage schedule increases by the same percentage in each period from $w^1 = 2/5$ (the wage of the cohort aged 20–23) to $w^8 = 1.5w^1$ (the wage of the cohort aged 48–51), stays the same in the ninth period of work ($w^9 = w^8$), and decreases to $w^{10} = w^7$ in the last period of work. The forty-year work phase ends at age 60 (the agent enters the workplace at age 20), and the agent receives no wage income during the last five periods (twenty years) of life. Figure 3 shows the representative agent's wage income during the working-life phase ($w^1, \ldots, w^{10}$).

Agents trade the equity contract, which pays a constant dividend D each period, where D is 19 percent of the average total wage income over the ten pyramids ($D = 12.74$). Agents can also borrow and lend at the riskless one-period interest rate r_t, and since the bond and the equity contract are perfect substitutes, the sequence $(q_t^b)_{t \geq 1}$ with $q_t^b = 1/(1 + r_t)$ and the sequence of equity prices $(q_t^e)_{t \geq 1}$ must satisfy

$$\frac{D + q_{t+1}^e}{q_t^e} = 1 + r_t = \frac{1}{q_t^b}.$$

As in the three-period case, there is a stationary equilibrium: let $c_i = (c_i^1, \ldots, c_i^{15})$ denote the equilibrium consumption stream, during the fifteen periods of (economically active) life, of the representative agent of a cohort entering the economy in pyramid Δ_i, and let (q_i^b, q_i^e) denote the equilibrium prices of the securities in pyramid Δ_i. Using k for the index of

Figure 3. Simulated Live Births and Lifetime Wages in Model with Four-Year Cohorts



Source: Authors' model described in the text.

age (an agent in the kth period of economic activity is called an agent of "age" k; for example, an agent of "age" 2 is between 24 and 27 years old), the consumption stream c_i must maximize $\sum_{k=1}^{15} \delta^{k-1} u(c_i^k)$, with $u(c) = c^{1-\alpha}/(1-\alpha)$, subject to the budget constraint

$$(16) \qquad c_i^1 - w^1 + q_i(c_i^2 - w^2) + \cdots + q_i q_{i+1} \cdots q_{i+14}(c_i^{15} - w^{15}) = 0,$$

where, to simplify, $q_i^b = q_i$, and all indices are taken modulo 10. Let Δ_i^k denote the number of agents of age k in pyramid Δ_i. Since these agents entered the economy $k - 1$ periods earlier, their number is n_{i-k+1}, where again the indices are taken modulo 10. The equilibrium prices must be such that in each pyramid Δ_i markets clear, that is,

$$\sum_{k=1}^{15} \Delta_i^k (c_{i-k+1}^k - w^k) = D, \quad i = 1, \ldots, 10.$$

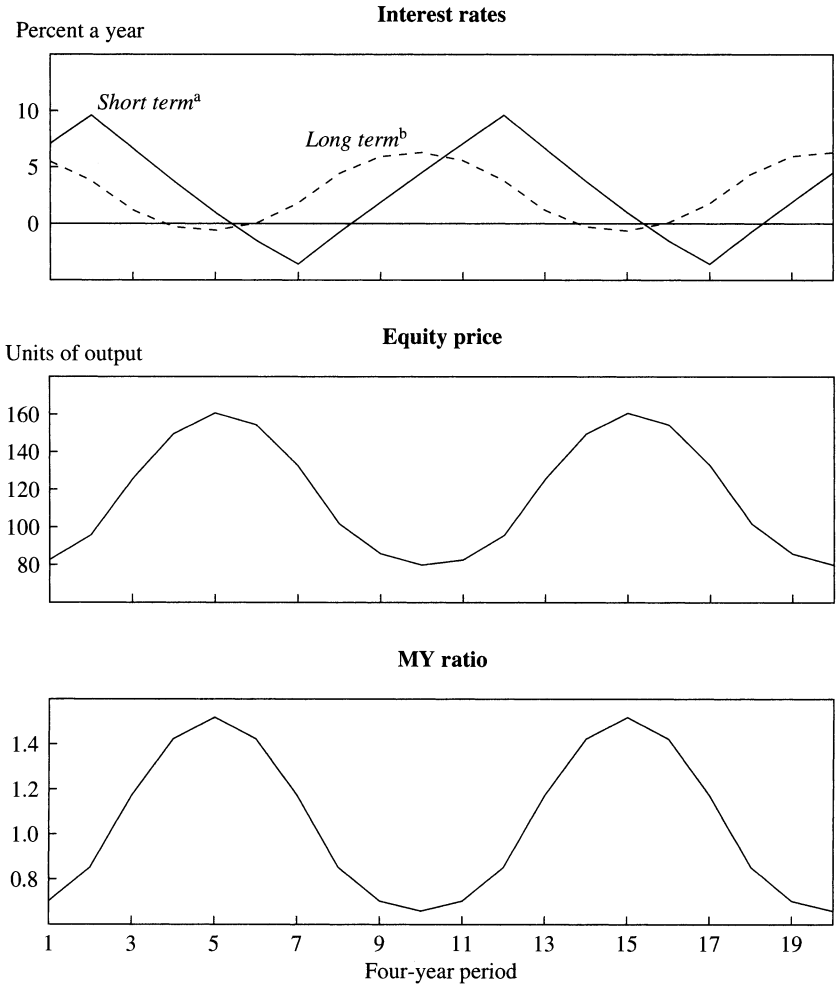
The equilibrium interest rates and equity prices for the case $\alpha = 4$ are shown in the top two panels of figure 4 as functions of the index i of the population pyramid Δ_i, as it runs through two cycles. The third panel shows a convenient index of the age composition of pyramid Δ_i reflecting the number of middle-aged relative to young agents: for pyramid Δ_i we take the ratio MY_i to be defined by

$$MY_i = \frac{\Delta_i^6 + \ldots + \Delta_i^{10}}{\Delta_i^1 + \ldots + \Delta_i^5}, \quad i = 1, \ldots, 10,$$

that is, the ratio of the number of agents aged 40–59 to the number of agents aged 20–39.

It is remarkable that the price q_i^e of equity is exactly in phase with a simple summary statistic of the age pyramid—the MY_i ratio—despite the fact that agents at the different phases of their youth, middle age, and retirement have different levels of income and different propensities to save. On the other hand, as figure 4 shows, in equilibrium the short-term interest rate is out of phase with the cycle of equity prices and the MY ratio. The interest rate, which coincides with the rate of return on equity, is the sum of the dividend yield and the capital gain yield. The dividend yield is inversely proportional to the equity price and thus co-moves negatively with it. However, the capital gain yield depends on the rate of change of the equity price, and, because of the cyclical pattern of the birth rate, this rate of change is maximal in the middle of the ascending phase of the equity prices and minimal in the middle of the descending phase: because of these capital gain terms, the turning points in the interest rate occur in the middle of the ascending and descending phases of the equity prices. Short-term interest rates begin to increase before equity prices have bottomed out, and they begin to decrease before equity prices have peaked. This synchronous behavior of equity prices and nonsynchronous behavior of rates of return with the MY ratio may help to explain one of the empirical findings reported by Poterba: although certain summary

Figure 4. Simulated Interest Rates, Equity Price, and MY Ratio in Model with Four-Year Cohorts



Source: Authors' model described in the text.

a. One-period interest rate.

b. Geometric mean of short-term rates in each of five periods into the future.

demographic statistics (similar to the MY ratio) correlate relatively well with the level of equity prices, they have essentially no significant correlation with rates of return on equity.[32]

Figure 4 also shows the behavior of the long-term (real) interest rate, defined as the interest rate on the twenty-year (five-period) bond, namely, the geometric mean of the short-term rates of return five periods into the future. The long-term interest rate is in (reverse) phase with the equity prices and the MY ratio. Thus the result of the model with three-period-lived agents—low interest rates associated with high equity prices, and conversely—holds true for the long-term real interest rate, which unfortunately is difficult to obtain from the data. The model also implies a changing term structure of (real) interest rates, with the long-term rate below the short-term rate on the ascending phase of equity prices and above it on the descending phase.

Introducing Business Cycle Shocks

If the real S&P 500 index is used as an approximate proxy for the level of stock prices, then the trough-to-peak variations observed over the past fifty years are more than twice those predicted by the simple demographic model presented in the previous sections (see figure 6). Demography cannot explain everything, nor should it. The long-term trends in equity prices over this period coincided not only with demographic trends but also with runs of luck: the 1970s and early 1980s saw mainly negative shocks (oil shortages, bursts of high inflation followed by restrictive monetary policy, leading to unemployment and low productivity), whereas the 1990s were characterized by aggregate shocks that were mainly positive (low inflation and energy prices, rapid technological progress resulting in low unemployment and high productivity). We thus add to the demographic model of the previous section the possibility of random shocks to income, to study the combined effect of demographic and business cycle fluctuations for asset prices.

Once uncertainty is introduced, risky equity and the riskless bond cease to be perfect substitutes. Equity must earn a risk premium relative

32. Poterba (2001).

to the bond to induce agents to hold it, and the model permits us to study the effect of the changing demographic structure on the risk premium.

The certainty model of the previous section showed that the qualitative results of the simplest model, with three-period-lived agents, exogenous dividends, and no bequests, are robust to the introduction of more-realistic features. We thus revert to this simplest model, adding the possibility of random wages and dividends, to study the combined effect for asset prices of demographic and business cycle fluctuations.

Risk Structure

We model the risk structure of the economy by assuming that the wage and the dividends on equity are subject to shocks. We use a highly simplified structure, assuming that at each date there are four possible states of nature (shocks): s_1, high wages, high dividends; s_2, high wages, low dividends; s_3, low wages, high dividends; and s_4, low wages, low dividends. Given the nature of the risks and the very extended length of time represented by a period (twenty years), we have chosen not to invoke a Markov structure, but rather to assume that the shocks are independent and identically distributed (i.i.d.). To reflect the fact that aggregate income and dividends are positively correlated, we assume that s_1 and s_4 are more likely (probability 0.4 each) than s_2 and s_3 (probability 0.1 each). This gives rise to a correlation coefficient between dividends and wages of 0.6.

Figure 2 shows that the maximum variability of the real annual wage income of the 45–54 cohort is about 4 percent: in the recession of 1990–91 the mean wage (in 1999 dollars) of this cohort fell from $65,000 to $60,000, a variability of (2.5/62.5) = 0.04; the variability of the wage income of the 25–34 cohort is somewhat lower. To take into account that some periods, such as 1970–83, experienced a sequence of negative shocks, in the calibration we increase the coefficient of variation of the wage income of the middle-aged to 20 percent and that of the young to 15 percent. Since the fluctuations of real (generalized) dividends are of the same order as those of wages, we take a coefficient of variation of 19 percent for dividends. This leads to a coefficient of variation of about 16 percent for aggregate income. In short, we assume four possible shocks with probabilities (0.4, 0.1, 0.1, 0.4), and wage income and dividends across the four states given by $w^y = (2.3, 2.3, 1.7, 1.7)$, $w^m = (3.6, 3.6, 2.4, 2.4)$, and $D = (74, 50, 74, 50)$.

Equilibrium

Since the financial markets in the model are incomplete—each date-event is followed by four possible income-dividend shocks, and agents can trade only two securities (equity and the bond)—the equilibrium cannot be solved, as in the previous section, in terms of the consumption variables with a single present-value budget constraint for each agent. We need to explicitly introduce the asset trades, portfolio optimization, and market-clearing asset prices. Let $z_t = (z_t^y, z_t^m) = (z_{b,t}^y, z_{e,t}^y, z_{b,t}^m, z_{e,t}^m)$ denote the lifetime portfolio of an agent born at date t, namely, the holdings of the bond and equity $z_t^y = (z_{b,t}^y, z_{e,t}^y)$ in youth and $z_t^m = (z_{b,t}^m, z_{e,t}^m)$ in middle age. Let $c_t = (c_t^y, c_t^m, c_t^r)$ denote the agent's lifetime consumption in youth, middle age, and retirement. Both z_t and c_t are stochastic, depending on the past history of shocks and on the shocks to wages and dividends during the agent's lifetime. The agent's consumption and portfolio holdings must satisfy the agent's budget constraints in each state, given by

$$\begin{aligned} c_t^y &= w_t^y - q_t z_t^y \\ c_t^m &= w_{t+1}^y + V_{t+1} z_t^y - q_{t+1} z_t^m \\ c_t^r &= V_{t+2} z_t^m, \end{aligned} \tag{17}$$

where $q_t = (q_t^b, q_t^e)$ denotes the vector of bond and equity prices at date t, and $V_{t+1} = [1, D_{t+1} + q_{t+1}^e]$ denotes the payoffs of the bond and equity at date $t + 1$. An equilibrium on the bond and equity markets is then a sequence $(z_t, q_t)_{t\geq 0}$ of portfolios and prices such that the representative agent born at date t maximizes lifetime expected utility in equation 1, subject to the budget equations 17, and such that the bond and equity markets clear at each date $t \geq 0$ for each state

$$\begin{cases} Nz_{bt}^y + nz_{b,t-1}^m = 0 \\ Nz_{et}^y + nz_{e,t-1}^m = 1 \end{cases} t \text{ odd} \qquad \begin{cases} nz_{bt}^y + Nz_{b,t-1}^m = 0 \\ nz_{et}^y + Nz_{e,t-1}^m = 1 \end{cases} t \text{ even.}$$

Our objective is to study how the alternating cohort sizes of young and middle-aged influence the equilibrium on the financial markets. In view of the alternating cohort structure and the assumption that the wage income and dividends are i.i.d., it is natural to look for a stationary equilibrium of the economy: in appendix B we define such an equilibrium and explain how it can be calculated.

Calibration Results

To study the properties of the equilibrium trajectories, we consider an economy with cohort sizes $(N, n) = (79, 52)$ and risk aversion parameter $\alpha = 4$. The characteristics of equity prices and interest rates on equilibrium trajectories are shown in table 4, and the characteristics of the consumption and portfolio strategies in table 5. A less detailed description is given in appendix table C1 for an economy with a smaller variation in cohort sizes $(N, n) = (79, 69)$, calibrated to the sizes of the cohorts born over the periods 1945–64 and 1965–84, for three different parameters of risk aversion ($\alpha = 2, 4, 6$).

As explained in appendix B, in order to find a Markov equilibrium, an endogenous state variable—the portfolio income that the middle-aged bring over from their youth—needs to be added to the exogenous state (k, s), where k is the population pyramid state ($k = 1, 2$, depending on whether the period is even or odd), and s is one of the four income-dividend shocks. Along every path, each pyramid-shock state (k, s) will occur infinitely often: in table 4 the standard deviations of the prices (the numbers in parentheses) about their means (the numbers not in parentheses) are given for each pyramid-shock state (k, s), averaged over all paths. An interesting feature of the equilibrium trajectories is that the standard deviations are very small, meaning that prices essentially depend only on the exogenous state (k, s). Thus the average values of the equity price (q^e) and of the interest rate (r^{an}) in the different states (k, s) give a rather precise description of the prices on the equilibrium trajectories. Table 4 also shows the price-dividend ratio for each state, which we have divided by 2 to make it comparable with the more familiar PE ratio, commonly used for evaluating the level of prices on the stock market.

A new variable that enters when uncertainty is introduced is the equity premium, namely, the amount by which the expected return on equity exceeds the return on bonds. The (annualized) equity premium is calculated on a trajectory as

$$rp^{an} = \text{average}(r_e^{an} - r^{an}),$$

where

$$r_{e,t}^{an} = \left(\frac{q_{t+1}^e + D_{t+1}}{q_t^e} \right)^{1/20} - 1$$

Table 4. Prices in Markov Equilibrium[a]

State	Pyramid Δ_1: MY ratio = 0.66				Pyramid Δ_2: MY ratio = 1.5			
	Equity price q^e	*Price-dividend ratio* $(PD)/2$	*Interest rate* r^{an}	*Equity premium* rp^{an}	*Equity price* q^e	*Price-dividend ratio* $(PD)/2$	*Interest rate* r^{an}	*Equity premium* rp^{an}
Case A: Standard equilibrium								
High wages, high dividends (s_1)	103	14	2.1	1.1	292	39.5	–5	0.9
	(3)	(0.4)	(0.05)	(2.8)	(27)	(3.6)	(0.19)	(1.65)
High wages, low dividends (s_2)	97.5	19.5	2.5	1.13	250	50	–4.3	0.87
	(2.5)	(0.5)	(0.04)	(2.8)	(22)	(4)	(0.17)	(1.65)
Low wages, high dividends (s_3)	37	5	7.9	1.14	80	11	1.2	0.96
	(0.9)	(0.13)	(0.05)	(3.1)	(7)	(1)	(0.2)	(1.69)
Low wages, low dividends (s_4)	34	7	8.6	1.15	61	12	2.6	0.93
	(0.6)	(0.13)	(0.03)	(3.1)	(5)	(1)	(0.2)	(1.74)
Average	68	10.7	5.4	1.13	175	27	–1.3	0.91
	(34)	(5)	(3.1)	(2.95)	(112)	(15)	(3.7)	(1.69)
Ratio of average equity price in pyramid Δ_1 to that in pyramid Δ_2 = 2.6								
Peak-trough ratio = 8.5								
Case B: 50 percent participation in equity market								
High wages, high dividends (s_1)	117	15.8	.75	2.18	297	40	–5.1	1.25
	(1.7)	(0.23)	(0.1)	(2.5)	(35)	(4.7)	(0.55)	(1.6)
High wages, low dividends (s_2)	110	22	1.1	1.96	259	52	–4.5	1.16
	(1)	(0.2)	(0.06)	(2.57)	(28)	(5.5)	(0.5)	(1.64)
Low wages, high dividends (s_3)	43	5.8	6.2	2.44	93	12.5	0.3	1.4
	(0.7)	(0.09)	(0.13)	(2.73)	(10)	(1.3)	(0.5)	(1.74)

Low wages, low dividends (s_4)	40	8.1	6.7	2.5	74.5	14.9	1.4	1.63
	(0.8)	(0.15)	(0.2)	(2.9)	(7)	(1.35)	(0.5)	(1.8)
Average	79	12.3	3.7	2.3	186	28.7	–2	1.4
	(37)	(5)	(2.9)	(2.7)	(109)	(15)	(3.2)	(1.7)

Ratio of average equity price in pyramid Δ_1 to that in pyramid $\Delta_2 = 2.4$
Peak-trough ratio = 7.3

Case C: 90 percent of young face borrowing constraints

High wages, high dividends (s_1)	101	13.7	2.75	1.4	259	35	–4.5	1.12
	(0.4)	(0.06)	(0.03)	(1.76)	(10)	(1.3)	(0.2)	(1.42)
High wages, low dividends (s_2)	101	20	2.8	1.31	240	48	–4.2	1.12
	(0.35)	(0.07)	(0.03)	(1.76)	(8)	(1.7)	(0.2)	(1.43)
Low wages, high dividends (s_3)	42.7	5.8	7.5	1.35	125	16.9	–1	1.06
	(0.15)	(0.02)	(0.04)	(1.85)	(5)	(0.7)	(0.2)	(1.48)
Low wages, low dividends (s_4)	42	8.4	7.6	1.46	110	22	–.35	1.13
	(0.1)	(0.06)	(0.03)	(1.88)	(3.6)	(0.7)	(0.16)	(1.49)
Average	72	11.4	5.2	1.41	184	29.2	–2.5	1.12
	(30)	(4)	(2.4)	(1.8)	(72)	(9.2)	(2)	(1.45)

Ratio of average equity price in pyramid Δ_1 to that in pyramid $\Delta_2 = 2.6$
Peak-trough ratio = 6.1

Source: Authors' calculations of the equilibrium values of the calibrated model described in the text.

a. Cohort sizes (N, n) are (79, 52), the coefficient of relative risk aversion is $\alpha = 4$, initial endowments of the young and middle-aged generations in states s_1 through s_4 are $w^y = (2.3, 2.3, 1.7, 1.7)$ and $w^m = (3.6, 3.6, 2.4, 2.4)$, dividends $D = (74, 50, 74, 50)$. Standard deviations are in parentheses.

Table 5. Consumption and Portfolio Strategies in Markov Equilibrium[a]

Generation	*Size*	*Consumption*	*Equity holding*	*Bond holding*	*Size*	*Consumption*	*Equity holding*	*Bond holding*
	Case A: Standard equilibrium							
	Pyramid Δ_1 ($r = 5.4$, $q^e = 68$)				*Pyramid Δ_2 ($r = -1.3$, $q^e = 175$)*			
Young	N	1.73 (0.21)	0.70 (0.33)	–0.44 (0.25)	n	2.41 (0.49)	1.38 (0.94)	–1.78 (1.13)
Middle-aged	n	2.02 (0.4)	0.24 (0.14)	0.66 (0.38)	N	2 (0.33)	1.31 (0.8)	1.17 (0.74)
Retired	N	1.69 (0.33)	0	0	n	2.31 (0.4)	0	0
	Case B: 50 percent participation in equity market							
	Pyramid Δ_1 ($r = 3.7$, $q^e = 78.5$)				*Pyramid Δ_2 ($r = -2$, $q^e = 186$)*			
Constrained young	0.5N	1.65 (0.21)	0	0.35 (0.09)	0.5n	2.35 (0.4)	0	–0.35 (0.14)
Unconstrained young	0.5N	1.82 (0.24)	1.50 (0.7)	–1.32 (0.6)	0.5n	2.44 (0.4)	3.16 (1.6)	–3.6 (1.8)
Constrained middle-aged	0.5n	1.79 (0.15)	0	0.98 (0.46)	0.5N	1.69 (0.17)	0	2.06 (0.78)
Unconstrained middle-aged	0.5n	2.24 (0.67)	0.71 (0.44)	0.49 (0.32)	0.5N	2.33 (0.78)	2.6 (1.7)	0.53 (0.49)
Constrained retired	0.5N	1.32 (0.33)	0	0	0.5n	1.8 (0.11)	0	0
Unconstrained retired	0.5N	2.08 (0.7)	0	0	0.5n	2.82 (1.04)	0	0

Case C: 90 percent of young face borrowing constraints

	Pyramid Δ_1 ($r = 5.2$, $q^e = 72$)				Pyramid Δ_2 ($r = -2.5$, $q^e = 184$)			
Constrained young	$0.9N$	1.65 (0.17)	0.22 (0.00)	0	$0.9n$	2 (0.3)	0	0
Unconstrained young	$0.1N$	1.7 (0.19)	0.79 (0.00)	–0.61 (0.00)	$0.1n$	2.49 (0.36)	1.59 (0.00)	–2.12 (0.00)
Constrained middle-aged	$0.9n$	2.14 (0.24)	0.39 (0.00)	0.09 (0.00)	$0.9N$	1.9 (0.15)	1.53 (0.00)	0.14 (0.00)
Unconstrained middle-aged	$0.1n$	2.18 (0.6)	0.31 (0.00)	0.07 (0.00)	$0.1N$	2.17 (0.6)	1.59 (0.00)	0.14 (0.00)
Constrained retired	$0.9N$	1.66 (0.44)	0	0	$0.9n$	2.74 (0.8)	0	0
Unconstrained retired	$0.1N$	1.86 (0.61)	0	0	$0.1n$	2.73 (0.9)	0	0

Source: Authors' calculations of the equilibrium values of the calibrated model described in the text.

a. Cohort sizes (N, n) are (79, 52), the coefficient of relative risk aversion is $\alpha = 4$, initial endowments of the young and middle-aged generations in states s_1 through s_4 are $w^y = (2.3, 2.3, 1.7, 1.7)$ and $w^m = (3.6, 3.6, 2.4, 2.4)$, dividends $D = (74, 50, 74, 50)$. Standard deviations are in parentheses.

is the (annualized) ex post rate of return on equity at date t. The ex ante equity premium is thus defined as the mean ex post equity premium and is given in table 4. The high variance of the ex post equity premium, even for a given pyramid-shock state (k, s), is natural, since the realized equity premium is large when a favorable state follows state s, and is small when an unfavorable state follows state s.

As is well known, the ex ante risk premia predicted by standard rational expectations models are significantly smaller than those obtained ex post from the data, at least for the United States. Several approaches have been proposed to obtain models with larger risk premia. One is to take into account the fact that agents face individual risks, which make their consumption significantly more variable than aggregate consumption. We cannot take into account individual risks without unduly complicating the model; to compensate, we have been generous in the calibration with the aggregate risk. Other solutions involve entering as constraints some observed deviations of the behavior of agents from that predicted by the model. One prediction of the model is that agents make use of all the available instruments to redistribute income and share risks. However, even though the proportion of U.S. households investing in the stock market has increased significantly over the last fifty years,[33] it still remains less than 50 percent. To take this into account, we solve for the equilibrium under the restriction that 50 percent of the agents in any cohort do not trade on the equity market and restrict their financial transactions to the bond market (case B in tables 4 and 5).

An alternative approach, recently proposed by Constantinides, Donaldson, and Mehra, is to impose a borrowing constraint on the young:[34] as shown in table 5, without such a constraint, the young typically borrow and use much of the proceeds to invest in the equity market, to take advantage of the equity premium. As Constantinides, Donaldson, and Mehra argue, this is not especially realistic. Although young agents can and do borrow significantly to buy houses (which serve as collateral), they do not typically borrow to invest in the stock market. The simplest way of preventing the young from taking leveraged positions on the equity market is to impose a borrowing constraint. Such a constraint on

33. Vissing-Jorgenson (2000) estimates the participation rate in the stock market at around 6 percent in the early 1950s and around 40 percent in 1995.

34. Constantinides, Donaldson, and Mehra (2002).

the young decreases the demand for the risky security and tends to increase the risk premium. However, in the simple model that we study, preventing every young agent from borrowing closes the bond market, and the interest rate is no longer well defined. To avoid this, while studying the effect on prices of reducing the demand for equity by the young, we solve for the equilibrium assuming that 90 percent of the young face borrowing constraints and the remaining 10 percent are unconstrained (case C in tables 4 and 5). In addition to the intrinsic interest and potentially greater realism of these two cases with restricted participation, they are also useful for checking the robustness of the results predicted by the standard model (case A in tables 4 and 5) to different assumptions about market participation.

Cyclical Fluctuations of Security Prices

The general principle that underlies the certainty model—namely, that aggregate demand for saving is high in even periods when there is a large middle-aged and a small young cohort, whereas it is low in odd periods where there is a small middle-aged and a large young cohort—carries over to the economy with uncertainty. In an economy with both demographic and business cycle shocks, the stochastic sequence of equilibrium security prices (q_t^e, q_t^b) co-moves with the MY ratio, being higher than average when the MY ratio is high and lower than average when it is low. Thus long-run fluctuations in demographic structure lead to long-run cyclical fluctuations in security prices over time. The order of magnitude of the demographic effect is indicated in table 4 by the ratio of the average prices in the two pyramid states, and this is approximately the same as in the certainty model.

Note that the average interest rate is high in odd periods, in which equity prices are low and rising, and low in even periods, in which the equity prices are high and falling. It is precisely this simultaneous adjustment of interest rates and equity prices that prevents arbitrage opportunities from arising.

Since, for a given population structure, an increase in income increases the demand for saving, equity prices covary positively with aggregate income. Thus adding shocks to income opens the possibility of greater variations in equity prices: the greatest increase occurs when the economy moves from $(1, s_4)$ to $(2, s_1)$, namely, from a period with a large young cohort and negative income shocks to a period with a large middle-aged cohort and favorable shocks. The ratio of these prices is given in table 4 by the peak-to-trough ratio, and its inverse, the trough-to-peak ratio, where we see that values of 6 or 7 are attained.

Equity Premium

The striking feature of the risk premium in the equilibria that we compute is that it is larger in pyramid Δ_1 than in pyramid Δ_2. At their initial endowment, the risk aversion of young agents is smaller than that of the middle-aged: they have the prospect of income in middle age, whereas the middle-aged have no income in retirement to help smooth the risk associated with buying a risky security. As a result, the young hold a higher percentage of stock in their portfolio and actually borrow to hold equity. One might have thought that the equilibrium risk premium would therefore be smaller in pyramid Δ_1, where there are many young and few middle-aged. Indeed, this is the standard prediction in the literature.

There are two reasons why we get the opposite conclusion. First, the risks are not the same. Agents investing in pyramid Δ_1 face a more risky—if more favorable—market than agents investing in pyramid Δ_2, because the return $D_{t+1} + Q_{t+1}$ depends more on the capital value term Q_{t+1} when the price-dividend ratio is expected to be high, and more on the dividend D_{t+1} when the price-dividend ratio is expected to be low. Dividends are less variable than capital values (in table 4 the coefficient of variation of equity prices is always more than 40 percent, whereas the coefficient of variation of dividends is 19 percent), and so the return on equity is more variable for agents investing in odd periods and expecting high equity prices next period than for those investing in even periods and expecting low prices. This can be seen from the standard deviation of the risk premium in table 4, which is essentially the same as the standard deviation of the rate of return, and is higher in pyramid Δ_1 than in pyramid Δ_2. The increase in risk from another dollar of equity is thus higher for the small generation of middle-aged than for the large generation of middle-aged.

Second, agents become more averse to additional risk as their consumption becomes riskier. The middle-aged are buyers of equity in every generation. Their risk aversion on the margin depends on how much risk they face in old age. The variability of consumption of the old agents in large generations is smaller than that of the old in small generations, precisely because their stock returns are less variable. Thus the middle-aged in the large generations may face less risk and be more risk-tolerant than the middle-aged of the small generations. This is sure to be the case if the young are prevented from holding much stock, as they are in cases B and C.

As can be seen from table 4, restricting the participation on the equity market to 50 percent of the agents (case B) is the most effective way of increasing the risk premium, because the risk of the equity is divided among a smaller number of agents.[35] Roughly speaking, the agents who are trading on the equity market (the unconstrained agents in table 5) hold twice as much equity as their counterparts in case A and expose themselves to more than twice the volatility of consumption. As a result, the equilibrium risk aversion is higher. Since the risk of equity is of the same order of magnitude, the risk premium is larger.

The last case, where most young (90 percent) cannot borrow, is perhaps more realistic in terms of portfolio behavior, although the borrowing constraint is too extreme, since it is not uncommon for a young agent to borrow to buy a house while at the same time investing a fraction of wage income in equity in a retirement account, but it is less effective at increasing the risk premium than case B. There are two reasons for this: The first is that the risk of equity decreases—because of the reduced participation of the young on the equity market, the variability of their income impinges less on the market, reducing the variability of equity prices. The second is that this reduced risk is shared among more agents than in case B.

The Favored Cohort Effect

As in the simple deterministic model, the long-run cyclical fluctuations in the demographic structure imply that agents in small cohorts receive more-favorable equilibrium lifetime consumption streams than do agents in large cohorts. The lifetime equilibrium consumption streams of agents

35. This is consistent with the findings of Heaton and Lucas (2000), who explore, in an overlapping-generations model with two-period-lived agents, the idea of using restricted participation as a way of increasing the equity premium. However, in our model participation has a bigger impact on the premium.

born into the small and large cohorts are shown in table 6 (they have been multiplied by 10,000 to make the comparison of the consumption streams more intuitive). Even though all agents begin with the same average lifetime wage income (20,000 as young, 30,000 in middle age, 0 in retirement), the average lifetime consumption stream of an agent born into a small cohort is significantly greater than that of an agent in a large cohort. This difference arises from the cyclical fluctuations in the security prices: the two columns to the right of the average consumption stream show the average prices (the equity price and the interest rate) that the corresponding agent faces during his or her lifetime, and the last entry in the interest rate column gives the expected utility (averaged over the possible income shocks when young) of an agent born into a large or a small cohort. In the constrained-participation cases, U^c and U^u denote the utility of the constrained and the unconstrained agents, respectively.[36]

Case B in table 6 shows the loss to their average lifetime consumption stream incurred by agents who are assumed not to participate on the equity market—as usual, the loss incurred by boomers is greater than that for Xers. Although there is a gain in terms of reduced variability of consumption, the loss to average consumption is substantial, especially in middle age and retirement. As a result, agents who for whatever reason—ignorance or fear—do not participate on the stock market do so at considerable cost to their lifetime consumption and utility.

The cost of nonparticipation is less marked in case C, where agents face borrowing constraints in youth. Constrained Xers lose only when they are young, because they cannot take advantage of the favorable terms for borrowing, whereas constrained baby-boomers lose throughout their life, since they cannot exploit the favorable terms for saving in youth, giving them less wealth in middle age and hence less consumption in both middle age and retirement.

Other authors, in particular Richard Easterlin,[37] have pointed out that the baby-boomers, being a large generation, face more competition on the labor market and thus should be expected to receive lower wages than the

36. It can be shown that the extent to which the small cohort is favored depends on the magnitude of the fluctuations in security prices: the greater the difference in cohort sizes, the greater the degree of relative risk aversion; or the greater the variability of agents' endowment streams, the greater the fluctuations in security prices, and the greater the extent to which capital markets favor the small cohort.

37. Easterlin (1987).

Table 6. Lifetime Consumption and the Favored Cohort Effect in Markov Equilibrium[a]

	Large cohort				Small cohort			
	Consumption				Consumption			
Generation	Constrained	Unconstrained	Prices and utilities[b]		Constrained	Unconstrained	Prices and utilities	
			Case A: standard equilibrium					
Young	. . .	17,300	68	5.4	. . .	24,100	175	−1.3
		(2,100)	(34)	(3.1)		(4,900)	(112)	(3.7)
Middle-aged	. . .	20,000	175	−1.3	. . .	20,200	68	5.4
		(2,300)	(112)	(3.7)		(4,000)	(34)	(3.1)
Retired	. . .	16,900	68	$U^u = -1.0$	. . .	23,100	175	$U^u = -0.6$
		(3,300)	(34)			(4,000)	(112)	
			Case B: 50 percent participation in equity market					
Young	16,500	18,200	79	3.7	23,500	24,400	186	−2.0
	(2,100)	(2,400)	(37)	(2.9)	(4,000)	(4,000)	(109)	(3.2)
Middle-aged	16,900	23,300	186	−2.0	17,900	22,400	79	3.7
	(1,700)	(7,800)	(109)	(3.2)	(1,500)	(6,700)	(37)	(2.9)
Retired	13,200	20,800	79	$U^c = -1.5$	18,000	28,200	186	$U^c = -0.69$
	(3,300)	(7,000)	(37)	$U^u = -0.95$	(1,100)	(10,400)	(109)	$U^u = -0.56$
			Case C: 90 percent of young face borrowing constraints					
Young	16,500	17,000	72	5.2	20,000	24,900	184	−2.5
	(1,700)	(1,900)	(30)	(2.4)	(3,000)	(3,600)	(72)	(2.0)
Middle-aged	19,000	21,700	184	−2.5	21,400	21,800	72	5.2
	(1,500)	(6,000)	(72)	(2.0)	(2,400)	(6,000)	(30)	(2.4)
Retired	16,600	18,600	72	$U^c = -1.2$	27,400	27,300	184	$U^c = -0.65$
	(4,400)	(6,100)	(30)	$U^u = -1.1$	(8,000)	(9,000)	(72)	$U^u = -0.51$

Source: Authors' calculations of the equilibrium values of the calibrated model described in the text.

a. Cohort sizes (N, n) are (79, 52), the coefficient of relative risk aversion is set at $\alpha = 4$, initial endowments of the young adult and middle-aged generations in states s_1 through s_4 are $w^y = (2.3, 2.3, 1.7, 1.7)$ and $w^m = (3.6, 3.6, 2.4, 2.4)$, and dividends $D = (74, 50, 74, 50)$. Standard deviations are in parentheses.

b. U is the expected utility of the representative agent of the cohort, with U^u indicating unconstrained agents and U^c constrained agents.

small generation that preceded them: this labor market cohort effect, which has been somewhat controversial,[38] is absent from our model, since we assume that agents have the same lifetime wage profile in both cohorts. Our model shows, however, that large cohorts face a second curse from the financial markets: by being so numerous, they drive the terms of trade against themselves, favoring the small cohorts on the other side of the market that follow or precede them.

Comparing Calibration with Observations

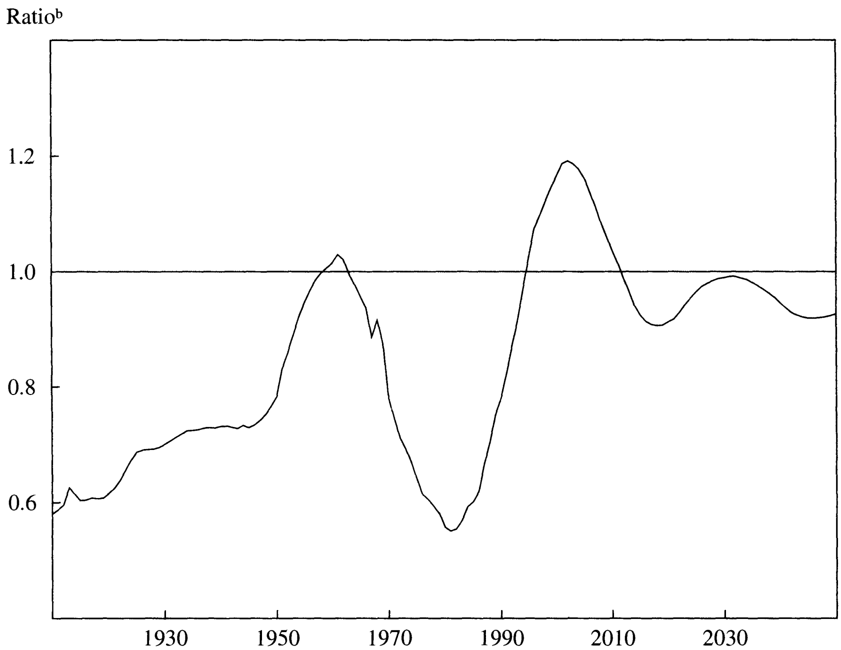
The model studied in the previous sections predicts relationships between demographic variables and asset prices. In this section we analyze in a stylized way whether the predictions of the model are consistent with data over the last century for the United States. The key demographic hypothesis of the model is that the birth rate is cyclical, with a period of forty years, which is a simplification of the observed birth rate in the United States during the twentieth century. As we have seen, leaving aside output shocks, the cyclical birth rate implies that equilibrium prices and quantities can be expressed as a function of a simple statistic of the population pyramid: the MY ratio. This ratio (shown in figure 5) is taken as the ratio of the size of the cohort aged 40–49 to the size of the cohort aged 20–29 for the U.S. population.[39] Note that the use of the MY ratio as a summary statistic of the population pyramid is justified only in the context of an intertemporal equilibrium of an economy with a cyclical birth rate: the MY ratio indicates where in the pyramid cycle the economy is located at a given time, but it does not imply that the young and the middle-aged cohorts that serve to define the ratio are the only cohorts whose trade influences the equilibrium—all cohorts trade, and all influence the equilibrium outcome.[40] The very weak cyclical movement in the

38. Welch (1979) found evidence that wages depend on cohort sizes for the period preceding 1980; for the period after 1980, as Macunovich (2002) has shown, additional variables are needed to explain the movements in wages.

39. The MY ratio obtained by using the size of the cohort aged 40–59 relative to that of the cohort aged 20–39 is approximately the same as the ratio we have chosen, with a phase shift (advance) of four years. The ratio chosen is slightly better related to the asset price data, but both indices give very similar results.

40. Empirical studies that have analyzed the influence of demography on asset markets without an equilibrium model have considered either several summary statistics of the

Figure 5. MY Ratio, 1910–2050[a]



Source: *Historical Statistics of the United States*, series A33-35, and Bureau of the Census data.
a. Ratio of the cohort aged 40–49 to the cohort aged 20–29.
b. Data from 2001 onward are projections.

MY ratio until 1945 indicates that there was only a weak cyclical component in the birth rate (and the immigration rate) at the end of the nineteenth and the beginning of the twentieth century: thus for the period 1910–45 we should expect to see a less systematic relationship between asset prices and the MY ratio than for the period 1945–2002.

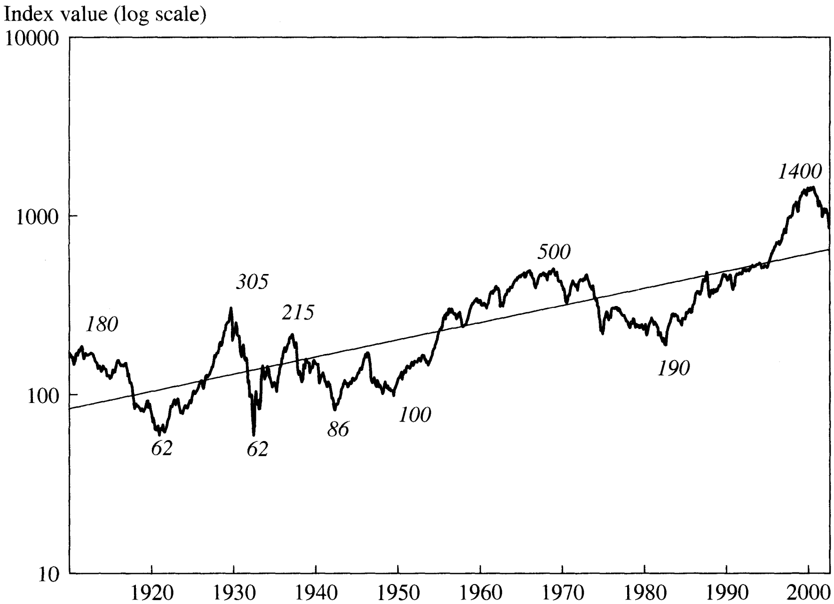
Equity Prices

Using the real Standard and Poor's index expressed in dollars of 2000 as the index of equity prices (figure 6), consider in broad outline the joint behavior of the MY ratio and equity prices.[41] Up to the late 1940s there

population pyramid (Poterba, 2001; Ang and Maddaloni, forthcoming) or the influence of all age cohorts on the financial variables (Poterba, 2001; Macunovich, 2002).

41. We are grateful to Robert Shiller for making the data set for the S&P index available to us.

Figure 6. Real Standard and Poor's Index of Common Stock Prices, 1910–2002[a]



Source: Data from Robert Shiller's website: www.econ.yale.edu/~shiller.
a. Base year for price index is 2000.

were no significant variations in the MY ratio, and this corresponds roughly with the lack of systematic long-run movement in the S&P index around its trend over this period. To be sure, there were large ten-year fluctuations up to the 1940s—for example, the ten-year boom of the Roaring Twenties—but we think of these as shorter-run business cycle fluctuations. Starting in the late 1940s and continuing all through the 1950s and early 1960s, the ratio of middle-aged to young agents was rising: the middle-aged agents had been born at the turn of the century, a period of relatively high birth rates (see figure 1) and immigration, and the young were the small generation born during the Great Depression and World War II. During this same period, equity prices were steadily rising. Stock market prices declined in real terms at the end of the 1960s and during the 1970s, during which the MY ratio also declined significantly: the small Great Depression generation became middle-aged, while the large generation of baby-boomers entered their active life. In the early 1980s equity prices began their remarkable ascent to their peak in 2000, and it was during this period that the plentiful baby-boomers moved into middle age, while the small cohort of Xers, born in the 1970s, entered their economic life, creating the equally dramatic surge in the MY ratio.

The price-earnings ratio is a normalized measure of the level of equity prices, which has the advantage of factoring out growth and is thus more directly comparable with the results of our model. As figure 7 shows, the PE ratio follows roughly the same pattern as the real S&P index and corresponds well with the long-run fluctuations in the MY ratio. The PE ratio increases from a low of 7 in 1949 to around 20 in the 1960s, then decreases in the 1970s and early 1980s to around 8, after which it increases to around 30 in 2000. These numbers correspond well with the predictions of tables 4 and C1 (with $\alpha = 4$): PE ratios (or half price-dividend ratios in the tables) vary between 7 and 8 in the bad state s_4 of pyramid Δ_1 and between 25 and 0 in the good state s_1 of pyramid Δ_2.

Table 7 shows the results of regressing the PE ratio on the MY cohort ratio,

$$PE_t = c + \beta MY_t + \varepsilon_t,$$

for different time periods. Since the series are slow moving and there is a danger of finding spurious correlations, we report the t statistics of the augmented Dickey-Fuller unit-root test on the residuals of the regression.[42] The regression tends to support the hypothesis of a systematic relationship between the PE ratio and the MY ratio: the regression coefficients are significant and stable, and the probability of a unit root in the residuals is low on the largest sample, that for 1910–2002.[43]

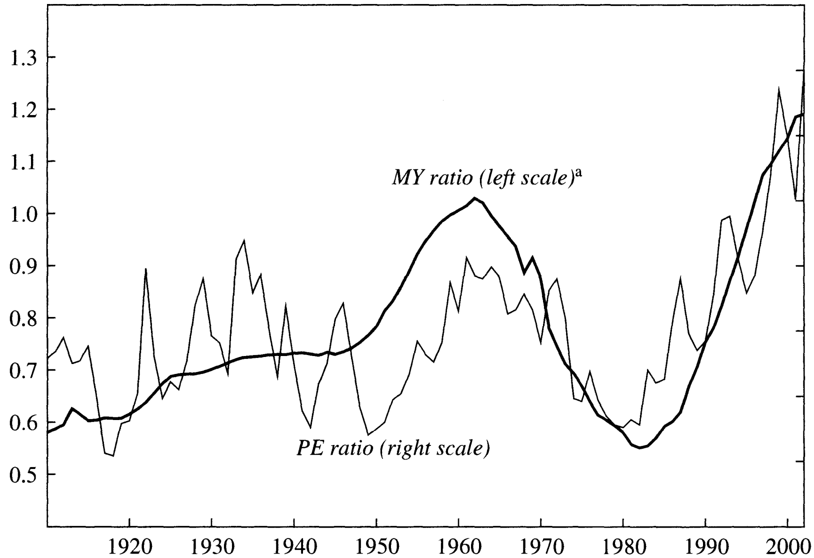
Rates of Return

A defect of the stochastic model with twenty-year time periods is that it cannot give insight into short-run rates of return. We were able to study

42. All the augmented Dickey-Fuller t statistics of residuals reported in this section are derived from the regression of the differenced residual on the residual without a constant and with one lagged variable. A critical value smaller than –3.39 leads to a rejection of the null hypothesis of a unit root in the residuals at the 99 percent confidence level. The critical levels for the 97.5 percent and 95 percent confidence levels are –3.05 and –2.76, respectively (Phillips and Ouliaris, 1990).

43. These results are consistent with those of Poterba (2001, table 9), who finds a significant relationship between the price-dividend ratio and demographic variables.

Figure 7. Average PE Ratio of Firms in Standard and Poor's Index and MY Ratio, 1910–2002



Source: *Historical Statistics of the United States*, series A33-35, Bureau of the Census data, and data from Robert Shiller's website: www.econ.yale.edu/~shiller.

a. Calculated by dividing the monthly S&P index by an average of earnings for the past twelve months and annualizing.

short-run rates of return only in the deterministic model, in which the rate of return on equity coincides with the interest rate. There we found that the rate of return (and hence the interest rate) is not synchronized with the MY ratio, because it is importantly influenced by capital gains or losses, which depend on the change in the equity price and hence on the change (and not the level) of the MY ratio. This suggests studying how annual

Table 7. Results of Regressions of PE Ratio on MY Ratio[a]

	Sample period		
Coefficient or test statistic	*1910–2002*	*1945–2002*	*1965–2002*
Constant	–3.5	–5.5	–7.1
	(3.2)	(3.7)	(2.6)
β	23.5	25.4	29.7
	(4.4)	(4.7)	(3.3)
R^2	0.48	0.55	0.78
ADF t statistic	–4.1	–2.8	–4.8

Source: Authors' regressions.

a. Newey-West standard errors (Newey and West, 1987) are in parentheses.

Table 8. Results of Regressing Rates of Return or Short-Term Interest Rates on Differenced MY Ratio[a]

	Dependent variable					
	Real rate of return on S&P 500 index			*Real short-term interest rate*		
Sample period	*Constant*	β	R^2	*Constant*	β	R^2
1910–2002	6.73	206	0.07	0.76	20.23	0.01
	(1.9)	(43)		(0.7)	(12.8)	
1945–2002	7.42	197	0.14	0.53	12.6	0.02
	(1.6)	(41)		(0.6)	(11.4)	
1965–2002	5.9	186	0.16	1.28	22.8	0.16
	(2.0)	(40)		(0.5)	(10.6)	

Source: Authors' regressions.
a. Newey-West standard errors are in parentheses.

interest rates and rates of return on equity covary with the differenced MY ratio. The results of the regression

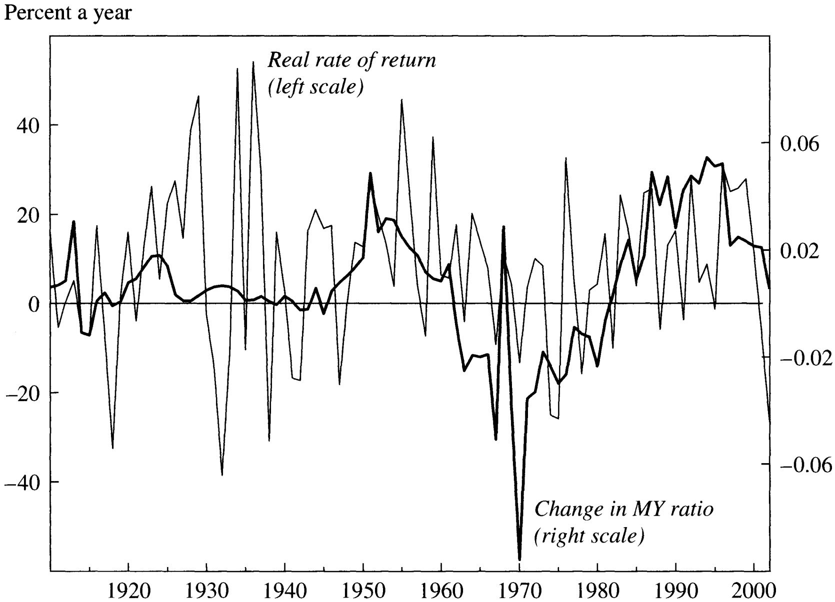
$$X_t = c + \beta D(MY)_t + \varepsilon_t$$

are shown in table 8 for different time periods, where X_t is either the rate of return on the S&P index or the real short-term interest rate, and $D(MY)_t = MY_t - MY_{t-1}$.

The results for the rate of return on equity are as expected: the rate of return is much more variable than the change in the MY ratio and is clearly affected by other shocks (to output). Nevertheless, demographic changes account for 14 percent of the variability of the rate of return between 1945 and 2002, which is nonnegligible. Figure 8 shows the relationship: rates of return tend to be higher in the late 1940s and 1950s and in the mid-1980s and the 1990s, when the MY ratio was increasing, and lower than average in the late 1960s and 1970s, when the MY ratio was decreasing.

On the other hand, the relationship between the short-term interest rate and the change in the MY ratio is weaker than expected during the period 1945–2002. The regression has to be restricted to 1965–2002 to obtain a significant relationship between the interest rate and the differenced MY ratio: indeed, as figure 9 shows, during this period the behavior of the interest rate is roughly compatible with the equilibrium behavior shown in figure 4: real interest rates declined after 1965 and were very low in the

Figure 8. Real Rate of Return on Standard and Poor's Index and Change in MY Ratio, 1910–2002

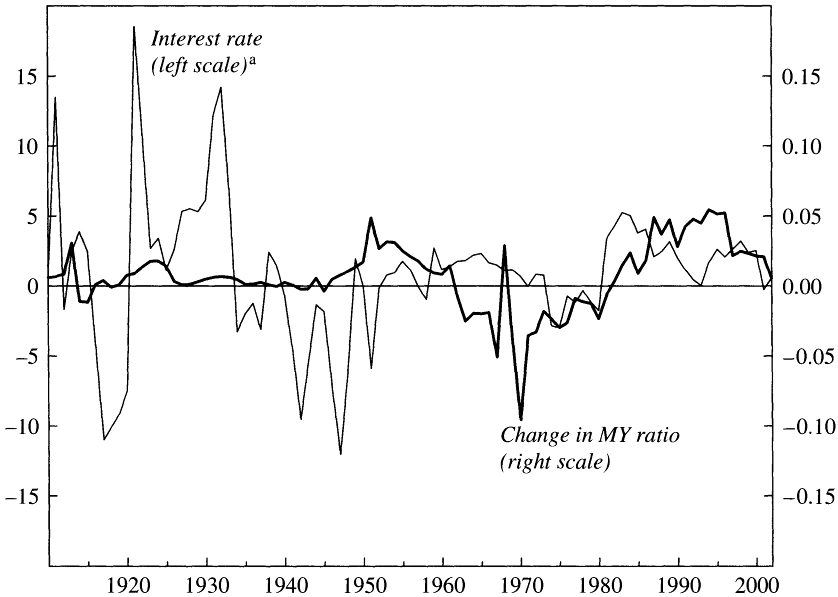


Source: Authors' calculations from Standard and Poor's, and *Historical Statistics of the United States* and Bureau of the Census data.

mid-1970s, when the MY ratio and real equity prices were declining rapidly. The turn in interest rates occurred in 1980, before the turn in equity prices, and interest rates were high in the early 1980s at the beginning of the rise in stock prices. They stayed relatively high until 2000, with a small intermission before and during the fall in equity prices accompanying the Gulf War recession. The period 1945–65 does not, however, fit the predictions of the model: the return on equity was consistently high during the bull market of the 1950s and early 1960s, while the interest rate was low, especially at the beginning of the rise in the late 1940s and early 1950s; this is difficult to reconcile with rational expectations. One hypothesis is that many investors, scared by the enormous losses incurred on the stock market during the Great Depression, fled to the relative safety of the bond market, leading to a period of low interest rates. As we have seen, restricted participation in the equity market decreases interest rates.

Figure 9. Real Short-Term Interest Rate and Change in MY Ratio, 1910–2002

Percent a year



Source: Authors' calculations from *Economic Report of the President*, various years, and *Historical Statistics of the United States*.

a. Series obtained by splicing the three-month Treasury bill rate series for the period 1939–2002 with the short-term commercial paper rate for the period 1910–38, with the latter decreased by 0.55 percent, the average premium on commercial paper for the period when both series are available.

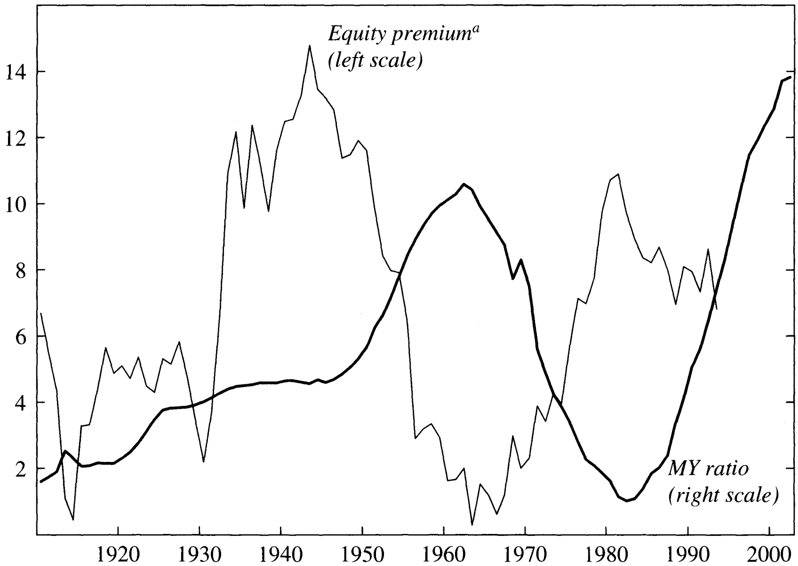
The Equity Premium

In the different equilibria that we calculated, real interest rates were between –5 percent and 9 percent. Although, as seen from figure 9, this interval is not exceptional by historical standards—before the 1950s the real interest rate fluctuated between –12 percent and 18 percent—the fluctuations in interest rates in the postwar period, in which the significant demographic changes occurred, have been smaller, between –3 percent and 5 percent. Part of the reason is that the change of regime from a gold standard to fiat money has increased the effectiveness of monetary policy aimed at reducing the variability of inflation and stabilizing real interest rates.

The smaller-than-predicted adjustment of interest rates to movements in equity prices implies that the high values of the risk premium are much

Figure 10. Equity Premium and MY Ratio, 1910–2002

Percentage points



Source: Authors' calculations from *Economic Report of the President*, various years, and *Historical Statistics of the United States*, and Standard and Poor's data.

a. Calculated as the geometric mean of the rate of return on the S&P twenty years forward minus the geometric mean of the interest rate over the period. The series is continued after 1983, which is the last year for which twenty observations forward are available, to 1993 by taking the forward geometric means over the available observations. The rate of return on the S&P is calculated as in figure 8. The short-term interest rate is calculated as in figure 9.

higher than that predicted by the model. The equity premium in figure 10 is calculated by taking the geometric mean rate of return on the S&P twenty years forward at each date and subtracting the geometric mean of the short-term interest rate over the same period; this gives the average equity premium that agents could have expected if they invested at this date with perfect foresight. The maximum occurred in the early to mid-1940s, reflecting the fact that the excess return on equity was high during the twenty years of rising prices from 1945 to 1965. The minimum occurred around 1965, which means that the equity premium was small during the declining market of the 1970s and early 1980s. Then there is a local maximum in 1980 arising from the high rate of return on equity from the beginning of the 1980s up to 2000.

The qualitative behavior of the equity premium fits the predictions of the model well: in equilibrium the excess return is higher on average for those agents who buy at low prices, when the MY ratio is low, than for those who buy at high equity prices and expect a low return, when the MY ratio is high. The equilibrium results on the equity premium are driven by the fact that returns are more variable when prices go up than when they go down. This is only partly supported by the data: with yearly data there is no marked change in the variability of the S&P index on the ascending and descending phases.[44] However, at the higher frequency of daily data, the market has been substantially riskier in the recent ascending phase (1982–2000) than it was in the preceding declining phase (1965–82): on these time intervals the standard deviation of the daily rate of change in the price index went from 0.83 percent to 1.1 percent, and the number of days when prices changed by more than 2 percent rose from 121 to 207. The bull market of the 1950s, on the other hand, did not exhibit more volatility than the ensuing bear market of the 1970s.

Note that, given the small variability of the short-term interest rate, the behavior of the average (geometric) *excess* return twenty years forward is close to that of the average (geometric) rate of return twenty years forward. This long-term rate of return on equity thus exhibits a cyclical behavior with a twenty-year phase shift from the MY ratio, which roughly fits the prediction of the deterministic and the stochastic models.

International Evidence

The three alternating twenty-year episodes of increasing and decreasing equity prices in the United States constitute a rather small sample for checking whether demographic forces were a significant causal element in these price changes. The experience of countries other than the United States may help to increase the number of observations for testing the demographic hypothesis. This section studies whether there is a relationship between equity prices and demography for Germany, France, Japan, and the United Kingdom.

The model that led to the tests for the United States rests on two assumptions—a cyclical live birth process and a closed economy—that

44. If we compute for each year the standard deviation of the rate of return on the S&P index during the following twenty years, the most obvious result is that, because of the Great Depression, the volatility in the rate of return experienced by investors at the beginning of the century was much larger than that experienced after World War II. For example, the standard deviation of the twenty-year-forward rate of return was between 24 and 28 percent from 1914 to 1932, whereas since 1940 it has varied between 13 and 17 percent.

may not be appropriate for other countries. The cyclical live birth process comes directly from the observation of the U.S. live birth process and justifies taking the MY ratio as a proxy for the composition of the population. Since the live births of the other countries just mentioned are less clearly cyclical, we study two proxies for the composition of the population: the MY ratio defined as for the United States, and the size of the cohort aged 35–59, which is a direct measure of the middle-aged group.

We have assumed a closed economy in order to explain asset prices in the United States by the country's own demographic structure. This assumption seems reasonable for studying the past, if not the future, behavior of the U.S. stock market, since until recently U.S. equity has been mostly owned by U.S. investors: up to 1975 foreigners held less than 4 percent of U.S. equity, and, despite the increase during the 1980s, foreigners still hold less than 11 percent.[45] The home bias phenomenon has been documented for other countries, but the closed-economy assumption may nevertheless be more appropriate for the United States and Japan, which have the two largest stock markets in the world, than for the three European markets, which seem to follow the U.S. market.

Table 9 presents results of the regression

$$RP_t = c + \beta M_t + \varepsilon_t,$$

where RP_t is the real stock price index of the country in question, and M_t is the demographic index: in the four left-hand columns M_t is the MY ratio for the cohort aged 40–49 to that aged 20–29, and in the remaining columns M_t is the size of the cohort aged 35–59. The regression is limited to the period 1950–2001, since the population data, which come from the United Nations, are available only since 1950.

The results are mixed. Germany shows little sign of a relationship between equity prices and demography: the R^2 is small, and the augmented Dickey-Fuller t statistic does not support cointegration. For France the real stock index has a relatively significant relationship with the MY ratio, but no convincing relationship with the cohort aged 35–59, and conversely the U.K. real stock index has no relationship with the MY ratio, but a relatively strong relationship with the 35–59 cohort. All the results improve significantly when the regression is restricted to 1980–2001: each of the

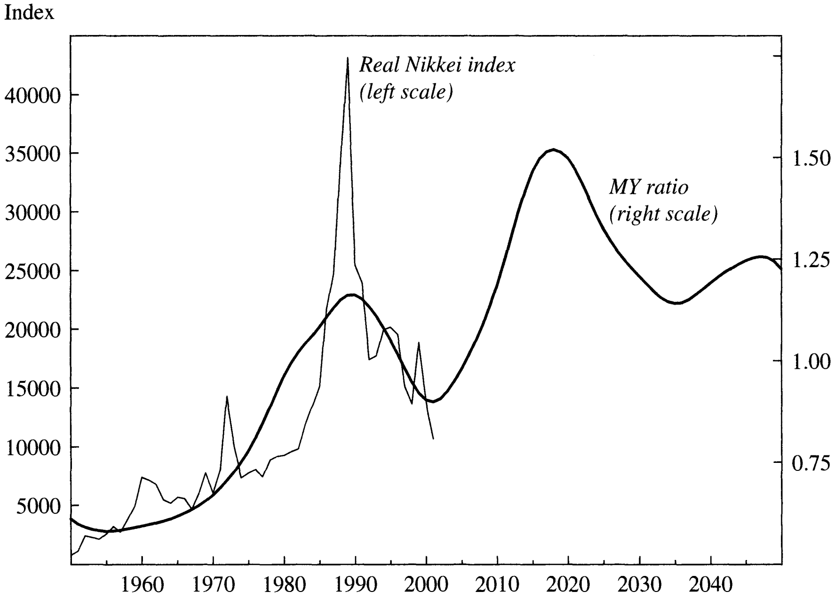
45. Board of Governors of the Federal Reserve System, Flow of Funds Accounts of the United States, 1945–2002.

Table 9. Results of Regressing Real Foreign Stock Price Indexes on Demographic Variables[a]

	Demographic variable							
	MY ratio				*Size of cohort ages 35–59*			
Country and price index	*Constant*	β	R^2	*ADF* t *statistic*	*Constant*	β	R^2	*ADF* t *statistic*
Germany: real CDAX	89	65	0.01	−1.03	−671	0.033	0.53	−2.47
	(75)	(80)			(109)	(0.004)		
France: real SBF	−2,000	3,958	0.32	−2.86	−2,400	0.23	0.37	−0.84
	(706)	(810)			(669)	(0.04)		
United Kingdom: real FTS	349	1,087	0.04	−0.77	−8,600	0.57	0.72	−3.14
	(695)	(747)			(894)	(0.05)		
Japan: real Nikkei 225	−16,481	33,555	0.70	−3.3	−14,493	0.75	0.63	−2.86
	(2,656)	(3,116)			(1,880)	(0.08)		

Source: Authors' regressions.
a. Sample period is 1950–2001.

Figure 11. Real Nikkei 225 Stock Price Index and Japanese MY Ratio, 1950–2050[a]



Source: Authors' calculations based on date from Global Financial Data, Inc., and from the United Nations (2001).
a. Data from 2001 onward are projections.

European countries had a baby boom after 1945, giving rise to a large and growing middle-aged cohort from 1980 to 2001, and each, like the United States, experienced a stock market boom over this period.

The most convincing evidence for the demographic hypothesis is provided by Japan. The Japanese market does not seem to follow the U.S. market: Japanese stock prices rose in the mid-1960s and the 1970s and fell during the 1990s, when the U.S. market was booming. Japan's live birth process has some of the same cyclical aspects as that in the United States, but with different dates for the peaks and troughs. As figure 11 shows, the turning point of the Nikkei index coincided almost exactly with the turning point of the MY ratio.

Concluding Remarks

The model studied in this paper has combined a demographic structure tailored to the demographic experience of the United States during the last century with a life-cycle behavior of the representative agent for each generation. The calculation of equilibrium shows that fluctuating cohort sizes induce substantial changes in equity prices, resulting in predictable rates of return on equity: high price-earnings ratios are followed on average by low rates of return, and low price-earnings ratios by high rates of return. The changes in equity prices are accompanied by changes in rates of return and interest rates that are linked to the change in, rather than the level of, the MY ratio. The equilibrium also exhibits some predictability of excess returns. When tested against the data, the model does not do too badly at predicting equity prices and rates of return on the stock market. However, the predictions of interest rates and excess returns are less satisfactory. On the whole, the fact that the turning points of stock prices and PE ratios are well synchronized with the demographic cycle, as measured by the MY ratio, seems to argue in favor of the demographic hypothesis.

Contrary to the conclusion of Poterba,[46] given the predicted future behavior of the MY ratio (figure 5), our model predicts a decline in the PE ratio in the U.S. equity market over the next twenty years; this conclusion is similar to that of John Campbell and Robert Shiller based on the historical mean reversion of the PE ratio process.[47] The predictions of our model should, however, be interpreted with caution in view of the ongoing globalization of equity markets. This study has been based on national (mainly U.S.) data for equity markets and demography—a restriction justified by the strong and well-documented home bias toward national equity issues.[48] However, a financial market model placed in an international setting predicts that agents will diversify across the equity issues of other countries. This discrepancy between theory and observation tends to disappear with the decrease in transactions and informational costs and the development of financial markets,[49] so that the future path of U.S. equity prices may well depend more on the joint demography of countries participating on the U.S. equity market than on U.S. demography alone. Most developed countries have similar demographic perspectives for the next thirty years, with a baby-boom generation going into retirement, low birth rates, and a lengthening of life expectancy—all factors leading to a

46. Poterba (2001).

47. Campbell and Shiller (2001).

48. French and Poterba (1991); Tesar and Werner (1995).

49. Recent papers (Lane and Milesi-Ferretti, 2003; Ahmadi, 2003) have documented an important decrease in the home bias.

high elderly dependency ratio (the ratio of retired to working agents). The only real prospect for offsetting the effect of a small generation of middle-aged agents buying the equity of a large retired generation comes from increased participation in the U.S. securities market by investors from the developing countries.

APPENDIX A

Correcting for Immigration

ANNUAL DATA FOR immigration were obtained from the *Historical Statistics of the United States* and from the *Statistical Yearbook* of the Immigration and Naturalization Service (INS, which has become the U.S. Citizenship and Immigration Service, USCIS). From these data we find that immigrants into the United States numbered approximately 2.4 million for the period 1925–44, 4.5 million in 1945–64, 8.8 million in 1965–84, and 16.8 million in 1985–2004. The USCIS statistics indicate that, between 1994 and 1996, on average 21 percent of immigrants were below age 15, 33.3 percent were aged 15–29, 26.3 percent were aged 30–44, 14.7 percent were aged 45–64, and 4.7 percent were 65 and older. These age groups do not correspond exactly to the age cohorts that we consider; therefore we estimate the number of immigrants below age 20 at 25 percent of the total, the number aged 20–39 at 50 percent, the number aged 40–59 at 20 percent, and the number 60 and over at 5 percent. We use the formulas

$$Y_t = LB_{t-1} + C_{t-1}^{im}, \quad M_t = Y_{t-1} + M_t^{im}$$

to correct the size of the cohorts, where Y_t and M_t denote the number of young and middle-aged at period t, respectively; LB_t denotes the number of live births; and C_t^{im}, Y_t^{im}, and M_t^{im} the numbers of immigrants who are children, young adults, and middle-aged, respectively.

The immigration-adjusted number of young (baby-boomers) and middle-aged (Depression generation) for the period 1965–84 becomes

$$Y_{65-84} = 79 + (0.25 \times 4.6) + (0.5 \times 8.8) = 85$$

$$M_{65-84} = 52 + (0.25 \times 2.4) + (0.5 \times 4.6) + (0.2 \times 8.8) = 57,$$

with a ratio of 85/57 = 0.67 instead of 79/52 = 0.66 as adopted in the text of the paper. Similarly, for the period 1985–2004, the corrected number of young (Xers) is $Y_{85-04} = 79.6$, and the corrected number of middle-aged (boomers) is $M_{85-04} = 88.4$, leading to a ratio of 88.4/79.6 = 1.11 instead of 79/69 = 1.14 as adopted in the text.

APPENDIX B

Markov Equilibrium

SINCE AGENTS' ECONOMIC lives span three periods, it can be shown that a Markov equilibrium that depends on the exogenous states—the pyramid and shock states—does not exist. What is needed is an endogenous variable that summarizes the dependence of the equilibrium on the past—the income that the middle-aged agents inherit from their portfolio decision in their youth. Thus we study equilibria with a state space $\Xi = G \times K \times S$, where G is a compact subset of $\mathfrak{R}_+$, $K = \{1, 2\}$ is the set of pyramid states (indexed by $k \in \{1, 2\}$), and $S = \{s_1, s_2, s_3, s_4\}$ is the set of shock states: we let $\xi = (\gamma, k, s)$ denote a typical element of the state space Ξ, with γ denoting the portfolio income inherited by the middle-aged agents from their youth. The pyramid state k determines the age pyramid $\Delta_k = (\Delta_k^y, \Delta_k^m, \Delta_k^r)$. If k is the population state at date t, let k^+ denote the pyramid state at period $t + 1$ and k^- the pyramid state at $t - 1$. Since the pyramid states alternate, if $k = 1$ then $k^+ = k^- = 2$. The output shock $s \in S$ determines the incomes $w^y = (w_s^y, s \in S)$ and $w^m = (w_s^m, s \in S)$ of the young and middle-aged agents, respectively, as well as the dividend $D = (D_s, s \in S)$ on the equity contract.

To find a Markov equilibrium, we note that the security prices only need to make the portfolio trades of the young and middle-aged agents compatible: the retired agents have no portfolio decision to make—they collect the dividends and sell their equity holdings. Thus we are led to study the portfolio problems of the young and the middle-aged agents, with the latter inheriting the income γ, and to look for security prices that clear the markets. This problem can be reduced to the study of a family of two-period portfolio problems in which middle-aged agents anticipate the consequences of their decisions for their retirement—they need to anticipate the next-period equity price Q^e—and young agents anticipate the portfolio income they will transfer into middle age (which also depends on Q^e) and the saving decision F that they will make in the next period to provide income for their retirement. A correct expectations equilibrium then has the property that the agents' expectations are fulfilled in the next period. Given that an equilibrium involves both current and anticipated variables, we introduce the convention that current variables are denoted by lowercase letters and anticipated variables are denoted by capitals. A stationary Markov equilibrium will be a function $\Phi : \Xi \to \Re^4 \times \Re^2_+ \times \Re^8_+$ with $\Phi = (z, q, Q^e, F)$, where $z = (z^y, z^m) = (z^y_b, z^y_e, z^m_b, z^m_e)$ is the vector of bond and equity holdings of the young and middle-aged agents, respectively, $q = (q^b, q^e)$ is the vector of current prices for the bond and equity, $Q^e = (Q^e_s, s \in S)$ is the vector of anticipated next-period equity prices, and $F = (F_s, s \in S)$ is the vector of anticipated next-period savings of the young. To express the condition on correct expectations, we need the following notation: if, in state ξ, young agents choose a portfolio $z^y(\xi)$ and anticipate equity prices $Q^e(\xi)$, then the income $\Gamma(\xi) = (\Gamma_s(\xi), s \in S)$ that they anticipate transferring into middle age is given by

$$\Gamma(\xi) = V(\xi) z^y(\xi), \quad \xi \in \Xi,$$

where $V(\xi) = [\mathbf{1}, D + Q^e(\xi)]$, $\mathbf{1} = (1, \ldots, 1) \in \Re^4$ denoting the sure payoff on the bond, and $D = (D_s, s \in S)$ the random dividend on equity. We let $f(\xi)$ denote the actual savings chosen by middle-aged agents when the state is ξ; thus

$$f(\xi) = q(\xi) z^m(\xi), \quad \xi \in \Xi.$$

Definition. A function $\Phi = (z, q, Q^e, F) : \Xi \to \Re^4 \times \Re^2_+ \times \Re^8_+$ is a stationary (Markov) equilibrium of the economy $E(u, w, D, \Delta)$ if, $\forall\ \xi = (\gamma, k, s) \in \Xi$,

$$\text{(B1)} \quad z^y(\xi) = \arg\max_{z^y \in \Re^2} \left\{ u(c^y) + \delta \sum_{s' \in S} \rho_{s'} u(C^m_{s'}) \,\middle|\, \begin{matrix} c^y = w^y_s - q(\xi) z^y \\ C^m = w^m + V(\xi) z^y - F(\xi) \end{matrix} \right\}$$

$$\text{(B2)} \quad z^m(\xi) = \arg\max_{z^m \in \Re^2} \left\{ u(c^m) + \delta \sum_{s' \in S} \rho_{s'} u(C^r_{s'}) \,\middle|\, \begin{matrix} c^m = w^m_s + \gamma - q(\xi) z^m \\ C^r = V(\xi) z^m \end{matrix} \right\}$$

$$\text{(B3)} \qquad \Delta^y_k z^y_b(\xi) + \Delta^m_k z^m_b(\xi) = 0, \quad \Delta^y_k z^y_e(\xi) + \Delta^m_k z^m_e(\xi) = 1$$

$$\text{(B4)} \qquad \begin{aligned} Q^e_{s'}(\xi) &= q^e\big(\Gamma_{s'}(\xi), k^+, s'\big), \ \forall\, s' \in S, \\ F_{s'}(\xi) &= f\big(\Gamma_{s'}(\xi), k^+, s'\big), \ \forall\, s' \in S. \end{aligned}$$

B1 and B2 are the conditions requiring maximizing behavior on the part of young and middle-aged agents who anticipate the equity prices $Q^e(\xi)$ and, in the case of the young agents, anticipate the saving $F(\xi)$. Note that the vector of consumption $C^m \in \Re^4_+$, which a young agent anticipates for middle age (hence the capital letter), must be distinguished from $c^m(\xi) \in \Re$, which is the current consumption of a middle-aged agent. B3 requires that the aggregate demands of the two cohorts for the bond and the equity contract clear the markets. B4 is the condition requiring that the agents' expectations be correct. In choosing their portfolio $z^y(\xi)$ in state ξ, young agents anticipate transferring the income $\Gamma(\xi) = V(\xi)z^y(\xi)$ to the next period, where $V(\xi)$ is the anticipated payoff of the securities depending on $Q^e(\xi)$. In order that $Q^e_{s'}(\xi)$ be a correct expectation, it must coincide with the price $q^e(\Gamma_{s'}(\xi), k^+, s')$, which is realized in output state s' when middle-aged agents receive the portfolio income $\gamma' = \Gamma_{s'}(\xi)$ and the pyramid state is k^+; in the same way the saving $F_{s'}(\xi)$ that the young anticipate doing in their middle age must coincide with the actual saving of a middle-aged agent with asset income $\gamma' = \Gamma_{s'}(\xi)$.

For given anticipation functions

$$(Q^e, F) : \Xi \rightarrow \Re^4_+ \times \Re^4_+$$

B1, B2, and B3 in the definition of a stationary equilibrium in the text, define a family of two-period equilibria indexed by $\xi = (\gamma, k, s) \in \Xi$. Assuming uniqueness of the equilibria, let

$$\big(z_{(Q^e,F)}(\xi), q_{(Q^e,F)}(\xi), \Gamma_{(Q^e,F)}(\xi), f_{(Q^e,F)}(\xi)\big)$$

denote the equilibrium portfolios, prices, and anticipated income transfers by the young, and the actual savings of the middle-aged, for each $\xi \in \Xi$. Finding a recursive equilibrium amounts to finding functions (Q^e, F) such that B4 is satisfied, that is,

$$\text{(B5)} \quad \begin{bmatrix} Q^e_{s'}(\xi) \\ F_{s'}(\xi) \end{bmatrix} = \begin{bmatrix} q^e_{(Q^e,F)}\big(\Gamma_{(Q^e,F)s'}(\xi), k^+, s'\big) \\ f_{(Q^e,F)}\big(\Gamma_{(Q^e,F)s'}(\xi), k^+, s'\big) \end{bmatrix} \ \forall\, s' \in S, \ \forall\, \xi = (\gamma, k, s) \in \Xi$$

Assuming that the anticipation functions as well as the equilibrium functions are continuous, an equilibrium is a fixed point on the space of continuous functions $C(\Xi, \Re^8_+)$ of the form $(Q^e, F) = \psi(Q^e, F)$, where $\psi(Q^e, F)$ is defined by the right-hand side of equation B5. We look for an approximate equilibrium in the space of piecewise linear functions on $G \times K \times S$, calculating "as if" ψ were a contraction.

We begin by choosing an interval $G = [\underline{\gamma}, \bar{\gamma}]$ and a grid $G_m = \{g_1, \ldots, g_m\}$ on this interval and then choose arbitrary initial anticipation functions $(Q^{e,0}, F^0)$ on $G_m \times K \times S$. By solving a sequence of two-period equilibrium problems, we can then compute the family of associated two-period equilibria $[z^0(\xi), q^0(\xi), \Gamma^0(\xi), f^0(\xi), \xi \in G_m \times K \times S]$, possibly modifying the interval G so that $\Gamma^0_s(\xi) \in G$ for all s and all $\xi \in G_m \times K \times S$.

Then by recursion we define for $n \geq 1$ the anticipation functions $(Q^{e,n}, F^n)$ by

$$\begin{bmatrix} Q^{e,n}_{s'}(\xi) \\ F^n_{s'}(\xi) \end{bmatrix} = \text{Lin} \begin{bmatrix} q^{e,n-1}\left(\Gamma^{n-1}_{s'}(\xi), k^+, s'\right) \\ f^{n-1}\left(\Gamma^{n-1}_{s'}(\xi), k^+, s'\right) \end{bmatrix} \quad \forall\, s' \in S, \ \forall\, \xi = G_m \times K \times S,$$

where $(z^{n-1}, q^{n-1}, \Gamma^{n-1}, f^{n-1})$ is the family of two-period equilibria associated with $(Q^{e,n-1}, F^{n-1})$, and Lin denotes the linear interpolation

$$\text{Lin } q^{e,n-1}\left(\Gamma^{n-1}_{s'}(\xi), k_+, s'\right) = \lambda q^{e,n-1}(g_j, k^+, s') + (1 - \lambda) q^{e,n-1}(g_{j+1}, k^+, s'),$$

if $\Gamma^{n-1}_{s'}(\xi) = \lambda g_j + (1 - \lambda) g_{j+1}$. At each step we modify G if necessary so that $\Gamma^n_s(\xi) \in G$ for all s and all $\xi \in G_m \times K \times S$. Although it seems difficult to prove formally that the properties of uniqueness and continuity of the two-period equilibria are satisfied, and that ψ is a contraction, in practice the algorithm converges in fewer than 1,000 iterations.

APPENDIX C

Table C1. Prices in Markov Equilibrium with Low Cohort Ratio[a]

	Pyramid Δ_1: MY ratio = 0.87				*Pyramid Δ_2: MY ratio = 1.14*			
State[b]	q^e	*PD/2*	r^{an}	rp^{an}	q^e	*PD/2*	r^{an}	rp^{an}
	Coefficient of relative risk aversion $\alpha = 2$							
s_1	97	11.4	1.96	0.27	117	14	0.6	0.28
	(0.9)	(0.1)	(0.01)	(1.3)	(1.9)	(0.23)	(0.01)	(1.2)
s_2	95	16.7	2.07	0.25	113	20	0.8	0.26
	(0.8)	(0.3)	(0.01)	(1.3)	(1.6)	(0.6)	(0.01)	(1.12)
s_3	51	6.5	5.3	0.23	61	7.3	3.96	0.27
	(0.6)	(0.13)	(0.01)	(1.3)	(1)	(0.3)	(0.01)	(1.2)
s_4	50	8.7	5.5	0.26	58	10.1	4.2	0.26
	(0.9)	(0.08)	(0.01)	(1.3)	(0.9)	(0.15)	(0.01)	(1.2)
Average	73	10.5	3.7	0.26	86	12.3	2.5	0.27
	(23)	(2.3)	(1.8)	(1.3)	(28)	(3.3)	(1.8)	(1.2)

Ratio of average equity price in pyramid Δ_1 to that in pyramid Δ_2 = 1.2
Peak-trough ratio = 2.3

	Coefficient of relative risk aversion $\alpha = 4$							
s_1	145	17.5	–0.16	1.04	207	25	–2.5	0.99
	(2.5)	(0.3)	(0.01)	(2.3)	(10)	(1.3)	(0.07)	(2.0)
s_2	134	23.4	0.34	1.06	183	32.2	–1.9	0.87
	(2)	(0.7)	(0.01)	(2.4)	(8.4)	(1.5)	(0.06)	(1.95)
s_3	46	5.5	5.8	0.96	60	7.2	3.6	1.09
	(0.7)	(0.09)	(0.01)	(2.5)	(3)	(0.7)	(0.07)	(2.0)
s_4	40	7.1	6.8	1.13	49	8.6	4.8	0.99
	(0.5)	(0.2)	(0.01)	(2.6)	(2)	(0.4)	(0.06)	(2.0)
Average	92	12.7	3.3	1.07	127	17.4	1.0	0.98
	(51)	(6.2)	(3.3)	(2.45)	(77)	(9.0)	(3.5)	(2.0)

Ratio of average equity price in pyramid Δ_1 to that in pyramid Δ_2 = 1.4
Peak-trough ratio = 5.1

	Coefficient of relative risk aversion $\alpha = 6$							
s_1	236	28.4	–2.6	2.16	301	47	–5.7	1.87
	(8.3)	(1.0)	(0.04)	(3.5)	(40)	(5)	(0.2)	(2.7)
s_2	202	35.5	–1.8	1.9	319	56	–4.8	1.93
	(6.3)	(1.1)	(0.03)	(3.6)	(30)	(11)	(0.2)	(2.8)
s_3	48	5.8	5.6	2.14	64	7.7	2.97	2.0
	(1.3)	(0.15)	(0.03)	(3.9)	(5.7)	(0.7)	(0.2)	(2.87)
s_4	37	6.5	7.4	2.3	43	7.5	5.2	2.7
	(0.8)	(0.1)	(0.03)	(4.2)	(3.3)	(0.8)	(0.2)	(3.0)
Average	134	18	2.3	2.2	211	28	–0.4	2.0
	(96)	(12)	(4.8)	(3.8)	(169)	(21)	(5.2)	(2.9)

Ratio of average equity price in pyramid Δ_1 to that in pyramid Δ_2 = 1.6
Peak-trough ratio = 10.6

Source: Authors' calculations of equilibrium values of the calibrated model.
a. Cohort sizes (N, n) are (79, 69), initial endowments of the young adult and middle-aged generations in states s_1 through s_4 are $w^y = (2.3, 2.3, 1.7, 1.7)$ and $w^m = (3.6, 3.6, 2.4, 2.4)$, and dividends $D = (83, 57, 83, 57)$. Standard deviations are in parentheses.
b. See table 4 in the text.