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ON THE CONVERGENCE OF FICTITIOUS PLAY

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We study the Brown-Robinson fictitious play process for non-zero sum games. We show that, in general, fictitious play cannot converge cyclically to a mixed strategy equilibrium in which both players use more than two pure strategies.

1. Introduction. This paper studies the "fictitious play" (FP) learning process due to Brown (1949) and (1951) and Robinson (1951). The FP process was originally proposed as a computational tool for determining the value of a two-person zero-sum game. However, it can also be interpreted as a learning process for boundedly rational agents in which each player plays a myopic best response in each period, on the assumption that the opponent's future actions will resemble the past.

Robinson (1951) established the result that in finite two-person zero-sum games every FP process converges to the set of equilibria of the game. Miyasawa (1963) showed the convergence of FP in 2×2 games. However, the convergence cannot be guaranteed in general non-zero sum games as an important 3×3 example due to Shapley (1964) shows. In Shapley's example the FP process follows a cycle in which runs of pure strategy combinations are repeated over and over but the run-lengths increase exponentially. The resulting mixed strategies then also cycle and are bounded away from the unique equilibrium. Shapley's example is generic in the sense that small perturbations of the payoffs do not affect this conclusion. More recently, Foster and Young (1996) have constructed an 8×8 generic coordination game in which an FP process does not converge to any of the equilibria.

In his original formulation of fictitious play Brown considered both a discrete time version and a continuous time version of fictitious play (see Brown 1949 in particular). The convergence results cited above hold for both the discrete and the continuous time versions, as does Shapley's counterexample. Indeed, many of the results are easier to derive in the continuous time model than in the discrete time model (Harris 1996, Hofbauer 1994 and Monderer et al. 1997). We also find it convenient to first study continuous time fictitious play (referred to as CFP) and to then extend our results to the discrete process (DFP).

Our main result (Theorem 1) is that CFP almost never converges cyclically to a mixed strategy equilibrium in which both players use more than two pure strategies. Thus, Shapley's example of nonconvergence is the norm rather than the exception. Mixed strategy equilibria appear to be generally unstable with respect to cyclical fictitious play processes. In a recent paper, Hofbauer (1994) has made a related conjecture: if CFP converges to a regular mixed strategy equilibrium, then the game is zero-sum.

We then show (in §10) how the main result may be extended to include DFPs also.

As is well known, the interpretation of mixed strategy equilibria is problematic (see, for instance, Rubinstein 1991). In two person zero-sum games a justification for mixed strategies is that the "correct" probabilities provide the best defense against the opponent.

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But in non-zero-sum games a justification on defensive grounds cannot be made. A point of view originating with Harsanyi (1973a) takes the position that the equilibrium probabilities represent only the subjective beliefs of other players about the behavior of a particular player; thus it is not necessary to assume that players actually choose randomized strategies. Fictitious play and associated learning procedures suggest a way in which such beliefs can form over time by means of a gradual process. However, learning procedures can serve to justify mixed strategy equilibria only in circumstances in which the procedures converge to an equilibrium. Our result shows that, in general non-zero-sum games, mixed strategy equilibria are inherently unstable.

The behavior of dynamical processes in the presence of mixed equilibria has previously been examined in a related context by Crawford (1985). Crawford studies a class of learning procedures in which players (a) have a finite memory and (b) play mixed strategies which are adjusted in response to the difference in payoffs from playing a particular pure strategy and the mixed strategy against the actual play in the recent past. Crawford then shows that mixed strategy equilibria are generally unstable. The procedures considered do not include the CFP; they are more akin to evolutionary processes like the so-called "replicator dynamics." Evolutionary dynamical systems are considered in more detail by Hofbauer and Sigmund (1988). Their results also suggest that mixed strategy equilibria are unstable in general (asymmetric) bimatrix games.

In other, more closely related work, Fudenberg and Kreps (1993) study interpretational issues concerning mixed strategies and learning processes like FP. They propose some alternative systems based on ideas stemming from Harsanyi's (1973a) purification theorem and derive convergence results for 2×2 games. Jordan (1993) points out other difficulties in interpreting the convergence of learning processes to mixed equilibria. In particular, he points out that the convergence concerns players' expectations and not strategies or payoffs.

2. Fictitious play. Let $G = (A, B)$ be a two-player *game* where A and B are $I \times J$ matrices. We will refer to $I = \{1, 2, \dots, I\}$ and $J = \{1, 2, \dots, J\}$ as the sets of pure strategies available to players 1 and 2, respectively. As usual, if player 1 chooses strategy i and player 2 chooses strategy j, the payoff to player 1 is a_{ij} and the payoff to player 2 is b_{ij}. The sets of mixed strategies are denoted by $\Delta(I)$ and $\Delta(J)$, respectively. Let $\delta_i \in \Delta(I)$ be the mixed strategy that assigns weight 1 to i. We will identify i with δ_i and write $i \in \Delta(I)$ instead of $\delta_i \in \Delta(I)$. For any finite set X, let $\#X$ denote the number of elements in X.

For all $q \in \Delta(J)$, let $BR(q)$ be the set of *pure strategy best responses* for player 1 and denote by supp $q = \{j : q_j > 0\}$ the *support* of q. The mixed strategy pair (p^*, q^*) is a *Nash equilibrium* if supp $p^* \subseteq BR(q^*)$ and supp $q^* \subseteq BR(p^*)$.

DEFINITION 1. For $t = 1, 2, 3, \dots$, the sequence $(p(t), q(t))$ is a *discrete time fictitious play process* (*DFP*) if

$$(p(1), q(1)) \in \Delta(I) \times \Delta(J);$$

and for all $t \geq 1$,

$$p(t+1) = \frac{tp(t) + i(t)}{t+1}, \qquad q(t+1) = \frac{tq(t) + j(t)}{t+1}$$

where $i(t) \in BR(q(t))$ and $j(t) \in BR(p(t))$.

The discrete time fictitious play process (DFP) is also known as the "Brown-Robinson Learning Process." In this paper, we find it convenient to work with both DFP and a continuous time version (CFP). Let $dp(t)/dt|_+$ denote the right-hand derivative of p at time t.

DEFINITION 2. For $t \geq 1$, the path $(p(t), q(t))$ is a *continuous time fictitious play process* (*CFP*) if $(p(t), q(t))$ are continuous functions of t satisfying:

$$(p(1), q(1)) \in \Delta(I) \times \Delta(J);$$

and for all $t \geq 1$,

$$\left.\frac{dp(t)}{dt}\right|_{+} = \frac{i(t) - p(t)}{t}, \qquad \left.\frac{dq(t)}{dt}\right|_{+} = \frac{j(t) - q(t)}{t}$$

where $i(t) \in BR(q(t))$ and $j(t) \in BR(p(t))$.

It is well known that if the DFP (or CFP) $(p(t), q(t))$ converges to (p^*, q^*), then (p^*, q^*) is a Nash equilibrium of G.

Recently, Harris (1996) and Hofbauer (1994) have looked at differential inclusions of the form: for almost all t,

$$\frac{dp}{dt} \in \frac{\overline{BR}(q(t)) - p(t)}{t}, \qquad \frac{dq}{dt} \in \frac{\overline{BR}(p(t)) - q(t)}{t}$$

where $\overline{BR}(q)$ and $\overline{BR}(p)$ are the (closed, convex and upper semi-continuous) correspondences consisting of *mixed strategy best responses*. The theory of differential inclusions guarantees the existence of a solution for each initial condition. Our definition of CFP is more restrictive in that we do not allow players to randomize when indifferent. But any CFP we consider will satisfy the differential inclusion.

Cyclic play. Under fictitious play, each player plays a best response against the empirical distribution of the opponent's play. Under CFP, therefore, when a player switches from one pure strategy to another he is precisely indifferent between these two strategies, a fact that is crucial for our analysis. We shall consider play which takes the following form. There is a sequence of times $(t_0, t_1, t_2, t_3, \cdots)$, with $t_0 = 1$, such that for each $n \geq 1$, $BR(q(t))$ and $BR(p(t))$ are singletons for all $t \in (t_{n-1}, t_n)$. The times $(t_0, t_1, t_2, t_3, \cdots)$ are the times when some player switches his strategy (in an exceptional case, both players may switch at the same time). Let

$$(i_{t_n}, j_{t_n}) \equiv (BR(q(t)), BR(p(t))) \quad \text{for } t \in (t_{n-1}, t_n)$$

denote the (uniquely determined) constant choices in the interval (t_{n-1}, t_n). The interval (t_{n-1}, t_n) consists of continuous play of (i_{t_n}, j_{t_n}), referred to as a *run*. The *run-length* is $t_n - t_{n-1}$. The *sequence of play* is the sequence of pure strategy combinations:

$$(i_{t_1}, j_{t_1}), (i_{t_2}, j_{t_2}), \ldots, (i_{t_n}, j_{t_n}), \cdots$$

A CFP *follows a cycle c* if there is a sequence of K pure strategy combinations

$$c = \langle (i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots, (i_K, j_K) \rangle$$

such that for all $n \geq 1$, $n \bmod K = k$ implies that $(i_{t_n}, j_{t_n}) = (i_k, j_k)$. In other words, CFP is cyclic if a particular sequence of K pure strategy combinations c is played over and over in the same order. We emphasize that cyclic play refers to the fact that pure strategy combinations are played in a fixed pattern, and not that the trajectory $(p(t), q(t))$ reaches a limit cycle. Cyclic play has been called "quasi-periodic" play by Rosenmüller (1971).

A cycle c is *robust* if there is an open set of initial conditions E such that for all $(p(1), q(1)) \in E$ there exists a CFP that follows the cycle c.

EXAMPLE 1. Matching Pennies:

	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

In this game, any initial condition (other than the equilibrium) generates the following cycle of play (see Metrick and Polak 1994 or Rosenmüller 1971):

$$c = \langle (H, H), (H, T), (T, T), (T, H) \rangle.$$

Thus, Matching Pennies has a robust cycle. Moreover, from every initial condition the resulting trajectory converges to the unique Nash equilibrium.

EXAMPLE 2. Shapley's Game:

	X	Y	Z
X	0, 0	1, 2	2, 1
Y	2, 1	0, 0	1, 2
Z	1, 2	2, 1	0, 0

In this game there are two possible cycles. Any initial condition $(p(1), q(1))$ satisfying $p(1) \neq q(1)$ generates the cycle of play (Shapley 1964):

$$c = \langle (X, Y), (Z, Y), (Z, X), (Y, X), (Y, Z), (X, Z) \rangle$$

and thus c is a robust cycle. The cycle c results since it is the unique "better response" path of pure strategies (see Monderer and Sela 1993). On the other hand, if $p(1) = q(1)$ the resulting cycle of play is

$$c' = \langle (X, X), (Y, Y), (Z, Z) \rangle$$

which is *not* robust since it is generated only by very special initial conditions. Thus, Shapley's game has both a robust and a nonrobust cycle of play. Moreover, a CFP that follows the (robust) cycle c does not converge (Shapley 1964) whereas every CFP that follows the (nonrobust) cycle c' converges (see §9 below for details).

3. The main result. Let Γ denote the set of all $I \times J$ games. Each game $G \in \Gamma$ can be associated with a point in the Euclidean space $R^{I \times J} \times R^{I \times J}$. A set of games $\Gamma' \subset \Gamma$ is *null* if it is contained in a closed set of Lebesgue measure zero. A statement is true for *almost all games* if the set of games for which it is not true is null (Harsanyi 1973b).

As argued above, in Matching Pennies a robust cycle leads to the convergence of every CFP. Moreover, a small perturbation of the payoffs of Matching Pennies will destroy neither the cycle nor the convergence, so there is an open set of games with a robust convergent cycle. However, in the case of Matching Pennies only two pure strategies are used by each player. Our main result is:

THEOREM 1. *For almost all games, if there is an open set of initial conditions and a cycle c such that (from these initial conditions) there exists a CFP that follows the cycle c and converges to a Nash equilibrium* (p^*, q^*), *then* $\# \operatorname{supp} p^* = \# \operatorname{supp} q^* \leq 2$.

The proof of Theorem 1 is somewhat involved and so we first present a brief outline of the argument.

3.1. An outline of the proof. Fictitious play (CFP) is a *continuous*, *nonlinear* and *nonautonomous* dynamical system. The first step is to reformulate the system so that the problem reduces to the study of an associated *discrete*, *linear*, and *autonomous* system. Once this is done, standard tools can be brought to bear on the problem. In the second step, these tools are employed to analyze the linear difference equation system and obtain the main result.

Step 1: Reduction. When the play is cyclic, a sequence of choices is

$$(i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots, (i_K, j_K)$$

repeated over and over in the same order. K consecutive runs corresponding to the choices $(i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots, (i_K, j_K)$ are a *round*. Thus, cyclic play consists of rounds $r = 1, 2, 3, \ldots$. A run corresponding to the choice (i_k, j_k) is referred to as a *k-run*. Let $n_k(r)$ denote the length of the k-run in round r, that is, $n_k(r)$ is the amount of time spent playing (i_k, j_k) in round r. Let $n(r) = (n_1(r), n_2(r), \ldots, n_K(r))$.

We will argue that if the CFP is cyclic, then there exists a $K \times K$ matrix F such that for all r,

$$n(r+1) = Fn(r). \tag{1}$$

Since CFP is completely determined by the associated system determining the run-lengths, the problem has been reduced to the study of a linear difference equation. A similar reduction appears in Rosenmüller (1971).

Step 2: Analysis of F. The behavior of the discrete linear dynamical system (1) is determined by eigen roots of F, and in the long run the evolution is determined by the dominant eigen root. The matrix F is singular, and so one of its eigen roots is 0, but the crucial fact (Lemma 2) is that the product of the nonzero eigen roots of F is one. Geometrically, this means that F is volume preserving in a $K - 1$ dimensional invariant subspace M, that is, if $S \subset M$ has volume V (relative to M) then the set $F(S) = \{Fn : n \in S\}$ also has volume V. Then there cannot exist an open set of starting positions such that the run-lengths decrease from round to round, for that would imply a reduction in volume. Run lengths must then be either constant or increasing.

The next step in the proof is to establish that if each player uses at least three pure strategies in the cycle, for almost all games, not all (nonzero) eigen roots of F can have absolute value equal to one. This involves a rather detailed analysis of the matrix F. In fact F always has some unit roots, and the crucial step is an *exact* determination of the number of unit roots of F (Lemma 3) and the dimension of the corresponding eigen space (Lemma 7). This, combined with Lemma 2, implies that there exists a real eigen root λ of F such that $\lambda > 1$. This shows that for almost all initial conditions the run-lengths increase exponentially as in Shapley's (1964) example, and CFP does not converge.

For nongeneric classes of games (such as zero-sum games) it may well happen that *all* nonzero eigen roots of F have absolute value equal to one, which allows for convergence.

We discuss this issue below in §9. Moreover, the result does not rule out convergence from exceptional starting positions, as illustrated by Example 2.

4. Determination of the run-lengths. We start by assuming that the game G is nondegenerate in the following sense.

DEFINITION OF Γ_1. Let $\Gamma_1 \subset \Gamma$ denote the set of games such that
(a) every Nash equilibrium (p^*, q^*) satisfies
(a1) $\#\operatorname{supp} p^* = \#\operatorname{supp} q^*$;
(a2) $i \in BR(q^*)$ implies $i \in \operatorname{supp} p^*$ and $j \in BR(p^*)$ implies $j \in \operatorname{supp} q^*$;
(b) if $i \neq i'$ then for all j, $a_{ij} \neq a_{i'j}$ and if $j \neq j'$ then for all i, $b_{ij} \neq b_{ij'}$.

It is well known that $\Gamma \backslash \Gamma_1$ is null. This is because all regular equilibria (p^*, q^*) satisfy (a1) and (a2), and the class of bimatrix games with a nonregular equilibrium is null (van Damme 1991, Chapter 3). Clearly, the set of games not satisfying (b) is also null. For the proof of our theorem we shall consider only games in Γ_1. Notice that this allows us to restrict attention to cycles where each player uses the same number of pure strategies, for if $(p(t), q(t)) \to (p^*, q^*)$ then by continuity, every pure strategy used along the cycle is a best response at (p^*, q^*). Thus, by (a2) only pure strategies in the support of (p^*, q^*) can be used along the cycle, but then by (a1) each player uses the same number of pure strategies.

Notation. Suppose CFP follows the cycle

$$c = \langle (i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots, (i_K, j_K) \rangle.$$

We will argue that there exists a $K \times K$ matrix F (which depends on the particular cycle and on the payoff matrices) such that for all r,

$$n(r+1) = Fn(r).$$

Let $(\alpha_1, \alpha_2, \ldots, \alpha_I)$ denote the I rows of A and let $(\beta_1, \beta_2, \ldots, \beta_J)$ denote the J columns of B. Let P^0 and Q^0 be vectors denoting the total amount of time each player has used each strategy prior to the start of round r. If $r = 1$ then $(P^0, Q^0) = (p(1), q(1))$ are the initial conditions at the start of the game. It is convenient to write $n = n(r)$ and $n' = n(r+1)$.

Define an $I \times K$ matrix P by:

$$P_{ik} = \begin{cases} 1 & \text{if } i_k = i, \\ 0 & \text{if } i_k \neq i, \end{cases}$$

and a $J \times K$ matrix Q by:

$$Q_{jk} = \begin{cases} 1 & \text{if } j_k = j, \\ 0 & \text{if } j_k \neq j. \end{cases}$$

Observe that $(Pn)_i$ is the amount of time player 1 played strategy i in round r and $(Qn)_j$ is the amount of time player 2 played strategy j in round r. Notice also that $\alpha_{i_k} Q = (a_{i_k j_1}, a_{i_k j_2}, \ldots, a_{i_k j_K})$ and $\beta_{j_k} P = (b_{i_1 j_k}, b_{i_2 j_k}, \ldots, b_{i_K j_k})$.

Let e_k denote the kth K-dimensional unit vector. It is convenient to define the $K \times K$ matrix:

$$E_k = (e_1, e_2, \ldots, e_k, 0, \ldots, 0)$$

whose first k columns are the first k unit vectors and the last $(K - k)$ columns are 0. By definition, $E_K = I$, the identity matrix. We also have $E_k n = (n_1, n_2, \ldots, n_k, 0, \ldots, 0)$.

Round *r* equations. Under CFP, when a player switches from one pure strategy to another he is precisely indifferent between these two strategies. Using this fact, we find that the players switch from (i_1, j_1) to (i_2, j_2) in round r when:

$$\alpha_{i_2} Q^0 + a_{i_2 j_1} n_1 = \alpha_{i_1} Q^0 + a_{i_1 j_1} n_1$$

and

$$\beta_{j_2} P^0 + b_{i_1 j_2} n_1 = \beta_{j_1} P^0 + b_{i_1 j_1} n_1 .$$

It is convenient to rewrite these as:

$$(\alpha_{i_2} - \alpha_{i_1}) Q E_1 n = -(\alpha_{i_2} - \alpha_{i_1}) Q^0 \tag{2}$$

and

$$(\beta_{j_2} - \beta_{j_1}) P E_1 n = -(\beta_{j_2} - \beta_{j_1}) P^0 . \tag{3}$$

We show below that in a robust cycle, only one player switches strategy in the transition from (i_1, j_1) to (i_2, j_2), and thus only one of Equations (2) or (3) will be nontrivial. For instance, if only player 1 switches strategies, that is, if $i_2 \neq i_1$ but $j_2 = j_1$, then (3) is trivially satisfied and hence redundant.

In general, for $k = 1, 2, \ldots, K$, when the players switch from (i_k, j_k) to (i_{k+1}, j_{k+1}) we have:

$$\alpha_{i_{k+1}} Q^0 + \sum_{s=1}^{k} a_{i_{k+1} j_s} n_s = \alpha_{i_k} Q^0 + \sum_{s=1}^{k} a_{i_k j_s} n_s$$

and

$$\beta_{j_{k+1}} P^0 + \sum_{s=1}^{k} b_{i_s j_{k+1}} n_s = \beta_{j_k} P^0 + \sum_{s=1}^{k} b_{i_s j_k} n_s$$

which can be rewritten as:

$$(\alpha_{i_{k+1}} - \alpha_{i_k}) Q E_k n = -(\alpha_{i_{k+1}} - \alpha_{i_k}) Q^0 \tag{4}$$

and

$$(\beta_{j_{k+1}} - \beta_{j_k}) P E_k n = -(\beta_{j_{k+1}} - \beta_{j_k}) P^0 \tag{5}$$

where we always write $K + 1 \equiv 1$.

Round $(r + 1)$ equations. By the earlier arguments, for $k = 1, 2, \ldots, K$, when the players switch from (i_k, j_k) to (i_{k+1}, j_{k+1}) in round $r + 1$ we have:

$$(6) \qquad (\alpha_{i_{k+1}} - \alpha_{i_k})QE_k n' = -(\alpha_{i_{k+1}} - \alpha_{i_k})Qn - (\alpha_{i_{k+1}} - \alpha_{i_k})Q^0$$

and

$$(7) \qquad (\beta_{j_{k+1}} - \beta_{j_k})PE_k n' = -(\beta_{j_{k+1}} - \beta_{j_k})Pn - (\beta_{j_{k+1}} - \beta_{j_k})P^0.$$

The basic difference equation. By substituting (4) and (5) into (6) and (7), respectively, we obtain for $k = 1, 2, \ldots, K$:

$$(8) \qquad (\alpha_{i_{k+1}} - \alpha_{i_k})QE_k n' = -(\alpha_{i_{k+1}} - \alpha_{i_k})Q(I - E_k)n,$$

$$(9) \qquad (\beta_{j_{k+1}} - \beta_{j_k})PE_k n' = -(\beta_{j_{k+1}} - \beta_{j_k})P(I - E_k)n,$$

where

$$I - E_k = [0, 0, \ldots, e_{k+1}, e_{k+2}, \ldots, e_K].$$

Let $\sigma(k) \in \{1, 2\}$ denote the player who switches after k:

$$(10) \qquad \sigma(k) = \begin{cases} 1 & \text{if } i_k \neq i_{k+1}, \\ 2 & \text{if } j_k \neq j_{k+1}. \end{cases}$$

For convenience, we usually assume player one is the first to switch in each round ($\sigma(1) = 1$) and player 2 switches next ($\sigma(2) = 2$). This is without loss of generality as there must always exist some k such that $\sigma(k) = 1$ and $\sigma(k+1) = 2$.

Consider the system of equations that results when out of Equations (8) and (9), for each k, only the kth equation corresponding to the switching player $\sigma(k)$ is considered. This results in a system of equations of the form

$$Cn' = Dn.$$

The kth row of C is

$$(11) \qquad (\alpha_{i_{k+1}} - \alpha_{i_k})QE_k = [a_{i_{k+1}j_1} - a_{i_k j_1}, a_{i_{k+1}j_2} - a_{i_k j_2}, \ldots, a_{i_{k+1}j_k} - a_{i_k j_k}, 0, 0, \ldots, 0],$$

if $\sigma(k) = 1$, and is

$$(12) \qquad (\beta_{j_{k+1}} - \beta_{j_k})PE_k = [b_{i_1 j_{k+1}} - b_{i_1 j_k}, b_{i_2 j_{k+1}} - b_{i_2 j_k}, \ldots, b_{i_k j_{k+1}} - b_{i_k j_k}, 0, 0, \ldots, 0],$$

if $\sigma(k) = 2$. Therefore, C is a *lower-triangular* matrix (that is, for all $k < l$, $c_{kl} = 0$).

The kth row of D is

$$(13) \qquad \begin{aligned} &-(\alpha_{i_{k+1}} - \alpha_{i_k})Q(I - E_k) \\ &\quad = -[0, 0, \ldots, 0, a_{i_{k+1}j_{k+1}} - a_{i_k j_{k+1}}, a_{i_{k+1}j_{k+2}} - a_{i_k j_{k+2}}, \ldots, a_{i_{k+1}j_K} - a_{i_k j_K}], \end{aligned}$$

if $\sigma(k) = 1$, and is

$$-(\beta_{j_{k+1}} - \beta_{j_k})P(I - E_k)$$
(14)
$$= -[0, 0, \ldots, 0, b_{i_{k+1}j_{k+1}} - b_{i_{k+1}j_k}, b_{i_{k+2}j_{k+1}} - b_{i_{k+2}j_k}, \ldots, b_{i_K j_{k+1}} - b_{i_K j_k}],$$

if $\sigma(k) = 2$. Thus, D is a *strictly upper-triangular* matrix (that is, for all $k \geq l$, $d_{kl} = 0$).

If $G \in \Gamma_1$ then the diagonal elements of C are all strictly positive (Monderer and Sela 1993 call this the "better response property"). Thus, C is invertible and we can write

$$n' = C^{-1}Dn. \tag{15}$$

This establishes that the run-lengths are determined by a first-order linear difference equation. The behavior of the difference Equation (15) is determined by the eigen roots of the matrix $C^{-1}D$. And clearly C and D are completely determined by the game G and the cycle c.

5. Robust cycles. A cycle $c = \langle (i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots, (i_K, j_K) \rangle$ has a *simultaneous switch* if there exists a k such that $i_k \neq i_{k+1}$ and $j_k \neq j_{k+1}$. An example is the cycle c' of Example 2. Recall that a cycle is robust if there is an open set of initial conditions E such that for all $(p(1), q(1)) \in E$ there exists a CFP that follows the cycle c.

LEMMA 1. *For all $G \in \Gamma_1$, if c is a robust cycle then c has no simultaneous switches.*

PROOF. Let E_0 be an open set of initial conditions that result in the cycle

$$c = \langle (i_1, j_1), \ldots, (i_k, j_k), (i_{k+1}, j_{k+1}), \ldots, (i_K, j_K) \rangle$$

and suppose both players switch simultaneously in c for the first time after run k, that is, both $i_{k+1} \neq i_k$ and $j_{k+1} \neq j_k$.

We first argue that there cannot be a simultaneous switch after the first run, (i_1, j_1). If there were, then the same n_1 would have to solve both (2) and (3). Since $G \in \Gamma_1$, this requires that

$$\frac{(\alpha_{i_2} - \alpha_{i_1})Q^0}{(a_{i_2 j_1} - a_{i_1 j_1})} = \frac{(\beta_{j_2} - \beta_{j_1})P^0}{(b_{i_1 j_2} - b_{i_1 j_1})}.$$

But since the cycle c is robust this condition would have to hold for all $(P^0, Q^0) \in E_0$ which is impossible since E_0 is an open subset of $\Delta(I) \times \Delta(J)$. (Every (P^0, Q^0) satisfies the normalization $\Sigma_i P_i^0 = \Sigma_j Q_j^0$ but since none of the strategies used in the cycle can be strictly dominated the condition given above is not the same as the normalization condition.) Thus, there cannot be a simultaneous switch after the first run and $k > 1$.

Let $l < k$ and without loss of generality suppose that player 1 makes a switch after run l, that is, $(i_l, j_l) \equiv (i, j)$ and $(i_{l+1}, j_{l+1}) \equiv (i', j)$. For any t and $t' > t$, define the mapping $\Phi_{t,t'}(p, q) = (p', q')$ by:

$$p' = \frac{(t' - s)i' + (s - t)i + tp}{t'},$$

$$q' = \frac{(t' - t)j + tq}{t'},$$

where

$$s = \frac{(a_{i'j} - a_{ij}) - (\alpha_{i'} - \alpha_i)q}{(a_{i'j} - a_{ij})}t.$$

This is well defined since $G \in \Gamma_1$ and thus $a_{i'j} - a_{ij} > 0$.

It may be verified that $\Phi_{t,t'}$ is an affine function. Its inverse $\Phi_{t,t'}^{-1}$ is defined by $\Phi_{t,t'}^{-1}(p', q') = (p, q)$ where

$$p = \frac{t'p' - (t' - s')i' - (s' - t)i}{t},$$

$$q = \frac{t'q' - (t' - t)j}{t},$$

and s' is defined by:

$$s' = \frac{(a_{i'j} - a_{ij})t - (\alpha_{i'} - \alpha_i)(t'q' - (t' - t)j)}{(a_{i'j} - a_{ij})}.$$

Then $\Phi_{t,t'}^{-1}$ is also affine.

Consider a CFP $(p(t), q(t))$ which follows the cycle c from some starting position $(p(1), q(1)) = (\bar{p}, \bar{q}) \in E_0$. We will write $(p(t), q(t))$ as $(p(t|\bar{p}, \bar{q}), q(t|\bar{p}, \bar{q}))$ to indicate the dependence of the CFP on the initial condition $(\bar{p}, \bar{q})$.

Consider the switch from i to i'. The mapping $\Phi_{t,t'}$ is defined so that if

$$(p(t), q(t)) \in \operatorname{int} BR^{-1}(i, j)$$

and

$$(p(t'), q(t')) \in \operatorname{int} BR^{-1}(i', j)$$

then $\Phi_{t,t'}(p(t), q(t)) = (p(t'), q(t'))$. Thus, there is an $\epsilon > 0$ such that for all (p, q) in an open ball E around $(p(t), q(t))$, $\Phi_{t,t'}(p, q) \in \operatorname{int} BR^{-1}(i', j)$. Moreover, since $\Phi_{t,t'}$ is affine and invertible, the image of E, $\Phi_{t,t'}(E)$ is also open. Because this is true at each $l = 1, \ldots, k$, we obtain that if $(\hat{p}, \hat{q}) \in E_0$ and $(p(t^*|\hat{p}, \hat{q}), q(t^*|\hat{p}, \hat{q})) \in \operatorname{int} BR^{-1}(i_k, j_k)$ for some t^*, then there is an open ϵ-ball $E_\epsilon \subset E_0$ around $(\hat{p}, \hat{q})$ such that for all $(\bar{p}, \bar{q}) \in E_\epsilon$ we have

$$(p(t^*|\bar{p}, \bar{q}), q(t^*|\bar{p}, \bar{q})) \in \operatorname{int} BR^{-1}(i_k, j_k).$$

Moreover, the set $E^* = \{(p(t^*|\bar{p}, \bar{q}), q(t^*|\bar{p}, \bar{q})) : (\bar{p}, \bar{q}) \in E_\epsilon\}$ is an open subset of $BR^{-1}(i_k, j_k)$.

Our hypothesis implies that for any starting position $(\bar{p}, \bar{q}) \in E_\epsilon$ both players switch at the same time after run k. The time of the switch $t_k(\bar{p}, \bar{q})$ must simultaneously satisfy

$$(a_{i_{k+1}j_k} - a_{i_k j_k})(t_k(\bar{p}, \bar{q}) - t^*) = -(\alpha_{i_{k+1}} - \alpha_{i_k})t^* q(t^*|\bar{p}, \bar{q}) \tag{16}$$

and

$$(b_{i_k j_{k+1}} - b_{i_k j_k})(t_k(\bar{p}, \bar{q}) - t^*) = -(\beta_{j_{k+1}} - \beta_{j_k})t^* p(t^*|\bar{p}, \bar{q}). \tag{17}$$

Equations (16) and (17) imply

$$\frac{\alpha_{i_{k+1}} - \alpha_{i_k}}{a_{i_{k+1}j_k} - a_{i_kj_k}} q(t^*|\bar{p}, \bar{q}) = \frac{\beta_{j_{k+1}} - \beta_{j_k}}{b_{i_kj_{k+1}} - b_{i_kj_k}} p(t^*|\bar{p}, \bar{q}). \tag{18}$$

As we vary $(\bar{p}, \bar{q}) \in E_\epsilon$, (18) must hold for all $(p(t^*|\bar{p}, \bar{q}), q(t^*|\bar{p}, \bar{q}))$ in the open set E^*, but this is obviously impossible. This contradiction proves the lemma. □

6. The eigen roots of $C^{-1}D$. The eigen roots of $C^{-1}D$ are determined by the solutions to the equation:

$$\lambda x = C^{-1}Dx$$

where $x \neq 0$ is an eigen vector, which are the same as the solutions to:

$$(\lambda C - D)x = 0.$$

Suppose the matrix $C^{-1}D$ has $S + 1$ distinct eigen roots $\lambda_0, \lambda_1, \ldots, \lambda_S$ so that we can write the characteristic polynomial of $C^{-1}D$ in the form:

$$|\lambda I - C^{-1}D| = \prod_{s=0}^{S} (\lambda_s - \lambda)^{\alpha_s}.$$

The number α_s is called the *algebraic multiplicity* of the root λ_s.

The number $\gamma_s \equiv \dim \ker(\lambda_s I - C^{-1}D)$ is called the *geometric multiplicity* of the root λ_s.

Note that for all s, $\gamma_s \leq \alpha_s$.

Now observe from (14) that the first column of D is 0. Thus $C^{-1}D$ is singular and $\lambda_0 \equiv 0$ is an eigen root of $C^{-1}D$. Furthermore for all $G \in \Gamma_1$, the algebraic multiplicity of λ_0 is exactly 1. To see this, notice that if we write

$$|\lambda I - C^{-1}D| = |C^{-1}|\,|\lambda C - D| = \lambda |M(\lambda)|,$$

then for $G \in \Gamma_1$, $|M(0)| \neq 0$.

We now establish an important result about the nonzero roots of $C^{-1}D$.

LEMMA 2. *For all $G \in \Gamma_1$ and all cycles without simultaneous switches, the product of the nonzero eigen roots of $C^{-1}D$ is* 1.

PROOF. Write:

$$C = \begin{bmatrix} c_{11} & 0 \\ c & \bar{C} \end{bmatrix}$$

where $\bar{C}$ is the triangular $(K - 1) \times (K - 1)$ submatrix of C in the lower right corner and $c = [c_{21}, c_{31}, \ldots, c_{K1}]^T$. Since $G \in \Gamma_1$, the diagonal elements of C are all strictly positive.

Similarly, write:

$$D = \begin{bmatrix} 0 & d \\ 0 & \bar{D} \end{bmatrix}$$

where $\bar{D}$ is the $(K - 1) \times (K - 1)$ submatrix of D in the lower right corner and $d = [d_{12}, d_{13}, \ldots, d_{1K}]$. Now:

$$C^{-1}D = \begin{bmatrix} 0 & \frac{1}{c_{11}} d \\ 0 & \bar{C}^{-1}\left(\bar{D} - \frac{1}{c_{11}} cd\right) \end{bmatrix}. \tag{19}$$

Let I_k denote the $k \times k$ identity matrix. The characteristic polynomial of $C^{-1}D$ is:

$$\begin{aligned} 0 &= \det(\lambda I_K - C^{-1}D) \\ &= \det \begin{bmatrix} \lambda & -\frac{1}{c_{11}} d \\ 0 & \lambda I_{K-1} - \bar{C}^{-1}\left(\bar{D} - \frac{1}{c_{11}} cd\right) \end{bmatrix} \\ &= \lambda \times \det\left(\lambda I_{K-1} - \bar{C}^{-1}\left(\bar{D} - \frac{1}{c_{11}} cd\right)\right) \end{aligned} \tag{20}$$

using (19).

Claim.

$$\det\left(\bar{C}^{-1}\left(\bar{D} - \frac{1}{c_{11}} cd\right)\right) = 1.$$

PROOF OF CLAIM. Observe that

$$\bar{D} - \frac{1}{c_{11}} cd$$

$$= \begin{bmatrix} 0 & d_{23} & d_{24} & \cdots & d_{2K} \\ 0 & 0 & d_{34} & \cdots & d_{3K} \\ 0 & 0 & 0 & & d_{4K} \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \frac{1}{c_{11}} \begin{bmatrix} c_{21}d_{12} & c_{21}d_{13} & c_{21}d_{14} & & c_{21}d_{1K} \\ c_{31}d_{12} & c_{31}d_{13} & c_{31}d_{14} & \cdots & c_{31}d_{1K} \\ c_{41}d_{12} & c_{41}d_{13} & c_{41}d_{14} & & c_{41}d_{1K} \\ & & \vdots & \ddots & \\ c_{K1}d_{12} & c_{K1}d_{13} & c_{K1}d_{14} & \cdots & c_{K1}d_{1K} \end{bmatrix}.$$

By repeated use of the rule that if a column of a matrix is the sum of two column vectors then the determinant is the sum of two determinants, we obtain:

$$\det\left(\bar{D} - \frac{1}{c_{11}} cd\right) = \frac{1}{c_{11}} \det \begin{bmatrix} -c_{21}d_{12} & d_{23} & d_{24} & & d_{2K} \\ -c_{31}d_{12} & 0 & d_{34} & \cdots & d_{3K} \\ -c_{41}d_{12} & 0 & 0 & & d_{4K} \\ & \vdots & & \ddots & \\ -c_{K1}d_{12} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Evaluating the determinant by expanding along the last row yields:

$$\det\left(\bar{D} - \frac{1}{c_{11}} cd\right) = (-1)^{K+1} \frac{1}{c_{11}} c_{K1} d_{12} d_{23} d_{34} \cdots d_{K-1,K}.$$

From Equations (11) to (14), for all k, $d_{k,k+1} = -c_{kk}$ and $c_{K1} = c_{KK}$. (Recall that if $\sigma(k) = 1$ then $j_k = j_{k+1}$ and if $\sigma(k) = 2$ then $i_k = i_{k+1}$.) Thus:

$$\det\left(\bar{D} - \frac{1}{c_{11}} cd\right) = (-1)^{K+1} \frac{1}{c_{11}} c_{K1}(-c_{11})(-c_{22})(-c_{33})\cdots(-c_{K-1,K-1})$$

$$= (-1)^{2K} c_{22} c_{33} \cdots c_{K-1,K-1} c_{KK}$$

$$= \det \bar{C}.$$

This establishes the claim.

By (20), $\lambda \neq 0$ is an eigen root of $C^{-1}D$ if and only if it is an eigen root of $\bar{C}^{-1}(\bar{D} - cd/c_{11})$. Thus, the claim implies that the product of the nonzero roots of $C^{-1}D$ is one. □

7. Unit roots of $C^{-1}D$.

Reversions. Suppose player 1 plays strategy i_{k_0} in run k_0, then plays some other strategies (but not i_{k_0}) for a while, and then returns to playing $i_{k_1} = i_{k_0}$ in run k_1 in the same round. That is, $k_0 < k_1 - 1$, $i_{k_0} = i_{k_1}$, and $i_k \neq i_{k_0}$ for all k such that $k_0 < k < k_1$. If this happens, we say that a *reversion* to strategy i_{k_0} occurs in run k_1. Suppose that there are ρ_i reversions for player i in each round. Let $\rho = \rho_1 + \rho_2$. Notice that there is always at least one reversion for each player, since $K + 1 \equiv 1$ by definition and hence $\rho \geq 2$.

7.1. Algebraic multiplicity of the unit root of $C^{-1}D$.

We now show that for almost all games, if each players uses at least three different pure strategies, the algebraic multiplicity of the unit root is equal to the number of reversions in the cycle.

LEMMA 3. *There exists $\Gamma_4 \subset \Gamma$ such that $\Gamma \backslash \Gamma_4$ is null, and for all $G \in \Gamma_4$, if c is a cycle in which each player uses at least three pure strategies and c has ρ reversions, then the algebraic multiplicity of the unit root of $C^{-1}D$ is ρ.*

The proof of Lemma 3 is by induction on ρ, the number of reversions in the cycle. We have broken the proof into two steps: the initial step for $\rho = 2$ is in Lemma 5 and the induction step is in Lemma 6.

DEFINITION OF Γ_2 AND Γ_3. Let $\Gamma_2 \subset \Gamma_1$ be the set of games in Γ_1 such that there do not exist l numbers $\eta_1, \eta_2, \ldots, \eta_l$, $2 < l \leq |I| + |J| + 1$, where for each $k = 1, 2, \ldots, l - 1$, $\eta_k \in \{-1, 1\}$, and $i_k j_k$ is some assignment and $i^* \neq i_{l-1}$, satisfying:

$$\sum_{k=1}^{l-1} \eta_k (a_{i_k j_k} - a_{i^* j_k}) = 0. \tag{21}$$

Let $\Gamma_3 \subset \Gamma_2$ be the set of games in Γ_2 such that there do not exist l numbers $\eta_1, \eta_2, \ldots, \eta_l$, $2 < l \leq |I| + |J| + 1$, where for each $k = 1, 2, \ldots, l - 1$, $\eta_k \in \{-1, 1\}$, and $i_k j_k$ is some assignment and $j^* \neq j_{l-1}$, satisfying:

$$\sum_{k=1}^{l-1} \eta_k (b_{i_k j_k} - b_{i_k j^*}) = 0. \tag{22}$$

Since (21) and (22) involve only a *finite* number of linear relationships among the payoffs, $\Gamma \backslash \Gamma_3$ is null.

7.1.1. Initial step ($\rho = 2$). In this step we show that Lemma 3 is true for $\rho = 2$. It is useful to define a *simple cycle* to be a cycle where the players alternate in switching strategies: for all k, $\sigma(k) \neq \sigma(k+1)$. After a relabeling of players and strategies, we can write the simple cycle as

$$c^* = \langle (1, 1), (2, 1), (2, 2), (3, 2), \ldots, (\kappa, \kappa), (1, \kappa) \rangle. \tag{23}$$

The number of pure strategies used by each player is $\kappa = K/2 > 2$, and $(i_k, j_k) = ((k+1)/2, (k+1)/2)$ if k is odd, $(i_k, j_k) = (k/2 + 1, k/2)$ if k is even. For a simple cycle, we have

(24)

$$C^* - D^* = \begin{bmatrix} (1,1) & (2,1) & (2,2) & & (i_k, j_k) & & (1,\kappa) \\ a_{21} - a_{11} & a_{21} - a_{11} & a_{22} - a_{12} & \cdots & a_{2j_k} - a_{1j_k} & \cdots & a_{2\kappa} - a_{1\kappa} \\ b_{12} - b_{11} & b_{22} - b_{21} & b_{22} - b_{21} & & b_{i_k 2} - b_{i_k 1} & & b_{12} - b_{11} \\ a_{31} - a_{21} & a_{31} - a_{21} & a_{32} - a_{22} & & a_{3j_k} - a_{2j_k} & & a_{3\kappa} - a_{2\kappa} \\ b_{13} - b_{12} & b_{23} - b_{22} & b_{23} - b_{22} & & b_{i_k 3} - b_{i_k 2} & & b_{13} - b_{12} \\ \vdots & & & \ddots & \vdots & \ddots & \vdots \\ b_{1\kappa} - b_{1\kappa-1} & b_{2\kappa} - b_{2\kappa-1} & b_{2\kappa} - b_{2\kappa-1} & & b_{i_k\kappa} - b_{i_k\kappa-1} & & b_{1\kappa} - b_{1\kappa-1} \\ a_{11} - a_{\kappa 1} & a_{11} - a_{\kappa 1} & a_{12} - a_{\kappa 2} & & a_{1j_k} - a_{\kappa j_k} & & a_{1\kappa} - a_{\kappa\kappa} \\ b_{11} - b_{1\kappa} & b_{21} - b_{2\kappa} & b_{21} - b_{2\kappa} & \cdots & b_{i_k 1} - b_{i_k\kappa} & \cdots & b_{11} - b_{1\kappa} \end{bmatrix}$$

where the first row of the matrix contains labels that indicate the strategy combination relevant for the column.

We now show how to "simplify" an arbitrary cycle with $\rho = 2$. Recall that if $G \in \Gamma_1$ then in a convergent cycle each player uses the same number of pure strategies, say κ, along the cycle.

LEMMA 4. *If c is any cycle such that $\rho = 2$ and each player uses κ pure strategies, then there exists a matrix Π^* such that $(C - D)\Pi^* = C^* - D^*$, where C^* and D^* correspond to the simple cycle c^*.*

PROOF. Consider a cycle

$$c = \langle (i_1, j_1), (i_2, j_2), (i_3, j_3), (i_4, j_4), \ldots, (i_K, j_K) \rangle$$

which is not simple, with $\rho = 2$. Choose any three successive plays in the cycle, (i_k, j_k), (i_{k+1}, j_{k+1}), (i_{k+2}, j_{k+2}) such that $\sigma(k) = 1$ and $\sigma(k+1) = 2$. It is convenient to rename these as $(1, 1)$, $(2, 1)$, $(2, 2)$. All this is without loss of generality, and thus c can be relabeled so that it is of the form:

$$c = \langle (1, 1), (2, 1), (2, 2), (i_4, j_4), \ldots, (i_{K-1}, j_{K-1}), (i_K, j_K) \rangle.$$

Notice that the cycle c is simple at least up to run 3, corresponding to the choices $(2, 2)$.

Suppose that the cycle c is simple up to run r, $r \geq 3$. In other words, the players alternate in switching until run r. Suppose, however, that $\sigma(r-1) = \sigma(r)$. This means that either c is of the form:

$$\langle(1,1),(2,1),\ldots,(k,k-1),\ (k,k)_{\text{Run } r},(k,k+1),\ldots,(k,l)_{\text{Run } r+l-k},(k+1,l),\ldots,(i_K,j_K)\rangle \tag{25}$$

or c is of the form:

$$\langle(1,1),(2,1),\ldots,(k,k),\ (k+1,k)_{\text{Run } r},(k+2,k),\ldots,(l,k)_{\text{Run } r+l-k},(l,k+1),\ldots,(i_K,j_K)\rangle. \tag{26}$$

Case I. Suppose c is of the form (25). Consider the $C-D$ matrix corresponding to the cycle c. The submatrix of the columns corresponding to the sequence of runs (k,k), $(k,k+1),\ldots,(k,l-1),(k,l),(k+1,l)$ is:

$$\begin{bmatrix} (k,k) & (k,k+1) & & (k,l-1) & (k,l) & (k+1,l) \\ a_{2k}-a_{1k} & a_{2,k+1}-a_{1,k+1} & & a_{2,l-1}-a_{1,l-1} & a_{2l}-a_{1l} & a_{2l}-a_{1l} \\ b_{k2}-b_{k1} & b_{k2}-b_{k1} & \cdots & b_{k2}-b_{k1} & b_{k2}-b_{k1} & b_{k+1,2}-b_{k+1,1} \\ a_{3k}-a_{2k} & a_{3,k+1}-a_{2,k+1} & & a_{3,l-1}-a_{2,l-1} & a_{3l}-a_{2l} & a_{3l}-a_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{kk}-a_{k-1,k} & a_{k,k+1}-a_{k-1,k+1} & & a_{k,l-1}-a_{k-1,l-1} & a_{kl}-a_{k-1,l} & a_{kl}-a_{k-1,l} \\ b_{k,k+1}-b_{k,k} & b_{k,k+1}-b_{k,k} & \cdots & b_{k,k+1}-b_{k,k} & b_{k,k+1}-b_{k,k} & b_{k+1,k+1}-b_{k+1,k} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_{k,l}-b_{k,l-1} & b_{k,l}-b_{k,l-1} & & b_{k,l}-b_{k,l-1} & b_{k,l}-b_{k,l-1} & b_{k+1,l}-b_{k+1,l-1} \\ a_{k+1,k}-a_{k,k} & a_{k+1,k+1}-a_{k,k+1} & \cdots & a_{k+1,l-1}-a_{k,l-1} & a_{k+1,l}-a_{k,l} & a_{k+1,l}-a_{k,l} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \end{bmatrix}. \tag{27}$$

In the matrix $C-D$ perform the following sequence of column operations. First replace column $(r+l-k)$ by (column $(r+l-k-1)$ − column $(r+l-k)$ + column $(r+l-k+1)$). Next replace column $(r+l-k-1)$ by (column $(r+l-k-2)$ − column $(r+l-k-1)$ + column $(r+l-k)$), and perform a similar operation sequentially on all columns $(r+l-k-2),(r+l-k-3),\ldots,r$. These operations are equivalent to post-multiplying $C-D$ by some matrix Π. It is easy to check that for all $j=k+1,\ldots,l-1$, in the column originally corresponding to the (k,j)-run, post-multiplication by Π replaces any entry pertaining to a switch by player 1 by the entry corresponding to a $(k,j-1)$-run, and replaces any entry pertaining to a switch by player 2 by the entry corresponding to a $(k+1,l)$-run.

As a result of post-multiplying $C-D$ by Π, the submatrix corresponding to (27) becomes:

$$\begin{bmatrix} (k,k) & (k+1,k) & & (k+1,l-2) & (k+1,l-1) & (k+1,l) \\ a_{2k}-a_{1k} & a_{2,k}-a_{1,k} & & a_{2,l-2}-a_{1,l-2} & a_{2,l-1}-a_{1,l-1} & a_{2l}-a_{1l} \\ b_{k2}-b_{k1} & b_{k+1,2}-b_{k+1,1} & \cdots & b_{k+1,2}-b_{k+1,1} & b_{k+1,2}-b_{k+1,1} & b_{k+1,2}-b_{k+1,1} \\ a_{3k}-a_{2k} & a_{3,k}-a_{2,k} & & a_{3,l-2}-a_{2,l-2} & a_{3,l-1}-a_{2,l-1} & a_{3l}-a_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{kk}-a_{k-1,k} & a_{k,k}-a_{k-1,k} & & a_{k,l-2}-a_{k-1,l-2} & a_{kl}-a_{k-1,l} & a_{kl}-a_{k-1,l} \\ b_{k,k+1}-b_{k,k} & b_{k+1,k+1}-b_{k+1,k} & \cdots & b_{k+1,k+1}-b_{k+1,k} & b_{k+1,k+1}-b_{k+1,k} & b_{k+1,k+1}-b_{k+1,k} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_{k,l}-b_{k,l-1} & b_{k+1,l}-b_{k+1,l-1} & & b_{k+1,l}-b_{k+1,l-1} & b_{k+1,l}-b_{k+1,l-1} & b_{k+1,l}-b_{k+1,l-1} \\ a_{k+1,k}-a_{k,k} & a_{k+1,k}-a_{k,k} & \cdots & a_{k+1,l-2}-a_{k,l-2} & a_{k+1,l-1}-a_{k,l-1} & a_{k+1,l}-a_{k,l} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \end{bmatrix}.$$

Up to a rearrangement of the rows, these columns correspond to the sequence of runs: $(k, k), (k+1, k), (k+1, k+1), (k+1, k+2), \ldots, (k+1, l-1), (k+1, l)$. If we let $\bar{\Pi}$ be the matrix which performs the required rearrangement of rows, and define $\hat{\Pi} \equiv \Pi\bar{\Pi}$, then the matrix $(C-D)\hat{\Pi}$ corresponds to the cycle:

$$\langle (1,1), \ldots, \underset{\text{Run } r}{(k,k)}, (k+1,k), \underset{\text{Run } r+2}{(k+1,k+1)}, \ldots,$$

$$(k+1, l-1), (k+1, l), \ldots, (i_K, j_K)\rangle$$

which is simple until run $r+2$.

Case II. This is treated in the same way as Case I. After post-multiplying $C-D$ by a suitable matrix $\hat{\Pi}$, we obtain a matrix $(C-D)\hat{\Pi}$ which corresponds to the cycle:

$$\langle (1,1), \ldots, \underset{\text{Run } r}{(k+1,k)}, (k+1,k+1), \underset{\text{Run } r+2}{(k+2,k+1)}, \ldots,$$

$$(l-1, k+1), (l, k+1), \ldots, (i_K, j_K)\rangle$$

which is simple until run $r+2$.

Thus, in either case we can, by row and column operations, extend the number of runs for which the relevant columns originate from a simple cycle. Because these operations can be applied repeatedly, there exists some matrix Π^* so that $(C-D)\Pi^* = C^* - D^*$, where $C^* - D^*$ corresponds to a simple cycle. □

LEMMA 5. *There exists $\Gamma_4 \subset \Gamma_3$ such that $\Gamma \backslash \Gamma_4$ is null, and for all $G \in \Gamma_4$, if c is a cycle with $\rho = 2$ in which each player uses $\kappa \geq 3$ pure strategies, the algebraic multiplicity of the unit root is* 2.

PROOF. Fix a cycle

$$c = \langle (1,1), (2,1), (2,2), (i_4, j_4), \ldots, (i_{K-1}, j_{K-1}), (i_K, j_K)\rangle$$

with $\rho = 2$. There are two cases. If $\sigma(K) = 1$ then $(i_K, j_K) = (\kappa, 1)$, and if $\sigma(K) = 2$ then $(i_K, j_K) = (1, \kappa)$.

Case 1. Suppose $(i_K, j_K) = (1, \kappa)$. Then we obtain

$$[\lambda C - D]$$

$$= \begin{bmatrix} \lambda(a_{21}-a_{11}) & a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2j_k}-a_{1j_k} & \cdots & a_{2\kappa}-a_{1\kappa} \\ \lambda(b_{12}-b_{11}) & \lambda(b_{22}-b_{21}) & b_{22}-b_{21} & & b_{i_k2}-b_{i_k1} & & b_{12}-b_{11} \\ \vdots & & & \ddots & \vdots & \ddots & \vdots \\ \lambda(b_{11}-b_{1\kappa}) & \lambda(b_{21}-b_{2\kappa}) & \lambda(b_{21}-b_{2\kappa}) & & \lambda(b_{i_k1}-b_{i_k\kappa}) & & \lambda(b_{11}-b_{1\kappa}) \end{bmatrix},$$

where the kth row of $\lambda C - D$ is

$$[\lambda(a_{i_{k+1}1}-a_{i_k1}), \ldots, \lambda(a_{i_{k+1}j_k}-a_{i_kj_k}), (a_{i_{k+1}j_{k+1}}-a_{i_kj_{k+1}}), \ldots, (a_{i_{k+1}\kappa}-a_{i_k\kappa})],$$

if $\sigma(k) = 1$, and

$$[\lambda(b_{1j_{k+1}} - b_{1j_k}), \ldots, \lambda(b_{i_k j_{k+1}} - b_{i_k j_k}), (b_{i_{k+2} j_{k+1}} - b_{i_{k+2} j_k}), \ldots, (b_{1j_{k+1}} - b_{1j_k})],$$

if $\sigma(k) = 2$.

Observe that for this matrix:

$$\sum_{\{k:\sigma(k)=1\}} \text{row } k$$

$$= (0, (1-\lambda)(a_{21} - a_{11}), \ldots, (1-\lambda)(a_{i_k j_k} - a_{1j_k}), \ldots, (1-\lambda)(a_{\kappa\kappa} - a_{1\kappa}), 0),$$

and

$$\sum_{\{k:\sigma(k)=2\}} \text{row } k$$

$$= (0, 0, (1-\lambda)(b_{22} - b_{21}), \ldots, (1-\lambda)(b_{i_k j_k} - b_{i_k 1}), \ldots, (1-\lambda)(b_{1\kappa} - b_{11})).$$

Therefore, replacing row 1 by ($\Sigma_{\{k:\sigma(k)=1\}}$ row k) and replacing row 2 by ($\Sigma_{\{k:\sigma(k)=2\}}$ row k), we obtain:

$$|\lambda C - D|$$

$$= \begin{vmatrix} 0 & (1-\lambda) \times (a_{21} - a_{11}) & \cdots & (1-\lambda) \times (a_{i_k j_k} - a_{1j_k}) & \cdots & 0 \\ 0 & 0 & & (1-\lambda) \times (b_{i_k j_k} - b_{i_k 1}) & & (1-\lambda) \times (b_{1\kappa} - b_{11}) \\ \vdots & & \ddots & & \ddots & \vdots \\ \lambda(b_{11} - b_{1\kappa}) & \lambda(b_{21} - b_{2\kappa}) & \cdots & \lambda(b_{i_k 1} - b_{i_k \kappa}) & \cdots & \lambda(b_{11} - b_{1\kappa}) \end{vmatrix}$$

$$= \lambda(1-\lambda)^2 \times |L_1(\lambda)|,$$

where

$$|L_1(\lambda)| \equiv \begin{vmatrix} 0 & (a_{21} - a_{11}) & (a_{22} - a_{12}) & \cdots & (a_{i_k j_k} - a_{1j_k}) & \cdots & 0 \\ 0 & 0 & (b_{22} - b_{12}) & & (b_{i_k j_k} - b_{i_k 1}) & & (b_{1\kappa} - b_{11}) \\ \vdots & & & \ddots & & \ddots & \vdots \\ (b_{11} - b_{1\kappa}) & \lambda(b_{21} - b_{2\kappa}) & \lambda(b_{21} - b_{2\kappa}) & \cdots & \lambda(b_{i_k 1} - b_{i_k \kappa}) & \cdots & \lambda(b_{11} - b_{1\kappa}) \end{vmatrix}.$$

Observe that only the first two rows of $\lambda C - D$ have been changed.

There exists a third unit root only if $|L_1(1)| = 0$. Now $|L_1(1)|$ is a function of the payoffs (a_{ij}, b_{ij}), which are points in $\mathbb{R}^{\kappa^2} \times \mathbb{R}^{\kappa^2}$.

Claim. The set of points in $\mathbb{R}^{\kappa^2} \times \mathbb{R}^{\kappa^2}$ such that $|L_1(1)| = 0$ is null.

PROOF OF CLAIM. Since $|L_1(1)|$ is a polynomial function of the payoffs (a_{ij}, b_{ij}), it suffices to show that it is not true that $|L_1(1)| = 0$ identically on $\mathbb{R}^{\kappa^2} \times \mathbb{R}^{\kappa^2}$. This can be shown by an example.

Consider the game defined by the payoff matrices A and B given by

$$a_{ij} = \begin{cases} 0 & \text{if } i = j = 1, \\ 1 & \text{if } i = j > 1, \\ x & \text{if } i = 2 \text{ and } j = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

$$b_{ij} = \begin{cases} x & \text{if } i = j = 1, \\ -1 & \text{if } i = j \neq 1, \\ 0 & \text{if } i \neq j, \end{cases} \tag{29}$$

where x is an arbitrary parameter which will chosen later. Since the cycle must end with a sequence of runs of the form:

$$\cdots(\kappa, l), (1, l), (1, l+1), \ldots, (1, \kappa)$$

we obtain that for this game

$$|L_1(1)| \equiv \begin{vmatrix} 0 & x & 1 & \cdots & (a_{i_k j_k} - a_{1 j_k}) & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & & (b_{i_k j_k} - b_{i_k 1}) & & 0 & -x & -x & & -x \\ \vdots & & & \ddots & & \ddots & & & & & \vdots \\ x & 0 & 0 & \cdots & (b_{i_k 1} - b_{i_k \kappa}) & \cdots & 0 & x & x & \cdots & x \end{vmatrix}.$$

Now consider the matrix $L_1(1)\Pi^*$ where Π^* is the matrix that simplifies the cycle, as given by Lemma 4. Only the first two rows of $L_1(1)$ differ from the corresponding rows of $C - D$. Thus, the last $K - 2$ rows of $L_1(1)\Pi^*$ are exactly the same as the last $K - 2$ rows of $C^* - D^*$, as defined in (24). For the game at hand:

(30) $C^* - D^*$

$$= \begin{vmatrix} x & x & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -x & -1 & -1 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & -x \\ -x & -x & -1 & -1 & 1 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 & 0 & 0 \\ \cdots & & & & & & & \ddots & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & -1 & -1 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & x \end{vmatrix}$$

In the second row of $L_1(1)$ there are $(\kappa - l - 1)$ x's which occur only in the columns corresponding to the runs $(1, l)$, $(1, l+1)$, $\ldots$, $(1, \kappa)$. Now consider the second row of $L_1(1)\Pi^*$. Recalling the sequence of column operations used to simplify the cycle, the operations that involve the columns corresponding to the runs $(1, l), (1, l+1), \ldots, (1, \kappa)$ occur when the cycle has been "simplified" until run r so that it ends in a sequence of runs of the form:

$$\cdots, \underset{\text{Run } r}{(l+1, l)}, (l+2, l), \ldots, (1, l), (1, l+1), \ldots, (1, \kappa).$$

In the next column operation, the entry in the second row corresponding to the run $(l + 2, l)$, which does not involve x, is replaced by the entry corresponding to the run $(l + 1, l)$, which does not involve x either. This remains true at each iteration since successive columns of the form $(1, k)$ and $(1, k+1)$ are always added with opposite signs, eliminating the x's. Thus, once the matrix is simplified, only the last entry in the second row, corresponding to the run $(1, \kappa)$, involves an x. The first two rows of $L_1(1)\Pi^*$ are therefore of the form:

$$\begin{bmatrix} 0 & x & \delta_1 & \delta_2 & \delta_3 & \cdots & \delta_{2\kappa-4} & \delta_{2\kappa-3} & 0 \\ 0 & 0 & -\delta_1 & -\delta_2 & -\delta_3 & & -\delta_{2\kappa-4} & -\delta_{2\kappa-3} & -x \end{bmatrix}$$

where the δ_k's are numbers which are independent of x.

The last $K - 2$ rows of $L_1(1)\Pi^*$ are exactly the same as those of $C^* - D^*$ (as given in (30)) and thus we obtain that:

$$|L_1(1)\Pi^*| = \Lambda_1(x)$$

where:

(31) $\Lambda_1(x)$

$$= \begin{vmatrix} 0 & x & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots & \delta_{2\kappa-6} & \delta_{2\kappa-5} & \delta_{2\kappa-4} & \delta_{2\kappa-3} & 0 \\ 0 & 0 & -\delta_1 & -\delta_2 & -\delta_3 & -\delta_4 & -\delta_5 & & -\delta_{2\kappa-6} & -\delta_{2\kappa-5} & -\delta_{2\kappa-4} & -\delta_{2\kappa-3} & -x \\ -x & -x & -1 & -1 & 1 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & \ddots & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & -1 & -1 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & x \end{vmatrix}$$

where, as above, the δ_k's are independent of x.

Now routine determinantal calculations show that

$$\Lambda_1(x) = \left[x + \sum_{j=1}^{2\kappa-3} (-1)^j \delta_j \right] \times (\kappa - 2)x^2.$$

Since the δ_j's are independent of x we can choose an $x \neq 0$ so that the term in the square brackets is not zero. Since $\kappa > 2$ this guarantees that $\Lambda_1(x) \neq 0$.

Thus, in case 1 $|L_1(1)|$ is not identically zero. Exactly the same argument works for case 2, but is omitted for brevity. Thus, for the given cycle c, the set of games for which $|L_1(1)| = 0$ is null. Denote this set by $\Gamma_c \subset \Gamma$. There are only a finite number of possible cycles with $\rho = 2$, and thus the set $\Gamma^* \equiv \bigcup \Gamma_c$ is also null, where the union is taken of all cycles c such that each player is using at least three strategies and $\rho = 2$. For any such cycle and any game not in Γ^*, the number of unit roots of $C^{-1}D$ is 2. Now let $\Gamma_4 = \Gamma_3 \backslash \Gamma^*$. □

7.1.2. The induction step. By construction, every $G \in \Gamma_4$ has the property that if c is any cycle with $\rho = 2$ such that each player is using at least three pure strategies, then the number of unit roots of $C^{-1}D$ is exactly 2. We now extend the result to all cycles.

LEMMA 6. *Fix a game $G \in \Gamma_4$. Suppose for all cycles $\bar{c}$ without simultaneous switches which have $\bar{\rho}$ reversions and where each player is using $\kappa \geq 3$ pure strategies, the corresponding matrix $\bar{C}^{-1}\bar{D}$ has $\bar{\rho}$ unit roots. Then, for any cycle c without simultaneous switches which has $\bar{\rho} + 1$ reversions and where each player is using $\kappa \geq 3$ pure strategies, the corresponding matrix $C^{-1}D$ has $\bar{\rho} + 1$ unit roots.*

PROOF. Suppose that for all cycles without simultaneous switches in which the number of reversions is $\bar{\rho} \geq 2$, the number of unit roots of $\bar{C}^{-1}\bar{D}$ is $\bar{\rho}$. Now consider an arbitrary cycle without simultaneous switches c of length K in which the number of reversions is $\bar{\rho} + 1$. Since $\bar{\rho} + 1 \geq 3$ and $\kappa \geq 3$, there is a player, say player 1, who plays i^* in (say) run 1, switches to i_2 in run 2, possibly plays other strategies in runs 2 to $l - 1$, then after run $l - 1$ switches from some strategy i_{l-1} "back" to i^*, then plays i^* for a number of runs, say l through m, and then switches to $i_{m+1} \neq i_{l-1}$. (It is possible that $l = m$ so that 1 switches away from i^* after run l.) The cycle c is then of the form:

$$c = \langle \underset{1}{i^* j_1}, \ldots, \underset{l-1}{i_{l-1} j_{l-1}}, \underset{l}{i^* j_l}, \ldots, \underset{m}{i^* j_m}, \underset{m+1}{i_{m+1} j_{m+1}}, \ldots, \underset{K}{i_K j_K} \rangle$$

where the numbers under the strategy labels are the runs. Notice that without loss of generality we can take $l \leq |I| + |J| + 1$, because a reversion has to occur as soon as each player has used all his pure strategies.

Consider the matrix $C - D$ that corresponds to the cycle c. We will construct a cycle $\bar{c}$ with one fewer reversion and study the corresponding matrix $\bar{C} - \bar{D}$.

For all k such that $l < k \leq m$, add column $l - 1$ of $C - D$ to column k and subtract column l from column k. Add row $l - 1$ to row m. Call the resulting matrix $\hat{X}$.

For all k such that $l < k \leq m$, and all $h \neq k$, the hth element in the kth column of $\hat{X}$ is (using the fact that $j_l = j_{l-1}$):

$$(32) \qquad (a_{i_{h+1} j_{l-1}} - a_{i_h j_{l-1}}) - (a_{i_{h+1} j_l} - a_{i_h j_l}) + (a_{i_{h+1} j_k} - a_{i_h j_k}) = (a_{i_{h+1} j_k} - a_{i_h j_k}),$$

if $\sigma(h) = 1$, and (using the fact that $i_k = i_l = i^*$):

$$(33) \qquad (b_{i_{l-1} j_{h+1}} - b_{i_{l-1} j_h}) - (b_{i_l j_{h+1}} - b_{i_l j_h}) + (b_{i_k j_{h+1}} - b_{i_k j_h}) = (b_{i_{l-1} j_{h+1}} - b_{i_{l-1} j_h}),$$

if $\sigma(h) = 2$.

Similarly, for $h \neq m$, the hth element in the mth row of $\hat{X}$ is

$$(a_{i_l j_h} - a_{i_{l-1} j_h}) + (a_{i_{m+1} j_h} - a_{i_m j_h}) = a_{i_{m+1} j_h} - a_{i_{l-1} j_h}.$$

Delete row $l - 1$ and column l from $\hat{X}$ and call the new matrix $\bar{X}$.

Now, consider the $K - 1$ cycle

$$\bar{c} = \langle \underset{1}{i^* j_1}, \ldots, \underset{l-1}{i_{l-1} j_{l-1}}, \underset{l}{i_{l-1} j_{l+1}}, \ldots, \underset{m-1}{i_{l-1} j_m}, \underset{m}{i_{m+1} j_{m+1}}, \ldots, \underset{K-1}{i_K j_K} \rangle$$

which is the same as c except that the sequence

$$(i_{l-1} j_{l-1}, i^* j_l, \ldots, i^* j_m, i_{m+1} j_{m+1})$$

in runs $l - 1$ to m has been replaced by the shorter sequence

$$(i_{l-1} j_{l-1}, i_{l-1} j_{l+1}, \ldots, i_{l-1} j_m, i_{m+1} j_{m+1}).$$

Note that the number of reversions in $\bar{c}$ is $\bar{p}$. Let $\bar{C} - \bar{D}$ denote the matrix corresponding to the cycle $\bar{c}$.

We now claim that $\bar{X} = \bar{C} - \bar{D}$. Indeed, the elements in (32) and (33) are the entries in the $(k-1)$th column of $\bar{X}$, and they correspond to a run where player 1 uses i_{l-1} and player 2 uses j_k. This is precisely the $(k-1)$th run in the cycle $\bar{c}$. Similarly, the lth row of the matrix $\bar{X}$ corresponds to a switch from strategy i_{l-1} to strategy i_{m+1} by player 1. The other rows and columns have not been disturbed, and hence $\bar{X} = \bar{C} - \bar{D}$.

We have thus shown that we can write

$$|C - D| = \pm \begin{vmatrix} \bar{C} - \bar{D} & \gamma \\ \alpha & \beta \end{vmatrix} \tag{34}$$

since the operations we have performed on $C - D$ do not affect the determinant, but interchanging rows and columns may affect the sign. In (34), $\bar{C} - \bar{D}$ is the matrix resulting from the smaller cycle $\bar{c}$, $[\begin{smallmatrix}\gamma \\ \beta\end{smallmatrix}]$ is the lth column of $\hat{X}$, and $[\alpha\ \beta]$ is the $(l-1)$th row of $\hat{X}$. Thus in particular,

$$\beta = a_{i_l j_{l-1}} - a_{i_{l-1} j_{l-1}}.$$

In the form (34), γ is a linear combination of the columns of $\bar{C} - \bar{D}$, and α is a linear combination of the rows. That is, there exist δ and η such that:

$$\begin{aligned} &\alpha + \delta(\bar{C} - \bar{D}) = 0 \\ &\gamma + (\bar{C} - \bar{D})\eta = 0. \end{aligned} \tag{35}$$

More precisely, recalling that $\sigma(1) = 1$, η is defined by:

$$\eta_k = \begin{cases} -1 & k = 1, \\ -1 & \text{if } 1 < k \le l-1,\ \sigma(k-1) = 2 \text{ and } \sigma(k) = 1, \\ 1 & \text{if } 1 < k \le l-1,\ \sigma(k-1) = 1 \text{ and } \sigma(k) = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and δ is defined by:

$$\delta_k = \begin{cases} 1 & \text{if } 1 \leq k < l - 1 \text{ and } \sigma(k) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It can then be checked that (35) holds.

Now consider the matrix $T(\lambda)$ that results when the row and column operations described above are performed on $\lambda C - D$.

The operations are: For all k such that $l < k \leq m$, add column $l - 1$ of $C - D$ to column k and subtract column l from column k. Add row $l - 1$ to row m. Finally, place row $l - 1$ as the last row and column l as the last column.

These operations preserve the determinant (up to a sign change) and thus $|T(\lambda)| = \pm|\lambda C - D|$. From (34) it follows that:

$$T(\lambda) = \begin{bmatrix} \bar{L}_1(\lambda) & \gamma(\lambda) \\ \alpha(\lambda) & \beta(\lambda) \end{bmatrix}$$

where $\bar{L}_1(\lambda)$ is a $(K - 1) \times (K - 1)$ matrix with the property that $\bar{L}_1(1) = \bar{C} - \bar{D}$.

Now add $\delta[\bar{L}_1(\lambda)\ \gamma(\lambda)]$ to the last row of $T(\lambda)$, resulting in the matrix

$$X = \begin{bmatrix} \bar{L}_1(\lambda) & \gamma(\lambda) \\ \alpha(\lambda) + \delta\bar{L}_1(\lambda) & \beta(\lambda) + \delta\gamma(\lambda) \end{bmatrix}.$$

It can be checked that

$$\begin{aligned} \alpha(\lambda) + \delta\bar{L}_1(\lambda) = (0, (1 - \lambda)(a_{i_2 j_2} - a_{i^* j_2}), \ldots, \\ (1 - \lambda)(a_{i_{k-1} j_{k-1}} - a_{i^* j_{k-1}}), (1 - \lambda)(a_{i_k j_k} - a_{i^* j_k}), \ldots, \\ (1 - \lambda)(a_{i_{l-2} j_{l-2}} - a_{i^* j_{l-2}}), 0, \ldots, 0, 0) \end{aligned} \tag{36}$$

where the jth entry is zero if $j \geq l - 1$, and

$$\beta(\lambda) + \delta\gamma(\lambda) = 0.$$

Add

$$\begin{bmatrix} \bar{L}_1(\lambda) \\ \alpha(\lambda) + \delta\bar{L}_1(\lambda) \end{bmatrix} \eta$$

to the last column of X. By (35), this results in a matrix:

$$T_1(\lambda) = \begin{bmatrix} \bar{L}_1(\lambda) & (1 - \lambda)\gamma_0 \\ (1 - \lambda)\alpha_0 & (1 - \lambda)\beta_0 \end{bmatrix}$$

where again we have that $|T_1(\lambda)| = |\lambda C - D|$.

Consider the form of β_0. Multiplying the expression in (36) by η, the result is

$$(1 - \lambda)\beta_0 = (1 - \lambda) \sum_{k=2}^{l-2} \eta_k(a_{i_k j_k} - a_{i^* j_k}).$$

Since we are assuming the game $G \in \Gamma_4 \subset \Gamma_3$, $\beta_0 \neq 0$.

We now investigate the unit roots of the matrix $C^{-1}D$. Since

$$|\lambda I - C^{-1}D| = |C^{-1}|\,|\lambda C - D|$$

we can investigate equally well the roots of the polynomial $|\lambda C - D| = 0$. We can write:

$$(37) \qquad \begin{aligned} |\lambda C - D| &= |T_1(\lambda)| \\ &= (1 - \lambda)|L_1(\lambda)| \end{aligned}$$

where

$$L_1(\lambda) = \begin{bmatrix} \bar{L}_1(\lambda) & (1 - \lambda)\gamma_0 \\ \alpha_0 & \beta_0 \end{bmatrix}.$$

Hence there is at least one unit root. If there is another one, then $|L_1(1)| = 0$ and there exists a vector $y = (\bar{y}, y_K) \neq 0$ such that

$$L_1(1)y = \begin{bmatrix} \bar{L}_1(1) & 0 \\ \alpha_0 & \beta_0 \end{bmatrix} y = 0.$$

Since $\bar{L}_1(1) = \bar{C} - \bar{D}$, this implies that

$$(38) \qquad \begin{aligned} &(\bar{C} - \bar{D})\bar{y} = 0 \\ &\alpha_0\bar{y} + y_K\beta_0 = 0. \end{aligned}$$

If $\bar{y} = 0$, then $y_K \neq 0$. Since $\beta_0 \neq 0$, this contradicts (38). Thus, $\bar{y} \neq 0$. Suppose without loss of generality that $\bar{y}_j = 1$ for some $j < K$.

Since $L_1(1)y = 0$, we have

$$L_1(\lambda)y = (1 - \lambda)v$$

for some vector v. Replace the jth column of $L_1(\lambda)$ by $(1 - \lambda)v$ and call the resulting matrix $T_2(\lambda)$. Let

$$L_2(\lambda) = \begin{bmatrix} \bar{L}_2(\lambda) & (1 - \lambda)\gamma_0 \\ \alpha(\lambda) & \beta_0 \end{bmatrix}$$

be the matrix that obtains when the jth column of $L_1(\lambda)$ is replaced by v. Then

$$(39) \qquad |\lambda C - D| = (1 - \lambda)|L_1(\lambda)| = (1 - \lambda)|T_2(\lambda)| = (1 - \lambda)^2|L_2(\lambda)|.$$

Suppose that there is a third unit root. Then $|L_2(1)| = 0$ and there is $z = (\bar{z}, z_K) \neq 0$ such that

$$L_2(1)z = \begin{bmatrix} \bar{L}_2(1) & 0 \\ \alpha(1) & \beta_0 \end{bmatrix} z = 0.$$

Again $\beta_0 \neq 0$ implies that $\bar{z} \neq 0$. Therefore, as in (39), we can use z to show that

$$|\lambda C - D| = (1 - \lambda)^3 |L_3(\lambda)|$$

for some matrix $L_3(\lambda)$. This procedure can be repeated until $|L_k(1)| \neq 0$ for some k.

Now we note that one unit root resulted directly from (37). After that each step of the procedure corresponds not only to a unit root of $|\lambda C - D| = 0$, but clearly also to a unit root of $|\lambda \bar{C} - \bar{D}| = 0$, where the matrix $\lambda \bar{C} - \bar{D}$ comes from the smaller cycle $\bar{c}$. By the induction hypothesis, $\bar{C}^{-1}\bar{D}$ has exactly $\bar{p}$ unit roots. Therefore, we can repeat the argument precisely $\bar{p}$ times. Therefore,

$$|\lambda C - D| = (1 - \lambda)^{\bar{p}+1} |L_{\bar{p}}(\lambda)|$$

and $|L_{\bar{p}}(1)| \neq 0$. This completes the induction step. □

Lemma 5 and Lemma 6 together complete the proof of Lemma 3.

7.2. Geometric multiplicity of the unit root of $C^{-1}D$. From (11) to (14), the kth row of $(C - D)$ is

$$(40) \qquad [a_{i_{k+1}j_1} - a_{i_k j_1}, a_{i_{k+1}j_2} - a_{i_k j_2}, \ldots, a_{i_{k+1}j_K} - a_{i_k j_K}] = (\alpha_{i_{k+1}} - \alpha_{i_k})Q,$$

if $\sigma(k) = 1$ and

$$(41) \qquad [b_{i_1 j_{k+1}} - b_{i_1 j_k}, b_{i_2 j_{k+1}} - b_{i_2 j_k}, \ldots, b_{i_K j_{k+1}} - b_{i_K j_k}] = (\beta_{j_{k+1}} - \beta_{j_k})P,$$

if $\sigma(k) = 2$.

If $\sigma(k) = 1$, then the kth element in the lth column of $(C - D)$ is

$$(C - D)_{kl} = (a_{i_{k+1}j_l} - a_{i_k j_l}).$$

Suppose $k_0 < k_1 - 1$, and suppose that a reversion to strategy i_{k_0} occurs in run k_1. Then $i_{k_0} = i_{k_1}$, and $i_k \neq i_{k_0}$ for all k such that $k_0 < k < k_1$. Consider the vector $s = (s_1, s_2, \ldots, s_K)$, where

$$s_k = \begin{cases} 1 & \text{if } \sigma(k) = 1 \text{ and } k_0 \leq k < k_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have, for each l,

$$\sum_{k=1}^{K} s_k(C - D)_{kl} = \sum_{\left\{ \substack{k:\sigma(1)=1 \\ \text{and } k_0 \leq k < k_1} \right\}} (a_{i_{k+1}j_l} - a_{i_k j_l}) = \sum_{\{k_0 \leq k < k_1\}} (a_{i_{k+1}j_l} - a_{i_k j_l})$$

in view of the fact that $a_{i_{k+1}j_l} = a_{i_k j_l}$ if $\sigma(k) = 2$. But $i_{k_1} = i_{k_0}$. Therefore,

$$\sum_{k=1}^{K} s_k(C-D)_{kl} = (a_{i_{k_0+1}j_l} - a_{i_{k_0}j_l}) + (a_{i_{k_0+2}j_l} - a_{i_{k_0+1}j_l}) + \cdots + (a_{i_{k_1}j_l} - a_{i_{k_1-1}j_l}) = 0$$

for each l. Hence,

$$s \in \ker(C-D)^T$$

where $(C-D)^T$ is the transpose of $(C-D)$ and $\ker M$ denotes the *kernel* (or null-space) of the matrix M.

By the same method, each reversion generates a vector in $\ker(C-D)^T$. It is clear that this set of vectors is linearly independent. A similar procedure works for player two, and we then use vectors of the form

$$s_k = \begin{cases} 1 & \text{if } \sigma(k) = 2 \text{ and } k_0 \leq k < k_1, \\ 0 & \text{otherwise.} \end{cases}$$

There are ρ_i reversions for player i in each round, say. This implies that there are $\rho_1 + \rho_2$ linearly independent vectors in $\ker(C-D)^T$. But

$$\dim\ker(C-D)^T = \dim\ker(C-D) = \dim\ker(I - C^{-1}D)$$

so that

$$\dim\ker(I - C^{-1}D) \geq \rho.$$

On the other hand, under the hypotheses of Lemma 3, $\dim\ker(I - C^{-1}D)$ is no greater than ρ, the algebraic multiplicity of the unit root, and hence

$$\dim\ker(I - C^{-1}D) = \rho,$$

a fact that we state for later reference as:

LEMMA 7. *For $G \in \Gamma_4$, if c is a cycle in which each player uses at least three pure strategies and c has ρ reversions then the geometric multiplicity of the unit root of $C^{-1}D$ is ρ.*

8. Nonconvergence.

We can now complete the proof of Theorem 1.

PROOF OF THEOREM 1. Suppose $G \in \Gamma_4$. Let the robust cycle be such that there are ρ reversions, and each player uses $\kappa \geq 3$ pure strategies. Let $\lambda_0, \lambda_1, \ldots, \lambda_S$ be the $S + 1$ *distinct* eigen roots of $C^{-1}D$. We know that $\lambda_0 \equiv 0$ is an eigen root of $C^{-1}D$ with algebraic multiplicity of 1 and from Lemma 3 we know that $\lambda_1 \equiv 1$ is an eigen root of $C^{-1}D$ with algebraic multiplicity $\alpha_1 = \rho$. But

$$\sum_{s=0}^{S} \alpha_s = K$$

where α_s is the algebraic multiplicity of λ_s, and since $K = 2\kappa + \rho - 2$, this implies that

$$\sum_{s=2}^{S} \alpha_s = K - \rho - 1 = 2\kappa - 3$$

which is an odd number.

Since the number of complex (nonreal) roots is always even, the multiplicity of the *real* roots is an odd number. It cannot be the case that the only real root is -1 since from Lemma 2

$$\prod_{s=2}^{S} \lambda_s^{\alpha_s} = 1.$$

This implies that there is a nonzero real root, say λ_2, such that $\lambda_2 \neq 1$ and $\lambda_2 \neq -1$ and hence $|\lambda_2| \neq 1$. Since the product of all nonzero roots is one, there exists a root λ_s such that $|\lambda_s| > 1$. Let λ_S be the dominant root, that is, the root with the largest absolute value. If λ_S is either negative or complex, the cycle cannot persist since run-lengths would become negative. Thus λ_S must be real and positive and hence $\lambda_S > 1$. (If there are many roots with the largest absolute value, there must be at least one real $\lambda_S > 1$.)

Finally, consider the vectors $P^0 = p(1)$ and $Q^0 = q(1)$ that describe the initial conditions before the cycle begins and let $n(0)$ be the vector of run lengths in the initial round that the cycle is played. We know from (4) and (5) that $n(0)$ satisfies

$$(\alpha_{i_{k+1}} - \alpha_{i_k})QE_k n(0) = -(\alpha_{i_{k+1}} - \alpha_{i_k})Q^0$$

and

$$(\beta_{j_{k+1}} - \beta_{j_k})PE_k n(0) = -(\beta_{j_{k+1}} - \beta_{j_k})P^0$$

which can be rewritten as:

$$n(0) = (C^{-1}D - I)m(0)$$

where $m(0) \in \mathbb{R}^K$ is a vector satisfying $Q^0 = Qm(0)$ and $P^0 = Pm(0)$. It is in fact convenient to take $m(0)$, normalized so that $\Sigma_{k=1}^{K} m_k(0) = 1$, as the initial condition. (For any such $m(0)$, there are initial beliefs $Q^0 = Qm(0)$ and $P^0 = Pm(0)$ satisfying $\Sigma_{i=1}^{K} P_i^0 = \Sigma_{j=1}^{K} Q_j^0 = 1$.)

Some notation: If λ_s is an eigenvalue x is a vector such that $(C^{-1}D - \lambda_s I)^{\nu_s - 1}x \neq 0$ and $(C^{-1}D - \lambda_s I)^{\nu_s}x = 0$ then x is a *generalized eigenvector* for λ_s with index $\nu_s \geq 1$. Let μ_s be the largest index of any generalized eigenvector for λ_s. The generalized eigenspace of the eigenvalue λ_s is

$$X_s = \{x : (C^{-1}D - \lambda_s I)^{\mu_s}x = 0\}$$

and by the Jordan decomposition theorem (Hirsch and Smale 1974) dim $X_s = \alpha_s$, the algebraic multiplicity of λ_s, and

$$\mathbb{R}^K = X_0 \oplus X_1 \oplus X_2 \oplus \cdots \oplus X_S \tag{42}$$

where $X \oplus Y$ denotes the direct sum of X and Y.

Claim. Range $(C^{-1}D - I) = X_0 \oplus X_2 \oplus \cdots \oplus X_S$.

Proof of Claim. Since $G \in \Gamma_4$, we know that if $\lambda_1 = 1$ then

$$\dim X_1 = \alpha_1 = \rho.$$

Moreover, as $\dim \ker(C^{-1}D - I) = \rho$ (Lemma 7) then $\mu_1 = 1$ and so there are no generalized eigenvectors associated with the unit root with index greater than 1. Thus,

$$X_1 = \ker(C^{-1}D - I).$$

Now $\dim \text{Range}(C^{-1}D - I) + \dim X_1 = K$; and Range $(C^{-1}D - I)$ and X_1 are disjoint subspaces; for if $0 \neq x = (C^{-1}D - I)u$ and $(C^{-1}D - I)x = 0$ then $(C^{-1}D - I)^2 u = 0$ contradicting the fact that $\mu_1 = 1$. Thus, $\mathbb{R}^K = X_1 \oplus$ Range $(C^{-1}D - I)$. But now (42) implies Range $(C^{-1}D - I) = X_0 \oplus X_2 \oplus \cdots \oplus X_S$. This proves the claim.

The claim implies that there cannot be an open set of initial conditions $m(0)$ such that when $n(0) \equiv (C^{-1}D - I)m(0)$ is written as a sum of $K - \rho$ vectors of the form $c_s^l x_s^l$ and $x_s^l \in X_s$ it is the case that for some $l = 0, 1, \ldots, \mu_s$ and $s = 0, 2, 3, \ldots, S$, $c_s^l = 0$. (The index l indicates that there could be more than one linearly independent generalized eigenvector corresponding to the eigenvalue λ_s.)

We have:

$$n(r) = (C^{-1}D)^r n(0).$$

Thus $n(r)$ has a component of the form $c_S r^j \lambda_S^r x_S^r$, where $j \geq 0$ and $\lambda_S > 1$ is the dominant root identified above (again see Hirsch and Smale 1974). Since $\lambda_S > 1$ the run-lengths grow exponentially as in Shapley (1964), and CFP does not converge for any starting position outside the set for which $c_S = 0$. This last set does not contain an open set.

This completes the proof of Theorem 1. □

9. Discussion. CFP converges for every zero-sum game, but the property of zero-sumness is nongeneric, and our main result is a statement about generic games. Furthermore, the class of 2×2 games is not covered by our result, and as we have already pointed out, there is an open set of 2×2 games with a unique equilibrium in mixed strategies for which every CFP is cyclically convergent. However, every 2×2 game with a unique equilibrium in mixed strategies has the same best-response correspondence as a zero-sum game. Since CFP depends only on the best-response correspondence, the 2×2 exception is a consequence of this equivalence.

Shapley's game (discussed in Example 2) is a 3×3 *nonzero sum* game where, for very special initial conditions, CFP converges cyclically to a mixed strategy equilibrium where all three pure strategies are used. Consider a slightly more general version:

EXAMPLE 3. Shapley's Game:

	X	Y	Z
X	$0, 0$	$1, x$	$x, 1$
Y	$x, 1$	$0, 0$	$1, x$
Z	$1, x$	$x, 1$	$0, 0$

where $x > 1$.

There exists a unique equilibrium $p^* = q^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It can be shown that any CFP starting from an initial condition where $p(1) = q(1)$, *converges* in a cyclical manner; both players play the same pure strategy at all times and follow the cycle: $(X, X) \to (Y, Y) \to (Z, Z)$. Of course, this cycle is not robust. For this cycle we have:

$$C^{-1}D = \begin{bmatrix} 0 & x^{-1} & 1 - x^{-1} \\ 0 & x^{-1} - x^{-2} & 1 - x^{-1} + x^{-2} \\ 0 & x^{-1} - x^{-2} + x^{-3} & 1 - x^{-1} + x^{-2} - x^{-3} \end{bmatrix}$$

and the eigen roots of this matrix are: 0, 1 and $-x^{-3}$. For any $x > 1$, the dominant root is 1. The run-lengths are, in the limit, constant and the CFP converges. The important point to notice is that if $x > 1$, the (absolute value of the) product of the nonzero roots is *less* than 1. Thus the conclusion of Lemma 2 does not hold and the run-lengths do not conserve volume. Recall that the assumption that the players do not switch simultaneously played an important role in the proof of Lemma 2.

Foster and Young (1996) have constructed an 8×8 coordination game in which the fictitious play does not converge to any of the equilibria. They show this by establishing that FP will follow a simple cycle in which each player uses 8 pure strategies. The cycle is robust and the game is generic in our sense. Thus Theorem 1 can be used to deduce that CFP will not converge in their game.

10. Discrete time process. Our main result concerns the continuous time version of fictitious play (CFP). We now indicate how the result may be applied to discrete time fictitious play processes (DFP).

Recall that a DFP is a sequence $(p(t), q(t))$, for $t = 1, 2, 3, \cdots$ such that

$$(p(1), q(1)) \in \Delta(I) \times \Delta(J)$$

and

$$p(t+1) = \frac{tp(t) + B(q(t))}{t+1}, \qquad q(t+1) = \frac{tq(t) + B(p(t))}{t+1}.$$

Thus $p(t+1)$ is a weighted average of $p(t)$ and $B(q(t))$ where the weights are $t/(t+1)$ and $1/(t+1)$ and players can adjust their strategies in each period, that is, after one unit of time has elapsed.

Now suppose $\Delta > 0$ is the time between adjustments, that is, players can adjust their strategies after Δ units of time have elapsed. Then using the weights $t/(t+\Delta)$ and $\Delta/(t+\Delta)$, we get:

$$p(t+\Delta) = \frac{tp(t) + \Delta B(q(t))}{t+\Delta}$$

or, equivalently:

$$\frac{p(t+\Delta) - p(t)}{\Delta} = \frac{B(q(t)) - p(t+\Delta)}{t}.$$

As the time between adjustments $\Delta \to 0$, we obtain that the right derivative of $p(t)$ for $t \geq 1$:

$$\left.\frac{dp(t)}{dt}\right|_{+} = \frac{B(q(t)) - p(t)}{t},$$

which is the CFP.

Thus a CFP approximates a DFP when the time between adjustments, Δ, is small. We now show how our main result applies to DFP's with small Δ.

DEFINITION 3. A Δ-DFP is a sequence $(\bar{p}(t), \bar{q}(t))$, for $t = 1, 1 + \Delta, 1 + 2\Delta, 1 + 3\Delta, \cdots$ *such that*

$$(\bar{p}(1), \bar{q}(1)) \in \Delta(I) \times \Delta(J)$$

and

$$\bar{p}(t + \Delta) = \frac{t\bar{p}(t) + \Delta B(\bar{q}(t))}{t + \Delta}, \qquad \bar{q}(t + \Delta) = \frac{t\bar{q}(t) + \Delta B(\bar{p}(t))}{t + \Delta}.$$

Of course, when $\Delta = 1$ we obtain the original Brown-Robinson process.

For $t \geq 1$ let $(p(t), q(t))$ be a cyclic CFP with the cycle c. Let $n(r) \in \mathbb{R}^K_+$, $r = 1, 2, \cdots$ be the run-lengths associated with the CFP $(p(t), q(t))$. Let $F \equiv C^{-1}D$ be the matrix associated with the cycle c, as defined earlier, so that for all r:

$$n(r + 1) = Fn(r). \tag{43}$$

For $t = 1, 2, 3, \cdots$ let $(\bar{p}(t), \bar{q}(t))$ be a cyclic Δ-DFP with the same cycle c. Let the amount of time spent playing the strategy combination (i_k, j_k) in round r be written as $\bar{n}^\Delta_k(r)\Delta$ so that $\bar{n}^\Delta_k(r) \in \mathbf{Z}_+$ is the number of "periods," each of length Δ, that (i_k, j_k) is played. Define $\bar{n}^\Delta(r) \in \mathbf{Z}^K_+$, $r = 1, 2, \cdots$ to be the vector of *run-lengths* associated with the Δ-DFP $(\bar{p}(t), \bar{q}(t))$. Also notice that given the initial condition $(\bar{p}(1), \bar{q}(1)) = (p_0, q_0)$ we have, as in the proof of Theorem 1, that $\bar{n}^\Delta(0)\Delta \equiv (C^{-1}D - I)m(0)$ where $m(0)$, as before, is a vector satisfying $Pm(0) = p_0$ and $Qm(0) = q_0$. Thus as $\Delta \to 0$,

$$\bar{n}^\Delta(0) = \frac{1}{\Delta}(C^{-1}D - I)m(0) \to \infty. \tag{44}$$

From now on we economize on notation by writing $\bar{n}(r)$ instead of $\bar{n}^\Delta(r)$.

For any vector x let $\|x\|$ denote the absolute value of the largest component of x. Similarly, for any matrix M let $\|M\|$ denote the absolute value of the largest entry in M. Observe that since in a Δ-DFP the run-lengths must all be positive integers (43) cannot hold precisely for the discrete process. However, (43) is approximately true:

LEMMA 8. *There exists a* γ *such that for all* r:

$$\|\bar{n}(r + 1) - F\bar{n}(r)\| < \gamma.$$

PROOF. Consider a cycle c and a switch from (i_1, j_1) to (i_2, j_2) by player 1, say. Then analogous to equation (2) in §4, we obtain for the Δ-DFP that player 1 switches when

$$(a_{i_2j_1} - a_{i_1j_1})\bar{n}_1\Delta \geq -(\alpha_{i_2} - \alpha_{i_1})Q^0,$$

$$(a_{i_2j_1} - a_{i_1j_1})(\bar{n}_1 - 1)\Delta \leq -(\alpha_{i_2} - \alpha_{i_1})Q^0,$$

which can be rewritten as

$$0 \leq (a_{i_2j_1} - a_{i_1j_1})\bar{n}_1\Delta + (\alpha_{i_2} - \alpha_{i_1})Q^0 \leq (a_{i_2j_1} - a_{i_1j_1})\Delta \leq \Omega\Delta,$$

where Ω is the largest payoff difference in the game.

In general, for $k = 1, 2, \ldots, K$ when the players switch from (i_k, j_k) to (i_{k+1}, j_{k+1}) in round r, we have, analogous to Equations (4) and (5), that

$$0 \leq (\alpha_{i_{k+1}} - \alpha_{i_k})QE_k\bar{n}\Delta + (\alpha_{i_{k+1}} - \alpha_{i_k})Q^0 \leq \Omega\Delta,$$

$$0 \leq (\beta_{j_{k+1}} - \beta_{j_k})PE_k\bar{n}\Delta + (\beta_{j_{k+1}} - \beta_{j_k})P^0 \leq \Omega\Delta.$$

In the switch from (i_k, j_k) to (i_{k+1}, j_{k+1}) in round $r + 1$, we have, analogous to Equations (6) and (7), that

$$0 \leq (\alpha_{i_{k+1}} - \alpha_{i_k})QE_k\bar{n}'\Delta + (\alpha_{i_{k+1}} - \alpha_{i_k})Q\bar{n}\Delta + (\alpha_{i_{k+1}} - \alpha_{i_k})Q^0 \leq \Omega\Delta,$$

$$0 \leq (\beta_{j_{k+1}} - \beta_{j_k})PE_k\bar{n}'\Delta + (\beta_{j_{k+1}} - \beta_{j_k})P\bar{n}\Delta + (\beta_{j_{k+1}} - \beta_{j_k})P^0 \leq \Omega\Delta.$$

Combining these we obtain:

$$-\Omega\Delta \leq (\alpha_{i_{k+1}} - \alpha_{i_k})QE_k\bar{n}'\Delta + (\alpha_{i_{k+1}} - \alpha_{i_k})Q(I - E_k)\bar{n}\Delta \leq \Omega\Delta,$$

$$-\Omega\Delta \leq (\beta_{j_{k+1}} - \beta_{j_k})PE_k\bar{n}'\Delta + (\beta_{j_{k+1}} - \beta_{j_k})P(I - E_k)\bar{n}\Delta \leq \Omega\Delta,$$

which is the same as:

$$-\Omega u \leq C\bar{n}' - D\bar{n} \leq \Omega u,$$

where $u = [1, 1, \ldots, 1]^T$ or equivalently

$$\|C\bar{n}' - D\bar{n}\| \leq \Omega,$$

which proves the result. □

Notice that F in the statement of the lemma above is the same matrix as in (43). We have shown in earlier sections that for almost all games, all cycles such that (a) there are no simultaneous switches; and (b) each player uses at least three strategies, have the property that the dominant root of the associated matrix F is $\lambda > 1$. We now show that these same conditions make cyclic convergence impossible for the Δ-DFP also, for almost all initial conditions once Δ is sufficiently small.

LEMMA 9. *For almost all (p_0, q_0) there exists a Δ_0 such that for all $\Delta < \Delta_0$ if $(\bar{p}(t), \bar{q}(t))$ is a cyclic Δ-DFP such that the dominant root of the associated matrix F is $\lambda > 1$ then $(\bar{p}(t), \bar{q}(t))$ does not converge starting from $(\bar{p}(1), \bar{q}(1)) = (p_0, q_0)$.*

PROOF. Since λ is the dominant eigen root of F, there exists a matrix F^* such that $\lim_{r\to\infty}(F^r/\lambda^r) = F^*$ and hence there exists a number M such that for all s, $\|F^s/\lambda^s\| < M$. Assume for simplicity that λ is real and has a multiplicity of 1 (the general case is similar, see Rosenmüller 1971). Then $F^*n(0) = 0$ only if $c_S = 0$ in the sense of the last paragraph of the claim at the end of the proof of Theorem 1. And the set of such $n(0)$ is a closed subspace of measure zero.

Thus, as in the proof of Theorem 1, for almost all initial conditions (p_0, q_0), $\bar{n}(0) \notin \ker F^*$. Fix such a (p_0, q_0). Recall that the amount of time spent playing (i_k, j_k) is $\bar{n}_k\Delta$. Therefore as $\Delta \to 0$, with a fixed $(\bar{p}(1), \bar{q}(1)) = (p_0, q_0)$, $\bar{n}_k(0) \to \infty$ as in (44). As $\bar{n}(0) \notin \ker F^*$ we can choose Δ_0 such that for all $\Delta < \Delta_0$,

$$\|F^*\bar{n}(0)\| > \gamma M \frac{1}{\lambda - 1} + 1. \tag{45}$$

Using the previous lemma we can write for all r:

$$\bar{n}(r) = F\bar{n}(r-1) + \epsilon(r)$$

where $\|\epsilon(r)\| < \gamma$ and thus for all r:

$$\bar{n}(r) = F^r\bar{n}(0) + \sum_{s=1}^{r} F^s\epsilon(r-s+1)$$

or that:

$$\frac{\bar{n}(r)}{\lambda^r} = \frac{F^r}{\lambda^r}\bar{n}(0) + \sum_{s=1}^{r} \frac{F^s}{\lambda^r}\epsilon(r-s+1).$$

Since $\lambda > 1$,

$$\left\|\frac{\bar{n}(r)}{\lambda^r} - \frac{F^r}{\lambda^r}\bar{n}(0)\right\| \leq \sum_{s=1}^{r} \frac{1}{\lambda^{r-s}}\left\|\frac{F^s}{\lambda^s}\right\|\|\epsilon(r-s+1)\| \leq \gamma M \sum_{s=1}^{r} \frac{1}{\lambda^{r-s}} \leq \gamma M \frac{1}{\lambda - 1}.$$

There exists an R such that for all $r > R$,

$$\left\|\frac{F^r}{\lambda^r} - F^*\right\| \leq \frac{1}{\|\bar{n}(0)\|}.$$

Thus for all $r > R$:

$$\left\|\frac{\bar{n}(r)}{\lambda^r} - F^*\bar{n}(0)\right\| \leq \left\|\frac{\bar{n}(r)}{\lambda^r} - \frac{F^r}{\lambda^r}\bar{n}(0)\right\| + \left\|\frac{F^r}{\lambda^r}\bar{n}(0) - F^*\bar{n}(0)\right\|$$

$$\leq \gamma M \frac{1}{\lambda - 1} + 1$$

and because of (45) there exists a $\mu > 0$ such that if $\Delta < \Delta_0$ then for all $r > R$:

$$\left\|\frac{\bar{n}(r)}{\lambda^r}\right\| > \mu.$$

Thus the sequence of run-lengths $\bar{n}(r)$ grows at least exponentially at the rate $\lambda > 1$. As before, this implies that the Δ-DFP sequence $(\bar{p}(t), \bar{q}(t))$ does not converge. □

Recently, Harris (1996) and Hofbauer (1994) have also investigated the relation between continuous and discrete time fictitious play using the theory of differential inclusions. By embedding the discrete time fictitious play in continuous time, the convergence of the continuous time fictitious play for zero-sum games can be used to show the convergence of the discrete time fictitious play for zero-sum games. The argument depends on the fact that in zero-sum games the convergence is uniform in the starting positions. It is then possible to show that limiting behavior of the discrete time process approximates that of the continuous time process.

11. Other related processes. CFP belongs to a more general class of continuous time processes which may be described as follows.

Suppose $\phi : [t_0, \infty) \to [0, \infty)$ is a one to one and onto function that is differentiable on (t_0, ∞) and satisfies $\phi' > 0$. Consider the system:

$$(46) \qquad \left.\frac{d\bar{p}}{dt}\right|_+ = \phi'(t)[B(q(t)) - \bar{p}(t)], \qquad \left.\frac{d\bar{q}}{dt}\right|_+ = \phi'(t)[B(p(t)) - \bar{q}(t)]$$

where $B(q(t)) \in BR(q(t))$ and $B(p(t)) \in BR(p(t))$. Suppose $(\bar{p}(t), \bar{q}(t))$ is a path that satisfies the system (46) from the initial conditions $(\bar{p}(t_0), \bar{q}(t_0)) = (p_0, q_0)$.

(Observe that CFP is just a special case of (46) when $\phi(t) = \ln t$.)

Define $s = \phi(t)$ and consider the path $p_b(s) = \bar{p}(\phi^{-1}(s))$ and $q_b(s) = \bar{q}(\phi^{-1}(s))$. We now obtain

$$\left.\frac{dp_b}{ds}\right|_+ = \left.\frac{d\bar{p}(\phi^{-1}(s))}{dt}\right|_+ \times \frac{dt}{ds}$$

$$= \phi'(\phi^{-1}(s))[B(\bar{q}(\phi^{-1}(s))) - \bar{p}(\phi^{-1}(s))] \times \frac{1}{\phi'(\phi^{-1}(s))}$$

$$= B(q_b(s)) - p_b(s)$$

and similarly for q_b.

Thus $(p_b(s), q_b(s))$ is a path satisfying the system

$$(47) \qquad \left.\frac{dp_b}{ds}\right|_+ = B(q_b(s)) - p_b(s), \qquad \left.\frac{dq_b}{ds}\right|_+ = B(p_b(s)) - q_b(s)$$

from the initial conditions $(p_b(0), q_b(0)) = (p_0, q_0)$ if and only if $(\bar{p}(t), \bar{q}(t))$ is a path that satisfies the system (46) from the initial conditions $(\bar{p}(t_0), \bar{q}(t_0)) = (p_0, q_0)$.

The system given by (47) is called the *continuous time best response* (CBR) dynamics by Hofbauer (1994). (Harris 1996 calls this "continuous time fictitious play of the second-kind.")

Since CBR is just CFP with a different time scale ($s = \ln t$) this does not affect the convergence properties of the system, and thus our results hold for CBR also. Now by extension they hold for all systems of the form (46).

Another useful equivalent process is the *continuous time weighted fictitious play* (CWFP):

$$\left.\frac{dp_w}{dt}\right|_+ = r[B(q_w(t)) - p_w(t)], \qquad \left.\frac{dq_w}{dt}\right|_+ = r[B(p_w(t)) - q_w(t)]$$

which is a system in which players use an exponentially weighted average of past plays to evaluate their beliefs about their opponent's play.

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