DISCUSSION PAPERS
Institute of Economics
University of Copenhagen

03-30

On the Continuity of Representations of Effectivity Functions

Hans Keiding and Bezalel Peleg

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

ON THE CONTINUITY OF REPRESENTATIONS OF EFFECTIVITY FUNCTIONS

Hans Keiding
University of Copenhagen[1]

Bezalel Peleg
Hebrew University of Jerusalem

May 2003

Abstract

An effectivity function assigns to each coalition of individuals in a society a family of subsets of alternatives such that the coalition can force the outcome of society's choice to be a member of each of the subsets separately. A representation of an effectivity function is a game form with the same power structure as that specified by the effectivity function. In the present paper we investigate the continuity properties of the outcome functions of such representation. It is shown that while it is not in general possible to find continuous representations, there are important subfamilies of effectivity functions for which continuous representations exist. Moreover, it is found that in the study of continuous representations one may practically restrict attention to effectivity functions on the Cantor set. Here it is found that general effectivity functions have representations with lower or upper semicontinuous outcome function.

[1] Communication with: Hans Keiding, Institute of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen K, Denmark.

Email: Hans.Keiding@econ.ku.dk

1. Introduction

In his famous address (Sen [1999]), Sen writes, "Impossibility results in social choice theory – led by the poineering work of Arrow [1951] – have been interpreted as being thoroughly destructive of the possibility of reasoned and democratic social choice, including welfare economics." Sen, of course, argued against the foregoing view in his Nobel lecture. However, in his closing section he concludes, "The possibility of constructive welfare economics and social choice (and their use in making social welfare judgements and in devising practical measures with normative significance) turns on the need for broadening the informational basis of such choice. Different types of informational enrichment have been considered in the literature. A crucial element in this broadening is the use of interpersonal comparisons of well-being and individual advantage."

While we do not question Sen's conclusions, we nevertheless point out that the pioneering work of Gärdenfors [1981] enables us to construct a social choice theory which avoids the Impossiblity Theorem of Arrow and the Liberal Paradox, and, at the same time, makes no interpersonal comparisons of utilities (which are not ruled out by Sen). We shall now very broadly outline Gärdenfors' approach [1981] and its recent developments (Peleg [1998], Keiding and Peleg [2002], Peleg et al. [2002]).

In order to avoid Arrow's Impossibility Theorem and the Liberal Paradox, Gärdenfors has chosen a new definition of constitution as "rights-system." In Gärdenfors' model a right of a set S of members of the society is a set B of social states such that S can enforce the final state to be member of B. In Arrow's model a constitution is a "well-behaved" social welfare function. Notice that, unlike Arrow's, Gärdenfors' notion of constitution does not depend on preferences. Rights-systems can be formally described by effectivity functions (see Moulin and Peleg [1982] for the definition of an effectivity function and Peleg [1998] for the linkage to Gärdenfors [1981]). While a rights-system is part of a general game-theoretic framework for the interaction between the members of the society, the model in Gärdenfors [1981] is somewhat remote from mainstream game theory. The next step has been taken in Peleg [1998].

A representation of an effectivity function (or a constitution) is a game form that endows the members of the society with precisely the same power as the effectivity function. Representations have been introduced and applied in Peleg [1998]. An effectivity function has a representation if it satisfies the mild assumptions of monotonicity and superadditivity. Thus, in particular, anonymous effectivity functions may have representations. Representations of the constitution enable the members of the society to exercise their rights simultaneously. Also, the standard theories of strategic games may be applied to representations whenever the preferences of the members of the society are specified.

The third step has been taken in Keiding and Peleg [2002] and Peleg et al. [2002]. Keiding and Peleg find necessary and sufficient conditions on a (discrete) effectivity function that guarantee the existence of a coalition proof

Nash consistent representation, that is a reprentation that has a coalition proof Nash equilibrium for each profile of preferences of the society. Peleg et al. [2002] investigate the existence of Nash consistent representations (i.e., representations that possess a Nash equilibrium for each profile of preferences), under various conditions (mainly topological). In particular, they prove the existence of weakly acceptable representations, that is representations that have a Pareto optimal Nash equilibrium for every profile of preferences under relatively mild conditions (thus avoiding the Liberal Paradox).

Our present work is motivated by this last work. Peleg et al. [2002] construct Nash consistent representations which are not continuous. Thus, the problem of existence of continuous representations of topological effectivity functions has remained open. This work is devoted to this basic question.

We shall now review the contents of our paper. Clearly, the continuity of the outcome function of a representation in the strategies played by the members of the society is very desirable. Unfortunately, in Section 3 we describe an effectivity function which admits no continuous representation. Fortunately, continuity properties of (topological) effectivity functions are latent everywhere in our model. Moreover, the results that we obtain and the techniques we use are novel and may be of some interest in their own.

In Section 4 we prove that every (monotonic and superadditive) effectivity function that is generated by a finite set of (closed) subsets of alternatives has a continuous representation. This leads to the result that effectivity functions with a continuous representation are dense in the set of all (topological) effectivity functions. All our results are obtained under the assumption that the set of alternatives is a compact metric space.

Let $\mathcal{C}$ be the Cantor set. It is well known that for every compact metric space A there exists a continuous surjection $f_A : \mathcal{C} \to A$. In Section 5 we observe that for all A and f_A as above, every effectivity function E on A can be "lifted" to an effectivity function $\tilde{E}$ on $\mathcal{C}$. Furthermore, every (continuous) representation of $\tilde{E}$ yields naturally a (continuous) representation of E. Thus, using a classical mathematical result we show that the general representation problem can be reduced to the representation problem on $\mathcal{C}$.

Section 6 contains the following important result: On $\mathcal{C}$ every effectivity function (with closed values) has an upper (or lower) semicontinuous representation. Using the techniques of Section 5, we show in Section 7 that the result in Section 6 implies that an effectivity function (over an arbitrary compact metric space) has a representation whose outcome function is a (modified) Baire function of order 2.

The short Section 8 is devoted to retractions of $\mathcal{C}$. In particular, every closed (nonempty) subset of $\mathcal{C}$ is a retract of $\mathcal{C}$. Using retractions of $\mathcal{C}$ we prove in Section 9 that every effectivity function that is majorized by a simple effectivity function (i.e., an effectivity function that is defined by a simple game) has a continuous representation.

We should mention here the closely related work on representation of committees (i.e., proper simple games), Peleg [1978], Ishikawa and Nakamura [1980], Holzman [1986a, 1986b], and Keiding and Peleg [2001]. In particular, in Keiding and Peleg [2001] the set of alternatives is a convex and compact subset of $\mathbb{R}^m$ and representations are strongly consistent (that is, they possess a strong Nash equilibrium for every profile of preferences of the members of the society). However, the representations constructed in that work are not continuous.

2. Definitions and notations

Throughout this paper, A denotes the set of alternatives. The set A may be finite or infinite; however, if A is finite, then A contains at least two alternatives. Further, we assume that A is a compact metric space. The metric on A will be denoted by d. If D is a set, then $P(D) = \{\tilde{D} \mid \tilde{D} \subseteq D\}$, and $P_0(D) = P(D) \backslash \{\emptyset\}$. Finally

$$\mathcal{K}(A) = \{B \in P_0(A) \mid B \text{ is closed }\}.$$

Let $N = \{1, \ldots, n\}$ be the set of players and let A be the (compact) metric space of alternatives. An *effectivity function* (EF) is a function $E : P(N) \to P(\mathcal{K}(A))$ that satisfies the following conditions: (i) $E(N) = \mathcal{K}(A)$; (ii) $E(\emptyset) = \emptyset$; and (iii) $A \in E(S)$ for every $S \in P_0(N)$.

As a general interpretation, $B \in E(S)$ means that the coalition S can force the final alternative to be an element of B. The interpretations of the three conditions are fairly obvious.

An EF E is *superadditive* if it satisfies the following condition: If $S_i \in P_0(N)$ and $B_i \in E(S_i)$, $i = 1, 2$, and $S_1 \cap S_2 = \emptyset$, then

$$B_1 \cap B_2 \in E(S_1 \cup S_2).$$

The EF is *monotonic* if

$$[B \in E(S),\ B^* \in \mathcal{K}(A),\ B \subseteq B^*, \text{ and } S \subseteq S^*] \Longrightarrow B^* \in E(S^*).$$

Monotonicity and superadditivity of EF's are natural properties in view of the foregoing interpretation. Moreover, EF's derived from game forms (see below) have these properties.

Let d_H be the Hausdorff metric on $\mathcal{K}(A)$ (see Hildenbrand [1974, p. 16]). We notice that $(\mathcal{K}(A), d_H)$ is a compact metric space (see, again, Hildenbrand [1974]), and we shall use this fact in the sequel. At one point we shall also use the *upper topology* τ_u on $\mathcal{K}(A)$. A basis for τ_u is given by

$$\{B \in \mathcal{K}(A) \mid B \subseteq U\},\ U \text{ an open subset of } A.$$

We also use some basic properties of simple games. A simple game is a pair (N, W), where $N = \{1, \ldots, n\}$ is the set of players and $W \subseteq P_0(N)$ is the set of winning coalitions, $W \neq \emptyset$. We always assume *monotonicity*

$$[S \in W \text{ and } S \subseteq T \subseteq N] \Longrightarrow T \in W.$$

A simple game $G = (N, W)$ is *proper* if

$$S \in W \Longrightarrow N \backslash S \notin W \text{ for all } S \in P_0(N).$$

We now turn to define some notions pertaining to game forms. A *game form* (GF) is an $(n+2)$-tuple $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$, where (i) for each i, Σ^i is the (nonempty) set of strategies of player $i \in N$; and (ii) $\pi : \Sigma^1 \times \cdots \times \Sigma^n \to A$ is the outcome function. We always assume that π is surjective. For $S \in P_0(N)$ we denote $\Sigma^S = \mathsf{X}_{i \in S} \Sigma^i$. Also, we denote $\Sigma = \Sigma^N$.

Let $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ be a GF. The EF E^Γ, associated with Γ, is defined in the following way: For $S \in P_0(N)$ and $B \in \mathcal{K}(A)$, S is effective for B if there exists $\sigma^S \in \Sigma^S$ such that $\pi(\sigma^S, \tau^{N \backslash S}) \in B$ for all $\tau^{N \backslash S} \in \Sigma^{N \backslash S}$. Now E^Γ is defined by $E^\Gamma(\emptyset) = \emptyset$ and

$$E^\Gamma(S) = \{B \in \mathcal{K}(A) \mid S \text{ is effective for } B\}, \text{ for } S \in P_0(N).$$

Clearly, E^Γ is a superadditive and monotonic EF.

Let $E : P(N) \to P(\mathcal{K}(A))$ be an EF. A GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ is a *representation* of E if $E^\Gamma(S) = E(S)$ for every $S \in P_0(N)$. Basically, this means that the GF distributes the same power among the players as does the EF. Γ is a *continuous* representation of E if $\Sigma^1, \ldots, \Sigma^n$ are compact metric spaces and $\pi : \Sigma \to A$ is continuous when Σ is endowed with the product topology.

3. An example

We shall present now an EF E which admits no continuous representation. Let

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0, \text{ and } x + y \leq 1\}.$$

Further, let $N = \{1, 2\}$,

$$E(\{1\}) = \{B \in \mathcal{K}(A) \mid B \supseteq [(1,0), (0,y)] \text{ for some } 0 \leq y \leq 1\},$$

and

$$E(\{2\}) = \{B \in \mathcal{K}(A) \mid B \supseteq [(x,0), (0,1)] \text{ for some } 0 \leq x \leq 1\}.$$

This completely specifies E as a monotonic and superadditive EF ($E(\emptyset) = \emptyset$ and $E(\{1,2\}) = \mathcal{K}(A)$). Assume now, on the contrary, that $\Gamma = (\Sigma^1, \Sigma^2; \pi; A)$ is a continuous representation of E. Let $\tilde{y}(k) \uparrow 1$. For each k there exists $\tilde{\sigma}^1(k) \in \Sigma^1$ such that

$$\pi(\tilde{\sigma}^1(k), \Sigma^2) = \{\pi(\tilde{\sigma}^1(k), \sigma^2) \mid \sigma^2 \in \Sigma^2\} = [(1,0), (0, \tilde{y}(k))]. \tag{3.1}$$

Let $\tilde{\sigma}^1(k_j) \to \sigma_0^1$ as $j \to \infty$. Then

$$\pi(\sigma_0^1, \Sigma^2) = [(1,0), (0,1)]. \tag{3.2}$$

Denote $\sigma^1(j) = \tilde{\sigma}^1(k_j)$ and $y(j) = \tilde{y}(k_j)$, $j = 1, 2, \ldots$.

Let now $\tilde{x}(k) \uparrow 1$. For each k there exists $\tilde{\sigma}^2(k) \in \Sigma^2$ such that

$$\pi(\Sigma^1, \tilde{\sigma}^2(k)) = [(\tilde{x}(k), 0), (0, 1)]. \tag{3.3}$$

Let $\tilde{\sigma}^2(k_j) \to \sigma_0^2$ as $j \to \infty$. Then

$$\pi(\Sigma^1, \sigma_0^2) = [(1, 0), (0, 1)]. \tag{3.4}$$

Denote $\sigma^2(j) = \tilde{\sigma}^2(k_j)$ and $\tilde{x}(k_j) = x(j)$, $j = 1, 2, \ldots$.

By (3.2) and (3.3)

$$\pi(\sigma_0^1, \sigma^2(j)) = (0, 1),\ j = 1, 2, \ldots$$

Hence, $\pi(\sigma_0^1, \sigma_0^2) = \lim_{j\to\infty} \pi(\sigma_0^1, \sigma^2(j)) = (0, 1)$. By (3.4) and (3.1)

$$\pi(\sigma^1(j), \sigma_0^2) = (1, 0),\ j = 1, 2, \ldots$$

Hence, $\pi(\sigma_0^1, \sigma_0^2) = \lim_{j\to\infty} \pi(\sigma^1(j), \sigma_0^2) = (1, 0)$. Thus, we have arrived at the desired contradiction.

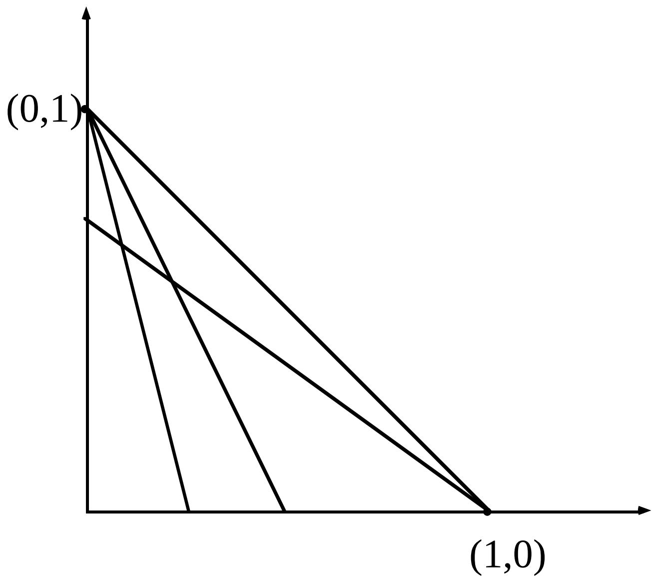

Fig. 1

The situation described in the exampe is illustrated in Fig.1. Each of the line segments from $(1, 0)$ belong to $E(\{1\})$, whereas all the line segments through $(0, 1)$ are in $E(\{2\})$. Consequently, the segment $[(0, 1), (1, 0)]$ belongs to both $E(\{1\})$ and $E(\{2\})$, and this is used to obtain a contradiction.

We conclude from the foregoing example that the main reason for the non-existence of continuous representation is the discontinuity of set intersection. For example, if $\alpha(k) \uparrow 1$, then $[(1,0),(0,\alpha(k))] \to [(1,0),(0,1)]$ and $[(\alpha(k),0),(0,1)] \to (1,0),(0,1)]$ in $(\mathcal{K}(A), d_H)$, whereas

$$[(1,0),(0,\alpha(k))] \cap [(\alpha(k),0),(0,1)] = \left(\frac{\alpha(k)}{1+\alpha(k)}, \frac{\alpha(k)}{1+\alpha(k)}\right) \to \left(\frac{1}{2}, \frac{1}{2}\right).$$

Thus, the intersection of the limits contains strictly the limit of the intersections. Indeed, let A be an arbitrary compact metric space, and let $B(t) \to B$ and $C(t) \to C$ in $(\mathcal{K}(A), d_H)$. Then all we can say is that

$$\limsup_{t\to\infty} B(t) \cap C(t) \subseteq B \cap C. \tag{3.5}$$

(If $B(t) \in \mathcal{K}(A)$, $t = 1, 2, \ldots$, then $x \in \limsup_{t\to\infty} B(t)$ if there exists a subsequence $t_1 < t_2 < \cdots$ and $x(t_j) \in B(t_j)$, $j = 1, 2\ldots$, such that $x = \lim_{j\to\infty} x(t_j)$.)

4. Finitely generated EFs and ε-representations

In view of the example in Section 3 we should impose extra conditions on an EF if we want to obtain for it a continuous representation. A possible simple condition is dependence on a finite number of sets of alternatives. This is made precise in the following.

Let A be a (compact) metric space of alternatives and let $N = \{1, \ldots, n\}$ be the set of players.

Definition 4.1. *An EF $E : P(N) \to P(\mathcal{K}(A))$ is* finitely generated *if for every $S \subseteq N$, $S \neq \emptyset, N$, there exists $B(j, S)$ in $\mathcal{K}(A)$, $j = 1, \ldots, k(S)$ such that*

$$E(S) = \{B \in \mathcal{K}(A) \mid B \supseteq B(j,S) \text{for some } 1 \leq j \leq k(S)\}.$$

Furthermore, $E(N) = \mathcal{K}(A)$ and $E(\emptyset) = \emptyset$ as usual.

The foregoing definition enables us to formulate our first existence result.

Theorem 4.2. *Let $E : P(N) \to P(\mathcal{K}(A))$ be a monotonic and superadditive EF. If E is finitely generated, then E has a continuous representation.*

PROOF: The sets $B(j,S)$, $j = 1, \ldots, k(S)$, $S \neq \emptyset, N$, generate a finite algebra $\mathcal{F}$. Denote by $\hat{A}$ the set of atoms of $\mathcal{F}$. For each $\hat{B} \subseteq \hat{A}$ denote further

$$\varphi(\hat{B}) = \cup\{\hat{a} \mid \hat{a} \in \hat{B}\} \in P_0(A).$$

This enables us to define a discrete EF $\hat{E} : P(N) \to P(P_0(\hat{A}))$, by

$$\hat{E}(S) = \{\hat{B} \subseteq \hat{A} \mid \varphi(\hat{B}) \in E(S)\}$$

for $S \neq \emptyset, N$, $\hat{E}(N) = P_0(\hat{A})$, and $\hat{E}(\emptyset) = \emptyset$. As the reader may easily verify, $\hat{E}$ is monotonic and superadditive. Hence, by Moulin and Peleg [1982], $\hat{E}$ has a (discrete) representation $\hat{\Gamma} = (\hat{\Sigma}^1, \ldots, \hat{\Sigma}^n; \hat{\pi}; \hat{A})$. Call $f : \hat{A} \to A$ a choice function if $f(\hat{a}) \in \hat{a}$ for all $\hat{a} \in \hat{A}$, and denote by Φ the set of all choice functions. Finally, define a new GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ by

$$(i)\ \Sigma^i = \hat{\Sigma}^i \times \Phi \times N \text{ for all } i \in N;$$

$$(ii)\ \pi((\hat{\sigma}^1, f^1, t^1), \ldots, (\hat{\sigma}^n, f^n, t^n)) = f^j(\hat{\pi}(\hat{\sigma}^1, \ldots, \hat{\sigma}^n)),$$

where $j \equiv \sum_{u=1}^{n} t^u\ (n)$.

If $\hat{\Sigma}^1, \ldots, \hat{\Sigma}^n$ and N are given the discrete topology, and $\Phi = \times_{\hat{a} \in \hat{A}} \hat{a}$ is given the product topology, then π is continuous.

It remains to show that Γ is a representation of E. Clearly, $E^\Gamma(N) = \mathcal{K}(A)$. Thus, let $S \subset N$, $S \neq \emptyset, N$. First, let $B = B(j, S)$ for some $1 \leq j \leq k(S)$. Then there is $\hat{B} \subseteq \hat{A}$ such that $B = \varphi(\hat{B})$. Hence S has a strategy $\hat{\sigma}^S \in \hat{\Sigma}^S$ such that $\hat{\pi}(\hat{\sigma}^S, \hat{\mu}^{N\setminus S}) \in \hat{B}$ for all $\hat{\mu}^{N\setminus S} \in \hat{\Sigma}^{N\setminus S}$. Therefore, it is clear that $\pi((\hat{\sigma}^S, f^S, t^S), (\hat{\mu}^{N\setminus S}, f^{N\setminus S}, t^{N\setminus S}) \in B$ for all $(\hat{\mu}^{N\setminus S}, f^{N\setminus S}, t^{N\setminus S}) \in \Sigma^{N\setminus S}$ (for arbitrary f^S and t^S). Thus, $E^\Gamma(S) \supseteq E(S)$.

Second, let $D \in \mathcal{K}(A)\setminus E(S)$. Then for every $\hat{B} \in \hat{E}(S)$, $\varphi(\hat{B})\setminus D \neq \emptyset$ (by monotonicity). Thus for every $\hat{\sigma}^S \in \hat{\Sigma}^S$ there exists $\hat{\mu}^{N\setminus S} \in \hat{\Sigma}^{N\setminus S}$ such that $\hat{\pi}(\hat{\sigma}^S, \hat{\mu}^{N\setminus S})\setminus D = \emptyset$. It is now clear that S cannot enforce D in Γ. □

Theorem 4.2 leads to an interesting approximation result. Indeed, the family of finitely generated effectivity functions plays an important role in relation to the set of all effectivity functions, being dense in this set in a sense to be made precise below. First we need the following definition.

Definition 4.3. *Let $E : P(N) \to P(\mathcal{K}(A))$ be an EF and let $\varepsilon > 0$. A game form $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ is an* ε-representation *of E if*

(i) $E^\Gamma(S) \subseteq E(S)$ for all $S \subseteq N$;

(ii) if $S \subseteq N$ and $B \in E(S)$, then $B(\varepsilon) \in E^\Gamma(S)$, where

$$B(\varepsilon) = \{y \in A \mid d(x, y) \leq \varepsilon \text{ for some } x \in B\}.$$

(Here d is the metric of A.)

We notice now that if Γ is an ε-representation of E, then E^Γ is an ε-approximation of E (in $(\mathcal{K}(A), d_H)$). Indeed, for every $S \subseteq N$, $S \neq \emptyset, N$, if $B \in E(S)$, then

$$d_H(B, E^\Gamma(S)) = \inf\{d_H(B, D) \mid D \in E^\Gamma(S)\} \leq d_H(B, B(\varepsilon)) \leq \varepsilon.$$

Let $E : P(N) \to P(\mathcal{K}(A))$ be a superadditive and monotonic EF. Then, as we shall prove, for every $\varepsilon > 0$ there exists a continuous ε-representation Γ_ε of E. Thus, in particular, E is approximated by the EF's E^{Γ_ε} which have (trivially) continuous representations.

Theorem 4.4. *Let $E : P(N) \to P(\mathcal{K}(A))$ be a monotonic and superadditive EF and let $\varepsilon > 0$. Then there exists a continuous ε-representation of E.*

Proof: We choose for each $S \subseteq N$, $S \neq \emptyset, N$ a finite set $\mathcal{D}^*(S) = \{B(1,S), \ldots, B(k(S),S)\} \subseteq E(S)$ such that for every $B \in E(S)$, there exists $1 \leq j \leq k(S)$ such that $B(\varepsilon) \supseteq B(j,S)$. Clearly, this is possible because $(\mathcal{K}(A), d_H)$ is compact. We now construct by induction on $|S|$ (the number of elements of S) a system $\mathcal{D}(S)$, $S \subseteq N$, $S \neq \emptyset, N$ such that:

(i) $\mathcal{D}^*(S) \subseteq \mathcal{D}(S) \subseteq E(S)$;

(ii) $\mathcal{D}(S)$ is finite;

(iii) $S \subseteq T \Longrightarrow \mathcal{D}(S) \subseteq \mathcal{D}(T)$; and

(iv) if $B_i \in \mathcal{D}(S_i)$, $i = 1, 2$, and $S_1 \cap S_2 = \emptyset$, then $B_1 \cap B_2 \in \mathcal{D}(S_1 \cup S_2)$.

Let now $\hat{E} : P(N) \to P(\mathcal{K}(A))$ be the EF which is finitely generated by the $\mathcal{D}(S)$, $S \neq \emptyset, N$. Clearly, $\hat{E}$ is superadditive and monotonic. By Theorem 4.2 $\hat{E}$ has a continuous representation Γ. Clearly, Γ is a continuous ε-representation of E. □

5. The reduction theorem

Let A be a compact metric space of alternatives and let $N = \{1, \ldots, n\}$ be a set of players. Assume further that M is another compact mectric space and $f : M \to A$ is a continuous surjection. Then, as we shall prove, every EF $E : P(N) \to P(\mathcal{K}(A))$ can be "lifted" to the space M to yield an EF $\tilde{E} : P(N) \to P(\mathcal{K}(M))$ such that every continuous representation of $\tilde{E}$ leads naturally to a continuous representation of E.

This procedure will turn out to be useful, in particular if applied to a particular compact metric space M, namely the Cantor set, which we introduce briefly (for a more detailed discussion, see, e.g. Willard [1970], section 30). The Cantor set may be defined as follows: Beginning with the unit interval $[0,1]$, define closed subsets $A_1 \supset A_2 \supset \cdots$ as follows: A_1 is obtained by removing the open interval $(\frac{1}{3}, \frac{2}{3})$ from $[0,1]$. A_2 is obtained by removing from A_1 the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, having defined A_{n-1}, A_n is obtained by removing the open middle thirds from each of the 2^{n-1} closed intervals that make up A_{n-1}. The Cantor set is the subspace $\mathcal{C} = \cap_{n \in \mathbb{N}} A_n$ of $[0,1]$. It is a nonempty compact metric space.

For later use, we note that the Cantor set has an alternative description. Each $x \in [0,1]$ has a ternary expansion $(x_1, x_2, \ldots)$ (so that each x_i belongs to $\{0,1,2\}$) with $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$. This expansion is unique except that any number $\neq 1$ with a ternary expansion ending in a series of 2's can alternatively be written as an expansion ending in 0's. The Cantor set $\mathcal{C}$ is the set of points in $[0,1]$ having a ternary expansion without 1's.

A famous result of general topology is the existence for each compact metric space A of a continuous surjection $f_A : \mathcal{C} \to A$ (Willard [1970], Theorem

30.7). Furthermore, the Cantor set is "simpler" than a general compact metric space, because it admits a continuous selection, that is, there exists a continuous function $\varphi : \mathcal{K}(\mathcal{C}) \to \mathcal{C}$ such that $\varphi(B) \in B$ for every B in $\mathcal{K}(\mathcal{C})$ (e.g., $\varphi(B) = \arg\max_{x \in B} x = \max(B)$, for $B \in \mathcal{K}(\mathcal{C})$). Indeed, we use this fact in the following sections.

Returning to the general theory, let, again, A be the compact metric space of alternatives, let M be an (arbitrary) compact metric space, and let $f : M \to A$ be a continuous surjection. For an EF $E : P(N) \to P(\mathcal{K}(A))$ we define a function $\tilde{E} : P(N) \to P(\mathcal{K}(M))$ by

$$\tilde{E}(S) = \{\tilde{B} \in \mathcal{K}(M) \mid \tilde{B} \supseteq f^{-1}(B) \text{ for some } B \in E(S)\} \tag{5.1}$$

for $S \neq \emptyset, N$, $\tilde{E}(N) = \mathcal{K}(M)$ and $\tilde{E}(\emptyset) = \emptyset$. Clearly, $\tilde{E}$ is an EF. We notice now the following result.

Lemma 5.1. *If E is monotonic and superadditive, then $\tilde{E}$ is also monotonic and superadditive.*

Proof: Clearly, we have only to prove superadditivity. Let $\tilde{B}_i \in \tilde{E}(S_i)$, $i = 1, 2$, and $S_1 \cap S_2 = \emptyset$. Then there exist $B_i \in E(S_i)$, $i = 1, 2$, such that $\tilde{B}_i \supseteq f^{-1}(B_i)$, $i = 1, 2$. As E is superadditive, $B_1 \cap B_2$ is in $E(S_1 \cup S_2)$. Hence

$$\tilde{B}_1 \cap \tilde{B}_2 \supseteq f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2),$$

and $\tilde{B}_1 \cap \tilde{B}_2 \in \tilde{E}(S_1 \cup S_2)$. □

We remark for future use that

$$\tilde{B} \in \tilde{E}(S) \Longrightarrow f(\tilde{B}) \in E(S), \text{ for all } S \in P(N) \text{ and } \tilde{B} \in \tilde{E}(S). \tag{5.2}$$

We are now ready for

Theorem 5.2 (the reduction theorem). *If $\tilde{\Gamma} = (\Sigma^1, \ldots, \Sigma^n; \pi; M)$ is a (continuous) representation of $\tilde{E}$, then $\Gamma = (\Sigma^1, \ldots, \Sigma^n; f \circ \pi; A)$ is a (continuous) representation of E.*

Proof. Clearly, we have only to prove that Γ is a representation of E. Let $S \in P_0(N)$ and $B \in E(S)$. Then $f^{-1}(B) \in \tilde{E}(S)$, and therefore there exists $\sigma^S \in \Sigma^S$ such that $\pi(\sigma^S, \mu^{N \setminus S}) \in f^{-1}(B)$ for all $\mu^{N \setminus S} \in \Sigma^{N \setminus S}$. Thus, $B \in E^{\Gamma}(S)$.

Let now $D \in \mathcal{K}(A) \setminus E(S)$. By (5.2), $f^{-1}(D) \notin \tilde{E}(S)$. Hence, for every $\sigma^S \in \Sigma^S$ there exists $\mu^{N \setminus S} \in \Sigma^{N \setminus S}$ such that $\pi(\sigma^S, \mu^{N \setminus S}) \notin f^{-1}(D)$. Thus, $f(\pi(\sigma^S, \mu^{N \setminus S})) \notin D$. □

An interesting corollary of the example in Section 3 and Theorem 5.2 is the existence of an EF $E : P(\{1, 2\}) \to P(\mathcal{K}(\mathcal{C}))$ that has no continuous representation.

6. Semicontinuous representations on $\mathbb{R}^1$

In the previous section, we have shown that effectivity functions on the Cantor set play a specific role in the analysis of effectivity functions having continuous representations. Once this importance of the Cantor set has been recognized, we may notice that the Cantor set has additional analytical advantages as a subset of the set of real numbers, in particular, it makes sense to consider upper or lower semicontinuity of (outcome) functions.

Let the set of alternatives A be a compact subset of the real line $\mathbb{R}^1$. If $E : P(N) \to P(\mathcal{K}(A))$ is an EF, then a representation $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ of E is semicontinuous if the (outcome) function $\pi : \Sigma \to A$ is upper semicontinuous or lower semicontinuous. It is now natural to enquire whether there exist semicontinuous representations of (monotonic and superadditive) EFs on A. We shall prove that if an EF $E : P(N) \to P(\mathcal{K}(A))$ is, in addition, closed-valued, that is $E(S)$ is closed in $(\mathcal{K}(A), d_H)$ for every $S \in P(N)$, then E has semicontinuous representations. We start with some general remarks on EFs.

Let A be a compact metric space. If an EF E is monotonic and closed-valued, then $E(S)$ is closed in the upper topology for every $S \subseteq N$ (the proof is straightforward). This fact has the following implication: Let, again, $E : P(N) \to P(\mathcal{K}(A))$ be an EF. The polar of E, E^*, is given by

$$E^*(S) = \{B \in \mathcal{K}(A) \mid B \cap \tilde{B} \neq \emptyset \text{ for all } \tilde{B} \in E(N \backslash S)\}$$

for $S \subseteq N$, $S \neq \emptyset, N$, $E^*(N) = E(N)$, and $E^*(\emptyset) = \emptyset$. If E has closed values in the upper topology, then E is reflexive, that is $E = E^{**}$ (see Abdou and Keiding (1991, p.46)).

We are now ready to prove existence of semicontinuous representations.

Theorem 6.1. *Let A be a compact subset of $\mathbb{R}^1$ and let $E : P(N) \to P(\mathcal{K}(A))$ be a monotonic and superadditive EF. If $E(S)$ is a closed set in $(\mathcal{K}(A), d_H)$ for every $S \subseteq N$, then E has a semicontinuous representation.*

Proof: For $i \in N$ let $N^i = \{S \subseteq N \mid i \in S\}$. Let

$$V^i = \{\nu : N^i \to N^i \times N \mid \nu_1(S) \subseteq S \text{ and } \nu_2(S) \in S\},$$

where $\nu = (\nu_1, \nu_2)$. Further, let $M^i = \{\varphi : N^i \to \mathcal{K}(A)) \mid \varphi(S) \in E(S) \text{ for all } S \in N^i\}$, and $M^i_* = \{\varphi_* : N^i \to \mathcal{K}(A) \mid \varphi(S) \in E^*(S) \text{ for all } S \in N^i\}$. Define a GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ by the following rules: Let $\Sigma^i = V^i \times M^i \times M^i_* \times N \times \{0, 1\}$ for all $i \in N$. Here M^i and M^i_* are given the product topology ($M^i = \mathsf{X}_{S \in N^i} E(S)$ and $M^i_* = \mathsf{X}_{S \in N^i} E^*(S)$), and V^i, N^i and $\{0, 1\}$ are given the discrete topology. Thus, Σ^i is a compact metric space for every $i \in N$.

It remains to define π. Let $\sigma^i = (\nu^i, \varphi^i, \varphi^i_*, t^i, q^i)$ for $i \in N$. Using $\nu^1, \ldots \nu^n$ we introduce the following partitions of N. First, for $S \in P_0(N)$, we define an equivalence relation $\sim_\sigma$ on S by

$$i \sim_\sigma j \Leftrightarrow \nu^i(S) = \nu^j(S), \text{ all } i, j \in S,$$

where $\sigma = (\sigma^1, \ldots, \sigma^n)$. Denote by $D(S, \sigma)$ the partition of S with respect to $\sim_\sigma$. Now let the first partition of N be $H_0(\sigma) = \{N\}$, and define inductively the following partitions. If $H_k(\sigma) = \{S_{k,1}, \ldots, S_{k,l}\}$ is the kth partition, where $k \geq 0$, then we define

$$H_{k+1}(\sigma) = \bigcup_{j=1}^{l} D(S_{k,j}, \sigma).$$

Clearly, there exists a minimal r such that $H_r(\sigma) = H_k(\sigma)$ for all $k \geq r$. Let $H_r = \{S_1, \ldots, S_l\}$. The coalitions $S_1, \ldots, S_l$ are called *final*. For each final coalition S_j, $j = 1, \ldots, l$, there exists $k_j \in S_j$ such that $\nu^i(S_j) = (S_j, k_j)$ for all $i \in S_j$. Further, a final coalition S_j is called *decided* if $q^{k_j} = 0$.

In defining π we distinguish the following cases. Let $\sigma = (\sigma^1, \ldots, \sigma^n)$ and $H_r(\sigma) = \{S_1, \ldots, S_l\}$.

(1) $l = 1$: $\pi(\sigma) = \max(\varphi^{k_1}(N))$.

(2) $l > 1$ and $S_1, \ldots, S_l$ are decided: $\pi(\sigma) = \max(\varphi^{k_1}(S_1) \cap \cdots \cap \varphi^{k_l}(S_l))$.

(3) $S_1, \ldots, S_h$ are undecided and $S_{h+1}, \ldots, S_l$ are decided, where $1 \leq h \leq l$: We choose $1 \leq j \leq h$ by the following rule: $j \equiv \sum_{u=1}^{h} t^{k_u}$ (h). Then $\pi(\sigma) = \max(\cap_{u \neq j} \varphi^{k_u}(S_u) \cap \varphi_*^{k_j}(S_j))$. This completes the definition of π.

We claim that π is upper semicontinuous. Indeed, if

$$\sigma_m^i = (\nu_m^i, \varphi_m^i, \varphi_{*m}^i, t_m^i, q_m^i) \to \sigma^i,$$

$i \in N$, then ν_m^i, t_m^i, q_m^i, $i = 1, \ldots, n$, are constant for $m \geq m_0$. So our claim follows from (1) – (3) (see (3.5)).

We shal now prove that Γ is a representation of E. Let $S \subseteq N$, $S \neq \emptyset, N$, let $B \in E(S)$, and let $u \in S$. Let σ^S satisfy $\nu^i(S') = (S, u)$ and $\varphi^i(S') = B$ for all $i \in S$ and $S \subseteq S'$, and, in addition, $q^u = 0$. Then $\pi(\sigma^S, \mu^{N \backslash S}) \in B$ for all $\mu^{N \backslash S} \in \Sigma^{N \backslash S}$. (For every $\mu^{N \backslash S} \in \Sigma^{N \backslash S}$, S is a decided coalition of $(\sigma^S, \mu^{N \backslash S})$; further $\varphi^u(S) = B$.)

Let $D \in \mathcal{K}(A) \backslash E(S)$. There exists $B \in E^*(N \backslash S)$ such that $B \cap D = \emptyset$ (E is reflexive). Let $u \in N \backslash S$, let $\mu^{N \backslash S} \in \Sigma^{N \backslash S}$ satisfy $\nu^i(T) = (N \backslash S, u)$ and $\varphi_*^i(T) = B$ for all $i \in N \backslash S$ and T with $N \backslash S \subseteq T$. Further, let $q^u = 1$. If σ^S is any strategy for S, then $N \backslash S$ is a final undecided coalition for $(\sigma^S, \mu^{N \backslash S})$ and $\phi_*^u(N \backslash S) = B$. By adjusting t^u, $N \backslash S$ can arrange that $\pi(\sigma^S, \mu^{N \backslash S}) \in B$. □

We see from the proof of Theorem 6.1 that if set intersection were continuous (in $(\mathcal{K}(A), d_H)$), then we could prove existence of continuous representations in $\mathcal{C}$. By the reduction theorem all (monotonic and superadditive) EFs with closed values (on arbitrary sets of alternatives) would have continuous representations. Thus, indeed, the discontinuity of the intersection is the sole reason for nonexistence of continuous representations.

7. Representations of EFs and modified Baire functions

The results about upper or lower semicontinuity of representations show that although continuity of outcome functions cannot be satisfied in general, it is still possible to obtain representations with properties which may well be useful in many contexts. In the present section, we consider another property of this type, since we show that the outcome function may be chosen as a modified Baire function of class 2. We start by introducing the family of such functions.

Let M be a metric space. A set $C \subseteq M$ is called a G_δ (F_σ) if it can be written as a countable intersection of open sets (a countable union of closed sets). It is a $G_{\delta\sigma}$ if has a representation as a countable union of sets, each of which is a G_δ. Now, a function $f : M \to N$ between metric spaces M and N is a (modified) Baire function of class 0 if it is continuous, that is if $f^{-1}(G)$ is open for each open set G. It is a (modified) Baire function of class 1 if $f^{-1}(G)$ is a F_σ for each open set G, and, finally, it is a (modified) Baire function of class 2 if $f^{-1}(G)$ is a $G_{\delta\sigma}$ for each open set G. For a further discussion of the Baire classes, the reader is referred to Hausdorff [1962].

Let M be a compact metric space and let $E : P(N) \to P(\mathcal{K}(M))$ be a monotonic and superadditive EF with closed values (in $(\mathcal{K}(M), d_H)$). By Lemma 5.1 E can be "lifted" to a monotonic and superadditive EF $\tilde{E} : P(N) \to P(\mathcal{K}(\mathcal{C}))$ (see (5.1)). As we shall prove below, $\tilde{E}$ may be chosen so that it has closed values (in $(\mathcal{K}(\mathcal{C}), d_H)$). Therefore, by Theorem 6.1, $\tilde{E}$ has an upper semicontinuous representation $\tilde{\Gamma} = (\Sigma^1, \dots, \Sigma^n; \pi; \mathcal{C})$. Applying now the reduction theorem we obtain that $\Gamma = (\Sigma^1, \dots, \Sigma^n; f \circ \pi; M)$ is a representation of E (where $f : \mathcal{C} \to M$ is a continuous surjection). Now, if $U \subseteq M$ is an open set, then $(f \circ \pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$ is, as we shall prove, in $G_{\delta\sigma}$ of $\Sigma = \Sigma^1 \times \dots \times \Sigma^n$. Hence, $f \circ \pi$ is a modified Baire function of class 2 (see Appendix D in Hausdorff (1962)).

We can now summarize the foregoing discussion in the following theorem.

Theorem 7.1. *Let M be a compact metric space and let $E : P(N) \to P(\mathcal{K}(M))$ be a monotonic and superadditive EF with closed values (in $(\mathcal{K}(M), d_H)$). Then E has a representation $\Gamma = (\Sigma^1, \dots, \Sigma^n; \hat{\pi}; M)$ such that for every open set $U \subseteq M$, $\hat{\pi}^{-1}(U) \in G_{\delta\sigma}$, that is $\hat{\pi}$ is a modified Baire function of class 2.*

The proof of Theorem 7.1. follows from the lemmas below.

Let M and E be as in Theorem 7.1 and let $f : \mathcal{C} \to M$ be a continuous surjection. Define an EF $\tilde{E} : P(N) \to P(\mathcal{K}(\mathcal{C}))$ in two steps. For $S \subseteq N$, $S \neq \emptyset, N$,

(i) $\hat{E}(S) = \text{cl}\{\hat{B} \in \mathcal{K}(\mathcal{C}) \mid \hat{B} = f^{-1}(B) \text{ for some } B \in E(S)\}$;

(ii) $\tilde{E}(S) = \{\tilde{B} \in \mathcal{K}(\mathcal{C}) \mid \tilde{B} \supseteq \hat{B} \text{ for some } \hat{B} \in \hat{E}(S)\}$.

As usual, $\tilde{E}(N) = \mathcal{K}(\mathcal{C})$ and $\tilde{E}(\emptyset) = \emptyset$.

Lemma 7.2. *$\tilde{E}$ is a monotonic and superadditive EF with closed values.*

Proof: Clearly, we have only to prove that $\tilde{E}$ is superadditive. Let $S_1, S_2 \in P_0(N)$, $S_1 \cap S_2 = \emptyset$, $\hat{B}_1 \in \hat{E}(S_1)$, and $\hat{B}_2 \in \hat{E}(S_2)$. Then there exist $B_{1k} \in E(S_1)$,

$B_{2k} \in E(S_2)$, $k = 1, 2, \ldots$, such that $f^{-1}(B_{1k}) \to \hat{B}_1$ and $f^{-1}(B_{2k}) \to \hat{B}_2$. Clearly, $C_k = B_{1k} \cap B_{2k} \in E(S_1 \cup S_2)$, $k = 1, 2, \ldots$. We may assume (by considering a subsequence if necessary) that $f^{-1}(C_k) \to \hat{C}$. By (i) $\hat{C} \in \hat{E}(S_1 \cup S_2)$. Also,

$$f^{-1}(C_k) = f^{-1}(B_{1k}) \cap f^{-1}(B_{2k})$$

for each k. Hence $\hat{C} \subseteq \hat{B}_1 \cap \hat{B}_2$ and $\hat{B}_1 \cap \hat{B}_2 \in \tilde{E}(S_1 \cup S_2)$. □

We now remark that

$$\tilde{B} \in \tilde{E}(S) \Longrightarrow f(\tilde{B}) \in E(S), \text{ for all } S \in P_0(N) \text{ and } \tilde{B} \in \tilde{E}(S). \tag{7.1}$$

Indeed, if $\hat{B} \in \hat{E}(S)$, then there is a sequence $(B_k)_{k \in \mathbb{N}}$ such that $f^{-1}(B_k) \to \hat{B}$, and by continuity of f, we have that $B_k = f(f^{-1}(B_k))$ converges to $f(\hat{B})$.

Lemma 7.2 and (7.1) imply the following result.

Lemma 7.3. *If $\tilde{\Gamma} = (\Sigma^1, \ldots, \Sigma^n; \pi; \mathcal{C})$ is a representation of $\tilde{E}$, then $\Gamma = (\Sigma^1, \ldots, \Sigma^n; f \circ \pi; M)$ is a representation of E.*

Lemma 7.3 is proved in the same way as the reduction theorem.

The last result is a standard exercise on semicontinuous (real) functions. The proof is included for completeness.

Lemma 7.4. *Let $\bar{M}$ be a metric space and let $\pi : \bar{M} \to \mathcal{C}$ be an upper semicontinuous function. Then for every open set $U \subseteq \mathcal{C}$, $\pi^{-1}(U) \in G_{\delta\sigma}$.*

Proof: First we compute the inverse image of a "ray" in $\mathcal{C}$. There are four possibilities.

(1) For every $a \in \mathcal{C}$, $\pi^{-1}(\{x \in \mathcal{C} \mid x \geq a\})$ is closed.

(2) If $D = \{x \in \mathcal{C} \mid x > a\}$, where $a \in \mathcal{C}$, then, by (1), we may assume that there are $a_s \in \mathcal{C}$, $a_s \downarrow a$. Hence

$$\pi^{-1}(D) = \pi^{-1}\left(\bigcup_s \{x \in \mathcal{C} \mid x \geq a_s\}\right) = \bigcup_s \pi^{-1}(\{x \in \mathcal{C} \mid x \geq a_s\}) \in F_\sigma.$$

(3) For every $b \in \mathcal{C}$, $\pi^{-1}(\{x \in \mathcal{C} \mid x < b\})$ is open.

(4) If $b \in \mathcal{C}$ and $D = \{x \in \mathcal{C} \mid x \leq b\}$, then we may assume $b < 1$ and the existence of a sequence $b_s \in \mathcal{C}$, $b_s \downarrow b$. Hence

$$\pi^{-1}(D) = \pi^{-1}\left(\bigcap_s \{x \in \mathcal{C} \mid x < b_s\}\right) = \bigcap_s \pi^{-1}(\{x \in \mathcal{C} \mid x < b_s\}) \in G_\delta.$$

By (1) – (4) the inverse image (by π) of every interval (closed, open, or half-closed) in $\mathcal{C}$ is the intersection of a set in F_σ with a set in G_δ, and hence it is in $G_{\delta\sigma}$. Finally, every open set in $\mathcal{C}$ is the union of countably many intervals. Hence, if $U \subseteq \mathcal{C}$ is open, then $\pi^{-1}(U) \in G_{\delta\sigma}$. □

Theorem 7.1 now follows from Lemma 7.2, Theorem 6.1, and Lemmas 7.3 and 7.4.

8. Retractions of the Cantor set

In the present section, we take a closer look at some properties of the Cantor set, which, as we have seen, play a central role in the theory of continuous representations of effectivity functions. As it was mentioned in Section 5, the Cantor set admits continuous selections of closed subsets, such as for example the function which to each closed set $B \subseteq \mathcal{C}$ assigns the maximal (for the usual ordering of numbers on $[0,1]$) element of B. This property may come in useful in the search for continuous representations of effectivity functions, since it allows us to construct continuous selections of multivalued functions. The construction is given below; as a by-product, we obtain that every closed subset of $\mathcal{C}$ is a rectract of $\mathcal{C}$; this is not new (cf., e.g., Willard [1970], p.197), but the particular retraction constructed has useful properties in connection with representation of effectivity functions and is derived by elementary methods.

Let $B \in \mathcal{K}(\mathcal{C})$. We shall now prove that $\arg\min_{t\in B}|t-y|$ has a continuous selection $\arg^*\min_{t\in B}|t-y| = r(B,y)$ for $y \in \mathcal{C}$. Notice that $\arg\min_{t\in B}|t-y|$ may be two-valued for some $y \in \mathcal{C}$. Also notice that $r(B,\cdot)$ is a retraction of $\mathcal{C}$ on B. The analysis of retractions in this section will enable us to prove an additional result on existence of continuous representations of EF's in Section 9.

Let $B \in \mathcal{K}(\mathcal{C})$ and let $\arg\min_{t\in B}|t-y| = \{x,z\}$. Then x, y, and z are (distinct) points of $\mathcal{C}$, $y \notin \{1,0\}$, and $y = \frac{x+z}{2}$. We shall prove that the point y is an endpoint of an open interval which is deleted from $I=[0,1]$ in the construction of $\mathcal{C}$. Indeed, let $x = \sum_{t=1}^{\infty}\frac{x(t)}{3^t}$, $y = \sum_{t=1}^{\infty}\frac{y(t)}{3^t}$, and $z(t) = \sum_{t=1}^{\infty}\frac{z(t)}{3^t}$, where $x(t)$, $y(t)$, and $z(t)$ belong to $\{0,2\}$, $t=1,2,\ldots$. As $x \neq z$ there is a $t_0 \geq 1$ such that $z(t_0) \neq x(t_0)$ and $z(t) = x(t)$ for $t < t_0$. Without loss of generality, $x(t_0)=0$ and $z(t_0)=2$. As $y \in \mathcal{C}$ there are only two possibilities:

(1) $x(t)=z(t)=0$ for $t>t_0$. Then y and z are indeed end points and (y,z) is deleted at the t_0th stage. Further, $(x-\varepsilon,x)\cap\mathcal{C}=\emptyset$ for some $\varepsilon>0$.

(2) $x(t)=z(t)=2$ for $t>t_0$. Then $y = x+\frac{1}{3^{t_0}}$ and $z = x+\frac{2}{3^{t_0}}$. Hence, (x,y) is deleted at the t_0th stage. Also, $(z,z+\varepsilon)\cap\mathcal{C}=\emptyset$ for some $\varepsilon>0$.

Now, let $B \subseteq \mathcal{C}$, $B \in \mathcal{K}(\mathcal{C})$, and let $y \in \mathcal{C}$. If $\left|\arg\min_{t\in B}|y-t|\,\right| = 1$, then we define $\arg^*\min_{t\in B}|y-t| = \arg\min_{t\in B}|y-t|$. And if $\arg\min_{t\in B}|y-t| = \{x,z\}$, then, by the previous discussion y is an endpoint of a deleted (open) interval. Then we define

$$\arg^*\min_{t\in B}|y-t| = \begin{cases} x & \text{if } (y,y+\varepsilon)\cap\mathcal{C}=\emptyset \text{ for some } \varepsilon>0,\\ z & \text{if } (y,y-\varepsilon)\cap\mathcal{C}=\emptyset \text{ for some } \varepsilon>0.\end{cases}$$

Lemma 8.1. $\arg^*\min_{t\in B}|y-t|$ *is continuous on* $\mathcal{C}$.

PROOF: We distinguish the following possibilities:

(1) $y \in B$. The proof is obvious.

(2) $y \notin B$ and $\arg\min_{t\in B}|y-t| = \{x\}$. Without loss of generality, $x < y$. Then $[y, 2y - x + \varepsilon] \cap B = \emptyset$ for some $\varepsilon > 0$. Hence if $y(k) \to y$, then $\arg\min_{t\in B}|y(k) - t| = \{x\}$ for k large enough.

(3) $\arg\min_{t\in B}|y-t| = \{x, z\}$ and $x \neq z$. Then $\arg^*\min_{t\in B}|y-t|$ is continuous at y by definition. □

The foregoing analysis will enable us to prove the following theorem.

Theorem 8.2. *Let $B(k) \in \mathcal{K}(\mathcal{C})$, $k = 1, 2, \ldots$, and let*

$$r_k(y) = r(B(k), y) = \arg^*\min_{t\in B(k)}|y-t|.$$

If $B(k) \to B$ and $y(k) \to y$, then $r_k(y(k)) \to r(y)(= r(B, y))$.

PROOF: We distinguish the following possibilities:

(1) $y \in B$. In this case the proof is obvious.

(2) $y \notin B$ and $\arg\min_{t\in B}|y-t| = \{x\}$. W.l.o.g. $x < y$. In this case $[y, 2y - x + \varepsilon] \cap B = \emptyset$ for some $\varepsilon > 0$. Hence $r_k(y(k)) \to x$.

(3) $\arg\min_{t\in B}|y-t| = \{x, z\}$, $x \neq z$. W.l.o.g. $(y, y + \varepsilon) \cap \mathcal{C} = \emptyset$ for some $\varepsilon > 0$. Then $r(B, y) = x$ and $(x - \varepsilon, x) \cap \mathcal{C} = \emptyset$ for some $\varepsilon > 0$. If $y(k) \to y$, then $y(k) < y$ for $k \geq k_0$. Hence, $r_k(y(k)) \to x$. □

Lemma 8.1 is included as it might be used in other discussions of the Cantor set. Formally, it is implied by Theorem 8.2.

9. Existence of continuous representations of subsimple EF's

The results derived in the previous section may now be put to use in our analysis of representations of effectivity functions, more specifically to show existence of continuous representations of a certain subclass of effectivity functions where the existence of a selections with suitable continuity properties is important.

Let $N = \{1, \ldots, n\}$ be a set of players and let A be a compact metric space. We recal that an EF $E : P(N) \to P(\mathcal{K}(A))$ is *simple* if there exists a monotonic and proper simple game (N, W) such that

$$E(S) = \begin{cases} \mathcal{K}(A), & S \in W, \\ \{A\}, & S \notin W,\ S \neq \emptyset, \\ \emptyset, & S = \emptyset. \end{cases}$$

A monotonic EF $\hat{E} : P(N) \to P(\mathcal{K}(A))$ is *subsimple* if there exists a simple EF E such that $\hat{E}(S) \subseteq E(S)$ for every $S \subseteq N$. As the reader may check, $\hat{E}$ is subsimple iff it is monotonic and satisfies, in addition,

$$\hat{E}(S) \neq \{A\} \Longrightarrow \hat{E}(N \backslash S) = \{A\} \text{ for all } S \neq \emptyset, N.$$

We shall now prove that every subsimple EF has a continuous representation. Clearly, we may restrict our discussion to $\mathcal{C}$ in the sequel.

Theorem 9.1. *If $E : P(N) \to P(\mathcal{K}(\mathcal{C}))$ is a subsimple EF, then E has a continuous representation.*

PROOF: For $i \in N$ let $N^i = \{S \subseteq N \mid i \in S\}$ and let

$$V^i = \{\nu^i : N^i \to N^i \times N \mid \nu_1(S) \subseteq S \text{ and } \nu_2(S) \in S\},$$

where $\nu = (\nu_1, \nu_2)$. Further, let $M_1^i = \{\varphi \mid \varphi : N^i \to \mathcal{K}(\mathcal{C}) \text{ and } \varphi(S) \in E(S) \text{ for all } S \in N^i\}$. Finally, $M_2^i = \{\varphi \mid \varphi : N^i \to \mathcal{C}\}$.

We define now a GF $\Gamma = (\Sigma^1, \ldots, \Sigma^n; \pi; A)$ in the following way. Let $\Sigma^i = V^i \times M_1^i \times M_2^i \times N \times \{0,1\}$, $i \in N$. Again, V^i, N^i and $\{0,1\}$ are given the discrete topology, and, therefore, Σ^i is a compact metric space for each i (M_1^i and M_2^i are given the (natural) product topology). It remains to define π.

Let $\sigma^i = (\nu^i, \varphi_1^i, \varphi_2^i, t^i, q^i) \in \Sigma^i$ for $i \in N$. As in the proof of Theorem 6.1, $\nu^1, \ldots \nu^n$ induce a sequence of partitions of N. Let $H_r(\sigma) = \{S_1, \ldots, S_l\}$ be the final partition and let $\nu^i(S_j) = (S_j, k_j)$ for all $i \in S_j$ and $j = 1, \ldots, l$. A final coalition S_j is called *decided* if $q^{k_j} = 0$. In defining π we distinguish the following possibilities.

(1) $l = 1$: $\pi(\sigma^N) = \max \varphi_1^{k_1}(N)$.

(2) $l > 1$ and $S_1, \ldots, S_l$ are decided: Let $\pi(\sigma^N) = \max(\varphi_1^{k_1}(S_1) \cap \cdots \cap \varphi_1^{k_l}(S_l))$. Notice that

$$|\{j \mid \varphi_1^{k_j}(S_j) \neq A\}| \leq 1. \tag{9.1}$$

(3) $S_1, \ldots, S_h$ are undecided and $S_{h+1}, \ldots, S_l$ are decided, where $1 \leq h \leq l$: We choose $1 \leq j \leq h$ by the following rule: $j \equiv \sum_{u=1}^{h} t^u(h)$. Define $B = \cap_{u \neq j} \varphi_1^{k_u}(S_u)$. Finally

$$\pi(\sigma^N) = \arg^* \min_{y \in B} |y - \phi_2^{k_j}(S_j)|.$$

Notice again that

$$|\{u \mid \varphi_1^{k_u}(S_u) \neq A\}| \leq 1. \tag{9.2}$$

This completes the definition of π.

Our first claim is that π is continuous. Let $\sigma_m^i = (\nu_m^i, \varphi_{1m}^i, \varphi_{2m}^i, t_m^i, q_m^i) \to \sigma^i$, $i \in N$. Then there exists m_0 such that ν_m^i, t_m^i, q_m^i are constant for $m \geq m_0$. $\pi(\sigma) = \lim_{m \to \infty} \pi(\sigma_m)$ is obvious in case (1), and it follows from (9.1) in case (2). In case (3) the continuity of π follows from (9.2) and the continuity of $\arg^* \min_{y \in B} |y - x|$ in both x and B (see Theorem 8.2).

We shall now prove that Γ is a representation of E. Let $S \subseteq N$, $S \neq \emptyset, N$, and let $B \in E(S)$. Choose $u \in S$ and let $\nu^i(S') = (S, u)$ for all $S' \supseteq S$ and $i \in S$. Further, let $\varphi_1^u(S) = B$ and $q^u = 0$. Then for all $\mu^{N\backslash S} \in \Sigma^{N\backslash S}$, S is a member of $H_r(\sigma^S, \mu^{N\backslash S})$, where $\sigma^S \in \Sigma^S$ is any S-strategy with the above properties. Furthermore, $\pi(\sigma^S, \mu^{N\backslash S}) \in B$ for all $\mu^{N\backslash S} \in \Sigma^{N\backslash S}$.

Consider now $D \in \mathcal{K}(\mathcal{C}) \backslash E(S)$. Let $N\backslash S$ choose $\mu^{N\backslash S} \in \Sigma^{N\backslash S}$ that satisfies the following:

$$\nu^i(T) = (N\backslash S, u)$$

for all $T \supseteq N \backslash S$ and $i \in N \backslash S$, where $u \in N \backslash S$ is fixed and $q^u = 1$. If $\sigma^S \in \Sigma^S$, then $H_r(\sigma^S, \mu^{N \backslash S}) = \{S_1, \ldots, S_h, N \backslash S\}$. Then $B = \varphi_1^{k_1}(S_1) \cap \cdots \cap \varphi_1^{k_h}(S_h) \in E(S)$ by superadditivity. By monotonicity $B \backslash D \neq \emptyset$. $N \backslash S$ can adjust t^u to arrange that $\pi(\sigma^S, \mu^{N \backslash S}) \in B \backslash D$. □

10. Concluding remarks

We offer a systematic study of continuity properties of representations of (topological) effectivity functions that are defined on a compact metric space of alternatives. We present first an effectivity function which admits no continuous representation. Then we proceed to show that, in spite of the example, there exist in many cases representations which possess continuity properties. Effectivity functions which are generated by a finite set of (closed) sets of alternatives have continuous representations. Hence, every effectivity function can be approximated by effectivity functions with continuous representations. When the set of alternatives is a subset of the real line it is posssible to define upper and lower semicontinuity of representations. And, indeed, we prove that in this case every effectivity function (with closed values) has an upper (or lower) semicontinuous representation.

Let $\mathcal{C}$ be the Cantor set and let A be the compact metric space of alternatives. Then there exists a continuous surjection $f : \mathcal{C} \to A$. Using f it is possible to "lift" every effectivity function E on A to an effectivity function $\tilde{E}$ on $\mathcal{C}$. Furthermore, if $\tilde{E}$ has a (continuous) representation, then E has an associated (continuous) representation. Using the foregoing technique we show that existence of semicontinuous representations on $\mathcal{C}$ implies the existence of representations whose outcome function is a (modified) Baire function of order 2 on arbitrary compact spaces.

We also investigate retractions of $\mathcal{C}$ and use our investigations to prove that every effectivity function that is majorized by an effectivity function of a simple game has a continuous representation.

As we have already mentioned in the introduction, Nash consistency and coalition proof Nash consistency have been fully characterized without reference to continuity (of representations). Therefore, this paper is devoted to continuity alone. The possibility of integrating the two theories is left for future study.

References

Abdou, J., and Keiding, H. [1991], Effectivity functions in social choice. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Arrow, K.J. [1951], Social choice and individual values, Wiley, New York.

Gärdenfors, P. [1981], Rights, games, and social choice, Noûs 15, 341 – 356.

Hausdorff, F. [1962], Set theory, Chelsea, New York.

Hildenbrand, W. [1974], Core and equilibria of a large economy, Princeton University Press, Princeton.

Holzman, R. [1986a], On strong representations of games by social choice functions, Journal of Mathematical Economics 15, 39 – 57.

Holzman, R. [1986b], The capacity of a committee, Mathematical Social Sciences 12, 139 – 157.

Ishikawa, S., and Nakamura, K. [1980], Representations of characteristic function games by social choice functions, International Journal of Game Theory 9, 191 – 199.

Keiding, H., and Peleg, B. [2001], Stable voting procedures for committees in economic environments, Journal of Mathematical Economics 36, 117 – 140.

Keiding, H., and Peleg, B. [2002], Representations of effectivity functions in coalition proof Nash equilibrium: A complete characterization, Social Choice and Welfare 19, 241 – 263.

Moulin, H., and Peleg, B. [1982], Cores of effectivity functions and implementation theory, Journal of Mathematical Economics 10, 115 – 145.

Peleg, B. [1978], Representations of simple games by social choice functions, International Journal of Game Theory 7, 81 – 94.

Peleg, B. [1998], Effectivity functions, game forms, games, and rights, Social Choice and Welfare 15, 67 – 80.

Peleg, B., Peters, H., Storcken, T. [2002], Nash consistent representations of constitutions: A reaction to the Gibbard paradox, Mathematical Social Sciences 43, 267 – 287.

Sen, A. [1999], The possibility of social choice, American Economic Review 89, 349 – 378.

Willard, S. [1970], General topology, Addison-Wesley, Reading, Massachusetts.