DISCUSSION PAPERS
Institute of Economics
University of Copenhagen

04-23

Inequality Preserving Rationing

Jens Leth Hougaard
Lars Peter Østerdal

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

Inequality preserving rationing

Jens Leth Hougaard
Institute of Economics
University of Copenhagen

Lars Peter Østerdal
Institute of Economics
University of Copenhagen

November 4, 2004

Abstract

The present paper considers rationing problems interpreted as e.g. bankruptcy problems or taxation problems. We demonstrate that among the continuous and order-preserving rationing methods, the proportional method is the only rationing method that preserves inequality in both gains and losses.

Keywords: Rationing, inequality preservation, taxation, manipulation, proportional method.
JEL classification: C71, D31, H24.

Corresponding author: Lars Peter Østerdal, Institute of Economics, University of Copenhagen, Studiestraede 6, 1455 Copenhagen K, Denmark.
E-mail: Lars.P.Osterdal@econ.ku.dk

Acknowledgements: We would like to thank Juan D. Moreno-Ternero and Bezalel Peleg for helpful comments.

1 Introduction

The present paper considers rationing problems as synonymous with bankruptcy and cost allocation problems, see e.g. Moulin (2002). With respect to solution methods for such rationing problems we shall focus on axioms of inequality preservation well-known from income inequality theory, see e.g. Moyes (1989, 1994). In terms of the rationing model, inequality preservation in gains (losses) means that whenever one claims vector is more equally distributed than another, in the sense of Lorenz-domination, then the corresponding gains (losses) vector is also more equally distributed.

We consider rationing methods that are continuous and order-preserving in the sense that gains and losses are ordered like the ordering of claims, and demonstrate that the proportional method is the only rationing method that preserves inequality in both gains and losses.

As noticed in Young (1987, 1988) the rationing model has a dual interpretation in terms of taxation where the sum of the taxes collected should equal a given revenue constraint. In this case the interpretation of inequality preservation is that when pre-tax incomes become more equally distributed then both post-tax incomes and taxes must become more equally distributed as well. Our result therefore also concerns a characterization of the flat tax by inequality preservation.

Inequality preservation may also be construed in terms of manipulability: If a given rationing method is inequality preserving in both gains and losses, then no lower coalition (ranking players according to claims) may gain by equalizing or spreading their claims respectively in the sense of the Lorenz ordering. Our result therefore relates to earlier results characterizing the proportional method by non-manipulability in Moulin (1987) and Chun (1988) who consider arbitrary reallocations, and to de Frutos (1999) and Ju (2003) who consider manipulation by merging and splitting of claims resulting in a variable number of agents.

2 The model

A *rationing problem* is given by a monetary value $E > 0$ that has to be shared among a fixed number $n \geq 3$ of agents with non-negative *claims* $c = (c_1, \ldots, c_n)$ where $E \leq C$, and $C = c_1 + \ldots + c_n$.

Given a rationing problem (c, E), a *solution* is a vector $x \in \mathbf{R}_+^n$ such that $x_1 + \ldots + x_n = E$ and $0 \leq x_i \leq c_i$ for all i. A *rationing method* is a function φ that assigns to every rationing problem a unique solution $x = \varphi(c, E)$. For example, the *proportional method* is defined as $\varphi_k^P(c, E) = c_k E/C$ for $k = 1, \ldots, n$. We restrict attention to rationing methods meeting the following two (standard) requirements:

> **Continuity.** A rationing method is *continuous* if it is continuous on every subdomain $\{(c, E) | c \in \mathbf{R}_+^n, 0 \leq E \leq C\}$.
>
> **Order-preservation.** A rationing method is *order-preserving* if $c_i \leq c_j$ implies $\varphi_i(c, E) \leq \varphi_j(c, E)$ and $c_i - \varphi_i(c, E) \leq c_j - \varphi_j(c, E)$, for all $i, j \in \{1, ..., n\}$.

Order-preservation was introduced in Aumann and Maschler (1985) and simply means that gains and losses should be ordered like the claims.[1]

Note that order-preservation implies the natural property of "equal treatment of equals" (ETE), i.e. $c_i = c_j \;\Rightarrow\; \varphi_i(c, E) = \varphi_j(c, E)$.

Let c and c' be (weakly) increasingly ordered claims vectors with $C = C'$ (where $C' = c'_1 + ... + c'_n$), then c *Lorenz dominates* c' if

$$c_1 + \ldots + c_k \geq c'_1 + \ldots + c'_k, \quad k = 1, \ldots, n-1,$$

Lorenz domination induces a partial ordering written as $c \succeq_L c'$. In economic terms, $c \succeq_L c'$ can be interpreted as the claims of c' being more

[1] In terms of the tax model order-preservation means that post-tax incomes as well as the taxes themselves should be ordered like the pre-tax incomes.

spread out (more unequally distributed) than the claims of c. Use of the Lorenz ordering in economics dates back to the beginning of the 20'th century, see e.g. Dalton (1920). Marshall and Olkin (1979) provide an elaborate mathematical treatment.

If c and c' are two distributions for which $C = C'$, $c_k = c'_k$ for $k \neq i, j$, $c_i \leq c_j$ and $c'_i \leq c'_j$ then if $c_i < c'_i$ we say that c' is obtained from c by an *equalizing bilateral transfer* and c is obtained from c' by a *spreading bilateral transfer*. For arbitrary c and c' where $C = C'$, it is well-known that c' can be obtained from c by a finite sequence of equalizing and spreading bilateral transfers.

3 Inequality preserving rationing

A rationing method φ is:

> **Inequality Preserving in Gains (IPG)** if, for any E and $c' \succeq_L c$ that $\varphi(c', E) \succeq_L \varphi(c, E)$.
>
> **Inequality Preserving in Losses (IPL)** if, for any E and $c' \succeq_L c$ that $c' - \varphi(c', E) \succeq_L c - \varphi(c, E)$.

The immediate interpretation of these axioms relates to inequality preservation (as indicated by the names). However, due to order-preservation, they may also be construed in terms of manipulation: Suppose some lower coalition of agents (in terms of the size of their claims relative to the other players') equalize their claims leading to a new claims vector that Lorenz dominates the original claims. Now, IPG requires that such a reallocation is not disadvantageous for this coalition. Consequently, if a rationing method satisfies IPG then it cannot be manipulated by any lower coalition spreading their claims (without changing the rank of the agents). Likewise, IPL concerns a spread of claims: If a rationing method satisfies IPL then its solution cannot be manipulated by any lower coalition equalizing their claims. Hence, if a rationing method satisfies both IPG and IPL then no lower coalition can manipulate the solution by spreading or equalizing their claims.[2] Note that the axiom of No-Advantageous-Reallocation (NAR) used in Moulin (1987) and Chun (1988) concerns reallocations within *arbitrary* coalitions. In this sense, IPL and IPG together are weaker than NAR.[3]

Theorem: *A continuous and order-preserving rationing method φ satisfies IPG and IPL if and only if φ is the proportional method.*

Proof: It is straightforward to show that the proportional method satisfies IPG and IPL. We prove sufficiency. We let E and C be fixed, and consider the restriction of φ to the domain $\{c \mid c_1 \leq \ldots \leq c_n, c_1 + \ldots + c_n = C\}$ and demonstrate below (Steps 1-5) that a method φ satisfying IPG and IPL on this domain is the proportional method. The theorem then follows immediately since E, C and the order of claims was arbitrarily chosen. Moreover, when IPG and IPL are invoked we implicitly use order-preservation.

Step 1: First we claim that for any $1 \leq k \leq n-2$, the gains $(\varphi_1(c,E), ..., \varphi_k(c,E))$ depend only on $(c_1, ..., c_k)$.

Indeed, let $1 \leq k \leq n-2$ and let c and c' be two claims vectors with $C = C'$ where $c_i = c_i'$, $i = 1, \ldots, k$. Since $c_i = c_i'$, $i = 1, \ldots, k$, c' can be obtained from c by a finite sequence of equalizing and spreading bilateral transfers between agents $\{k+1, ..., n\}$. It is therefore sufficient to show that an equalizing or spreading bilateral transfer between agents $\{k+1, ..., n\}$ leaves the gains of agents $\{1, ..., k\}$ unchanged. Hence, assume that c' is obtained from c by either such transfer.

[2] Concerning the rationing model, axioms of inequality preservation were originally introduced in Hougaard and Thorlund-Petersen (2001) where it was demonstrated that the Constrained-Equal-Awards method satisfies IPG and the Constrained-Equal-Loss method satisfies IPL whereby the Talmud method satisfies neither.

[3] We do, however, focus on order-preserving methods which is not the case in Moulin (1987) and Chun (1988).

Consider an arbitrary h where $1 \leq h \leq k$, and an equalizing transfer. By IPG we have $\varphi_1(c', E) + ... + \varphi_h(c', E) \geq \varphi_1(c, E) + ... + \varphi_h(c, E)$. By IPL we have $(c'_1 - \varphi_1(c', E)) + ... + (c'_h - \varphi_h(c', E)) \geq (c_1 - \varphi_1(c, E)) + ... + (c_h - \varphi_h(c, E))$. If $h = 1$ we have $\varphi_1(c', E) = \varphi_1(c, E)$, hence assume that $1 < h$. Likewise by IPG and IPL we get that $\varphi_1(c', E) + \ldots + \varphi_{h-1}(c', E) = \varphi_1(c, E) + \ldots + \varphi_{h-1}(c, E)$, implying that $\varphi_h(c', E) = \varphi_h(c, E)$. Hence $(\varphi_1(c, E), ..., \varphi_k(c, E)) = (\varphi_1(c', E), ..., \varphi_k(c', E))$ which proves the claim.

If the transfer is spreading, all inequalities are reversed and the same conclusion is obtained.

Step 2: Likewise, we claim that for any $3 \leq k \leq n$, the gains $(\varphi_k(c, E), ..., \varphi_n(c, E))$ depend only on $(c_k, ..., c_n)$.

Let c and c' be two claims vectors with $C = C'$ where $c_i = c'_i$, $i = k, \ldots, n$. Since $c_i = c'_i$, $i = k, \ldots, n$, c' can be obtained from c by a finite sequence of equalizing and spreading bilateral transfers between agents $\{1, ..., k-1\}$. It is therefore sufficient to show that an equalizing or spreading bilateral transfer between agents $\{1, ..., k-1\}$ leaves the gains of agents $\{k, ..., n\}$ unchanged. Hence, assume in the following that c' is obtained from c by either equalizing or a spreading bilateral transfer.

Consider an arbitrary h where $k \leq h \leq n$, and an equalizing transfer. By IPG, $\varphi_1(c', E) + \ldots + \varphi_h(c', E) \geq \varphi_1(c, E) + \ldots + \varphi_h(c, E)$. By IPL, $(c'_1 - \varphi_1(c', E)) + \ldots + (c'_h - \varphi_h(c', E)) \geq (c_1 - \varphi_1(c, E)) + \ldots + (c_h - \varphi_h(c, E))$. Since $c'_1 + \ldots + c'_h = c_1 + \ldots + c_h$ we get $\varphi_1(c', E) + \ldots + \varphi_h(c', E) = \varphi_1(c, E) + \ldots + \varphi_h(c, E)$. Likewise by IPG and IPL we get that $\varphi_1(c', E) + \ldots + \varphi_{h-1}(c', E) = \varphi_1(c, E) + \ldots + \varphi_{h-1}(c, E)$, implying that $\varphi_h(c', E) = \varphi_h(c, E)$. Hence $(\varphi_k(c, E), ..., \varphi_n(c, E)) = (\varphi_k(c', E), ..., \varphi_n(c', E))$ which proves the claim.

If the transfer is spreading, all inequalities are reversed and the same conclusion is obtained.

Step 3: We now claim that for $1 \leq k \leq n - 2$, if for $h < k$, $\varphi_h = \varphi_h^P$, then we have $\varphi_k = \varphi_k^P$.

By contradiction: Assume that for some claims vector c there is k such that $c_k > 0$, $\varphi_h = \varphi_h^P$ for all $h < k$ and $\varphi_k(c, E) \neq \varphi_k^P(c, E)$.

Since $\varphi_h = \varphi_h^P$ for all $h < k$, then since C is fixed, let

$$\hat{c} = (c_1, \ldots, c_{k-1}, \frac{c_k + \ldots + c_n}{n-k+1}, \ldots, \frac{c_k + \ldots + c_n}{n-k+1}).$$

Consequently, $\varphi(\hat{c}, E) = \varphi^P(\hat{c}, E)$ by ETE.

Now, let $(c_1, \ldots c_{k-1})$ be fixed. By Step 1, φ_k depends only on c_k (written $\varphi_k(c_k)$) and define $e_k(c_k) \equiv \varphi_k^P(c, E) - \varphi_k(c, E)$, i.e. the excess of player k relative to proportional allocation.

It was shown above that $e_k(\frac{c_k+\ldots+c_n}{n-k+1}) = 0$ and by ETE $\varphi_k(c_{k-1}) = \varphi_{k-1}(c_{k-1}) = \varphi_{k-1}^P(c_{k-1})$, i.e. $e_k(c_{k-1}) = 0$ (with the convention that $c_{k-1} = 0$ if $k = 1$). By continuity of φ, $e_k(c_k)$ is continuous in c_k, hence there is $c_{k-1} \leq c_k' \leq c_k'' \leq \frac{c_k+\ldots+c_n}{n-k+1}$ such that $e_k(c_k') = e_k(c_k'') = 0$ and either $e_k(c_k) > 0$ for all $c_k' < c_k < c_k''$ or $e_k(c_k) < 0$ for all $c_k' < c_k < c_k''$. In the following we restrict attention to the case $e_k(c_k) > 0$ for all $c_k' < c_k < c_k''$; the other case can be dealt with in a similar manner.

We claim that $c_k'' \neq \frac{c_k+\ldots+c_n}{n-k+1}$. Indeed, assume to the contrary that $c_k'' = \frac{c_k+\ldots+c_n}{n-k+1}$, and consider the distribution

$$\widetilde{c} = (c_1, .., c_{k-1}, c_k'' - \frac{1}{n-k}[c_k'' - c_k'], \ldots, c_k'' - \frac{1}{n-k}[c_k'' - c_k'], c_k'' + [c_k'' - c_k']).$$

By Step 2, φ_n depends only on E, C and c_n and define $e_n(c_n) \equiv \varphi_n^P(c, E) - \varphi_n(c, E)$. By ETE we have $\varphi_k(\widetilde{c}, E) = \ldots = \varphi_{n-1}(\widetilde{c}, E) < \varphi_k^P(\widetilde{c}, E)$, hence $e_n(c_n) < 0$ for all c_n where $c_k'' < c_n < c_k'' + (n-k)(c_k'' - c_k')$.

Now, consider the distribution

$$\widetilde{c}' = (c_1, .., c_{k-1}, c_k', c_k'', \ldots, c_k'', c_k'' + [c_k'' - c_k']).$$

Since $e(c_k') = 0$ and $e_n(c_k'' + [c_k'' - c_k']) < 0$, and $\varphi_h = \varphi_h^P$ for all $h < k$ we have $\varphi_h(\widetilde{c}', E) < \varphi_h^P(\widetilde{c}', E)$ for all $k < h < n$. However, for the distribution

$$\widetilde{c}'' = (c_1, .., c_{k-1}, c_k', c_k'' + \frac{[c_k'' - c_k']}{n-k}, \ldots, c_k'' + \frac{[c_k'' - c_k']}{n-k}),$$

obtained from *widetildec*$'$ by equalizing the claims among agents $\{k+1, ..., n\}$ we have $\varphi_h = \varphi_h^P$ for all $h \leq k$. By ETE and the fact that $e_n(c_n) < 0$ for all $c''_k < c_n < C - c_1 - ... - c_{k-1} - (n-k)c'_k$ we have $\varphi_h > \varphi_h^P$ for all $h > k$ – a contradiction.

Therefore, let $c''_k < \frac{c_k + ... + c_n}{n-k+1}$. In the following let c''_k be the highest c_k for which there exists a pair (c'_k, c_k) satisfying $e_k(c'_k) = 0,\ e_k(c_k) = 0,\ e_k(\hat{c}_k) > 0$, for all $c'_k < \hat{c}_k < c_k$. Hence, if $c_n < C - c_1 - ... - c_{k-1} - (n-k)c''_k$ we have $e_n(c_n) = 0$. Now consider the distribution

$$\widetilde{c} = (c_1, .., c_{k-1}, c''_k - \varepsilon, ..., c''_k - \varepsilon, C - c_1 - ... - c_{k-1} - (n-k)(c''_k - \varepsilon)).$$

Then we can select ε sufficiently small such that $c''_k > c''_k - \varepsilon > c'_k$ and for the distribution

$$\begin{aligned} \widetilde{c}' \;&=\; (c_1, .., c_{k-1}, c''_k - \varepsilon, \frac{(n-k-1)(c''_k - \varepsilon) + C - c_1 - .. - c_{k-1} - (n-k)(c''_k - \varepsilon)}{n-k}, \\ &\quad ..., \frac{(n-k-1)(c''_k - \varepsilon) + C - c_1 - ... - c_{k-1} - (n-k)(c''_k - \varepsilon)}{n-k}), \end{aligned}$$

obtained from $\widetilde{c}$ by equalizing the claims of agents $\{k+1, ..., n\}$ we have

$$\begin{aligned} \frac{c_k + ... + c_n}{n-k+1} \;&<\; \frac{(n-k-1)(c''_k - \varepsilon) + C - c_1 - ... - c_{k-1} - (n-k)(c''_k - \varepsilon)}{n-k} \\ &<\; C - c_1 - ... - c_{k-1} - (n-k)c''_k. \end{aligned}$$

By ETE we have $\varphi_k(\widetilde{c}, E) = ... = \varphi_{n-1}(\widetilde{c}, E) < \varphi_k^P(\widetilde{c}, E)$, and $\varphi_n(\widetilde{c}, E) > \varphi_n^P(\widetilde{c}, E)$. However, for the distribution $\widetilde{c}'$ since

$$\frac{c_k + ... + c_n}{n-k+1} < \widetilde{c}'_n < C - c_1 - ... - c_{k-1} - (n-k)c''_k,$$

we have $e_n(\widetilde{c}'_n) = 0$ hence by ETE $\varphi_h(\widetilde{c}', E) = \varphi_h^P(\widetilde{c}', E)$ for all $h > k$. But then $\varphi_k(\widetilde{c}', E) \neq \varphi_k^P(\widetilde{c}', E)$, i.e. agent k is the only agent for which the gain is not equal to proportional allocation – a contradiction. We therefore conclude that $\varphi_k(c, E) = \varphi_k^P(c, E)$ for all c and E, $1 \leq k \leq n-2$.

Step 4: We now claim that if $\varphi_k = \varphi_k^P$ for all $1 \leq k \leq n-2$, then $\varphi_n = \varphi_n^P$.

By Step 2, φ_n depends only on E, C and c_n. If $\varphi_n \neq \varphi_n^P$ then there is $c = (c_1, ..., c_n)$ and E such that $\varphi_n(c, E) \neq \varphi_n^P(c, E)$. Now, define $c_- = \frac{c_1+...+c_{n-1}}{n-1}$, and consider the distribution $c' = (c_-, ..., c_-, c_n)$. By ETE $\varphi_1(c', E) = ... = \varphi_{n-1}(c', E)$, and because $\varphi_n(c', E) = \varphi_n(c, E) \neq \varphi_n^P(c, E)$, we have $\varphi_1(c', E) \neq \varphi_1^P(c', E)$ contradicting Step 3.

Step 5: Finally, by Step 3, for $1 \leq h \leq n-2$ we have $\varphi_h(c, E) = \varphi_h^P(c, E)$ for all c and E, and by Step 4, $\varphi_n(c, E) = \varphi_n^P(c, E)$ for all c and E. Hence $\varphi_{n-1} = \varphi_{n-1}^P(c, E)$ for all c and E. We therefore have $\varphi = \varphi^P$ which concludes the proof. □

Remark: The characterizations based on inequality preservation found in Moyes (1989, 1994) restrict attention to tax methods where the post-tax income of agent i is independent of the other agents pre-tax incomes. For example, in this context, Moyes (1989) shows that an axiom similar to IPG uniquely characterizes linear tax methods (Theorem 2.1).

4 References

Aumann, R. and M. Maschler (1985). Game theoretic analysis of a bankruptcy problem from the Talmud, *Journal of Economic Theory* 36, 195-213.

Chun, Y. (1988). The proportional solution for rights problems, *Mathematical Social Sciences* 15, 231-246.

Dalton, H. (1920). The measurement of the inequality of incomes, *The Economic Journal* 30, 348-361.

de Frutos, M.A. (1999). Coalitional manipulations in a bankruptcy problem, *Review of Economic Design* 4, 255-272.

Hougaard, J.L. and L. Thorlund-Petersen (2001). Bankruptcy rules, inequality, and uncertainty, *Working paper* 4/01, Department of Operations Management, Copenhagen Business School.

Ju, B-G. (2003). Manipulation via merging and splitting in claims problems, *Review of Economic Design* 8, 205-215.

Marshall, A.W. and I. Olkin (1979). Inequalities: Theory of Majorization and Its Application. Academic Press.

Moulin, H. (1987). Equal or proportional division of a surplus and other methods, *International Journal of Game Theory* 16, 161-186.

Moulin, H. (2002). Axiomatic cost and surplus-sharing. Ch. 6 in Handbook of Social Choice and Welfare, Vol 1, 289-357.

Moyes, P. (1989). Some classes of functions that preserve the inequality and welfare orderings of income distributions, *Journal of Economic Theory* 49, 347-359.

Moyes, P. (1994). Inequality reducing and inequality preserving transformations of incomes: Symmetric and individualistic transformations, *Journal of Economic Theory* 63, 271-298.

Young, H.P. (1987). On dividing an amount according to individual claims and liabilities, *Mathematics of Operations Research* 12, 398-414.

Young, H.P. (1988). Distributive justice in taxation, *Journal of Economic Theory* 44, 321-335.