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THE IDENTIFICATION OF PREFERENCES FROM MARKET DATA UNDER UNCERTAINTY[1]

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Abstract

We show that even under incomplete markets, the equilibrium manifold identifies aggregate demand and individual demands everywhere in their domains. Moreover, under partial observation of the equilibrium manifold, we we construct maximal domains of identification. For this, we assume conditions of smoothness, interiority and regularity, but avoid implausible observational requirements. It is crucial that there be date-zero consumption. As a by-product, we develop some duality theory under incomplete markets.

Key words: Identification, General equilibrium, Incomplete Markets.

JEL classification: D50, D52, D11

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IDENTIFICACIÓN DE PREFERENCIAS CON BASE EN DATOS OBSERVADOS Y BAJO INCERTIDUMBRE

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Resumen

Mostramos como, aún en mercados incompletes, la variedad de equilibrio identifica la demanda agregada y las demandas individuales en todas partes de su dominio. Más aún, bajo observación parcial de la variedad de equilibrio, construimos dominios máximos de identificación. Utilizamos condiciones de suavidad y regularidad pero evitamos hacer hipótesis poco plausibles sobre los observables de la economía. Es crucial la existencia de consumo en el primer período. Como un subproducto desarrollamos la teoría dual básica de la teoría del consumidor en mercados incompletos.

Palabras clave: Identificación, Equilibrio General y Mercados Incompletos.

Clasificación JEL: D50, D52, D11

When there exist uninsurable risks, competitive equilibrium is typically inefficient in a strong sense: a planner could use the existing insurance possibilities to make every individual better off (Geanakoplos and Polemarchakis [10]). The question immediately arises of how much information a planner needs to have in order to figure out an improving policy intervention. The question is not trivial: the transfer paradox, first pointed out by Leontief, and generalized by Donsimoni and Polemarchakis [8], illustrates how ambiguous the welfare effects of a policy can be when the fundamentals of the economy are unknown.

Under the hypothesis of general equilibrium, the aggregate demand function cannot be assumed to be observed: at equilibrium prices aggregate demand is, by definition, equal to aggregate endowment. Demand, either individual or aggregate, cannot be observed for out-of-equilibrium prices. One can observe, however, equilibrium prices and individual incomes. In this paper we address the problem of identifying individual preferences from the equilibrium manifold of a dynamic economy with financial markets.

For the standard Arrow-Debreu model, positive results have been obtained by Balasko [1], Chiappori et al. [6] and [7], Matzkin [17] and Carvajal and Riascos [5]. Balasko's result has been criticized for making very strong observational assumptions: that one can observe equilibrium prices in situations in which endowment is zero for all individuals but one. Under regularity assumptions, Chiappori et al obtain local identification of individual demands, but their argument has been criticized by Balasko, who has pointed out that it requires extreme smoothness assumptions. Matzkin determines the largest class of fundamentals for which identification is possible, by excluding translations of the income expansion paths of individual demands. Carvajal and Riascos obtain local (maximal) and global identification of individual demands, by combining the methods of Balasko and Chiappori et al. in a way that avoids their weaknesses: it does not use boundary information, nor does it require analyticity of preferences.

The case of uncertainty is more cumbersome. Kubler et al. [13] extend the results of Chiappori et al.[6]: they use the implicit function theorem to identify individual demands (locally) from the equilibrium correspondence, and then use Geanakoplos and Polemarchakis [11] to identify preferences from individual demand functions.

This paper extends the results of Carvajal and Riascos [5] -and hence of Balasko and Chiappori et al.- to the case of uncertainty. We assume an economy with numeraire assets and show that we can identify individual demands locally (moreover, we find maximal domains in which local identification holds). As a corollary, it follows that identification holds globally, if there is global equilibrium information. For general real assets structures, we conjecture that our results hold generically in the space of prices and endowments. We extend Balasko's idea on how to recover the aggregate demand function from the equilibrium manifold to the case of (possibly incomplete) asset markets, hence we avoid using the implicit function theorem. We then use a slightly different argument from Kubler et al.'s to identify individual demands from the aggregate demand function and we also avoid using Balasko's strong observational assumption. In contrast to Kubler et al., our results do not assume that preferences are additively separable across states. If that assumption is made, however, our result suffices to imply, by Geanakoplos and Polemarchakis [11], that the equilibrium manifold locally identifies individual preferences. As a necessary by-product, we develop some basic duality theory for incomplete markets.

1 No-arbitrage equilibrium

Consider the canonical, two-period, incomplete markets model with financial assets. There are $S+1$ states of nature, $s = 0, ..., S$,[1] I individuals, $i = 1, ..., I$, and $L \geqslant 2$ commodities available in each state, $l = 1, ..., L$. Denote $L(S+1)$ by N and define the commodity space as $\mathbb{R}^N_+$.

A financial asset is a contract $v \in \mathbb{R}^S$ that promises delivery of an amount $v_s \in \mathbb{R}$ of commodity $l = 1$ at state of nature $s = 1, ..., S$. Let $p = (p_s)_{s=0}^S \in \mathbb{R}^N_{++}$ denote the vector of commodity prices, where $p_s = (p_{s,l})_{l=1}^L \in \mathbb{R}^L_{++}$ and $p_{s,l}$ denotes the price in state s of one unit of good l. Date-1 prices are denoted by $\mathbf{p_1} = (p_s)_{s=1}^S$.

Let $v^1, ..., v^J$ be $J \geq 1$ financial assets, define $V(\mathbf{p_1})$ as the matrix of income transfers:

$$V(\mathbf{p_1}) = \begin{bmatrix} p_{1,1} & 0 & \cdots & 0 \\ 0 & p_{2,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{S,1} \end{bmatrix} V = \begin{bmatrix} V_1(p_{1,1}) \\ V_2(p_{2,1}) \\ \vdots \\ V_S(p_{S,1}) \end{bmatrix},$$

where

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_S \end{bmatrix} = \begin{bmatrix} v_1^1 & \cdots & v_1^J \\ \vdots & \ddots & \vdots \\ v_S^1 & \cdots & v_S^J \end{bmatrix}.$$

The space of income transfers is the column span of the matrix of income transfers: $\langle V(\mathbf{p_1})\rangle = \left\{ t \in \mathbb{R}^S \middle| \left(\exists z \in \mathbb{R}^J\right) : V(\mathbf{p_1}) z = t \right\}$. In general, as $\mathbf{p_1}$ changes, $\langle V(\mathbf{p_1})\rangle$ changes. By construction, however, for $\mathbf{p_1} \in \mathbb{R}^{LS}_{++}$, the dimension of $\langle V(\mathbf{p_1})\rangle$ is always equal to the rank of V.

[1] State $s = 0$ is used to denote date zero.

Assume the following:

Condition 1 *V has full column rank.*

Take p to denote date-zero present-value prices (see Magill and Shafer, [15, p. 1534]), and let $w = (w_s)_{s=0}^{S} \in \mathbb{R}_{++}^{N}$ represent an endowment of commodities.

Given p and w, define the budget[2]

$$B(p,w) = \left\{ x \in \mathbb{R}_{+}^{N} \middle| \sum_{s=0}^{S} p_s \cdot (x_s - w_s) \leq 0 \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle \right\},$$

where $x_{\mathbf{1}} = (x_s)_{s=1}^{S}$ and $w_{\mathbf{1}} = (w_s)_{s=1}^{S}$. Future consumption $x_{\mathbf{1}}$ is financially feasible at future prices and endowments $(p_{\mathbf{1}}, w_{\mathbf{1}})$ if the second condition in the definition of $B(p,w)$ is satisfied: there is a portfolio of assets, $z \in \mathbb{R}^{J}$, that delivers the transfers necessary to finance $x_{\mathbf{1}}$.[3]

Since there is one degree of nominal indeterminacy, we normalize prices to lie in $\left\{ p \in \mathbb{R}_{++}^{N} \middle| p_{0,1} = 1 \right\}$, a set that we denote by $\mathcal{S}_{++}^{N-1}$.

Assume that there are $I \in \mathbb{N}$ individuals. Individual $i \in \mathcal{I} = \{1, ..., I\}$ has preferences $u^i : \mathbb{R}_{+}^{N} \longrightarrow \mathbb{R}$, which are assumed to satisfy the following condition:

Condition 2 *u^i is continuous, monotone and strongly quasi-concave, and satisfies that for all $x \in \mathbb{R}_{++}^{N}$, $\left\{ x' \in \mathbb{R}_{+}^{N} \middle| u^i(x') \geq u^i(x) \right\} \subseteq \mathbb{R}_{++}^{N}$.*[4]

Define the individual demands, $f^i : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{N} \to \mathbb{R}_{++}^{N}$, by $f^i(p,w) = \arg\max_{x \in B(p,w)} u^i(x)$, and the aggregate demand function $F : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{N} \to \mathbb{R}_{++}^{N}$, as $F(p, (w^i)_{i=1}^{I}) = \sum_{i \in \mathcal{I}} f^i(p, w^i)$. Functions $(f^i)_{i \in \mathcal{I}}$ and F are well defined, since for $(p,w) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{N}$, $B(p,w)$ is nonempty, compact and convex, and each u^i is continuous and strongly quasi-concave. Condition 2 guarantees that the range of f^i is contained in $\mathbb{R}_{++}^{N}$.

A **no-arbitrage equilibrium** for the economy $\left((u^i, w^i)_{i \in \mathcal{I}}, V \right)$ is a pair $(x,p) \in \mathbb{R}_{+}^{NI} \times \mathcal{S}_{++}^{N-1}$ such that:

1. For every i, $x^i = f^i(p, w^i)$;

2. $F(p, (w^i)_{i \in \mathcal{I}}) = \sum_{i \in \mathcal{I}} w^i$.

[2] For any $(\rho, \gamma) \in \mathbb{R}^{LS} \times \mathbb{R}^{LS}$, we denote $\rho \boxdot \Delta = \begin{bmatrix} \rho_1 \cdot \Delta_1 & \cdots & \rho_S \cdot \Delta_S \end{bmatrix}^{\top}$.

[3] If $\dim \langle V(P_{\mathbf{1}}) \rangle = S$, or, equivalently, $\dim \langle V \rangle = S$, the second condition that defines $B(P, w; V)$ is nonbinding. This is the case of complete markets.

[4] This condition excludes additively separable preferences of the form $u^i(x) = \sum_{s=0}^{S} u_s^i(x_x)$, for $\left(u_s^i : \mathbb{R}_{+}^{L} \longrightarrow \mathbb{R}\right)_{s=0}^{S}$. In this case, our analysis still holds if we introduce the following assumption: for every sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}_{++}^{L}$, if it converges to some x in $\partial \mathbb{R}_{+}^{L}$, then it is true that $\left\| Du_s^i(x_n) \right\|^{-1} Du_s^i(x_n) \cdot x_n \longrightarrow 0$ and $\left\| Du_s^i(x_n) \right\|^{-1} \longrightarrow \infty$.

Notice that, since V has full column rank, in the previous definition we need not explicitly consider market clearing of portfolios.

Given $\left(u^i\right)_{i\in\mathcal{I}}$ and V, the **no-arbitrage equilibrium manifold** (for short, **equilibrium manifold**) is

$$M = \left\{ (p,w) \in \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++} \,\middle|\, F(p, \left(w^i\right)_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} w^i \right\}.$$

Since F is continuous, it is straightforward that M is closed.

Henceforth, we maintain the assumption that there are an asset structure that satisfies condition 1 and a profile of preferences that satisfies condition 2, and assume that some subset of the equilibrium manifold is observed.[5] We study whether unobserved preferences can be uniquely determined from that subset, but we do not test their existence. Thus, our question is one of identification and not one of testability or refutability, which has been dealt with elsewhere.[6] Since, under our assumptions, equilibrium is known to exist for every profile of preferences and endowments, our observational assumption is not vacuous.

2 From the Equilibrium Manifold to Aggregate Demand

Under our assumptions, F is continuous, satisfies Walras's law (that is, $p \cdot F(p, \left(w^i\right)_{i\in\mathcal{I}}) = p \cdot \left(\sum_{i\in\mathcal{I}} w^i\right)$), $p_{\mathbf{1}} \boxdot \left(F_{\mathbf{1}}(p, \left(w^i\right)_{i\in\mathcal{I}}) - \sum_{i\in\mathcal{I}} w^i_{\mathbf{1}}\right) \in \langle V(p_{\mathbf{1}})\rangle$ and satisfies that

$$\left(p \cdot \widehat{w}^i = p \cdot w^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) \in \langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}} \Longrightarrow F(p,w) = F(p,\widehat{w}).$$

For any $\Phi : \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++} \longrightarrow \mathbb{R}^{N}_{++}$ and $D \subseteq \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++}$, denote by $\Phi_{|D}$ the restriction of Φ to D.

We say that $E \subseteq M$ **identifies** F **over** $D \subseteq \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++}$ if for every continuous function $\Phi : \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++} \longrightarrow \mathbb{R}^{N}_{++}$ that satisfies Walras's law, $p_{\mathbf{1}} \boxdot \left(\Phi_{\mathbf{1}}(p, \left(w^i\right)_{i\in\mathcal{I}}) - \sum_{i\in\mathcal{I}} w^i_{\mathbf{1}}\right) \in \langle V(p_{\mathbf{1}})\rangle$ and is such that

$$\left(p \cdot \widehat{w}^i = p \cdot w^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) \in \langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$$

[5] So far, we have defined the manifold in terms of present-value prices of commodities only. Section 4 shows how to identify subsets of the manifold so defined, from observation of the manifold defined in terms of spot commodity prices and asset prices.

[6] For the standard Arrow-Debreu model, see Brown and Matzkin [3]. For the case of uncertainty, see Kubler [12]. There are several extensions of this literature; for a survey, see Carvajal et al. [4].

and

$$\left\{ (p,w) \in \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++} \middle| \, \Phi(p,w) = \sum_{i \in \mathcal{I}} w^i \right\} \supseteq E,$$

it is true that $\Phi_{|D} = F_{|D}$.

If in the previous definition we drop the requirement that Φ be continuous, then we say that E **strongly identifies** F **over** D. Clearly, strong identification implies identification.

Intuitively, we say that E identifies F over D if, for any function that cannot be ruled out as aggregate demand function, we have that, on the restricted domain D, that function is identical to the true aggregate demand function. A function cannot be ruled out as aggregate demand function when (i) it satisfies the properties that are necessary for it to be an aggregate demand, and (ii) it is consistent with the observed data (because all the observed equilibria are equilibrium according to it).

We say that **the equilibrium manifold identifies (strongly identifies) aggregate demand globally** if M identifies (strongly identifies) F over $\mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++}$. It is straightforward that the equilibrium manifold identifies aggregate demand globally if, and only if, F is the only continuous function $\Phi : \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++} \longrightarrow \mathbb{R}^{N}_{++}$, satisfying Walras's law, $p_{\mathbf{1}} \boxdot \left(\Phi_{\mathbf{1}}(p, \left(w^i\right)_{i \in \mathcal{I}}) - \sum_{i \in \mathcal{I}} w^i_{\mathbf{1}} \right) \in \langle V(p_{\mathbf{1}}) \rangle$ and Φ is such that

$$\left(p \cdot \widehat{w}^i = p \cdot w^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}} \right) \in \langle V(p_{\mathbf{1}}) \rangle \right)_{i \in \mathcal{I}} \Longrightarrow \Phi(p,w) = \Phi(p,\widehat{w})$$

and

$$\left\{ (p,w) \in \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++} \middle| \, \Phi(p,w) = \sum_{i \in \mathcal{I}} w^i \right\} = M.$$

In the case of global identification, every function that satisfies the properties of an aggregate demand can be ruled out, with the only exception of the true aggregate demand.

Some properties of the concept of identification, that straightforwardly apply here, are shown by Carvajal and Riascos [5]. For example, (i) identification over a set implies identification over its subsets; (ii) identification from a subset of the manifold suffices to imply identification from its supersets (over the same, given, set); (iii) identification over a set implies identification over its closure.

For any $E \subseteq M$, define

$$D_E = \left\{ (p, w) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI} \middle| \left(\exists \widehat{w} \in \mathbb{R}_{++}^{NI}\right) : \left(\begin{array}{c} (p, \widehat{w}) \in E \\ \left(p \cdot \widehat{w}^i\right)_{i \in \mathcal{I}} = \left(p \cdot w^i\right)_{i \in \mathcal{I}} \\ \left(p_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) \in \langle V(p_{\mathbf{1}}) \rangle\right)_{i \in \mathcal{I}} \end{array} \right) \right\}.$$

The key result is the following theorem, which generalizes the idea of Balasko to less-than-global observation under uncertainty.

Theorem 1 *$E \subseteq M$ identifies F over any $D \subseteq \overline{D_E}$, strongly over any $D \subseteq D_E$.*

Proof. It suffices to show that E strongly identifies F over D_E.

Let Φ be such that if $p \cdot \widehat{w}^i = p \cdot w^i$ and $p_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) \in \langle V(p_{\mathbf{1}}) \rangle$, for all i, then $\Phi(p, w) = \Phi(p, \widehat{w})$, and $E \subseteq \left\{ (p, w) \middle| \Phi(p, w) = \sum_{i \in \mathcal{I}} w^i \right\}$. Fix $(p, w) \in D_E$. By definition, we can fix $\widehat{w} \in \mathbb{R}_{++}^{NI}$ such that $(p, \widehat{w}) \in E$, and $p \cdot \widehat{w}^i = p \cdot w^i$ and $p_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) \in \langle V(p_{\mathbf{1}}) \rangle$ for all i. Then, by construction, $\Phi(p, w) = \Phi(p, \widehat{w})$ and $F(p, \widehat{w}) = F(p, w)$. Since $E \subseteq \left\{ (\widetilde{p}, \widetilde{w}) \middle| \Phi(\widetilde{p}, \widetilde{w}) = \sum_{i \in \mathcal{I}} \widetilde{w}^i \right\}$, it follows that $\Phi(p, \widehat{w}) = \sum_{i \in \mathcal{I}} \widehat{w}^i$, whereas since $E \subseteq M$, $F(p, \widehat{w}) = \sum_{i \in \mathcal{I}} \widehat{w}^i$. It follows that $\Phi(p, w) = F(p, w)$. ■

Corollary 1 *The equilibrium manifold strongly identifies aggregate demand globally.*

Proof. It suffices to show that $D_M = \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$. Let $\widehat{w}^i = f^i(p, w^i) \in \mathbb{R}_{++}^N$. By monotonicity, $p \cdot \widehat{w}^i = p \cdot w^i$, whereas, by definition of individual demands, $p_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) \in \langle V(p_{\mathbf{1}}) \rangle$. That $\left(p, \left(\widehat{w}^i\right)_{i \in \mathcal{I}}\right) \in M$ is straightforward, since $f^i(p, \widehat{w}^i) = \widehat{w}^i$. ■

Lemma 1 *If $E \subseteq M$ is closed and the closure of E in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{+}^{NI}$ is contained in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$, then D_E is closed.*

Proof. Let $(p_n, w_n)_{n=1}^{\infty}$ be a sequence defined in D_E such that $(p_n, w_n)_{n=1}^{\infty} \longrightarrow (p, w) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$. By definition, there exists $(\widehat{w}_n)_{n=1}^{\infty}$ such that $(p_n, \widehat{w}_n)_{n=1}^{\infty}$ is a sequence in E such that, for all $n \in \mathbb{N}$,

$$\left(\left(p_n \cdot \widehat{w}_n^i\right)_{n=1}^{\infty} = \left(p_n \cdot w_n^i\right)_{n=1}^{\infty} \text{ and } p_{n,\mathbf{1}} \boxdot \left(\widehat{w}_{n,\mathbf{1}}^i - w_{n,\mathbf{1}}^i\right) \in \langle V(p_{n,\mathbf{1}}) \rangle\right)_{i \in \mathcal{I}}$$

For each i, since $\left(p_n w_n^i\right)_{n=1}^{\infty}$ is bounded and $(p_n)_{n=1}^{\infty}$ is bounded away from 0, it follows that $(\widehat{w}_n)_{n=1}^{\infty}$ is bounded: for every $(i, n, s, l) \in \mathcal{I} \times \mathbb{N} \times \{0, ..., S\} \times \{1, ..., L\}$,

$$0 \leq \widehat{w}_{n,s,l}^i \leq \sup_{n \in \mathbb{N}} \left\{p_n w_n^i\right\} / \inf_{n \in \mathbb{N}} \left\{p_{n,s,l}\right\}.$$

It follows that we can take a convergent subsequence $\left(p_{n(k)}, \widehat{w}_{n(k)}\right)_{k=1}^{\infty}$ such that $\left(p_{n(k)}, \widehat{w}_{n(k)}\right) \longrightarrow (p, \widehat{w})$. Since the closure of E in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{+}^{NI}$ is contained in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$, then $(p, \widehat{w}) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$, and, since E is closed (in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$), $(p, \widehat{w}) \in E$. Let $\left(z_n^i\right)_{n=1}^{\infty}$ in $\mathbb{R}^{JI}$ be such that $p_{n,\mathbf{1}} \boxdot \left(\widehat{w}_{n,\mathbf{1}}^i - w_{n,\mathbf{1}}^i\right) = V(p_{n,\mathbf{1}}) z_n^i$ for all $n \in \mathbb{N}$. Since $\left(p_{n(k)}, \widehat{w}_{n(k)}\right) \to (p, \widehat{w})$ and $w_n \longrightarrow w$, then $V(p_{n(k)),\mathbf{1}}) \to V(p_{\mathbf{1}})$ and, therefore, since V has full column rank, $z_{n(k)} \to z$ for some z in $\mathbb{R}^{JI}$. In conclusion, $(p, \widehat{w}) \in E$, $p \cdot \widehat{w}^i = p \cdot w^i$ for all i and $p_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) = V(p_{\mathbf{1}}) z^i$, and, therefore, $(p, w) \in D_E$. ■

As in Carvajal and Riascos [5], if the manifold is known only up to a lower bound on the value of wealth, then aggregate demand is identified subject to the same restriction. The largest domain on which, given $E \subseteq M$, identification is possible, is determined next.[7]

Theorem 2 *1. Suppose $E \subseteq M$ is closed and the closure of E in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{+}^{NI}$ is contained in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$. If E identifies F over D, then $D \subseteq D_E$.*

2. If $E \subseteq M$ strongly identifies F over D, then $D \subseteq D_E$.

Proof. For the first part, by lemma 1 D_E is closed. Now, denote by $\delta : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{+}^{NI} \longrightarrow \mathbb{R}_{+}$ the function distance-to-E, which, since E is closed, is defined by

$$\delta(p, w) = \min_{(\widetilde{p}, \widetilde{w}) \in E} \|(p, w) - (\widetilde{p}, \widetilde{w})\|$$

Closedness also implies that $\delta(p, w) = 0 \Longleftrightarrow (p, w) \in E$.

Let

$$\overline{B}(p, w) = \left\{ x \in \mathbb{R}_{+}^{N} \Big| \sum_{s=0}^{S} p_s \cdot (x_s - w_s) = 0 \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle \right\}$$

and define the function $\Delta : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI} \longrightarrow [0, 1]$ by

$$\Delta(p, w) = \min \left\{ \min_{\widetilde{w} \in \prod_{i \in \mathcal{I}} \overline{B}(p, w^i)} \delta(p, \widetilde{w}), 1 \right\},$$

which is well defined (because δ is continuous and each $\overline{B}\left(p, w^i\right)$ is compact) and is continuous. Moreover, suppose that $\Delta(p, w) = 0$; this implies that $\min_{\widetilde{w} \in \prod_{i \in \mathcal{I}} \overline{B}(p, w^i)} \delta(p, \widetilde{w}) = 0$ and hence, again by compactness of $\prod_{i \in \mathcal{I}} \overline{B}\left(p, w^i\right)$, that for some $\widetilde{w} \in \prod_{i \in \mathcal{I}} \overline{B}\left(p, w^i\right)$, $\delta(p, \widetilde{w}) = 0$; it follows that $(p, \widetilde{w}) \in E$ and hence, since $\widetilde{w} \in \prod_{i \in \mathcal{I}} \overline{B}\left(p, w^i\right)$, it follows that $(p, w) \in D_E$; then, it is

[7] The following theorem is stronger and its proof considerably simplifies the one reported in [5] for complete markets.

immediate that $\Delta(p,w)=0 \Longleftrightarrow (p,w)\in D_E$. Also, by construction,

$$\begin{aligned}\left(\left(p\cdot\widehat{w}^i\right)=\left(p\cdot w^i\right) \text{ and } p_{\mathbf{1}}\boxdot\left(\widehat{w}^i_{\mathbf{1}}-w^i_{\mathbf{1}}\right)\in\langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}} &\implies \prod_{i\in\mathcal{I}}\overline{B}\left(p,w^i\right)=\prod_{i\in\mathcal{I}}\overline{B}\left(p,\widehat{w}^i\right)\\ &\implies \Delta\left(p,\widehat{w}\right)=\Delta\left(p,w\right)\end{aligned}$$

Now, define the function $\Phi:\mathcal{S}^{N-1}_{++}\times\mathbb{R}^{NI}_{++}\longrightarrow\mathbb{R}^N_{++}$, by

$$\Phi\left(p,w\right)=F\left(p,w\right)+\Delta\left(p,w\right)\frac{F_{0,2}\left(p,w\right)}{2}\begin{bmatrix} p_{0,2} & -1 & 0 & \dots & 0\end{bmatrix}^{\top}_{N\times 1}.$$

Function Φ is well defined: it is continuous, maps into $\mathbb{R}^N_{++}$, satisfies Walras's law and, by construction: $p_{\mathbf{1}}\boxdot\left(\Phi_{\mathbf{1}}(p,w)-\sum_{i\in\mathcal{I}}w^i_{\mathbf{1}}\right)\in\langle V(p_{\mathbf{1}})\rangle$ and

$$\left(\left(p\cdot\widehat{w}^i\right)=\left(p\cdot w^i\right) \text{ and } p_{\mathbf{1}}\boxdot\left(\widehat{w}^i_{\mathbf{1}}-w^i_{\mathbf{1}}\right)\in\langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}}\Longrightarrow\Phi\left(p,\widehat{w}\right)=\Phi\left(p,w\right)$$

Moreover, for every $(p,w)\in E$, $\Phi\left(p,w\right)=F\left(p,w\right)=\sum_{i\in\mathcal{I}}w^i$, so $\left\{(p,w)\middle|\,\Phi(p,w)=\sum_{i\in\mathcal{I}}w^i\right\}\supseteq E$. Now, assume that $D\nsubseteq D_E$ and let $(\overline{p},\overline{w})\in D\backslash D_E$. By identification over D, $\Phi\left(\overline{p},\overline{w}\right)=F\left(\overline{p},\overline{w}\right)$, so $\Delta\left(\overline{p},\overline{w}\right)=0$ and, hence, $(\overline{p},\overline{w})\in D_E$, an obvious contradiction.

For the second part, define $\widehat{\Phi}:\mathcal{S}^{N-1}_{++}\times\mathbb{R}^{NI}_{++}\longrightarrow\mathbb{R}^N_{++}$ simply by

$$\widehat{\Phi}\left(p,w\right)=\begin{cases} F\left(p,w\right), \text{ if } (p,w)\in D_E\\ F\left(p,w\right)+\frac{F_{0,2}(p,w)}{2}\begin{bmatrix} p_{0,2} & -1 & 0 & \dots & 0\end{bmatrix}^{\top}_{N\times 1}, \text{ if } (p,w)\notin D_E\end{cases}.$$

Function $\widehat{\Phi}$ is well defined, satisfies Walras law and, by construction $p_{\mathbf{1}}\boxdot\left(\Phi_{\mathbf{1}}(p,w)-\sum_{i\in\mathcal{I}}w^i_{\mathbf{1}}\right)\in\langle V(p_{\mathbf{1}})\rangle$ and

$$\left(\left(p\cdot\widehat{w}^i\right)=\left(p\cdot w^i\right) \text{ and } p_{\mathbf{1}}\boxdot\left(\widehat{w}^i_{\mathbf{1}}-w^i_{\mathbf{1}}\right)\in\langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}}\Longrightarrow\widehat{\Phi}\left(p,\widehat{w}\right)=\widehat{\Phi}\left(p,w\right).$$

Also, $\left\{(p,w)\middle|\,\widehat{\Phi}(p,w)=\sum_{i\in\mathcal{I}}w^i\right\}\supseteq E$, because $D_E\cap M\supseteq E$. Now suppose that there is $(\overline{p},\overline{w})\in D\backslash D_E$. Then, by definition, $\widehat{\Phi}_{0,2}\left(\overline{p},\overline{w}\right)=\frac{F_{0,2}(p,w)}{2}$, and, since $(\overline{p},\overline{w})\in D$ and E identifies F over D then $\Phi\left(\overline{p},\overline{w}\right)=F\left(\overline{p},\overline{w}\right)$, implying that $F_{0,2}\left(\overline{p},\overline{w}\right)=0$, which is impossible. ■

3 From Aggregate Demand to Individual Demands

If equilibrium prices are observable for situations in which the incomes of all individuals but one are zero, the argument above still holds and, then, it is straightforward that aggregate demand identifies individual demands. That result, however, is not surprising: observation of situations in which the endowments of all consumers but one are pegged at zero amounts, in effect, to assuming that individual information is available.

We now show that, under an additional assumption, one can identify individual demands, without resorting to boundary analysis. Our proof is somewhat similar to the one presented by Kubler et al., but simpler: it does not require separability of preferences; it does not require us to claim uniqueness of the solution to a system of partial differential equations; and it requires a weaker regularity condition than the one used by Kubler et al.

For the sake of simplicity in the presentation, we initially study the global setting introduced in the previous section. The case in which the aggregate demand is not globally known is presented afterwards.

3.1 The global case:

In this case, we weaken Kubler et al.'s Regularity assumption as follows:

Condition 3 (Regularity in present value prices) *For every individual i, $u^i \in \mathbf{C}^4\left(\mathbb{R}^N_{++}\right)$ and is differentiably strictly monotone and differentiably strongly concave, and for every $p \in \mathcal{S}^{N-1}_{++}$, there exist $w \in \mathbb{R}^N_{++}$, and $(s,l), (s',l') \in \{0,...,S\} \times \{1,...,L\} \setminus \{(0,1)\}$, such that $\frac{\partial^2 f^i_{s,l}}{\partial \left(w^i_{0,1}\right)^2} \neq 0$ and*

$$\begin{vmatrix} \frac{\partial^2 f^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}(p,w) & \frac{\partial^2 f^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}(p,w) \\ \frac{\partial^3 f^i_{s,l}}{\partial\left(w^i_{0,1}\right)^3}(p,w) & \frac{\partial^3 f^i_{s'l'}}{\partial\left(w^i_{0,1}\right)^3}(p,w) \end{vmatrix} \neq 0.$$

The previous condition is weaker because w only needs an existential, and not a universal, quantifier, because it is not assumed state-by-state and because it is not assumed for asset demands. The condition is indeed restrictive as it requires, for example, that income effects do not vanish. Intuitively, the condition requires that preferences be "complex enough" so as to generate the independence of income effects. As Chiappori et al. have pointed out, under complete markets it suffices that individual demands have rank at least two for the condition to be met everywhere. Appendix 1 illustrates the assumption.

Theorem 3 (Slutsky symmetry) *For each i, $f^i \in \mathbf{C}^3\left(\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^N\right)$ and for every $(p, w) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^N$ and $(s, l), (s', l') \in (\{0, ..., S\} \times \{1, ..., L\}) \setminus \{(0, 1)\}$,*

$$\frac{\partial f_{s,l}^i}{\partial p_{s',l'}}(p, w) + \left(f_{s',l'}^i(p, w) - w_{s',l'}\right)\frac{\partial f_{s,l}^i}{\partial w_{0,1}}(p, w) = \frac{\partial f_{s',l'}^i}{\partial p_{s,l}}(p, w) + \left(f_{s,l}^i(p, w) - w_{s,l}\right)\frac{\partial f_{s',l'}^i}{\partial w_{0,1}}(p, w).$$

Proof. That $f^i \in \mathbf{C}^3$ follows from Duffie and Shafer [9]. Symmetry follows from theorems 4 and 6 in appendix 2. ■

We say that **aggregate demand identifies individual demands globally** if, given F, $\left(f^i\right)_{i\in\mathcal{I}}$ is the only profile of functions $\left(\varphi^i : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^N \to \mathbb{R}_{++}^N \in \mathbf{C}^3\left(\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^N\right)\right)_{i\in\mathcal{I}}$, satisfying Walras's law and Slutsky symmetry, and such that $\sum_{i\in\mathcal{I}} \varphi^i = F$.

Theorem 4 *Aggregate demand identifies individual demands globally.*

Proof. Let $\left(\varphi^i\right)_{i\in\mathcal{I}}$ satisfy Walras's law and Slutsky symmetry and be such that $F = \sum_{i\in\mathcal{I}} \varphi^i$. Fix $i \in \mathcal{I}$ and define $\theta^i : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^N \longrightarrow \mathbb{R}_{++}^N$ and $\gamma^i : \mathcal{S}_{++}^{N-1} \longrightarrow \mathbb{R}^N$ by $\theta^i(p, w) = F(p, (\mathbf{1}, \mathbf{1}, ..., w, ..., \mathbf{1}))$, where w occupies the i^{th} position, and $\gamma^i(p) = -\sum_{j\in\mathcal{I}\setminus\{i\}} \varphi^j(p, \mathbf{1})$.

By Slutsky symmetry, for every $(s, l), (s', l') \in \{0, ..., S\} \times \{1, ..., L\} \setminus \{(0, 1)\}$, everywhere in $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^N$, $\frac{\partial \varphi_{s,l}^i}{\partial p_{s',l'}} + \left(\varphi_{s',l'}^i - w_{s',l'}^i\right)\frac{\partial \varphi_{s,l}^i}{\partial w_{0,1}^i} = \frac{\partial \varphi_{s',l'}^i}{\partial p_{s,l}} + \left(\varphi_{s,l}^i - w_{s,l}^i\right)\frac{\partial \varphi_{s',l'}^i}{\partial w_{0,1}^i}$. Since $\varphi^i\left(p, w^i\right) = \theta^i\left(p, w^i\right) + \gamma^i(p)$, substituting,

$$\frac{\partial \theta_{s,l}^i}{\partial p_{s',l'}} + \frac{\partial \gamma_{s,l}^i}{\partial p_{s',l'}} + \left(\theta_{s',l'}^i + \gamma_{s',l'}^i - w_{s',l'}^i\right)\frac{\partial \theta_{s,l}^i}{\partial w_{0,1}^i} = \frac{\partial \theta_{s',l'}^i}{\partial p_{s,l}} + \frac{\partial \gamma_{s',l'}^i}{\partial p_{s,l}} + \left(\theta_{s,l}^i + \gamma_{s,l}^i - w_{s,l}^i\right)\frac{\partial \theta_{s',l'}^i}{\partial w_{0,1}^i}.$$

Fix $p \in \mathcal{S}_{++}^{N-1}$. Taking that $(s, l), (s', l') \neq (0, 1)$ and deriving once and twice with respect to $w_{0,1}^i$ gives

$$\frac{\partial^2 \theta_{s,l}^i}{\partial w_{0,1}^i \partial p_{s',l'}} + \left(\theta_{s',l'}^i + \gamma_{s',l'}^i - w_{s',l'}^i\right)\frac{\partial^2 \theta_{s,l}^i}{\partial\left(w_{0,1}^i\right)^2} = \frac{\partial^2 \theta_{s',l'}^i}{\partial w_{0,1}^i \partial p_{s,l}} + \left(\theta_{s,l}^i + \gamma_{s,l}^i - w_{s,l}^i\right)\frac{\partial^2 \theta_{s',l'}^i}{\partial\left(w_{0,1}^i\right)^2}$$

and

$$\frac{\partial^3 \theta_{s,l}^i}{\partial\left(w_{0,1}^i\right)^2 \partial p_{s',l'}} + \frac{\partial \theta_{s',l'}^i}{\partial w_{0,1}^i}\frac{\partial^2 \theta_{s,l}^i}{\partial\left(w_{0,1}^i\right)^2} + \left(\theta_{s',l'}^i + \gamma_{s',l'}^i - w_{s',l'}^i\right)\frac{\partial^3 \theta_{s,l}^i}{\partial\left(w_{0,1}^i\right)^3} =$$

$$\frac{\partial^3 \theta_{s',l'}^i}{\partial\left(w_{0,1}^i\right)^2 \partial p_{s,l}} + \frac{\partial \theta_{s,l}^i}{\partial w_{0,1}^i}\frac{\partial^2 \theta_{s',l'}^i}{\partial\left(w_{0,1}^i\right)^2} + \left(\theta_{s,l}^i + \gamma_{s,l}^i - w_{s,l}^i\right)\frac{\partial^3 \theta_{s',l'}^i}{\partial\left(w_{0,1}^i\right)^3},$$

which rewrites as (recall that p is fixed)

$$\Delta_{(s,l),(s',l')}\left(w^i\right)\begin{bmatrix}\gamma^i_{s',l'}(p)\\ \gamma^i_{s,l}(p)\end{bmatrix}=\Gamma_{(s,l),(s',l')}\left(w^i\right),$$

where

$$\Delta_{(s,l),(s',l')}\left(w^i\right)=\begin{bmatrix}\frac{\partial^2\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}\left(p,w^i\right) & -\frac{\partial^2\theta^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}\left(p,w^i\right)\\ \frac{\partial^3\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^3}\left(p,w^i\right) & -\frac{\partial^3\theta^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^3}\left(p,w^i\right)\end{bmatrix}$$

and $\Gamma_{(s,l),(s',l')}\left(w^i\right)$ has first component

$$\frac{\partial^2\theta^i_{s',l'}}{\partial w^i_{0,1}\partial p_{s,l}}-\frac{\partial^2\theta^i_{s,l}}{\partial w^i_{0,1}\partial p_{s',l'}}+\left(\theta^i_{s,l}-w^i_{s,l}\right)\frac{\partial^2\theta^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}-\left(\theta^i_{s',l'}-w^i_{s',l'}\right)\frac{\partial^2\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}$$

and second component

$$\begin{array}{c}\frac{\partial^3\theta^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2\partial p_{s,l}}-\frac{\partial^3\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2\partial p_{s',l'}}+\frac{\partial\theta^i_{s,l}}{\partial w^i_{0,1}}\frac{\partial^2\theta^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^2}-\frac{\partial\theta^i_{s',l'}}{\partial w^i_{0,1}}\frac{\partial^2\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}\\ +\left(\theta^i_{s,l}-w^i_{s,l}\right)\frac{\partial^3\theta^i_{s',l'}}{\partial\left(w^i_{0,1}\right)^3}-\left(\theta^i_{s',l'}-w^i_{s',l'}\right)\frac{\partial^3\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^3}\end{array}.$$

Notice that the resulting $\Delta_{(s,l),(s',l')}\left(w^i\right)$ and $\Gamma_{(s,l),(s',l')}\left(w^i\right)$ do not contain $\gamma^i_{s,l}$ or $\gamma^i_{s',l'}$, but only θ^i, which is determined by F.

By regularity, for some $w^i\in\mathbb{R}^N_{++}$ and $(s,l),(s',l')\in\{0,...,S\}\times\{1,...,L\}$, $\Delta_{(s,l),(s',l')}\left(w^i\right)$ is invertible, so

$$\begin{bmatrix}\gamma^i_{s',l'}(p)\\ \gamma^i_{s,l}(p)\end{bmatrix}=\left(\Delta_{(s,l),(s',l')}\left(w^i\right)\right)^{-1}\Gamma_{(s,l),(s',l')}\left(w^i\right), \tag{*}$$

whereas, for every other $(s'',l'')\in\{0,...,S\}\times\{1,...,L\}\setminus\{(0,1)\}$, by Slutsky symmetry, $\gamma^i_{s'',l''}(p)$ equals

$$\frac{\frac{\partial^2\theta^i_{s'',l''}}{\partial w^i_{0,1}\partial p_{s,l}}-\frac{\partial^2\theta^i_{s,l}}{\partial w^i_{0,1}\partial p_{s'',l''}}+\left(\theta^i_{s,l}\left(p,w^i\right)+\gamma^i_{s,l}(p)-w^i_{s,l}\right)\frac{\partial^2\theta^i_{s'',l''}}{\partial\left(w^i_{0,1}\right)^2}-\left(\theta^i_{s'',l''}\left(p,w^i\right)-w^i_{s'',l''}\right)\frac{\partial^2\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}}{\frac{\partial^2\theta^i_{s,l}}{\partial\left(w^i_{0,1}\right)^2}},$$

and, by Walras's law, $\gamma^i_{0,1}(p)=-\left(\sum_{l=2}^L p_{0,l}\gamma^i_{0,l}(p)+\sum_{s=1}^S\sum_{l=1}^L p_{s,l}\gamma^i_{s,l}(p)\right)$.

Since $\varphi^i\left(p,w^i\right)=\theta^i\left(p,w^i\right)+\gamma^i(p)$ and the expression on the right hand side of equation (*) depends only on F, it follows that $\varphi^i_{s,l}=f^i_{s,l}$, which implies that $\varphi^i=f^i$. ■

3.2 Restricted observation

Notice that the choice of $\left(w^j = \mathbf{1}\right)_{j\in\mathcal{I}\setminus\{i\}}$ in the proof of theorem 4 is arbitrary and that when only $F_{|D}$ is available, such choice can be modified as needed. If only local information of F is available, one must strengthen the second part of the assumption so that, for given prices, the profile of endowments at which the condition is met lies in the observed domain. For the sake of simplicity, assume that we strengthen condition 3 by substituting the existential quantifier of w by the universal quantifier.

Let $D \subseteq \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++}$ and, for every i, let $D^i \subseteq \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{N}_{++}$. We say that $F_{|D}$ **identifies** $\left(f^i\right)_{i\in\mathcal{I}}$ **over** $\left(D^i\right)_{i\in\mathcal{I}}$ if for every $\left(\varphi^i : \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{N}_{++} \to \mathbb{R}^{N}_{++} \in \mathbf{C}^3\left(\mathcal{S}^{N-1}_{++} \times \mathbb{R}^{N}_{++}\right)\right)_{i\in\mathcal{I}}$, satisfying Walras's law and Slutsky symmetry, and such that:

$$\left(p \cdot \widehat{w}^i = p \cdot w^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) \in \langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}} \Longrightarrow \left(\varphi^i(p,w)\right)_{i\in\mathcal{I}} = \left(\varphi^i(p,\widehat{w})\right)_{i\in\mathcal{I}}$$

$$\sum_{i\in\mathcal{I}} \varphi^i\left(p, w^i\right)_{|D} = F\left(p,w\right)_{|D}$$

it is true that $\left(\varphi^i_{|D^i}\right)_{i\in\mathcal{I}} = \left(f^i_{|D^i}\right)_{i\in\mathcal{I}}$.

Intuitively, we say that $F_{|D}$ **identifies** $\left(f^i\right)_{i\in\mathcal{I}}$ **over** $\left(D^i\right)_{i\in\mathcal{I}}$ if, for any profile of functions that cannot be ruled out as individual demands, we have that, on the restricted domains $\left(D^i\right)_{i\in\mathcal{I}}$, those functions are identical to the true demand functions. A profile of functions cannot be ruled out as individual demands when (i) it satisfies the properties that are necessary for a profile of individual demands, and (ii) it is (locally) consistent with aggregate demand.

Theorem 5 *Let $D \subseteq \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{NI}_{++}$ and, for each $i \in \mathcal{I}$, denote*

$$D^i = \left\{ \left(p, w^i\right) \in \mathcal{S}^{N-1}_{++} \times \mathbb{R}^{N}_{++} \middle| \left(\exists\left(\widehat{p},\widehat{w}\right) \in D_0\right) : \begin{pmatrix} \widehat{p} = p \\ \left(p \cdot \widehat{w}^i\right)_{i\in\mathcal{I}} = \left(p \cdot w^i\right)_{i\in\mathcal{I}} \\ \left(p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) \in \langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}} \end{pmatrix} \right\},$$

where D_0 is the interior of D. $F_{|D}$ identifies $\left(f^i\right)_{i\in\mathcal{I}}$ over $\left(\overline{D^i}\right)_{i\in\mathcal{I}}$.

Proof. Let $\left(\varphi^i\right)_{i\in\mathcal{I}}$, satisfy Walras's law, Slutsky symmetry and such that

$$\left(p \cdot \widehat{w}^i = p \cdot w^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) \in \langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}} \Longrightarrow \left(\varphi^i(p,w)\right)_{i\in\mathcal{I}} = \left(\varphi^i(p,\widehat{w})\right)_{i\in\mathcal{I}}$$

and

$$\sum_{i\in\mathcal{I}}\varphi^i\left(p,w^i\right)_{|D}=F\left(p,w\right)_{|D}$$

Fix $i\in\mathcal{I}$ and $\left(p,w^i\right)\in D_0^i$, where D_0^i is the projection of D_0 into the space of $\left(p,w^i\right)$. By definition, there exists $\left(w^j\right)_{j\in\mathcal{I}\setminus\{i\}}$ such that $\left(p,\left(w^j\right)_{j=1}^I\right)\in D_0$. Since D_0 is open, there exists $\epsilon>0$ such that

$$\left\{(\widetilde{p},\widetilde{w})\in\mathcal{S}_{++}^{N-1}\times\mathbb{R}_{++}^{NI}\,\middle|\,\|(\widetilde{p},\widetilde{w})-(p,w)\|<\epsilon\right\}\subseteq D_0.$$

Denote $O=\left\{\widetilde{p}\in\mathcal{S}_{++}^{L-1}\,\middle|\,\|\widetilde{p}-p\|<\epsilon\right\}$ and

$$O^i=\left\{\left(\widetilde{p},\widetilde{w}^i\right)\in\mathcal{S}_{++}^{L-1}\times\mathbb{R}_{++}^{L}\,\middle|\,\left\|\left(\widetilde{p},\widetilde{w}^i\right)-\left(p,w^i\right)\right\|<\epsilon\right\},$$

and define the functions $\theta^i:O^i\longrightarrow\mathbb{R}_+^L$ and $\gamma^i:O\longrightarrow\mathbb{R}_+^L$, as $\theta^i\left(\widetilde{p},\widetilde{w}^i\right)=F\left(\widetilde{p},\left(w^1,w^2,...,\widetilde{w}^i,...,w^I\right)\right)$, where $\widetilde{w}^i$ occupies the i^{th} position, and $\gamma^i\left(\widetilde{p}\right)=-\sum_{j\in\mathcal{I}\setminus\{i\}}\varphi^j\left(\widetilde{p},w^j\right)$. By Slutsky symmetry and regularity, as in the proof of theorem 4, $\varphi^i\left(p,w^i\right)=f^i\left(p,w^i\right)$. That is $\varphi^i_{|D_0^i}=f^i_{|D_0^i}$

Now, let $\left(\overline{p},\overline{w}^i\right)\in\overline{D^i}$. By definition, there exists $\left(p_n,w_n^i\right)_{n=1}^{\infty}$, in D^i, such that $\left(p_n,w_n^i\right)\to\left(\overline{p},\overline{w}^i\right)$. Then, for each $n\in\mathbb{N}$, there exists $\widetilde{w}_n\in\mathbb{R}_{++}^{NI}$ such that $\left(p_n,\widetilde{w}_n\right)\in D_0$, $\left(p\cdot w_n^i\right)_{i\in\mathcal{I}}=\left(p\cdot\widetilde{w}_n^i\right)_{i\in\mathcal{I}}$ and $\left(p_{\mathbf{1}}\boxdot\left(w_{n,\mathbf{1}}^i-\widetilde{w}_{n,\mathbf{1}}^i\right)\in\langle V(p_{\mathbf{1}})\rangle\right)_{i\in\mathcal{I}}$ therefore, $\varphi^i\left(p_n,\widetilde{w}_n^i\right)=\varphi^i\left(p_n,w_n^i\right)$ and $f^i\left(p_n,\widetilde{w}_n^i\right)=f^i\left(p_n,w_n^i\right)$ and, by continuity, $\varphi^i\left(p_n,\widetilde{w}_n^i\right)=\varphi^i\left(p_n,w_n^i\right)\to\varphi^i\left(\overline{p},\overline{w}^i\right)$ and $f^i\left(p_n,\widetilde{w}_n^i\right)=f^i\left(p_n,w_n^i\right)\to f^i\left(\overline{p},\overline{w}^i\right)$. Since $\left(p_n,\widetilde{w}_n^i\right)\in D_0^i$, it follows from our previous argument that $\varphi^i\left(p_n,\widetilde{w}_n^i\right)=f^i\left(p_n,\widetilde{w}_n^i\right)$, and hence that $\varphi^i\left(\overline{p},\overline{w}^i\right)=f^i\left(\overline{p},\overline{w}^i\right)$. ■

4 Observability: Financial Markets Equilibrium Manifold

Identification results are useful when based on observable data. In real life one does not observe date-zero present-value equilibrium prices, but, instead, one observes (financial) equilibrium spot prices for commodities and asset prices.

4.1 From the Equilibrium Manifold to Aggregate Demand

Let $q\in\mathbb{R}^J$ be the price vector at which assets can be bought at $s=0$.

For $(p,q)\in\mathbb{R}_{++}^N\times\mathbb{R}^J$ and $w\in\mathbb{R}_{++}^N$, let

$$\mathbf{B}(p,q,w)=\left\{x\in\mathbb{R}_+^N\,\middle|\,\left(\exists z\in\mathbb{R}^J\right):\begin{pmatrix}p_0\cdot(x_0-w_0)\leq-qz\\ p_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})=V\left(p_{\mathbf{1}}\right)z\end{pmatrix}\right\}.$$

Since there are $S+1$ degrees of nominal indeterminacy, we normalize prices, in each state, to lie in

$$\mathcal{S}^{L-1}_{++} = \left\{ p_s \in \mathbb{R}^L_{++} \middle| \, p_{s,1} = 1 \right\}.$$

Asset prices $q \in \mathbb{R}^J$ are a no-arbitrage price vector if $Vz > 0$ implies that $q \cdot z > 0$. It is well known that no-arbitrage is a necessary condition for optimization, and that q is a no-arbitrage price only if for some $\pi \in \mathbb{R}^S_{++}$, $q = \pi V$. Let $\mathcal{Q}$ denote the set (positive cone) of no-arbitrage price vectors.

Define the individual demand function in financial markets, $\mathbf{f}^i : \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}^N_{++} \to \mathbb{R}^N_{++}$, as $\mathbf{f}^i(p,q,w) = \arg\max_{x \in \mathbf{B}(p,q,w)} u(x)$, which is well defined since $\mathbf{B}(p,q,w)$ is nonempty, compact (because q is a no-arbitrage price vector) and convex, and u is continuous and strongly quasi-concave. Define also the aggregate demand function in financial markets, $\mathbf{F} : \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \to \mathbb{R}^N_{++}$, as $\mathbf{F}(p,q,w) = \sum_{i \in \mathcal{I}} \mathbf{f}^i(p,q,w^i)$.

A **financial markets equilibrium** for economy $\left(\left(u^i, w^i\right)_{i \in \mathcal{I}}, V\right)$ is $(x,z,p,q) \in \mathbb{R}^{NI}_{+} \times \mathbb{R}^J \times \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q}$ such that:

1. For every i, $x^i = \mathbf{f}^i(p,q,w^i)$, $p_0 \cdot (x^i_0 - w^i_0) = -qz^i$ and $p_{\mathbf{1}} \boxdot \left(x^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) = Vz^i$;

2. $\mathbf{F}(p,q,\left(w^i\right)^I_{i=1}) = \sum_{i \in \mathcal{I}} w^i$.

Since V has full column rank, $\sum_{i \in \mathcal{I}} z^i = 0$ is unnecessary in the previous definition.

Given $\left(u^i\right)_{i \in \mathcal{I}}$ and V, the **financial markets equilibrium manifold** is

$$\mathbf{M} = \left\{ (p,q,w) \in \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \middle| \, \mathbf{F}(p,q,\left(w^i\right)^I_{i=1}) = \sum_{i \in \mathcal{I}} w^i \right\}.$$

Under our assumptions, $\mathbf{F}$ is continuous and satisfies:

1. $\exists z \in \mathbb{R}^J : p_0 \cdot \left(\mathbf{F}_0(p,q,\left(w^i\right)^I_{i=1}) - \sum_{i \in \mathcal{I}} w^i_0\right) = -qz$ and $p_{\mathbf{1}} \boxdot \left(\mathbf{F}_{\mathbf{1}}(p,q,\left(w^i\right)^I_{i=1}) - \sum_{i \in \mathcal{I}} w^i_{\mathbf{1}}\right) = Vz$.

2.

$$\begin{aligned} &\left(\exists z^i \in \mathbb{R}^J : p_0 \cdot \left(\widehat{w}^i_0 - w^i_0\right) = -qz^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) = Vz^i\right)_{i \in \mathcal{I}} \\ \Longrightarrow \quad & \mathbf{F}(p,q,w) = \mathbf{F}(p,q,\widehat{w}) \end{aligned}$$

For any $\mathbf{\Phi} : \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \longrightarrow \mathbb{R}^N_{++}$ and $\mathbf{D} \subseteq \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++}$, denote by $\mathbf{\Phi}_{|\mathbf{D}}$ the restriction of $\mathbf{\Phi}$ to $\mathbf{D}$.

We say that $\mathbf{E} \subseteq \mathbf{M}$ **identifies F over D** $\subseteq \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI}$ if for every continuous function $\mathbf{\Phi} : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI} \longrightarrow \mathbb{R}_{++}^{N}$ such that:

1. $\exists z \in \mathbb{R}^J : p_0 \cdot \left(\mathbf{\Phi}_0(p, q, \left(w^i\right)_{i=1}^I) - \sum_{i \in \mathcal{I}} w_0^i\right) = -qz$ and $p_{\mathbf{1}} \boxdot \left(\mathbf{\Phi}_{\mathbf{1}}(p, q, \left(w^i\right)_{i=1}^I) - \sum_{i \in \mathcal{I}} w_{\mathbf{1}}^i\right) = Vz$

2. $$\begin{aligned} & \left(\exists z^i \in \mathbb{R}^J : p_0 \cdot \left(\widehat{w}_0^i - w_0^i\right) = -qz^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) = Vz^i\right)_{i \in \mathcal{I}} \\ \Longrightarrow \quad & \mathbf{\Phi}(p, q, w) = \mathbf{\Phi}(p, q, \widehat{w}) \end{aligned}$$

   and
   $$\left\{ (p, q, w) \in \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI} \middle| \mathbf{\Phi}(p, q, w) = \sum_{i \in \mathcal{I}} w^i \right\} \supseteq \mathbf{E},$$

   it is true that $\mathbf{\Phi}_{|\mathbf{D}} = \mathbf{F}_{|\mathbf{D}}$.

If in the previous definition we drop the requirement that $\mathbf{\Phi}$ be continuous, then we say that **E strongly identifies** $\mathcal{F}$ **over D**. Clearly, strong identification implies identification.

Since, what may be observed in the real world is subsets of $\mathbf{M}$, we now show how to derive subsets E of M from subsets $\mathbf{E}$ of $\mathbf{M}$ and functions Φ on $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$ from function $\mathbf{\Phi}$ on $\left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI}$.

For every $\mathbf{E} \subseteq \mathbf{M}$ define

$$E = \left\{ (p, w) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI} \middle| \left(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^{S}, \sum_{s=1}^{S} V_s\left(p_{s,1}\right), w \right) \in \mathbf{E} \right\},$$

and for every function $\mathbf{\Phi} : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI} \longrightarrow \mathbb{R}_{++}^{N}$ define $\Phi : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI} \longrightarrow \mathbb{R}_{++}^{N}$ by:

$$\Phi(p, w) = \mathbf{\Phi} \left(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^{S}, \sum_{s=1}^{S} V_s\left(p_{s,1}\right), w \right).$$

Theorem 6 *Let* $\mathbf{E} \subseteq \mathbf{M}$.

1. $E \subseteq M$.

2. *If* $\mathbf{\Phi} : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI} \longrightarrow \mathbb{R}_{++}^{N}$ *satisfies the properties that characterize identification (strong identification) in financial markets, then* $\Phi : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI} \longrightarrow \mathbb{R}_{++}^{N}$ *satisfies the properties that characterize identification (strong identification) in the no-arbitrage equilibrium manifold.*

3. Define

$$\mathbf{D_E} = \left\{ \begin{array}{c} (p,q,w) \in \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \Big| \left(\exists \left(\widehat{w}, z\right) \in \mathbb{R}^{NI}_{++} \times \mathbb{R}^{JI}\right) : \\ \left(\begin{array}{c} (p,q,\widehat{w}) \in \mathbf{E} \\ \left(p_0 \left(\widehat{w}^i_0 - w^i_0\right) = -qz^i\right)_{i\in\mathcal{I}} \\ \left(p_{\mathbf{1}} \boxdot \left(\widehat{w}^i_{\mathbf{1}} - w^i_{\mathbf{1}}\right) = Vz^i\right)_{i\in\mathcal{I}} \end{array} \right) \end{array} \right\}.$$

Then, $\mathbf{E}$ *identifies* $\mathbf{F}$ *over* $\overline{\mathbf{D_E}}$*, strongly over* $\mathbf{D_E}$.

Proof. It is well known that if $p \in \mathbb{R}^N_{++}$ and $\pi = (\pi_0, \pi_{\mathbf{1}}) \in \{1\} \times \mathbb{R}^S_{++}$, then $\mathbf{B}\left(p, \pi_{\mathbf{1}} V\left(p_{\mathbf{1}}\right), w\right) = B\left(\left(\pi_s p_s\right)_{s=0}^S, w\right)$. By substitution, then, for every $p \in \mathcal{S}^{N-1}_{++}$ and $\pi = (\pi_0, \pi_{\mathbf{1}}) \in \{1\} \times \mathbb{R}^S_{++}$, $\mathbf{B}\left(\left(\pi_s^{-1} p_s\right)_{s=0}^S, \mathbf{1}^\top V\left(p_{\mathbf{1}}\right), w\right) = B\left(p, w\right)$.

Then, for part 1, let $(p,w) \in E$. By definition, $\left(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^S, \sum_{s=1}^S V_s\left(p_{s,1}\right), w\right) \in \mathbf{E}$. Let $\pi = \left(1, \left(p_{s,1}\right)_{s=1,\ldots S}\right)$ then

$$\sum_{i\in\mathcal{I}} w^i = \sum_{i\in\mathcal{I}} \mathbf{f}^i \left(\left(\pi_s^{-1} p_s\right)_{s=0}^S, \left(\sum_{s=1}^S V_s\left(p_{s,1}\right)\right), w^i\right) = \sum_{i\in\mathcal{I}} f^i(p, w^i) = F\left(p, w\right).$$

For part 2, suppose $\mathbf{\Phi} : \left(\mathcal{S}^{L-1}_{++}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}^{NI}_{++} \longrightarrow \mathbb{R}^N_{++}$ satisfies properties (1) and (2) in the definition of identification in financial markets (clearly, if $\mathbf{\Phi}$ is continuous, then Φ is cotinuous). We now check that Φ satisfies all the properties that characterize identification in the no-arbitrage equilibrium manifold.

Let us check Walras law. By the second part of property (1):

$$\overline{p}_{\mathbf{1}} \boxdot \left(\mathbf{\Phi_1}(\overline{p}, q, \left(w^i\right)_{i=1}^I) - \sum_{i\in\mathcal{I}} w^i_{\mathbf{1}}\right) = Vz.$$

Let $\overline{p} = \left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^S$ then the previous equation implies which implies

$$\begin{array}{rcl} \left(\frac{1}{p_{s,1}} p_s\right)_{s=1}^S \boxdot \left(\mathbf{\Phi_1}(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^S, q, \left(w^i\right)_{i=1}^I) - \sum_{i\in\mathcal{I}} w^i_{\mathbf{1}}\right) & = & Vz \\ & \Rightarrow & \\ p_{\mathbf{1}} \boxdot \left(\mathbf{\Phi_1}(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^S, q, \left(w^i\right)_{i=1}^I) - \sum_{i\in\mathcal{I}} w^i_{\mathbf{1}}\right) & = & V\left(p_{\mathbf{1}}\right) z, \end{array}$$

and by the first part of property (1) :

$$p_0 \cdot \left(\mathbf{\Phi}_0(\left(\frac{1}{p_{s,1}} p_s \right)_{s=0}^{S}, q, \left(w^i\right)_{i=1}^{I}) - \sum_{i \in \mathcal{I}} w_0^i \right) = -qz$$

Summing up over s the last two equations we obtain:

$$p \cdot \left(\mathbf{\Phi}(\left(\frac{1}{p_{s,1}} p_s \right)_{s=0}^{S}, q, \left(w^i\right)_{i=1}^{I}) - \sum_{i \in \mathcal{I}} w^i \right) = \left(V\left(p_{\mathbf{1}}\right) - q\right) z.$$

Let $q = \sum_{s=1}^{S} V_s\left(p_{s,1}\right)$, then we just proved that:

$$p \cdot \left(\mathbf{\Phi}(\left(\frac{1}{p_{s,1}} p_s \right)_{s=0}^{S}, \sum_{s=1}^{S} V_s\left(p_{s,1}\right), \left(w^i\right)_{i=1}^{I}) - \sum_{i \in \mathcal{I}} w^i \right) = 0,$$

which, by definition, is the same as:

$$p \cdot \left(\Phi(p, w) - \sum_{i \in \mathcal{I}} w^i \right) = 0$$

For the next property notice that, by definition,

$$\begin{aligned} & p_{\mathbf{1}} \boxdot \left(\Phi_{\mathbf{1}}(p, \left(w^i\right)_{i \in \mathcal{I}}) - \sum_{i \in \mathcal{I}} w_{\mathbf{1}}^i \right) \\ = \quad & p_{\mathbf{1}} \boxdot \left(\mathbf{\Phi}_{\mathbf{1}} \left(\left(\frac{1}{p_{s,1}} p_s \right)_{s=0}^{S}, \sum_{s=1}^{S} V_s\left(p_{s,1}\right), \left(w^i\right)_{i \in \mathcal{I}} \right) - \sum_{i \in \mathcal{I}} w_{\mathbf{1}}^i \right) \\ = \quad & \left(p_{s,1}\right)_{s=1}^{S} \boxdot \overline{p}_{\mathbf{1}} \boxdot \left(\mathbf{\Phi}_{\mathbf{1}} \left(\overline{p}, q, \left(w^i\right)_{i \in \mathcal{I}} \right) - \sum_{i \in \mathcal{I}} w_{\mathbf{1}}^i \right) \end{aligned}$$

where $\overline{p}_{\mathbf{1}} = \left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^{S}$ and $q = \sum_{s=1}^{S} V_s\left(p_{s,1}\right)$. By the second part of property (1) , $\overline{p}_{\mathbf{1}} \boxdot \Big(\mathbf{\Phi}_{\mathbf{1}} \left(\overline{p}, q, \left(w^i\right)_{i \in \mathcal{I}}\right) - \sum_{i \in \mathcal{I}}$
$\langle V \rangle$ therefore,

$$\left(p_{s,1}\right)_{s=1}^{S} \boxdot \overline{p}_{\mathbf{1}} \boxdot \left(\mathbf{\Phi}_{\mathbf{1}} \left(\overline{p}, q, \left(w^i\right)_{i \in \mathcal{I}} \right) - \sum_{i \in \mathcal{I}} w_{\mathbf{1}}^i \right) \in \langle V(p_{\mathbf{1}}) \rangle$$

For part 3, it suffices to prove that $\mathbf{E} \subseteq \mathbf{M}$ strongly identifies $\mathbf{F}$ over $\mathbf{D_E}$. Let $\mathbf{\Phi} : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI} \longrightarrow \mathbb{R}_{++}^{N}$ such that it satisfies the properties that characterize strong identification and $(p, q, w) \in \mathbf{D_E}$. By definition, there exists $(\widehat{w}, z) \in \mathbb{R}_{++}^{NI} \times \mathbb{R}^{JI}$ such that $(p, q, \widehat{w}) \in \mathbf{E}$, $p_0\left(\widehat{w}_0^i - w_0^i\right) = -qz^i$ and $p_s\left(\widehat{w}_s^i - w_s^i\right) = Vz^i$, for all i. By no-arbitrage, for some $\pi \in \{1\} \times \mathbb{R}_{++}^{S}$, $q = \pi_{\mathbf{1}} V$, which means that $\left(p, \mathbf{1}^{\top} V\left(\left(\pi_s p_s\right)_{s=1}^{S}\right), \widehat{w}\right) \in \mathbf{E}$, and hence that $\left(\left(\pi_s p_s\right)_{s=0}^{S}, \widehat{w}\right) \in E \subseteq M$. By parts (1) and (2) of this theorem, Φ satisfies all the properties that characterize strong identification in the no-arbitrage equilibrium manifold. Therefore, by theorem 1, $\Phi\left((\pi_s p_s)_{s=0}^S, \widehat{w}\right) = F\left((\pi_s p_s)_{s=0}^S, \widehat{w}\right)$ but $\mathbf{\Phi}(p,q,w) = \mathbf{\Phi}(p,q,\widehat{w}) = \mathbf{\Phi}\left((\pi_s p_s)_{s=0}^S, \widehat{w}\right)$ and $\mathbf{F}(p,q,w) = \mathbf{F}(p,q,\widehat{w}) = F\left((\pi_s p_s)_{s=0}^S, \widehat{w}\right)$ and hence $\mathbf{\Phi}(p,q,w) = \mathbf{F}(p,q,w)$. ■

The following corollary stablishes the global result.

Corollary 2 *Global knowledge of the financial markets manifold identifies aggregate demand globally.*

Proof. It suffices to show that $\mathbf{D_E} = \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI}$. Let $(p,q,w) \in \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI}$ and define $\left(\widehat{w}^i\right)_{i\in\mathcal{I}} = \left(f^i(p,q,w^i)\right)_{i\in\mathcal{I}}$ then obviously, there exists $z \in \mathbb{R}^{JI}$ such that $(\widehat{w}, z) \in \mathbb{R}_{++}^{NI} \times \mathbb{R}^{JI}$, $(p,q,\widehat{w}) \in \mathbf{M}$, $\left(p_0\left(\widehat{w}_0^i - w_0^i\right) = -qz^i \text{ and } p_{\mathbf{1}} \boxdot \left(\widehat{w}_{\mathbf{1}}^i - w_{\mathbf{1}}^i\right) = Vz^i\right)_{i\in\mathcal{I}}$. Therefore $(p,q,w) \in \mathbf{D_E}$. ■

For completness we include the following theorem. Knowledge of the financial markets manifold identifies the no-arbitrage manifold.

Theorem 7 *Define*

$$\overline{M} = \left\{ (p,w) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI} \middle| \left(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^S, \sum_{s=1}^S V_s(p_{s,1}), w \right) \in \mathbf{M} \right\}.$$

Then $\overline{M} = M$.

Proof. That $\overline{M} \subseteq M$ follows immediately from theorem 4.1. Now, let $(p,w) \in M$. By construction,

$$\begin{aligned} \sum_{i\in\mathcal{I}} w^i &= \sum_{i\in\mathcal{I}} f^i(p,w^i) \\ &= \sum_{i\in\mathcal{I}} \mathbf{f}^i \left(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^S, \left(\sum_{s=1}^S V_s(p_{s,1})\right), w^i \right) \\ &= \mathbf{F}\left(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^S, \left(\sum_{s=1}^S V_s(p_{s,1})\right), w \right), \end{aligned}$$

where the third equality follows from the fact pointed out at the beginning of the previous proof. ■

This means that, given two profiles of preferences $\left(u^i\right)_{i\in\mathcal{I}}$ and $\left(\widetilde{u}^i\right)_{i\in\mathcal{I}}$ and an asset structure V, if $\mathbf{M}_{(u^i)_{i\in\mathcal{I}}} = \mathbf{M}_{(\widetilde{u}^i)_{i\in\mathcal{I}}}$, then $M_{(u^i)_{i\in\mathcal{I}}} = M_{(\widetilde{u}^i)_{i\in\mathcal{I}}}$, which is to say that the equilibrium manifold is identified.

We know identify the largest domain on which, given $\mathbf{E} \subseteq \mathbf{M}$, identification is possible.

Lemma 2 *Suppose* $\mathbf{E} \subseteq \mathbf{M}$ *is closed and the closure of* $\mathbf{E}$ *in* $\left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{+}^{NI}$ *is contained in* $\left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI}$. *Then*

1. $E \subseteq M$ *is closed and the closure of* E *in* $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{+}^{NI}$ *is contained in* $\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI}$.

2. $\mathbf{D}_{\mathbf{E}}$ *is closed.*

Proof. This is straightforward. ■

We are now ready for the main theorem regarding the identification of the aggregate demand from the financial markets equilibrium manifold. For this, we will need to strenghten condition (2).

Condition 4 u^i *is continuous,* $\mathbf{C}^1\left(\mathbb{R}_{++}^N\right)$*, monotone, strongly quasi-concave, differentiable strictly monotone (i.e.,* $\forall x \in \mathbb{R}_{++}^N$*,* $Du^i(x) >> 0$*) and diferenciable strictly concave (*$\forall x \in \mathbb{R}_{++}^N$*,* $D^2 u^i(x)$ *is negative definite) and satisfies that for all* $x \in \mathbb{R}_{++}^N$*,* $\left\{x' \in \mathbb{R}_{+}^N \middle| u^i(x') \geq u^i(x)\right\} \subseteq \mathbb{R}_{++}^N$.

Theorem 8 *1. Suppose* $\mathbf{E} \subseteq \mathbf{M}$ *is closed and the closure of* $\mathbf{E}$ *in* $\left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{+}^{NI}$ *is contained in* $\left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI}$. *If* $\mathbf{E}$ *identifies* $\mathbf{F}$ *over* $\mathbf{D}$*, then* $\mathbf{D} \subseteq \mathbf{D}_E$.

2. *If* $\mathbf{E} \subseteq \mathbf{M}$ *strongly identifies* $\mathbf{F}$ *over* $\mathbf{D}$*, then* $\mathbf{D} \subseteq \mathbf{D}_E$.

Proof. For the first part, suppose it's not true. Let $(\overline{p}, \overline{q}, \overline{w}) \notin \mathbf{D}_E$. Define $\left(\widehat{w}^i = \mathbf{f}^i(\overline{p}, \overline{q}, \overline{w}^i)\right)_{i \in \mathcal{I}}$, then $(\overline{p}, \overline{q}, \widehat{w}) \notin \mathbf{E}$. Now, define $\pi : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^N \to \mathbb{R}_{++}^{S+1}$ by

$$\pi(p, q, w) = \left(\frac{\frac{\partial u^1\left(\mathbf{f}^i(p,q,w)\right)}{\partial x_{s,1}}}{\frac{\partial u^1\left(\mathbf{f}^1(p,q,w)\right)}{\partial x_{0,1}}} \right)_{s=0,1..S},$$

then π is continuous and it is well know that $q = \pi(p, q, w) V$. Let $\overline{\pi} = \pi(\overline{p}, \overline{q}, \overline{w}^1)$ hence $\overline{q} = \pi(\overline{p}, \overline{q}, \overline{w}^1) V$ and then $\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^S, \widehat{w}\right) \notin E \Rightarrow \left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^S, \widehat{w}\right) \notin D_E$. By lemma (***) we can define the following function $\Phi(p, q, w)$ as in the prove of Theorem (***).

$$\Phi(p, w) = F(p, w) + \Delta(p, w) \frac{F_{0,2}(p, w)}{2} \begin{bmatrix} p_{0,2} & -1 & 0 & \ldots & 0 \end{bmatrix}_{N \times 1}^{\top}.$$

By construction, Φ satisfies all the properties that characterize identification from the no-arbitrage equilibrium, $\left\{(p, w) \in \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{NI} \middle| \Phi(p, w) = \sum_{i \in \mathcal{I}} w^i\right\} \supseteq E$ and $\Phi\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^S, \widehat{w}\right) \neq F\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^S, \widehat{w}\right)$ (otherwise, $\Delta\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^S, \widehat{w}\right) = 0 \Rightarrow \left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^S, \widehat{w})) \in D_E$.

Now define the following function $\mathbf{\Phi} : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI} \to \mathbb{R}_{++}^N$ from Φ:

$$\mathbf{\Phi}(p, q, w) = \Phi\left(\left(\pi_s\left(p, q, w^1\right) p_s\right)_{s=0}^S, w\right)$$

Then $\mathbf{\Phi}$ satisfies all properties that characterize identification in the financial markets manifold (since there is nothing new in this prove we differ it to an Appendix). Moreover,

$$\left\{ (p, q, w) \in \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{NI} \middle| \, \mathbf{\Phi}(p, q, w) = \sum_{i \in \mathcal{I}} w^i \right\} \supseteq \mathbf{E}.$$

To prove this, take $(p, q, w) \in \mathbf{E}$ which is the same as:

$$\left(\left(\frac{1}{\pi_s\left(p, q, w^1\right)} \pi_s\left(p, q, w^1\right) p_s \right)_{s=0}^{S}, \pi\left(p, q, w^1\right) V, w \right) \in \mathbf{E}$$

and by definition, this implies $\left(\pi_s\left(p, q, w^1\right) p_s\right)_{s=0}^{S}, w) \in E$ (recall the normalization, $p \in \left(\mathcal{S}_{++}^{L-1}\right)^{S+1}$) therefore $\Phi\left(\left(\pi_s\left(p, q, w^1\right) p_s\right)_{s=0}^{S}, w\right) = \sum_{i \in \mathcal{I}} w^i \Rightarrow \mathbf{\Phi}(p, q, w) = \sum_{i \in \mathcal{I}} w^i$.

Finally, by hyphotesis $\mathbf{E} \subseteq \mathbf{M}$ strongly identifies $\mathbf{F}$ over $\mathbf{D}$ therefore, $\mathbf{\Phi}(\overline{p}, \overline{q}, \overline{w}) = \mathbf{F}(\overline{p}, \overline{q}, \overline{w})$ but by construction, $\mathbf{\Phi}(\overline{p}, \overline{q}, \overline{w}) = \mathbf{\Phi}\left((\overline{p}, \overline{q}, \widehat{w}\right)$ and $\mathbf{F}(\overline{p}, \overline{q}, \overline{w}) = \mathbf{F}(\overline{p}, \overline{q}, \widehat{w})$ and by definition $\mathbf{\Phi}\left(\overline{p}, \overline{q}, \widehat{w}\right) = \Phi\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^{S}, \widehat{w}\right)$ and $\mathbf{F}(\overline{p}, \overline{q}, \widehat{w}) = F\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^{S}, \widehat{w}\right)$ therefore $\Phi\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^{S}, \widehat{w}\right) \neq F\left(\left(\overline{\pi}_s \overline{p}_s\right)_{s=0}^{S}, \widehat{w}\right)$ a contradiction.

For the second part the argument is similar. ■

Remark 1 *Strictly speaking, part* (2) *does not require strenghtening condicion* (2) *on utility fucntions.*

4.2 From Aggregate Demand to Individual Demands

In this section we briefly discuss, for simplicity, the case of global identification.

We say that **aggregate demand identifies individual demands globally** if, given $\mathbf{F}$, $\left(\mathbf{f}^i\right)_{i \in \mathcal{I}}$ is the only profile of functions $\left(\boldsymbol{\varphi}^i : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{N} \rightarrow \mathbb{R}_{++}^{N}\right)_{i \in \mathcal{I}}$, such that $\sum_{i \in \mathcal{I}} \boldsymbol{\varphi}^i = F$.and the profile of functions $\left(\varphi^i : \mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{N} \rightarrow \mathbb{R}_{++}^{N} \in \mathbf{C}^3\left(\mathcal{S}_{++}^{N-1} \times \mathbb{R}_{++}^{N}\right)\right)_{i \in \mathcal{I}}$ defined by $\left(\varphi^i(p, w) = \boldsymbol{\varphi}^i\left(\left(\frac{1}{p_{s,1}} p_s\right)_{s=0}^{S},\right.\right.$ Walras's law and Slutsky symmetry.

Theorem 9 *Aggregate demand identifies individual demands globally.*

Proof. Suppose it doesn't. Then there exists a profile of function $\left(\boldsymbol{\varphi}^i : \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{N} \rightarrow \mathbb{R}_{++}^{N}\right)_{i \in \mathcal{I}}$ that satisfy all conditions that characterize individual identification from the aggregate demand in financial markets, such that for some $j \in \mathcal{I}$, $\boldsymbol{\varphi}^j \neq \mathbf{f}^j$.In particular, there exists $(p, q, w) \in \left(\mathcal{S}_{++}^{L-1}\right)^{S+1} \times \mathcal{Q} \times \mathbb{R}_{++}^{N}$ such that $\boldsymbol{\varphi}^j\left(p, q, w\right) \neq \mathbf{f}^j\left(p, q, w\right)$. Let $\pi = (\pi_0, \pi_{\mathbf{1}}) \in \{1\} \times \mathbb{R}_{++}^{S}$ be such that $q = \pi_{\mathbf{1}} V$ then by definition $\varphi^j(\left(\frac{p_s}{\pi_s}\right)_{s=0}^{S}, w) \neq f^j(\left(\frac{p_s}{\pi_s}\right)_{s=0}^{S}, w)$ but f^j defined by $f^j(p, w) =$

$\mathbf{f}^j\left(\left(\frac{1}{p_{s,1}}p_s\right)_{s=0}^S, \sum_{s=1}^S V_s\left(p_{s,1}\right), w\right)$ is the same as the individual demands introduced in the first part of the paper (in terms of present value prices). Therefore, by theorem (***) $\varphi^j(\left(\frac{p_s}{\pi_s}\right)_{s=0}^S, w) = f^j(\left(\frac{p_s}{\pi_s}\right)_{s=0}^S, w)$ a contradiction. ■

5 Concluding remarks

We have shown that enough information on how prices respond to income shocks can pin down, in a unique manner, unobserved individual information.

When there is global information about the equilibria, given unobserved preferences and observed asset structure, one can identify the aggregate demand function. This is a remarkable property of the competitive model: the roots of a function (the aggregate excess demand) contain as much information as the whole function itself. Less than global knowledge will, obviously, give less comprehensive information about the aggregate demand. These results obtain as a consequence of simple properties of the model, namely (i) Walras's law, (ii) the fact that, for each individual, if two possible endowments have the same market value and give the same financial constraint, then they give the same demand, and (iii) no-trade equilibria immediately inform about the aggregate demand.

Then we show that aggregate demand can be used to recover, uniquely, individual demands. This requires that income effects be different across commodities, which in turn requires that observed domains allow for perturbations of all possible endowments, which is indeed a restrictive assumption (in particular, as it requires that some income effects do not vanish). Again, local information gives local identification, while the same is true for global information.

The results do not require that preferences be separable across states, but, if one is willing to assume that they are, then they have the implication that equilibrium prices contain the same information as the profile of individual preferences (by Geanakoplos and Polemarchakis [11]). Namely, when going from preferences to equilibrium prices, we first optimize, then aggregate and then solve for market clearing; the results imply that after all this transformations we still have essentially the same information as at the beginning, which contrasts with the anything-goes intuition that was derived from the Sonnenschein-Mantel-Debreu literature.

From a more practical perspective, one would hope that these results implied that readily available market information suffices for the unambiguous determination of the welfare effects of economic policy, something that is desirable (Geanakoplos and Polemarchakis [10]) but far from obvious (Donsimoni and Polemarchakis [8]). But, for this to be true, identification results based on more realistic data are necessary. On the one hand, identification of Pareto improving policies has been shown to fail when price effects across commodities are unknown (Geanakoplos and Polemarchakis [11]) and when only a finite data set is available for a nonstationary economy (work in progress by A. Carvajal and H.M. Polemarchakis). Moreover, identification, or lack thereof, in the presence of production, or for stationary economies, remains an open problem.

6 Appendix 1: Regularity

It follows from Lewbel [14] and Blanks, Blundell Lewbel [2] that there exists $u_0 : \mathbb{R}^3_+ \longrightarrow \mathbb{R}$ such that $f : \mathbb{R}^3_{++} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}^3_{++}$ is given by

$$f_l(p,m) = \frac{m}{p_l}\left(A_l(p) + B_l(p)\log\left(\frac{m}{a(p)}\right) + C_l(p)\log\left(\frac{m}{a(p)}\right)^2\right),\ l = 1,2,3,$$

are the solutions to the problem

$$\max_x u_0(x) \text{ s.t. } p \cdot x = m,$$

where A_l, B_l and C_l are homogeneous of degree zero in p and a is homogeneous of degree 1 in p (these two conditions guarantee that f_l is homogeneous of degree zero in p and m) and

$$\sum_{l=1}^{3} A_l(p) + \sum_{l=1}^{3} B_l(p)\log\left(\frac{m}{a(p)}\right) + \sum_{l=1}^{3} C_l(p)\log\left(\frac{m}{a(p)}\right)^2 = 1,$$

for all p and m (which is necessary for Walras's law).[8]

Now, suppose that there are three commodities and two states of nature so that, for some functions u_1 and u_2, $u(x_0, x_1, x_2) = u_0(x_0) + \min\{u_1(x_1), u_2(x_2)\}$.

Suppose that there is only one asset, $V = [1 \quad -1]^\top$ and define

$$v_s(p,m) = \max_x u_s(x) \text{ s.t. } p \cdot x = m$$

and

$$v(p,w) = \max_x u(x) \text{ st. } \left\{ \begin{array}{c} p \cdot x = p \cdot w \\ \begin{bmatrix} p_1 \cdot (x_1 - w_1) \\ p_2 \cdot (x_2 - w_2) \end{bmatrix} \in \langle V(\mathbf{p_1}) \rangle \end{array} \right. .$$

By construction,

$$\begin{aligned} v(p,w) &= \max_{m_0,m_1,m_2} \left(v_0(p_0,m_0) + \min\{v_1(p_1,m_1), v_2(p_2,m_2)\}\right) \\ & \quad \text{st.} \left\{ \begin{array}{c} m_0 + m_1 + m_2 = p \cdot w \\ \begin{bmatrix} m_1 - p_1 \cdot w_1 \\ m_2 - p_2 \cdot w_2 \end{bmatrix} \in \langle V(\mathbf{p_1}) \rangle \end{array} \right. \end{aligned}$$

[8] For example, take: $A_i(P) = 1/3$, $B_l(P) = b_l$, $C_l(P) = c_i P_1 \|P\|^{-1}$, with $b_1 + b_2 + b_3 = 0$ and $c_1 + c_2 + c_3 = 0$.

Claim 1 *Let $m_0(p,w)$, $m_1(p,w)$ and $m_2(p,w)$ denote the solution of*

$$\max_{m_0,m_1,m_2} \left(v_0(p_0,m_0) + \min\left\{v_1(p_1,m_1), v_2(p_2,m_2)\right\}\right)$$
$$st. \left\{\begin{array}{l} m_0+m_1+m_2 = p\cdot w \\ \left[\begin{array}{c} m_1 - p_1\cdot w_1 \\ m_2 - p_2\cdot w_2 \end{array}\right] \in \langle V(p_{\mathbf{1}})\rangle \end{array}\right.$$

and let $(p,\widetilde{w})$ be such that $v_1(p_1,m_1(p,\widetilde{w})) = v_2(p_2,m_2(p,\widetilde{w}))$. Then, for every $w_0 > \widetilde{w}_0$,

$$\frac{\partial m_0}{\partial w_{0,1}}(p,(w_0,\widetilde{w}_{\mathbf{1}})) = 1.$$

Proof. Since $w_0 > \widetilde{w}_0$, $v_1(p_1,m_1(p,(w_0,\widetilde{w}_{\mathbf{1}}))) = v_2(p_2,m_2(p,(w_0,\widetilde{w}_{\mathbf{1}})))$. Consider a perturbation $dw_{0,1}$ to $w_{0,1}$. Notice that by construction of $V(p_{\mathbf{1}})$, $dm_1 > 0 \Longrightarrow dm_2 < 0 \Longrightarrow (dv_1 > 0 \text{ and } dv_2 < 0)$, whereas $dm_2 > 0 \Longrightarrow dm_1 < 0 \Longrightarrow (dv_1 < 0 \text{ and } dv_2 > 0)$, which cannot be optimal, given that v_s is increasing in m. ■

Now, suppose that for every p, there exists $\widetilde{w}$ such that $v_1(p_1,m_1(p,\widetilde{w})) = v_2(p_2,m_2(p,\widetilde{w}))$, and consider only w with $w_0 > \widetilde{w}_0$ and $w_{\mathbf{1}} = \widetilde{w}_{\mathbf{1}}$.

Let $f_{s,l}(p,w)$ denote optimal demands. By construction, for all l, $f_{0,l}(p,w) = f_l(p_0,m_0(p,w))$, so

$$\frac{\partial f_{0,l}}{\partial w_{0,1}}(p,w) = \frac{\partial f_l}{\partial m}(p_0,m_0(p,w))\frac{\partial m_0}{\partial w_{0,1}}(p,w) = \frac{\partial f_l}{\partial m}(p_0,m_0(p,w)),$$

where the second equality follows from the claim. It then follows (using the claim again) that $\frac{\partial^2 f_{0,l}}{\partial w_{0,1}^2}(p,w) = \frac{\partial^2 f_l}{\partial m^2}(p_0,m_0(p,w))$ and $\frac{\partial^3 f_{0,l}}{\partial w_{0,1}^3}(p,w) = \frac{\partial^3 f_l}{\partial m^3}(p_0,m_0(p,w))$.

Now, the rank of system $f(p_0,m)$ is, by definition, the rank of:

$$\left[\begin{array}{ccc} A_1(p_0) & B_1(p_0) & C_1(p_0) \\ A_2(p_0) & B_2(p_0) & C_2(p_0) \\ A_3(p_0) & B_3(p_0) & C_3(p_0) \end{array}\right].$$

If B_l and C_l are zero then the system is of rank 1, the utility function is homothetic and clearly the regularity condition does not hold. If $B_2(p_0)C_3(p_0) - B_2(p_0)C_3(p_0) \neq 0$ the system has rank at least 2. Below, we prove that, for this case, the regularity condition holds.

Set $p_{0,1} = 1$. Then,

$$\frac{\partial f_l}{\partial m}(p_0, m) = \frac{1}{p_{0,l}}\left(A_l(p_0) + B_l(p_0)\log\left(\frac{m}{a(p_0)}\right) + C_l(p_0)\log\left(\frac{m}{a(p_0)}\right)^2\right) + \frac{1}{p_{0,l}}\left(B_l(p_0) + 2C_l(p_0)\log\left(\frac{m}{a(p_0)}\right)\right),$$

$$\frac{\partial^2 f_l}{\partial m^2}(p_0, m) = \frac{1}{p_{0,l}m}\left(B_l(p_0) + 2C_l(p_0)\left(\log\left(\frac{m}{a(p_0)}\right) + 1\right)\right)$$

and

$$\frac{\partial^3 f_1}{\partial m^3}(p_0, m) = -\frac{1}{p_{0,l}\,(m)^2}\left(B_l(p_0) + 2C_l(p_0)\log\left(\frac{m}{a(p_0)}\right)\right).$$

It follows that the regularity condition is satisfied if

$$\begin{vmatrix} \frac{1}{p_{0,2}m}\left(B_2(p_0) + 2C_2(p_0)\left(\log\left(\frac{m}{a(p_0)}\right) + 1\right)\right) & \frac{1}{p_{0,3}m}\left(B_3(p_0) + 2C_3(p_0)\left(\log\left(\frac{m}{a(p_0)}\right) + 1\right)\right) \\ -\frac{1}{p_{0,2}m^2}\left(B_2(p_0) + 2C_2(p_0)\log\left(\frac{m}{a(p_0)}\right)\right) & -\frac{1}{p_{0,3}m^2}\left(B_3(p_0) + 2C_3(p_0)\log\left(\frac{m}{a(p_0)}\right)\right) \end{vmatrix} \neq 0$$

$$\Leftrightarrow \begin{vmatrix} \left(B_2(p_0) + 2C_2(p_0)\left(\log\left(\frac{m}{a(p_0)}\right) + 1\right)\right) & \left(B_3(p_0) + 2C_3(p_0)\left(\log\left(\frac{m}{a(p_0)}\right) + 1\right)\right) \\ \left(B_2(p_0) + 2C_2(p_0)\log\left(\frac{m}{a(p_0)}\right)\right) & \left(B_3(p_0) + 2C_3(p_0)\log\left(\frac{m}{a(p_0)}\right)\right) \end{vmatrix} \neq 0$$

$$\Leftrightarrow \begin{vmatrix} C_2(p_0) & C_3(p_0) \\ \left(B_2(p_0) + 2C_2(p_0)\log\left(\frac{m}{a(p_0)}\right)\right) & \left(B_3(p_0) + 2C_3(p_0)\log\left(\frac{m}{a(p_0)}\right)\right) \end{vmatrix} \neq 0$$

$$\Leftrightarrow C_2(p_0)B_3(p_0) - C_3(p_0)B_2(p_0) \neq 0$$

A rank 3 system clearly satisfies this condition and, if the condition is satisfied, then the rank is at least 2.[9]

7 Appendix 2: Duality in Incomplete Markets

Fix an individual i, but ignore its superindex.

Define $U = \left\{\mu \in \mathbb{R} : \left(\exists x \in \mathbb{R}^N_{++}\right) : u(x) = \mu\right\}$.

For each $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$, define

$$D(w_{\mathbf{1}}, \mu) = \left\{p \in \mathcal{S}^{N-1}_{++} \middle| \left(\exists x \in \mathbb{R}^N_{++}\right) : u(x) = \mu \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}})\rangle\right\}.$$

[9] For the special case considered in previous footnote, the regularity condition is equivalent to:

$$(c_1 b_2 - c_2 b_1)\, P_1 \neq 0$$

which is true for all (P_1, P_2) as long as the demand system has rank 3 or rank 2, with $c_1 b_2 - c_2 b_1 \neq 0$.

Proposition 1 *For each* $(w_{\mathbf{1}},\mu)\in\mathbb{R}^{LS}_{++}\times U$, $D(w_{\mathbf{1}},\mu)$ *is diffeomorphic to*

$$\left\{((p_{0,2},...,p_{0,L}),p_{\mathbf{1}})\in\mathbb{R}^{N-1}_{++}\middle|\left(\exists x\in\mathbb{R}^{N}_{++}\right):u(x)=\mu \text{ and } p_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(p_{\mathbf{1}})\rangle\right\},$$

which is open.

Proof. Let D denote the latter set. That $D(w_{\mathbf{1}},\mu)$ and D are diffeomorphic is straightforward. We now show that D is open. Let $p\in D$. By definition, for some $x\in\mathbb{R}^{N}_{++}$, $u(x)=\mu$ and $p_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(p_{\mathbf{1}})\rangle$, whereas using the implicit function theorem, since $\partial_{x_0}(u(x))\in\mathbb{R}^{L}_{++}$, for some $\varepsilon>0$, $B_{\varepsilon}(x_{\mathbf{1}})\subseteq\mathbb{R}^{LS}_{++}$ and

$$(\forall\widetilde{x}_{\mathbf{1}}\in B_{\varepsilon}(x_{\mathbf{1}}))\left(\exists\widetilde{x}_0\in\mathbb{R}^{L}_{++}\right):u(\widetilde{x}_0,\widetilde{x}_{\mathbf{1}})=u(x).$$

Given that $\forall(s,l)\in\{1,...,S\}\times\{1,...,L\}$, $\lim_{\delta\longrightarrow 0}\frac{\delta(w_{s,l}-x_{s,l})}{\frac{p_{s,l}}{p_{s,1}}+\delta}=0$, there exists $\overline{\delta}_{s,l}>0$ such that

$$|\delta|<\overline{\delta}_{s,l}\Longrightarrow\frac{|\delta|\,|w_{s,l}-x_{s,l}|}{\left|\frac{p_{s,l}}{p_{s,1}}+\delta\right|}<\frac{\varepsilon}{\sqrt{LS}}.$$

Define $\overline{\delta}=\min_{(s,l)\in\{1,...,S\}\times\{1,...,L\}}\left\{\overline{\delta}_{s,l}\right\}$, and consider the function $h:\mathbb{R}^{N-1}_{++}\to\mathbb{R}^{N-1}_{++}$, $h(p)=\left((p_{0,2},...,p_{0,L}),\frac{p_1}{p_{1,1}},...,\frac{p_S}{p_{S,1}}\right)$. The function h is continuous, therefore there is a $\delta>0$ such that for all $p'\in B_{\delta}(p)$, $\|h(p')-h(p)\|<\overline{\delta}$, in particular $\left|\frac{p'_{s,l}}{p'_{s,1}}-\frac{p_{s,l}}{p_{s,1}}\right|<\overline{\delta}$.

Define $x'_{\mathbf{1}}\in\mathbb{R}^{LS}$ as follows: $\forall(s,l)\in\{1,...,S\}\times\{1,...,L\}$,

$$x'_{s,l}=\frac{\frac{p_{s,l}}{p_{s,1}}x_{s,l}+\left(\frac{p'_{s,l}}{p'_{s,1}}-\frac{p_{s,l}}{p_{s,1}}\right)w_{s,l}}{\frac{p'_{s,l}}{p'_{s,1}}}.$$

Then, $\left|x'_{s,l}-x_{s,l}\right|=\left|\frac{p'_{s,l}}{p'_{s,1}}-\frac{p_{s,l}}{p_{s,1}}\right||w_{s,l}-x_{s,l}|\left|\frac{p'_{s,l}}{p'_{s,1}}\right|^{-1}$, and, since $p'\in B_{\delta}(p)$, it follows that $\left|\frac{p'_{s,l}}{p'_{s,1}}-\frac{p_{s,l}}{p_{s,1}}\right|<\overline{\delta}\leqslant\overline{\delta}_{s,l}$, from where $\left|x'_{s,l}-x_{s,l}\right|<\frac{\varepsilon}{\sqrt{LS}}$ and, hence, $\|x'_{\mathbf{1}}-x_{\mathbf{1}}\|<\varepsilon$. This implies that $x'_{\mathbf{1}}\in B_{\varepsilon}(x_{\mathbf{1}})$ and, therefore, that there exists $x'_0\in\mathbb{R}^{L}_{++}$ such that $u(x'_0,x'_{\mathbf{1}})=u(x)$.

Finally, by construction, $(\frac{p'_1}{p'_{1,1}},...\frac{p'_S}{p'_{S,1}})\boxdot(x'_{\mathbf{1}}-w_{\mathbf{1}})=(\frac{p_1}{p_{1,1}},...\frac{p_S}{p_{S,1}})\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V\rangle$, and, hence, $p'\in D$. ■

For each $(w_{\mathbf{1}},\mu)\in\mathbb{R}^{LS}_{++}\times U$ such that $D(w_{\mathbf{1}},\mu)\neq\varnothing$, define the Hicksian demand function $h(\cdot;w_{\mathbf{1}},\mu):D(w_{\mathbf{1}},\mu)\longrightarrow\mathbb{R}^{N}_{++}$, as

$$h(p;w_{\mathbf{1}},\mu)=\arg\min\left\{\sum_{s=0}^{S}p_s\cdot x_s\middle|u(x)\geq\mu\text{ and }p_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(p_{\mathbf{1}})\rangle\right\},$$

and the expenditure function $e(\cdot; w_{\mathbf{1}}, \mu) : D(w_{\mathbf{1}}, \mu) \longrightarrow \mathbb{R}$, as $e(p; w_{\mathbf{1}}, \mu) = p \cdot h(p; w_{\mathbf{1}}, \mu)$.

By condition 2, $h(p; w_{\mathbf{1}}, \mu)$ is well defined into $\mathbb{R}^N_{++}$.

Now, for each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, define $\mathbf{D}(w_{\mathbf{1}}) \subseteq \mathcal{S}^{N-1}_{++} \times \mathbb{R}_{++}$ as follows:

$$\mathbf{D}(w_{\mathbf{1}}) = \left\{ (p, m) \in \mathcal{S}^{N-1}_{++} \times \mathbb{R}_{++} \middle| \left(\exists x \in \mathbb{R}^N_{++}\right) : \begin{pmatrix} \sum_{s=0}^{S} p_s \cdot x_s \leqslant m \\ p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle \end{pmatrix} \right\}.$$

Proposition 2 *For each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, $\mathbf{D}(w_{\mathbf{1}})$ is diffeomorphic to*

$$\left\{ \left(\left(\left(p_{0,2}, ..., p_{0,L}\right), p_{\mathbf{1}}\right), m\right) \in \mathbb{R}^{N-1}_{++} \times \mathbb{R}_{++} \middle| \left(\exists x \in \mathbb{R}^N_{++}\right) : \begin{pmatrix} \sum_{s=0}^{S} p_s \cdot x_s \leqslant m \\ p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle \end{pmatrix} \right\},$$

which is nonempty and open.

Proof. This is straightforward. ■

For each $w_{\mathbf{1}} \in \mathbb{R}^{LS}_{++}$, define the conditional individual demand function $\widetilde{f}(\cdot, \cdot; w_{\mathbf{1}}) : \mathbf{D}(w_{\mathbf{1}}) \longrightarrow \mathbb{R}^N_{++}$ as

$$\widetilde{f}(p, m; w_{\mathbf{1}}) = \arg\max \left\{ u(x) \middle| \sum_{s=0}^{S} p_s \cdot x_s \leq m \text{ and } p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle \right\}.$$

By condition 2, any solution to the maximization problem above lies in $\mathbb{R}^N_{++}$ and is unique. Obviously, $f(p, \sum_{s=0}^{S} p_s \cdot w_s; w_{\mathbf{1}}) = f(p, w)$.

Proposition 3 *1. For every $w = (w_0, w_{\mathbf{1}}) \in \mathbb{R}^N_{++}$ and every $p \in \mathcal{S}^{N-1}_{++}$, $u(f(p, w)) \in U$, $p \in D(w_{\mathbf{1}}, u(f(p, w)))$ and $h(p; w_{\mathbf{1}}, u(f(p, w))) = f(p, w)$;*

2. *Given $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$, for every $p \in D(w_{\mathbf{1}}, \mu)$, $f(p, h(p; w_{\mathbf{1}}, \mu)) = h(p; w_{\mathbf{1}}, \mu)$;*

3. *Given $(w_{\mathbf{1}}, \mu) \in \mathbb{R}^{LS}_{++} \times U$, for every $p \in D(w_{\mathbf{1}}, \mu)$, $(p, e(p, w, \mu)) \in \mathbf{D}(w_{\mathbf{1}})$ and $\widetilde{f}(p, e(p, w, \mu); w_{\mathbf{1}}) = h(p; w_{\mathbf{1}}, \mu)$.*

Proof. Part (1) is straightforward: argue by contradiction and use strict monotonicity of the utility function.

Given that u is continuous, for parts (2) and (3) it suffices to prove that $u(h(p; w_{\mathbf{1}}, \mu)) = \mu$. For this, suppose not: $u(h(p; w_{\mathbf{1}}, \mu)) > \mu$. Define $x = h(p; w_{\mathbf{1}}, \mu) - (\varepsilon, 0, ..., 0)$, where $\varepsilon \in \mathbb{R}_{++}$. By construction, $x_{\mathbf{1}} = h_{\mathbf{1}}(p; w_{\mathbf{1}}, \mu)$, from where $p_{\mathbf{1}} \boxdot (x_{\mathbf{1}} - w_{\mathbf{1}}) \in \langle V(p_{\mathbf{1}}) \rangle$, and $\sum_{s=0}^{S} p_s \cdot x_s < e(p; w_{\mathbf{1}}, \mu)$, whereas since $h(p;w_{\mathbf{1}},\mu)\in\mathbb{R}^N_{++}$, for ε small enough $x\in\mathbb{R}^N_+$ and, by continuity, $u(x)\geqslant\mu$, which is a contradiction. ■

Proposition 4 (Shepard's Lemma) *For every* $(w_{\mathbf{1}},\mu)\in\mathbb{R}^{LS}_{++}\times U$*, the function* $e(\cdot;w_{\mathbf{1}},\mu):D(w_{\mathbf{1}},\mu)\longrightarrow\mathbb{R}_{++}$ *is differentiable and* $\partial_p e(p;w_{\mathbf{1}},\mu)=h(p;w_{\mathbf{1}},\mu)$.

Proof. This is an immediate consequence of the Duality Theorem (see Mas-Colell et al. [16, Proposition 3.F.1]): let $K=\left\{x\in\mathbb{R}^N_+\middle|\, u^i(x)\geq\mu \text{ and } p_{\mathbf{1}}\boxdot(x_{\mathbf{1}}-w_{\mathbf{1}})\in\langle V(p_{\mathbf{1}})\rangle\right\}$; then, K is closed and $e(p;w_{\mathbf{1}},\mu)$ is the support function of K. ■

Proposition 5 *For every* $w_{\mathbf{1}}\in\mathbb{R}^{LS}_{++}$*, the function* $\widetilde{f}(\cdot,\cdot;w_{\mathbf{1}}):\mathbf{D}(w_{\mathbf{1}})\longrightarrow\mathbb{R}^N_{++}$ *is differentiable.*

Proof. This can be argued in the same way as fact 5 in Duffie and Shafer [9]. ■

Proposition 6 (Slutsky Equation in incomplete markets) *Let* $(p,w)\in\mathcal{S}^{N-1}_{++}\times\mathbb{R}^N_+$ *and* $\mu=u(f(p,w))$*. Then,* $h(\cdot;w_{\mathbf{1}},\mu):D(w_{\mathbf{1}},\mu)\longrightarrow\mathbb{R}^N_{++}$ *is differentiable and for all* $(s,l),(s',l')\in(\{0,...,S\}\times\{1,...,L\})\setminus\{(0,1)\}$,

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}=\frac{\partial f_{s,l}(p,w)}{\partial p_{s',l'}}+\frac{\partial f_{s,l}(p,w)}{\partial w_{0,1}}(f_{s',l'}(p,w)-w_{s',l'}).$$

Proof. That $h(\cdot;w_{\mathbf{1}},\mu)$ is differentiable follows from propositions 3 and 5.

Also from proposition 3, $h(p;w_{\mathbf{1}},\mu)=\widetilde{f}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})$. Therefore,

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}=\frac{\partial\widetilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial p_{s',l'}}+\frac{\partial\widetilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial m}\frac{\partial e(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}.$$

By proposition 4,

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}}=\frac{\partial\widetilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial p_{s',l'}}+\frac{\partial\widetilde{f}_{s,l}(p,e(p;w_{\mathbf{1}},\mu);w_{\mathbf{1}})}{\partial m}h_{s',l'}(p;w_{\mathbf{1}},\mu).$$

Now, since $f(p,w)=\widetilde{f}\left(p,\sum_{s=0}^S p_s\cdot w_s;w_{\mathbf{1}}\right)$, then

$$\frac{\partial f_{s,l}(p,w)}{\partial p_{s',l'}}=\frac{\partial\widetilde{f}_{s,l}\left(p,\displaystyle\sum_{s=0}^S p_s\cdot w_s;w_{\mathbf{1}}\right)}{\partial p_{s',l'}}+\frac{\partial\widetilde{f}_{s,l}\left(p,\displaystyle\sum_{s=0}^S p_s\cdot w_s;w_{\mathbf{1}}\right)}{\partial m}w_{s',l'}.$$

Under monotonicity, at $\mu=u^i\left(f^i(p,w)\right)$, $e(p;w_{\mathbf{1}},\mu)=\sum_{s=0}^S p_s\cdot w_s$ and, hence,

$$\frac{\partial f_{s,l}(p,w)}{\partial p_{s',l'}}=\frac{\partial\widetilde{f}_{s,l}(p,e(p,w,\mu);w_{\mathbf{1}})}{\partial p_{s',l'}}+\frac{\partial\widetilde{f}_{s,l}(p,e(p,w,\mu);w_{\mathbf{1}})}{\partial m}w_{s',l'}.$$

Solving for $\frac{\partial \widetilde{f}_{s,l}(p,e(p,w,\mu);w_{\mathbf{1}})}{\partial p_{s',l'}}$ and replacing gives

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}} = \frac{\partial f_{s,l}\left(p,w\right)}{\partial p_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(p,e\left(p,w,\mu\right);w_{\mathbf{1}}\right)}{\partial m}(h_{s',l'}(p;w_{\mathbf{1}},\mu) - w_{s',l'}).$$

By proposition 3, since $\mu = u(f(p,w))$,

$$\frac{\partial h_{s,l}(p;w_{\mathbf{1}},\mu)}{\partial p_{s',l'}} = \frac{\partial f_{s,l}\left(p,w\right)}{\partial p_{s',l'}} + \frac{\partial \widetilde{f}_{s,l}\left(p,\sum_{s=0}^{S} p_s \cdot w_s;w_{\mathbf{1}}\right)}{\partial m}(f_{s',l'}\left(p,w\right) - w_{s',l'}).$$

Finally, notice that $\frac{\partial f_{s,l}(p,w)}{\partial w_{0,1}} = \frac{\partial \widetilde{f}_{s,l}\left(p,\sum_{s=0}^{S} p_s \cdot w_s;w_{\mathbf{1}}\right)}{\partial m}$, so substitution gives the desired result. ■

8 Appendix 3: Complements prove of Theorem (***)

In this appendix we prove that, if we define $\mathbf{\Phi}$ by

$$\mathbf{\Phi}\left(p,q,w\right) = \Phi\left(\left(\pi_s\left(p,q,w^1\right)p_s\right)_{s=0}^{S},w\right),$$

and where Φ satisfies all the properties that characterize identification in the no-arbitrage equilibrium manifold then $\mathbf{\Phi}$ satisfies all properties that characterize aggregate demand in the financial markets equilibrium manifold.

We know that Φ satisfies Walras's law and $\exists z \in \mathbb{R}^J$ such that

$$\begin{aligned}
\left(\pi_s\left(p,q,w^1\right)\right)_{s=1}^{S} \boxdot p_{\mathbf{1}} \boxdot \left(\Phi_{\mathbf{1}}(\left(\pi_s\left(p,q,w^1\right)p_s\right)_{s=0}^{S},w) - \sum_{i\in\mathcal{I}} w_{\mathbf{1}}^i\right) &= V(\left(\pi_s\left(p,q,w^1\right)\right)_{s=1}^{S} \boxdot p_{\mathbf{1}})z \\
&\Rightarrow \\
\left(\pi_s\left(p,q,w^1\right)\right)_{s=1}^{S} \boxdot p_{\mathbf{1}} \boxdot \left(\mathbf{\Phi}_{\mathbf{1}}\left(p,q,w\right) - \sum_{i\in\mathcal{I}} w_{\mathbf{1}}^i\right) &= V(\left(\pi_s\left(p,q,w^1\right)\right)_{s=1}^{S} \boxdot p_{\mathbf{1}})z \qquad (*) \\
&\Rightarrow \\
p_{\mathbf{1}} \boxdot \left(\mathbf{\Phi}_{\mathbf{1}}\left(p,q,w\right) - \sum_{i\in\mathcal{I}} w_{\mathbf{1}}^i\right) &= V(p_{\mathbf{1}})z = Vz
\end{aligned}$$

(recall the normalization of $p \in \left(\mathcal{S}_{++}^{L-1}\right)^{S+1}$) which is one thing we had to prove. By Walras law:

$$\left(\pi_s\left(p,q,w^1\right)p_s\right)_{s=0}^{S} \cdot \left(\Phi\left(\left(\pi_s\left(p,q,w^1\right)p_s\right)_{s=0}^{S},w\right) - \sum_{i\in\mathcal{I}} w^i\right) = 0$$

and summing over $s = 1, ...S$ in equation $(*)$ and subtracting the result from the previous equation we get:

$$p_0 \left(\mathbf{\Phi}_0 \left(p, q, w \right) - \sum_{i \in \mathcal{I}} w_0^i \right) = -qz$$

which isthe last thing we had to prove.

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