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The harmonic sequence paradox reconsidered

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Abstract

According to the harmonic sequence paradox (Blavatskyy 2006), an expected utility decision maker's willingness to pay for a gamble whose expected payoffs evolve according to the harmonic series is finite if and only if his marginal utility of additional income becomes eventually zero. Since the assumption of zero marginal utility is implausible, expected utility theory (as well as cumulative prospect theory) does apparently do a bad job in describing this decision behavior. The present note demonstrates that the harmonic sequence paradox only applies to time-patient but not to time-impatient (risk-neutral) expected utility decision makers.

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1 The harmonic sequence paradox

The so-called harmonic sequence paradox, due to Blavatskyy (2006), is based on the following hypothetical gamble G:

> "Consider an urn that initially contains one white and one black ball. An individual draws one ball from this urn and receives one dollar (nothing) should the ball be white (black). Whatever the drawn ball happens to be, it is subsequently put back into the urn. Additionally, one more black ball is added to the urn. The individual then draws one ball again and the cycle continues ad infinitum. At each iteration, the drawn white ball pays off one dollar and the number of black balls is increased by one." Blavatskyy (2006, p. 28)

Formally, consider the state space

$$X = \times_{t=1}^{\infty} \{0, 1\} \tag{1}$$

with generic element $x = (x_1, x_2, ...)$ and define the coordinate (random) variable Y_t, $t \in \mathbb{N}$, such that

$$Y_t (x_1, x_2, ...) = x_t.$$

Observe that the gamble G is given as a sequence of independently distributed random variables $(Y_t)_{t\in\mathbb{N}}$ with distribution

$$P_t (Y_t = 1) = \frac{1}{t+1} \tag{2}$$

and define by

$$P_{t,...,t+k} = P_t \cdot ... \cdot P_{t+k} \text{ for } k \geq 0 \tag{3}$$

the so-called *finite-dimensional distributions* of the stochastic process $(Y_t)_{t\in\mathbb{N}}$ on $(X, \mathcal{F})$ whereby $\mathcal{F}$ denotes the σ-algebra generated by the following partition of X

$$\mathcal{X} = \{\{x\} \mid x \in X\}. \tag{4}$$

By Kolmogorov's existence theorem (cf. theorem 36.1 in Billingsley 1995 or chapter 15.6 in Aliprantis and Border 2006), there exists a unique additive probability measure P^* on $(X, \mathcal{F})$ such that $(Y_t)_{t\in\mathbb{N}}$ has (3) as finite dimensional distributions.

Given the measure space $(P^*, X, \mathcal{F})$ the monetary outcomes that can be earned by an individual who participates in G for T iterations are described by the $\mathcal{F}$-measurable random variable $Z_T = \sum_{t=1}^{T} Y_t$. Define now the $\mathcal{F}$-measurable (theorem 13.4 in Billingsley 1995) random variable $Z_\infty = \sum_{t=1}^{\infty} Y_t$ with extended range space $\mathbb{R}_+ \cup \{+\infty\}$ and recall that Blavatkyy (proposition 1, 2006) reports the interesting result that

$$P^* (Z_\infty = +\infty) = 1. \tag{5}$$

That is, any individual participating forever in the gamble G will earn with certainty an infinite amount of money.[1]

In spite of this extremly attractive feature of G real-life people are only willing to pay up to some finite amount of money, say M (Blavatskyy considers $M = \$10$ as plausible), for a participation in G. In order to describe such preferences by an expected utility functional we therefore have to assume that

$$\begin{aligned} E[U(Z_\infty), P^*] &= \int_X U(Z_\infty(x))\, dP^*(x) && (6) \\ &= U(+\infty) \cdot 1 && (7) \\ &= U(M) && (8) \end{aligned}$$

whereby (weak) monotonicity of the von Neumann Morgenstern (=vNM) utility U of money would imply that the individual is indifferent between all amounts of money greater or equal than M. This indifference, however, is quite implausible and apparently at odds with the fact that individuals are not willing to pay more than M for a participation in the gamble G. In a nutshell, the harmonic sequence paradox therefore states:

> Suppose that an individual regards a participation in gamble G as equivalent to the random variable Z_∞. If this individual is an expected utility decision maker–or for that matter a cumulative prospect theory decision maker (Tversky and Kahneman 1992)–her willingness to pay for G is finite if and only if her marginal utility of additional monetary income becomes eventually zero.

While the famous St. Petersburg paradox (Bernoulli 1954), respectively the super St. Petersburg paradox (Menger 1934), can be resolved within the expected utility under risk framework by the assumption of a sufficiently concave vNM utility function, respectively by the assumption of a bounded vNM utility function, the above argument demonstrates that such a resolution is not at hand for the harmonic sequence paradox. Two alternative explanations for the harmonic sequence paradox come instead to mind:

[1] Observe that this feature makes G much more attractive to risk-averse individuals than just an infinite expected value of Z_∞.

Either real-life people have a totally wrong (subjective) notion about the "objective" probability measure P^* generated by G or they do not regard the participation in G as equivalent to the random variable Z_∞.

In the following section we offer an explanation of the harmonic sequence paradox based on the latter alternative whereby we consider subjective expected utility decision-makers whose subjective probability measure coincides with P^* but who identify the possible outcomes of the gamble G as infinite streams of monetary payoffs rather than as payoff-points in the range of Z_∞. We further illustrate our argument in section 3 by a simple example involving expected utility decision makers with additivly separable time-preferences. Whenever these decision-makers are sufficiently time-impatient, their willigness to pay for a participation in G is rather moderate. Since time-impatience appears as a rather natural assumption for an individual who is confronted with a gamble that goes on forever, our approach might contribute towards an explanation of why real-life people only offer a finite amount for participating in the gamble G. Moreover, the fact that an individual's willingness to pay for participating in G is rather low might then result from her correct understanding that the expected monetary payoff increases very slowly over time.

2 The general argument

The original statement of the harmonic sequence paradox considers a situation of decision making under risk where the individual has preferences over lotteries with given objective probability distributions. For our purpose it is convenient to slightly change the perspective and consider an individual who is a subjective expected utility decision maker in the sense of Savage (1954). That is, we consider a probability space (P^s, Ω, Σ) where P^s is a subjective probability measure derived from the individual's preferences over Savage-acts which are mappings from the state space Ω into some set of consequences C. Under Savage's axioms such preferences are representable by an expected utility functional, i.e., for any two Savage-acts f, g,

$$f \succeq g \text{ if and only if } E\left[U\left(f\right), P^s\right] \geq E\left[U\left(g\right), P^s\right]$$

for a unique P^s and a unique (up to positive affine transformations) vNM utility function $U : C \to \mathbb{R}$.

In order to recast the harmonic sequence paradox within this Savage framework, we assume that the set of consequences is given as the possible payoff-sequences of G, i.e.,

$$C \equiv X = \times_{t=1}^{\infty} \{0, 1\}. \tag{9}$$

Furthermore, we suppose that the Savage state- and event space are sufficiently rich in the sense that they can reflect the previous section's definitions of X and $\mathcal{F}$. To this end we assume that

$$\Omega = X \times U, \tag{10}$$

with generic element $\omega = ((x_1, x_2, ...), u)$, for some space U whereby the event space

$$\Sigma = \mathcal{F} \otimes \mathcal{U} \tag{11}$$

is given as the product σ-algebra of $\mathcal{F}$ and some σ-algebra $\mathcal{U}$ on U. Observe that within this Savage framework a participation in the gamble G can be defined as the Savage act $f^G : \Omega \to C$ such that

$$f^G((x_1, x_2, ...), u) = (x_1, x_2, ...) \tag{12}$$

whereby the consequence $(x_1, x_2, ...)$ is interpreted as one possible outcome of the stochastic process $(Y_t)_{t\in\mathbb{N}}$ as introduced in the previous section. The following assumption finally connects our subjective model to the harmonic sequence paradox by stating that the individual's subjective probability measure P^s coincides in an apropriate way with the "objective" probability measure P^*.

Assumption 1. *For all $A \in \mathcal{F}$,*

$$P^s(A \times U) = P^*(A).$$

Given this re-interpretation of the harmonic sequence gamble within a Savage framework, the next proposition follows easily.

Proposition 1.

Consider a (weakly) increasing vNM utility function, i.e., $U(c) \geq U(c')$ whenever $c \geq c'$ in the standard vector order.

Then the individual's subjective expected utility of participating in the gamble G is finite if U is bounded from above, i.e., $\sup_C U(c) < +\infty$, whereby

$$E\left[U\left(f^G\right), P^s\right] \leq U(1, 1, ..).$$

Proof. Observe that

$$\begin{aligned}
E\left[U\left(f^G\right), P^s\right] &= \int_\Omega U\left(f^G(\omega)\right) dP^s(\omega) \\
&= \int_{X\times U} U\left(f^G\left(\left(x_1, x_2, ...\right), u\right)\right) dP^s\left(\left(x_1, x_2, ...\right), u\right) \\
&= \int_X U\left(x_1, x_2, ...\right) dP^s\left(\left(x_1, x_2, ...\right)\times U\right) \text{ by (12)} \\
&= \int_X U\left(x_1, x_2, ...\right) dP^*\left(x_1, x_2, ...\right) \text{ by assumption 1} \\
&\leq U(1,1,..)\cdot 1 \text{ by (weak) monotonicity} \\
&< +\infty.
\end{aligned}$$

The final line thereby follows from

$$U(1,1,..) = \sup_C U(c),$$

which is finite by assumption.□

By the above proposition, the paradox is trivially avoided as long as an individual evaluates an infinite stream of one dollar payoffs by some finite vNM utility. It is obvious from the formal proof that this result holds for any subjective probability measure and therefore as well for the harmonic sequence probability distribution P^* according to which some consequence $c = (x_1, x_2, ...)$ such that $\sum_{t=1}^{\infty} x_t = +\infty$ occurs with probability one.

3 An illustrative example

As an illustrative example we consider in this section the special case of a vNM utility function that results from additively time-separable preferences whereby we are in line with the standard assumption of a time-impatient individual.[2]

Assumption 2. *The individual's preferences*[3] *over infinite payoff-sequences $x \in X$ are additively time-separable with discount factor $\beta \in (0,1)$, i.e., for all $c = (x_1, x_2, ...)$,*

$$U(c) = \sum_{t=1}^{\infty} \beta^{t-1} u(x_t) \tag{13}$$

[2] The reader can easily verify that the assumption of time-*patient* individual, i.e., $\beta = 1$, in our model would bring us immediately back to the original formulation of the harmonic sequence paradox.

[3] Recall that in a Savage-framework preferences over consequences are interpreted as preferences over constant Savage-acts, i.e., acts that give in every state of the world the same consequence.

for some strictly increasing function $u : \{0,1\} \to \mathbb{R}$.

Observe that we have, by assumption 2,

$$\begin{aligned} U(1,1,..) &= \sum_{t=1}^{\infty} \beta^{t-1} u(1) \\ &= \frac{u(1)}{1-\beta}, \end{aligned}$$

so that proposition 1 immediatly implies the following result.

Corollary. *If an individual is time-impatient with additively time-separable preferences, her willigness to pay for participating in the harmonic sequence gamble G will never exceed the amount*

$$\frac{u(1)}{1-\beta}.$$

Recall that a risk-averse (risk-loving) expected utility decison-maker is modelled through a concave (convex) vNM utility function U whereby concavity (convexity) of U results for the additive separable case from concavity (convexity) of u. Whenever we speak of a risk-neutral individual, i.e., linear U, we noramlize w.o.l.o.g. $u(0) = 0$ and $u(1) = 1$. For example, in case of a risk-neutral individual the corollary determines $\frac{1}{1-\beta}$ as an upper bound for the individual's willigness to pay to participate in G. While the corollary characterizes only an upper-bound for an individual's evaluation of the gamble G, the following proposition gives the exact expected utility.

Proposition 2.

If an individual is time-impatient with additively time-separable preferences, she evaluates a participation in the harmonic sequence gamble G by the following expected utility

$$E\left[U\left(f^G\right), P^s\right] = \left(\sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{t+1} u(1) + \frac{t}{t+1} u(0)\right)\right).$$

If, in addition, the individual is risk-neutral, her expected utility coincides with the expected value of the discounted random payment-stream, i.e.,

$$E\left[U\left(f^G\right), P^s\right] = \sum_{t=1}^{\infty} \frac{\beta^{t-1}}{t+1}.$$

Proof. Observe that

$$\begin{aligned}
E\left[U\left(f^{G}\right), P^{s}\right] &= \int_X U\left(x_1, x_2, ...\right) dP^{*}\left(x_1, x_2, ...\right) \\
&= \int_X \left(\sum_{t=1}^{\infty} \beta^{t-1} u\left(x_t\right)\right) dP^{*}\left(x_1, x_2, ...\right) \\
&= \sum_{t=1}^{\infty} \beta^{t-1} \int_X u\left(x_t\right) dP^{*}\left(x_1, x_2, ...\right) \\
&= \sum_{t=1}^{\infty} \beta^{t-1} E\left[u\left(Y_t\right), P^{*}\right] \\
&= \sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{1}{t+1} u\left(1\right) + \frac{t}{t+1} u\left(0\right)\right)
\end{aligned}$$

The first line follows from the proof of proposition 1, the third line results from the linearity of the expectations operator, and the final line uses the probability distribution of the harmonic sequence gamble.□

Suppose that an individual's willigness to pay for participating in the harmonic sequence gamble G coincides with the rather fast converging series (+++). In that case we can easily come up with good approximations for the amount a risk-neutral individual of our model would be willing to pay in order to participate a reasonable number of rounds, say 1000, in the harmonic sequence gamble. For example, for $\beta = 0.99$ we obtain as willigness to pay (in dollars)

$$\sum_{t=1}^{1000} \frac{\beta^{t-1}}{t+1} \simeq 3.69$$

(which is also a good approximation for the infinite case) and for $\beta = 0.999$ we have

$$\sum_{t=1}^{1000} \frac{\beta^{t-1}}{t+1} \simeq 5.70.$$

To sum: For already small degrees of time-impatience a risk-neutral individual of our model is willing to pay only moderate amounts of dollars in order to participate in the harmonic sequence gamble (even considerably less than the ten dollars mentioned by Blavatskyy). Obviously, this effect would be even more pronounced for risk-averse individuals.

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