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WORLD INCOME COMPONENTS: MEASURING AND EXPLOITING RISK-SHARING OPPORTUNITIES

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World Income Components: Measuring and Exploiting Risk-Sharing Opportunities[1]

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Abstract

We provide a method for decomposing the variance of changes in incomes in the world into components, world income components (WICs), in such a way as to indicate the most important risk-sharing opportunities among people of the world. We develop a constant absolute risk premium model, an intertemporal general equilibrium model of the world that facilitates consideration of optimal contract design. We show that for a contract designer maximizing a social welfare function, the optimal risk-management contracts maximize the equilibrium world real interest rate. That is the contract designer achieves the risk-optimal interest rate. We show that these WIC securities are defined in terms of eigenvectors of a transformed variance matrix of income changes. The method is applied with a variance matrix estimated using Penn World Table data on the G-7 countries, 1950-92.

Keywords: Constant Absolute Risk Premium, risk-optimal interest rate, three-level income model, WIC securities, contract design, macro markets, hedging. JEL Code: G20

Since most people's incomes originate primarily from untradable sources such as labor and difficult-to-trade sources such as real estate, and since over long intervals of time the real value of each individual's income is substantially uncertain, the dominant concern in the design of risk management contracts ought to be allowing people to share these income risks as much as possible. To make such risk sharing as effective as possible, it is logical to define risk management contracts in the form of securities based on these incomes themselves; these securities could be defined by contracts that represent long-term or perpetual claims on incomes or income aggregates.[1] Such securities can be designed to correlate better than existing assets with major risks and thus serve better as hedging vehicles.

We cannot, however, have a market for contracts on each individual's income. There would be far too many markets: the markets will have to be markets for contracts on aggregates (or indexes) of individual incomes. The question then arises, and is the subject of this paper, how shall we decide which aggregates to trade on risk markets?

With billions of people in the world today, there are myriad's of ways of defining aggregates of incomes that could be traded on our financial markets. Our national income statisticians have already chosen some aggregates, the simple sums of incomes of people within nations, but they did so based purely on political boundaries and without any concerns for risk management. How do we know which aggregates would be most important to create markets for? Is it possible to be systematic in our means of defining new markets, seeking out the new markets that maximize world welfare without regard for traditional definitions of contracts, letting an estimated economic model define them?

We develop a constant absolute risk premium model, an intertemporal general equilibrium model, that is designed to allow us to see how we might answer these questions and provides a framework for rigorous econometric research on market design. The model represents a large number of people as trading in riskless one period bonds and in a small number of risky (risk-management) contracts. It is a risky endowment economy, where each individual has an exogenously given random income, at first with no risk management beyond borrowing and lending possible. Then new contracts representing claims on linear income combinations (CLICs) will be injected. The prices of the CLICs, their equilibrium expected returns, the real interest rate, and the welfare gain to creating markets are all derived. The model then yields a method to define the welfare-optimizing CLICs, the world income components (WICs), to be used in the definition

[1] See Shiller [1993a] for a discussion of how such contracts could be implemented.

of new securities. As an example, we apply the model to the readily available data on world incomes of advanced countries, data on national incomes of the G-7 countries, as measured by the real gross domestic product data of Summers and Heston (1950-1992), based on a simple model of the relation of individual incomes to national incomes. In the future, further econometric work combining the fragmentary data we have on individuals around the world in conjunction with this model might be used to define better aggregates of individual incomes on which to base the definitions of new securities.

Our method of identifying risk-sharing arrangements (based on Athanasoulis [1995] and Shiller and Athanasoulis [1995]) is related to principal components analysis; see also Demange and Laroque [1995]. Some of the securities that our method produces can be described as insurance policies for certain groups of people; calling a security an insurance policy is most appropriate when the variation in the component is highly negatively correlated with the income of one group of people, and those people buy the security to reduce their income risk. Some of the securities can also be described as swaps of income of certain groups of people for income of other groups; calling one of our securities a swap is most appropriate when the component gives negative weights to roughly half of the incomes. But our analysis does not start from any preconceived notions whether the securities will look like insurance policies or like swaps; our method will derive the optimal form of the contracts from the variance matrix of incomes.

Our study of risk-sharing opportunities among individual incomes is potentially very important. There is very little effective risk sharing of individual income risks. There is not even much diversification across nations today (see, for example, Obstfeld [1994a,b], Tesar and Werner [1995]). It is obvious that national governments do not make significant Risk-sharing arrangements with each other; even within the European Union, Sala-i-Martin and Sachs [1992] estimate, a one dollar adverse shock to the national income of one country creates, all things considered, less than a 0.005 dollars reduction of that country's tax payment to the European Community. While the family or village units may share risks within a country, there are still important unexploited risk-sharing opportunities within countries, see for example Townsend [1994].

Because there do not now exist any markets for claims on any large income aggregates, and because there is very little income risk sharing, when we set up any such contracts we must consider how they would work pretty much in isolation. Existing markets for stocks, bonds, and readily-marketable real estate are markets for claims on the rents of factors of production other than labor or are residual claims, minor components of national incomes, and there is no reason to expect dividends in these markets to correlate well with labor income. There is instead some evidence that they do not correlate well, see Shiller [1993b] and Bottazzi, Pesenti and van Wincoop [1996].

In attempting to define a small number of WIC securities, we are essentially seeking to define the best first world risk sharing market to set up, as well as the best second and/or third markets. We confine our attention to only one or a few contracts as it is useful for us to be able to prescribe in simple terms the most important risk management actions that should be taken by large groups of people. Simple prescriptions are what most people take from existing models. The capital asset pricing model (CAPM) in finance, to which our method is related, is most often used by practitioners not to arrive at complicated definitions of optimal portfolios, but just for the simple prescription that investors should hold the market portfolio of investable assets, and we now have many indexed funds that are designed to allow them to do just this. The problem with this commonly given prescription is that it is not really the logical consequence of the foundations of the CAPM, since it disregards the correlation of investment returns with innovations in other income, other income which is much larger in the aggregate than income from existing investable assets. We seek here to devise a method to replace this simple prescription associated with the CAPM with a more sensible simple prescription, though any such prescription cannot be taken until the new securities are created.

We derive, Section 2 below, an expression for the risk premia of the CLICs in equilibrium. The CLICs are then chosen so as to maximize social welfare (Section 3); this then defines our WIC securities. It turns out that the WICs are defined in terms of eigenvectors of a variance matrix of deviations of individual incomes from *per capita* world income. We also show that with these securities, the world interest rate achieves a maximum, the risk-optimal interest rate. With CARA utility, if the variance matrix of incomes does not change through time, then, even if there are major shifts in endowments across groups, the WICs will not change through time, nor will the individuals' optimal investments change. The markets for WIC securities need to be set up only once and people have to decide only once how much to invest in each security. Thus, the creation of the new WIC securities is a relatively simple, and potentially very important, step to achieve welfare gains. Having made a specification of utility functions, we are able to derive estimates of the welfare increase in dollars generated by the creation of the new contracts.

In Section 4 below, we show some illustrative examples of the theory, and in Section 5 we discuss how to apply our method of defining the WIC securities to the data. We present a model of individual income, the three-level income model, that allows estimation of a restricted variance matrix of individual incomes for the G-7 countries. A discussion follows to interpret these results as suggesting opportunities for potentially important new contracts. In section 6 we conclude.

1 Definition of Contracts and Risk Structure

There are two kinds of securities that are traded in the economy: CLICs which are long-lived securities and riskless one-period bonds. Both kinds of securities are in zero net supply: for every long there is a short. All CLICs are assumed to be traded (constructed) for the first time at time 0, and people immediately make optimal use of these contracts from that date onward. The construction of the CLIC securities at time zero is unanticipated by all individuals in the economy. In contrast, bonds are assumed already traded: the bond market is the one pre-existing market that we represent in our model. Some methods to take account of other pre-existing securities are outlined in Athanasoulis and Shiller [2000]. There are N kinds of perpetual claims, and I individuals, presumably I much larger than N. Individuals are infinite lived, and in each period t, from 0 onward, re-evaluate their holdings of these securities in view of the prices of the perpetual claims, $P_{tn}, n = 1, ..., N$, and the interest rate r_t from t to $t + 1$.

The model takes as exogenous the income process $\{y_{ti}\}_{t_0}^{\infty}$ of the ith individual at time t, $i = 1, ..., I$. The date that bonds began to be traded is $t_0 < 0$. Income, y_{ti}, is derived from sources other than the risk-management contracts or bonds, let us say labor; it may also be called the endowment of individual i at time t. Let $\mathbf{y}_t$ be the I-element column vector whose ith element is y_{ti}. We will assume that $\mathbf{y}_t$ is a Gaussian random walk:

$$\mathbf{y}_t = \mathbf{y}_{t-1} + \boldsymbol{\epsilon}_t \tag{1}$$

where $\boldsymbol{\epsilon}_t$ is i.i.d. normal, with zero mean and with variance matrix $\boldsymbol{\Sigma}$ that is constant through time. In the rest of the paper we use the convention that vectors are either I element column vectors, N element row vectors and when appropriate matrices are $I \times N$.

Each CLIC specifies both a riskless payment and a risky payment that must be made from the short in the CLIC to the long each time period starting with time $t = 1$. We will call the sum of the riskless and the risky payment the dividend of the nth CLIC, D_{tn}. We call the riskless payment the risk premium, $\bar{D}_n$, paid on the nth CLIC in each period $t \geq 1$. We will choose $\bar{D}_n$ below so that the CLIC has a zero price at time 0, $P_{0n} = 0$. $\bar{D}_n$ may thus be thought of as a regular insurance premium paid by the shorts in the CLIC or regular risk premium received by the longs in the CLIC; no other payment or compensation for risk-bearing is expected at time 0. The assumption that the risk premium is constant through time is natural given our assumption that the variance matrix $\mathbf{\Sigma}$ is constant through time.

We will denote the risky payment made from the short to the long in each period from time $t = 1$ onward by X_{tn} which may be positive or negative. X_{tn} is defined in terms of individual endowments in such a way that $\mathrm{E}_0(X_{tn}) = 0$ and $\mathrm{Var}_{t-1}(X_{tn}) = 1$. Setting this variance to one is just a normalization rule to define the size of one CLIC. To define the aggregates we assume that an I-element column vector $\mathbf{A}_n$ is defined in the nth CLIC at time 0, and the CLIC specifies that $X_{tn} = (\mathbf{y}_t - \mathbf{y_0})'\mathbf{A}_n$. The risky payment is defined as a linear combination of the (unexpected) changes in incomes since risk management began at time 0. When $\mathbf{A}_n$ is chosen optimally, as will be defined below, the risky payment X_{tn} will be our nth world income component (WIC). Our objective below will be to define this random payment stream optimally by choosing the best vector $\mathbf{A}_n$. We adopt the convention that $\mathbf{A}_n$ is defined so that the risk premium $\bar{D}_n$ is nonnegative. If $\bar{D}_n$ were negative we would multiply $\mathbf{A}_n$ by minus one. This is just a convention defining who is called long and who short. We may write the processes for the dividend, the risky and the riskless payment as:

$$\bar{D}_{tn} = \bar{D}_n \quad \text{for } t \geq 1 \tag{2}$$

$$X_{tn} = (\mathbf{y}_t - \mathbf{y}_0)'\mathbf{A}_n \quad \text{for } t \geq 1 \tag{3}$$

$$D_{tn} = \bar{D}_n + X_{tn} \quad \text{for } t \geq 1. \tag{4}$$

Commitments to make payments specified in the CLICs last forever, and at each time period t one can avoid making future payments only by selling the CLIC one is long in or buying a CLIC one is short in; there is no free disposal and no default. Thus, the price P_{tn} of the nth CLIC at time t can be either positive or negative after time 0, depending on which way incomes turn. If price becomes positive after time 0 it means that the CLIC is valuable to longs since it is expected to make positive risky payments to them. If price becomes negative after time 0 it means that the CLIC is valuable to shorts since it is expected to make negative risky payments. Knowing this, people will at time 0 try to take positions in the CLICs that help them to offset their endowment risks, and each period thereafter will re-evaluate the ability of the CLICs to do this; prices of the CLICs after time 0 and the interest rate will be determined by the market clearing condition.

Our timing convention is as follows: In each period t individual i's endowment, y_{ti}, is realized. She also receives payments from her securities holdings, that is she receives both dividends and interest from the CLICs and the one-period bond. Individual i then decides how much to consume and how many of the CLICs and the riskless bonds to buy or sell, taking account of expected future payoffs of the securities as well as the prices, $P_{tn},\ n = 1, ..., N$, and interest rate r_t that are simultaneously determined in the competitive market. At time 0, in contrast, no dividends are received, and individual i chooses quantities of the CLICs and the bond to buy or sell. The risk premium $\bar{D}_n$ is chosen by the contract designer so that equilibrium prices of the CLICs at time zero, $P_{0n},\quad n = 1, ..., N$, are zero. Thus, at time zero the very important risk-sharing commitments are made and the only resources that change hands are those for the purchase and sale of bonds as well as payment of interest and principal for bonds issued at time $t - 1$.

Let $\mathbf{X}_t$ denote the N-element row vector whose nth element is X_{tn}. Then, $\mathbf{X}_t = (\mathbf{y}_t - \mathrm{E}_0\mathbf{y}_t)'\mathbf{A}$ where $\mathbf{A}$ is the $I \times N$ matrix whose nth column is $\mathbf{A}_n$. Let $\mathbf{P}_t$ denote the N-element row vector whose nth element is P_{tn}; $\bar{\mathbf{D}}$ denotes the N-element row vector whose nth element is $\bar{D}_n$; $\mathbf{D}_t$ denotes the N-element row vector whose nth element is D_{tn}. Note that $\mathbf{D}_t = \bar{\mathbf{D}} + \mathbf{X}_t$. We also adopt the normalization that $\mathbf{A}'\boldsymbol{\Sigma}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the $N \times N$ identity matrix. This is just a normalization which does not restrict our contracts.

2 The Constant Absolute Risk Premium Model

In this section we develop a general equilibrium model of the world that has the property that the risk premia are constant through time in absolute (dollar) terms. This property yields important simplifications for our purposes because it makes possible a closed form solution with a constant investment opportunity set in dollars rather than in percent, the latter used in Merton [1971] and others. Thus, we will see that there are simple relations such as $P_{tn} = \frac{X_{tn}}{r}$, where r is a constant riskless interest rate.

We will assume that each individual i at time t maximizes:

$$U_{ti} = \mathrm{E}_t[\sum_{\tau=0}^{\infty} u_t(c_{t+\tau i})/(1+\rho)^{\tau}] \tag{5}$$

where $u_t(c_{t+\tau i})$ is felicity, or instantaneous utility, of individual i at time t of consumption at time $t+\tau$, and ρ is the discount rate, i.e., the subjective rate of time preference. We assume that the felicity, $u_t(c_{t+\tau i})$, is defined by a negative exponential (Constant Absolute Risk Aversion or CARA) function:

$$u_t(c_{t+\tau i}) = -\exp(-\gamma c_{t+\tau i}), \qquad \gamma > 0. \tag{6}$$

This felicity function will allow us to compute an explicit closed-form general equilibrium solution.[2] We assume that $\ln(1+\rho) > \frac{1}{2}\gamma^2 \frac{1}{I}\sum_{i=1}^{I}\sigma_{\epsilon_i}^2$, where $\sigma_{\epsilon_i}^2$ is the variance of the innovation to income for agent i. As will be seen below, this ensures a positive market-clearing real interest rate. We also assume the interest rate process is bounded.

Let us define $\mathbf{q}_{ti}$ as the N-element row vector whose nth element is the number of securities n owned by individual i at time t. Let us also define bond demand (in units of the consumption good) of individual i at time t earning interest from time t to time $t+1$ as B_{ti}. The budget constraint of individual i at time t is:

$$c_{ti} = y_{ti} + (1+r_{t-1})B_{t-1i} - B_{ti} + \mathbf{q}_{t-1i}(\mathbf{P}_t + \mathbf{D}_t)' - \mathbf{q}_{ti}\mathbf{P}_t' \tag{7}$$

which holds for all t, though we require that $\mathbf{q}_{ti}$ is zero for $t<0$.[3] Individual i maximizes expected lifetime utility, (5), subject to the budget constraint, (7), initial conditions, B_{t-1i} and y_{ti} are given, and the no-ponzi-game condition which is:

$$\lim_{\tau\to\infty} \mathrm{E}_{ti}^*[(\mathbf{q}_{t+\tau i}\mathbf{P}_{t+\tau}' + B_{t+\tau i})\prod_{k=0}^{\tau}(1+r_{t+k-1})^{-1}] \geq 0 \quad \forall t \geq 0, \tag{8}$$

where $\mathbf{q}_{ti}\mathbf{P}_t' + B_{ti}$ is non-labor wealth at time t and E_{ti}^* is the expectation under a probability measure which is equivalent to the subjective probability measure of indi-

[2] The only known closed form solution to an individual's problem, with expected utility preferences which are increasing everywhere, for a discrete time intertemporal economy where there is investment in one-period bonds is for individuals with CARA utility, see Caballero [1990]. Willen [1999] also studies such an economy with risky financial assets included.

[3] We assume that agents are not allowed to pursue doubling strategies. One can formulate precise conditions on trading strategies so that they are ruled out, see for example Duffie [1996] for conditions in a continuous time economy. It has been shown by Omberg [1989] that for negative exponential utility, in the Balck Scholes economy, pursuing a doubling strategy leads to negative infinite expected utility.

vidual i used by individual i to evaluate the present value of future income streams.[4] The no-ponzi-game condition simply states that the present value of ones wealth at time T as T approaches infinity must be non-negative. If it were allowed to be negative, then one would always incur debts and never repay them. In order to write the no-ponzi-game condition in this way, we follow Magill and Quinzii [1994] who use the concept of competitive price perceptions of Grossman and Hart [1979]. Competitive price perceptions means that agents use their own marginal rate of substitution to evaluate future income streams. We can then show that the equivalent probability measure under which individual i takes expectations in equation (8), are consistent with her first order conditions.[5] We can further show that under this new probability measure, prices plus total accumulated dividends after a normalization follows a martingale, which is well known.[6]

A solution to individual i's problem must satisfy the Euler equation for bonds:

$$\exp(-\gamma c_{ti}) = \frac{1+r_t}{1+\rho}\mathrm{E}_t[\exp(-\gamma c_{t+1i})] \tag{9}$$

the Euler equation for the I risky securities:

$$\mathbf{P}_t \exp(-\gamma c_{ti}) = \frac{1}{1+\rho}\mathrm{E}_t[\exp(-\gamma c_{t+1i})(\mathbf{P}_{t+1} + \mathbf{D}_{t+1})], \quad t \geq 0 \tag{10}$$

and the following condition so that the transversilty condition is satisfied:

$$\lim_{\tau\to\infty} \mathrm{E}_t[u'(c_{t+\tau i})(\mathbf{q}_{t+\tau i}\mathbf{P}'_{t+\tau} + B_{t+\tau i})(\frac{1}{1+\rho})^{\tau}] = 0 \quad \forall t. \tag{11}$$

We can rewrite this condition, equation (11), as;[7]

[4]We assume that agents trade at equilibrium prices and there are no arbitrage opportunities. However, when markets are incomplete, there is not a unique way to evaluate one's borrowing opportunities. Thus we must define for each individual how she evaluates future income streams, see Magill and Quinzii [1994].

[5]See Huang and Litzenberger [1988] for an example of how to construct this measure.

[6]For the general theory of equivalent martingale measures and no arbitrage, see Harrison and Kreps [1979]. Duffie [1996] also covers this though in a different fashion. For a less terse handling of these issues see Huang and Litzenberger [1988]. For a discussion of equivalent probability measures in an infinite horizon setting, see Huang and Pagès [1992]. See the appendix for a short discussion on the budget constraint and the no-ponzi-game condition.

[7]For the general theory and discussion of a transversality condition equilibrium for infinite horizon economies see Magill and Quinzii [1994].

$$\lim_{\tau\to\infty} \mathrm{E}^*_{ti}[(\mathbf{q}_{t+\tau i}\mathbf{P}'_{t+\tau} + B_{t+\tau i}) \prod_{k=0}^{\tau}(1 + r_{t+k-1})^{-1}] = 0 \quad \forall t \geq 0. \tag{12}$$

We conjecture that the equilibrium interest rate is constant through time, $r_t = r$, that the quantities demanded of the CLICs are constant through time, $\mathbf{q}_{ti} = \mathbf{q}_i$, that price equals X_{tn} discounted by the riskless interest rate, $P_{tn} = X_{tn}/r$, $\{B_{ti}\}_0^\infty$ is a linear deterministic function of time for all i, and that consumption follows a random walk with a drift. These conjectures can be summarized as follows:

$$r_t = r \quad \text{for } t \geq 0 \tag{13}$$

$$P_{tn} = \frac{X_{tn}}{r} \quad \text{for } t \geq 0,\ n = 1, ..., N \tag{14}$$

$$\mathbf{q}_{ti} = \mathbf{q}_i \quad \text{for } t \geq 0 \tag{15}$$

$$B_{ti} = a_i + b_i t \quad \text{for } t \geq 0 \tag{16}$$

where a_i and b_i are constants, and

$$c_{ti} = c_{t-1i} + \mu_i + v_{ti} \quad \text{for } t \geq 1 \tag{17}$$

where μ_i is a constant and v_{ti} is an error term with zero conditional expected mean and constant variance, $v_{ti} \sim N(0, \sigma^2_{v_i})$.[8] Moreover, we conjecture that v_{ti} is jointly normally distributed with the innovation $\boldsymbol{\epsilon}_t$ and uncorrelated with lagged values. We shall establish that these conjectures are consistent with general equilibrium by assuming them, and then checking consistency with conditions for individual maximization and with market clearing at all times.

Note that if the conjectures hold up and the interest rate r turns out to be a "small" number, then prices of the CLICs will make large swings from period to period relative to the change in income, reflecting the changed expected present value of incomes out to infinity, and thus large capital gains or losses on existing holdings of the CLICs. But the changes pose no crisis for individuals, since they have no incentive or need to re-trade

[8]A disadvantage of our negative exponential utility model is that, for some individuals, consumption will eventually become negative; we assume that this will not occur until the distant future, and so disregard this problem. There is no tractable or simple model that avoids all problems of approximation.

the CLICs. The situation is somewhat analogous to that of homeowners who say that they do not care about the swings in the value of their homes since they will live in them forever. Despite the possibly large price changes, people are always in a very comfortable situation: our model represents people as in effect paying a regular insurance premium, $-\bar{D}_n$, at all times, receiving a payment X_{tn} as a sort of insurance claim reflecting their changed economic circumstances since time 0. If they chose their investments in the CLICs right, offsetting the change in their income since time 0, they can forget about making any further adjustments in their investment positions.

Using the first-order condition, equation (9), the conjectured interest rate process, equation (13), and using the process for consumption, equation (17), we find that

$$\mu_i = \frac{1}{\gamma} \ln \left[\frac{1+r}{1+\rho} \mathrm{E}_t \exp(-\gamma v_{t+1i}) \right]. \tag{18}$$

We now find the optimal investments of the individuals, $\mathbf{q}_{ti}$. We can, using equation (9) and equation (13), rewrite the first-order condition, equation (10), as:[9]

$$\mathbf{P}_t(1+r) - \mathrm{E}_t[\mathbf{P}_{t+1} + \mathbf{X}_{t+1} + \bar{\mathbf{D}}] = -\gamma \mathrm{Cov}_t[c_{t+1i}, \mathbf{P}_{t+1} + \mathbf{X}_{t+1}]. \tag{19}$$

Moreover, using our conjectured price processes, $\mathbf{P}_t = \frac{\mathbf{X}_t}{r}$, we may rewrite equation (19) as:

$$\bar{\mathbf{D}} = \gamma \mathrm{cov}_t(c_{t+1i}, \mathbf{P}_{t+1} + \mathbf{X}_{t+1}) \tag{20}$$

In order to use equation (20) to find the individual's optimal investments, $\mathbf{q}_i$, we need to solve for v_{ti} to obtain an explicit solution for the covariance term. Then using the budget constraint, equation (7), along with our conjectured price processes, equation (14), our conjecture of constant investments, equation (15), and our conjectured bond demand, equation (16), we see that the innovation in consumption is linear in the innovation in income and in price, which is in turn linear in income, and so:

$$v_{ti} = \epsilon_{ti} + \mathbf{q}_i \mathbf{A}' \boldsymbol{\epsilon}_t \quad \text{for } t \geq 1 \tag{21}$$

and so our conjecture that v_{ti} is jointly normally distributed with $\boldsymbol{\epsilon}_t$ is confirmed. v_{ti} will have a smaller variance than ϵ_{ti} because of the success of the risk management contracts. Note that $v_{ti} \sim N(0, \sigma^2_{v_i})$ where:

[9] We use the fact that for any two random variables x and y, $\mathrm{E}(xy) = \mathrm{E}(x)\mathrm{E}(y) + \mathrm{cov}(x, y)$ and, assuming joint normality, we use Stein's Lemma that $\mathrm{cov}(\mathrm{f}(x), y) = E(\mathrm{f}'(x))\mathrm{cov}(x, y)$.

$$\sigma_{v_i}^2 = \Sigma_{ii} + \mathbf{q}_i \mathbf{q}_i' + 2\boldsymbol{\Sigma}_i \mathbf{A} \mathbf{q}_i' \tag{22}$$

where $\boldsymbol{\Sigma}_i$ is the i^{th} row of $\boldsymbol{\Sigma}$ and Σ_{ii} is the i^{th} diagonal element of $\boldsymbol{\Sigma}$.

Using equations (3), (14), (17), (20), (21) and (22) we obtain:

$$\mathbf{q}_i = \frac{1}{\gamma} \frac{r}{1+r} \bar{\mathbf{D}} - \boldsymbol{\Sigma}_i \mathbf{A} \tag{23}$$

Let $\mathbf{q}$ be an $I \times N$ matrix with $\mathbf{q}_i$ in the i^{th} row. We can write $\mathbf{q}$ as:

$$\mathbf{q} = \frac{1}{\gamma} \frac{r}{1+r} \boldsymbol{\iota} \bar{\mathbf{D}} - \boldsymbol{\Sigma} \mathbf{A}. \tag{24}$$

where $\boldsymbol{\iota}$ is a $I \times 1$ vector of ones. This confirms our conjecture that $\mathbf{q}_t$ does not depend on t.

In equilibrium we must have $\boldsymbol{\iota}'\mathbf{q} = 0$, that is these contracts are in zero net supply. Using this equilibrium condition and equation (24), we obtain the following equilibrium risk premia:

$$\bar{\mathbf{D}} = \frac{1}{I} \gamma \frac{1+r}{r} \boldsymbol{\iota}' \boldsymbol{\Sigma} \mathbf{A}. \tag{25}$$

If we substitute equation (25) into equation (24), we obtain $\mathbf{q} = -\mathbf{M}\boldsymbol{\Sigma}\mathbf{A}$ where $\mathbf{M} = \mathbf{I} - \frac{1}{I}\boldsymbol{\iota}\boldsymbol{\iota}'$. $\mathbf{M}$ is symmetric and idempotent, $\mathbf{M}\mathbf{M} = \mathbf{M}$, and $\mathbf{M}\boldsymbol{\iota} = 0$. For any vector $\mathbf{A}$ which defines the risky payment stream, and with the corresponding defined price processes, we have derived the optimal demands for individuals, and the equilibrium risk premia. We have also confirmed that the risk premia are constant over time, hence the constant absolute risk premium model. We still need to derive the equilibrium interest rate and the equilibrium borrowing and lending by individuals. We begin with the equilibrium borrowing and lending by individuals. In order to do this we take budget constraint, equation (7), rearrange it and use the constant investment conjecture, equation (15), to obtain:

$$B_{t-1i} = (c_{ti} - y_{ti} - \mathbf{q}_i(\mathbf{X}_t + \bar{\mathbf{D}})') \left(\frac{1}{1+r} \right) + B_{ti} \left(\frac{1}{1+r} \right) \quad \text{for } t > 0. \tag{26}$$

Equation (26) does not hold for time $t = 0$ since there are no dividend payments at time zero and since the interest rate prior to time zero, r_{-1}, need not equal r, the equilibrium interest rate from time zero on. Instead, the budget constraint at $t = 0$ is:

$$B_{0i} = y_{0i} - c_{0i} + B_{0i}^{-} \tag{27}$$

where B_{0i}^{-} is the initial value of bonds at time 0 just before trade, equal to $B_{-1i}(1+r_{-1})$. Once we solve for c_{0i}, then from equations (17), (27) and (26) we can obtain the path for borrowing and lending for each individual i. If we substitute equation (17) into equation (26), take expectations conditional on information at time t and solve forward we obtain:

$$B_{t-1i} = \left(\frac{1}{r}\right)^2 \mu_i + \frac{1}{r}(c_{ti} - y_{ti} - \mathbf{q}_i \mathbf{X}_t' - \mathbf{q}_i \bar{\mathbf{D}}') \quad \text{for } t > 0. \tag{28}$$

Substituting equation (27) into equation (28) for $t = 1$ and taking expectation at $t = 0$ and solving for c_{0i} we obtain:

$$c_{0i} = y_{0i} + \frac{1}{1+r}\mathbf{q}_i \bar{\mathbf{D}}' - \frac{1}{r}\mu_i + \frac{r}{1+r}B_{0i}^{-} \tag{29}$$

and for $t \geq 1$;

$$c_{ti} = y_{ti} + \mathbf{q}_i \bar{\mathbf{D}}' + \mathbf{q}_i \mathbf{X}_t' + rB_{t-1i} - \frac{1}{r}\mu_i \tag{30}$$

From (27) and (29) we have;

$$B_{0i} = \frac{1}{r}\mu_i - \frac{1}{1+r}\mathbf{q}_i \bar{\mathbf{D}}' + \frac{1}{1+r}B_{0i}^{-}. \tag{31}$$

We can further show, see the appendix, that;

$$B_{ti} = B_{0i} + \frac{\mu_i}{r}t \tag{32}$$

and thus, if μ_i is constant, we confirm our conjecture that bond demands are a linear deterministic function of time with $a_i = B_{0i}$, and $b_i = \frac{\mu_i}{r}$. We still need to derive what the value of μ_i is in equilibrium. We do this when we find the equilibrium interest rate. To solve for the equilibrium interest rate we note that all bond demands by all individuals must sum to zero or by Walras' law we can simply use the equilibrium condition in the goods market, namely:

$$\sum_{i=1}^{I} c_{ti} = \sum_{i=1}^{I} y_{ti}. \tag{33}$$

Thus we have:

$$\sum_{i=1}^{I} \mu_i = 0. \tag{34}$$

Now substitute equation (18) into (34) to obtain the equilibrium interest rate:

$$\ln(1+r) = \ln(1+\rho) - \frac{1}{2}\gamma^2 \frac{1}{I}\sum_{j=1}^{I}\sigma_{v_j}^2. \tag{35}$$

The interest rate r will always be less than or equal to the subjective discount rate ρ, and strictly less so long as individuals still bear some risk in equilibrium. Note that the interest rate r_{-1} before the CLIC securities are constructed is given by the same expression but with $\sigma_{\epsilon_j}^2$ in place of $\sigma_{v_j}^2$.

Now if we substitute equation (35) into (18) we obtain

$$\mu_i = \frac{1}{2}\gamma(\sigma_{v_i}^2 - \frac{1}{I}\sum_{j=1}^{I}\sigma_{v_j}^2) \tag{36}$$

which confirms our conjecture that μ_i is constant through time.

2.1 Discussion

The equilibrium in the constant absolute risk premium model can be summarized in just nine equations. Consumption is given by a simple difference equation, equation (17), where the right hand term v_{ti} is given by equation (21) and the term μ_i is given by equation (36). The initial condition for the difference equation, at time 0, is given by equation (29). The demand for the two kinds of securities are given by equations (32) for bonds (with bond demand at time 0 given by equation (31)) and (24) for the CLICs. The risk premia are given by equation (25) and the interest rate is given by equation (35).

We can see from the consumption processes, equation (17), and the error term, equation (21) that one of the benefits from investing in the CLICs is the possible risk reduction in one's consumption process. Let us for the moment assume that there are no CLICs in the economy and only bonds are traded. Then of course one will be unable to reduce the riskiness of one's consumption process though one is able to save some of her income in case there is a bad shock in the future. Since all of the shocks to income are permanent shocks there is no possibility of "smoothing" shocks over time through borrowing and lending. This particular motive for savings is the precautionary motive for savings.[10]

[10] Caballero [1990] covers this precautionary motive for savings for an individual when income follows general ARMA processes.

To interpret this savings motive in this model we notice that the shock to consumption will be identical to the shock to income when there are no CLIC securities to trade the risks in the economy. As such, given the consumption process, we have that with only a bond market and no CLIC securities, the drift term μ_i will depend on one's own consumption risk, which in this case is equal to income risk, relative to the average consumption risk in the economy, equation (36). In particular, individuals whose risk is larger than average will lend to individuals whose risk is less than average.

Our intuition tells us that this should be the case; that individuals with a riskier consumption path will borrow since there is a higher probability that a bad shock will be worse for them than it will be for someone with a less risky consumption path. We must keep in mind our assumptions which lead to this result; all individuals have the same coefficient of absolute risk aversion, all individuals have the same rate of time preference, and the third derivative of the utility function is positive, and utility is additive separable. It is known that when utility is separable and the third derivative is positive and one cannot insure their income shocks, then an increase in the volatility of the income shocks will reduce consumption and increase savings today, see for example Caballero [1990].

Now once CLIC markets are included in the model, what will change is the variance of the consumption process and the drift term in the consumption process (since it is a function of the variance in the consumption process), as well as consumption at time zero. Thus when there is only a bond market, an individual in the economy may be a lender because she is riskier than average, while the same individual in the economy when there is a bond market and one CLIC market may be a borrower if her consumption process is less risky than average after taking positions in the CLIC market. The reason why this may occur is that by taking positions in the CLICs, if the CLICs are well correlated with her endowment process, then she will be able to hedge much of her risk. As such, if she lays off much of her risk, her precautionary motive for saving (lending) declines.

We also see that consumption at time zero changes when we include CLIC markets, not only because the precautionary motive for saving changes, μ_i, but also because we expect future payments from the CLIC securities as dividends, equation (29), $\frac{1}{1+r}\mathbf{q}_i\bar{\mathbf{D}}'$. When we take positions in the available CLICs, we expect at time zero to receive from time $t = 1, ..., \infty$ a dividend stream equal to the risk premium, which is $\mathbf{q}_i\bar{\mathbf{D}}'$. This part of the dividend component is riskless. The individuals will also consume the risky part of their dividends as well. In order to smooth the riskless part of her dividend stream over her entire lifetime, individual i consumes $\frac{1}{1+r}\mathbf{q}_i\bar{\mathbf{D}}'$ of it in each period of her lifetime.

Thus the savings decision is affected by the new CLIC securities. From equation (32) and (31) we see that savings decisions (buying bonds) will depend on the precautionary motive, μ_i, and on a "smoothing" motive, the second term on the right hand side in equation (31). Let us look at the demands by individuals for the CLICs, equation (24). We see that an individual will demand more of a CLIC the higher is its risk premium $\bar{\mathbf{D}}$. An individual will demand less of a CLIC the less the CLIC covaries with the individual's endowment, $\mathbf{\Sigma}_i\mathbf{A}$. This is as we expect, the higher the expected payoff of a security, the higher will be the demand for that security. The higher the hedging services of a security, the more attractive is the security to an investor and the more the investor will demand.

The risk premia will depend on the covariance of the aggregate consumption processes with the values of the CLIC security next period, see equation (20) and aggregate over $i = 1, ..., I$. This is a standard result in intertemporal Capital Asset Pricing, which was first clearly exposited by Breeden [1979]; the generality and robustness of this result was indicated by Grossman and Shiller [1982]. Our conjectured processes for prices and the risky component of dividends assures that the risk premia, $\bar{\mathbf{D}}$, will be constant over time. In particular we have devised securities in which individuals will take positions at time zero, consume the dividends forever and never rebalance portfolios. Given these prices and dividends, individuals will always maintain the same investment positions in the CLICs.

The equilibrium interest rate will depend in part on the rate of time preference in the economy, equation (35). One way to see how is to take a world where there is no risk. Then the interest rate will equal the rate of time preference and there will be no borrowing or lending in the economy. We can see this since the drift term will necessarily be zero in the consumption processes for all individuals and thus the consumption and income processes will coincide in this case and both will be riskless. Since the rates of time preferences are the same across individuals, no one will want to borrow or lend due to patience or impatience in this world as we have effectively assumed this away by assuming the same rate of time preference across individuals.

Once we introduce risk in the economy, the interest rate will still depend on the rate of time preference but will also depend on the average risk of consumption in the economy. As an example take a world where there is no risk in the aggregate but each individual has some risk in their endowment. Then in a world with only a bond there is still a motive to save for precautionary reasons and the average risk in the economy will also determine the interest rate. If there were CLICs in the economy which allowed all individuals to hedge their income risk perfectly, then the interest rate would once again equal the rate of time preference and there would once again be no borrowing or lending.

If we are in a world where there is some aggregate risk, market risk, then the interest rate will always be less than the rate of time preference. Even if everyone is perfectly hedged the interest rate could never equal the rate of time preference. The reason for this is as long as individuals have some risk in their consumption processes, they will have a precautionary motive for saving. Thus if the interest rate equaled the rate of time preference, and all individuals held the market portfolio (world portfolio), then all individuals would want to save as a precautionary motive. But the market for bonds would not clear as all individuals would demand bonds. Thus the interest rate would have to be lower than the rate of time preference in order to offset this precautionary motive for savings to clear the bond market.

3 Contract Designer's Problem: The Risk-Optimal Interest Rate

Let us now turn to the contract designer's problem, which is to define a small number, $N \ll I$, of optimal securities, WICs, and show the designer wishes to maximize the interest rate. We assume that the contract designer wishes to choose $\mathbf{A}$ to maximize a social welfare function which we assume is the negative sum of the log of negative lifetime expected utilities over all individuals, where the individual's utilities are maximized given equilibrium prices $\mathbf{P}_t$, dividends $\mathbf{D}_t$ and interest rate r. The contract designer maximizes:

$$S_0 = -\sum_{i=1}^{I} \ln(-U_{0i}) \tag{37}$$

subject to

$$\mathbf{A}'\mathbf{\Sigma}\mathbf{A} = \mathbf{I}. \tag{38}$$

By assuming that the social welfare function is loglinear, we achieve much simplification of expressions. It is also the social welfare function which results in the same contract design problem as in the one period mean-variance case under similar assumptions.[11]

In order to solve the problem for the contract designer, we must derive an expression for the lifetime expected utility for the individuals in the economy. We can show, see the appendix for derivation, that lifetime expected utility at time zero, U_{0i}, can be expressed as:

$$U_{0i} = -\frac{1+r}{r}\exp(-\gamma c_{0i}) \tag{39}$$

and so, using equation (33) and (38) we can rewrite the social welfare function as:

$$S_0 = -\sum_{i=1}^{I}(\ln(1+r) - \ln(r) - \gamma y_{0i}). \tag{40}$$

One will notice that the social welfare function is monotonically increasing in r. Thus to define new contracts all we need to do is to maximize r or $\ln(1+r)$ with respect to $\mathbf{A}$ subject to the normalization constraint, equation (38). Equivalently, using equation (35), we can maximize $tr(\mathbf{q}'\mathbf{q})$ or in terms of the $\mathbf{A}$ matrix we can maximize the following expression:[12]

$$S_1 = tr(\mathbf{A}'\mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}). \tag{41}$$

We set up the Lagrangian that represents the constraint that diagonal elements of $\text{Var}(\mathbf{X}_t) = \mathbf{A}'\mathbf{\Sigma}\mathbf{A}$ equal 1 and off-diagonal elements equal 0. The Lagrangian is,

$$L = tr(\mathbf{A}'\mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}) - \sum_{n=1}^{N}\sum_{m=1}^{n}(\mathbf{A}_m'\mathbf{\Sigma}\mathbf{A}_n - e(m,n))\lambda_{mn} \tag{42}$$

where

$$e(m,n) = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n. \end{cases} \tag{43}$$

[11] In social welfare theory, it is normally assumed that the social welfare function is an increasing and concave function of individual utilities, see for example Mas-Colell, Whinston and Green [1995]. Ours being convex in individual utilities, and utilities being negative exponential, effectively offsets diminishing marginal utility and results in our maximizing risk sharing, i.e. minimizing consumption variance, without regard to income levels. In practice when we experimented with concave social welfare functions, optimal contracts were in effect designed for people in the poorest regions of the world. These are the regions where we suspect that innovative risk-sharing institutions are least likely to succeed in practice.

[12] To obtain this particular expression see the appendix.

First order conditions for a maximum are:

$$\frac{\partial L}{\partial \mathbf{A}_n} = \mathbf{\Sigma M \Sigma A}_n - \mathbf{\Sigma A}_n \lambda_{nn} - \sum_{n=1}^{N} \sum_{m=1}^{n} \mathbf{\Sigma A}_m \lambda_{mn} = 0, \quad n = 1, ..., N. \tag{44}$$

$$\frac{\partial L}{\partial \lambda_{nn}} = \mathbf{A}_n' \mathbf{\Sigma A}_n - 1 = 0, \quad n = 1, ..., N. \tag{45}$$

$$\frac{\partial L}{\partial \lambda_{mn}} = \mathbf{A}_n' \mathbf{\Sigma A}_m = 0, \quad m \neq n. \tag{46}$$

In this case, we need consider explicitly only the diagonal constraints in $\mathbf{A}'\mathbf{\Sigma A} = \mathbf{I}$ as the off diagonal elements will be zero even if unconstrained. This particular result is shown in Darroch [1965] and in Okamoto and Kanazawa [1968]. In matrix form, the first order conditions reduce to:

$$\mathbf{M\Sigma A} = \mathbf{A\Lambda} \tag{47}$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose nth diagonal element is λ_{nn}. Thus, the columns of the desired matrix $\mathbf{A}$ are determined as N eigenvectors of the matrix $\mathbf{M\Sigma}$, whose ijth element is the covariance between the deviation of individual i's income from world-average income and individual j's income. To find which eigenvectors should be included in $\mathbf{A}$, we pre-multiply (47) by $\mathbf{A}'\mathbf{\Sigma}$ and using $\mathbf{A}'\mathbf{\Sigma A} = \mathbf{I}$, find that $\mathbf{A}'\mathbf{\Sigma M \Sigma A} = \mathbf{\Lambda}$ which is diagonal. Comparing this with (41) we find that the objective function equals the sum of the eigenvalues of the included eigenvectors. Thus the social planner chooses the N eigenvectors with the largest eigenvalues to define the WICs. We will arrange the columns of $\mathbf{A}$ in order of decreasing eigenvalues, so that the more important contracts are towards the left.

We now show that all of our optimal WIC securities will be essentially swaps. Post-multiplying (47) by $\mathbf{\Lambda^{-1}}$, one finds that $\mathbf{A} = \mathbf{M\Sigma A\Lambda}^{-1}$. Since $\mathbf{M}$ is idempotent, $\mathbf{MA} = \mathbf{MM\Sigma A\Lambda}^{-1} = \mathbf{A}$. It follows, since $\boldsymbol{\iota}'\mathbf{M} = \mathbf{0}$, that $\boldsymbol{\iota}'\mathbf{A} = \mathbf{0}$. Thus, the sum of the elements in each column of $\mathbf{A}$ equals zero.

Since $\mathbf{MA} = \mathbf{A}$, we can rewrite (47) as:

$$\mathbf{M\Sigma MA} = \mathbf{A\Lambda}. \tag{48}$$

$\mathbf{A}$ is just the matrix of the N eigenvectors of $\mathbf{M\Sigma M}$ corresponding to the highest eigenvalues. $\mathbf{M\Sigma M}$ is the variance matrix of deviations of individual endowments from the world endowment. From (47) we also see that $\mathbf{q} = -\mathbf{A\Lambda}$. Note also that since

$\mathbf{A}$ is an eigenmatrix of a symmetric positive-semi-definite matrix, $\mathbf{A'A}$ is a diagonal matrix. Since $\mathbf{A'\Sigma A} = \mathbf{I}$, the diagonal element of $\mathbf{A'A}$ are inversely proportional to the corresponding eigenvalues.

We can now, using the risk-optimal interest rate model, produce measures that place a dollar value on the availability of these WIC securities. The amount of (certain) endowment increase per period for individual i to be as well off without the N WIC securities as with the N WIC securities is given by F_{iN}. Define c_{ti}^- as the consumption stream for individual i if WIC securities were never constructed, only a bond market exists. Also, define r_-, in contrast to the risk-optimal interest rate, as the interest rate that would obtain if there were no WIC securities, it is the interest rate prior to the construction of the new WIC securities. Then F_{iN} is the solution to:

$$\mathrm{E}_0[\sum_{\tau=0}^{\infty} \mathrm{u}_0(\mathrm{c}_{\tau\mathrm{i}}^- + \mathrm{F}_{\mathrm{iN}})/(1+\rho)^\tau] = \mathrm{E}_0[\sum_{\tau=0}^{\infty} \mathrm{u}_0(\mathrm{c}_{\tau\mathrm{i}})/(1+\rho)^\tau] \tag{49}$$

where F_{iN} is a constant. We can solve for F_{iN} as:[13]

$$F_{iN} = (c_{0i} - c_{0i}^-) + \frac{1}{\gamma}(\ln(r) - \ln(1+r) - \ln(r_-) + \ln(1+r_-)). \tag{50}$$

We see that the welfare gain measure depends on two terms, the first being the change in time zero consumption from before to after risk sharing. The second term takes into account the change in the interest rate which will be positive here since $r > r_-$. Note that for some individuals the first terms will necessarily have to be negative so that the goods market will clear at time zero.

One of the interesting results in this section is that the contract design problem under the assumption that the welfare function is the sum of the logs of the utilities results in the contract design problem which is identical to that in a one period mean variance world where agents have identical coefficients of absolute risk aversion, compare equation (47) with Athanasoulis and Shiller [2000] Theorem 1. As such we are able to make simple prescriptions on how optimal contracts are to be chosen. One estimates a variance matrix and use equation (47) to choose the $\mathbf{A}$ matrix. One will notice that under the assumptions in this model, if the contract designer were allowed to change the $\mathbf{A}$ matrix after time zero, she would not as that same $\mathbf{A}$ matrix would still be optimal. Thus the contracts designed here are time consistent.

[13] See appendix for derivation.

4 Some Illustrative Examples

To clarify what we have done let us consider some very simple examples of our theory and their implications for the WIC securities. The first example, example A, illustrates the effect of unequal risk on contract design in the simplest two-individual case. The second example, example B, is used to illustrate how our theory creates groupings of individuals of the world, how it decides which individuals should be grouped with each other as having positive weights in the **A** matrix and which should be grouped together as having negative weights. The third example, example C, illustrates how the population of these groups affects the optimal contracts.

4.1 Example A

In this example we assume there are two individuals and that the first individual has income change variance of 1, the second a much larger variance σ^2 and the covariance is zero. Thus the $\mathbf{M\Sigma M}$ matrix is:

$$\mathbf{M\Sigma M} = \begin{bmatrix} 1/4 + \sigma^2/4 & -1/4 - \sigma^2/2 \\ -1/4 - \sigma^2/4 & 1/4 + \sigma^2/2 \end{bmatrix} \tag{51}$$

It is clear the optimal contract, WIC, will be a one-for-one swap of each individual's demeaned (at time 0) endowment. That the swap is one-for-one may seem strange given the unequal risk of endowments across the two individuals but one must remember that the contract must be constructed so that each individual in the world can take a position in the contract and hold a share of the world (market portfolio). Thus there must be a one-for-one swap of risk. Given there are only two individuals in this example, individual 2 will pay a premium each period for this contract while individuals in country 1 receive a premium each period. In the end all individuals hold a share of the world endowment risk.

4.2 Example B

In this example we move to a four-individual case where we assume that the world divides naturally, in terms of correlations, into two blocks: within each block the individuals are highly correlated with each other, but there is no correlation between blocks. We write the $\mathbf{\Sigma}$ matrix as follows:

$$\mathbf{\Sigma} = \begin{bmatrix} 1.0 & 0.9 & 0 & 0 \\ 0.9 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0.9 \\ 0 & 0 & 0.9 & 1.0 \end{bmatrix}. \tag{52}$$

We can then obtain $\mathbf{M\Sigma M}$:

$$\mathbf{M\Sigma M} = \begin{bmatrix} .525 & .425 & -.475 & -.475 \\ .425 & .525 & -.475 & -.475 \\ -.475 & -.475 & .525 & .425 \\ -.475 & -.475 & .425 & .525 \end{bmatrix}. \tag{53}$$

This matrix has one eigenvalue equal to 1.9 and two eigenvalues both equal to 0.1. The vector $\mathbf{A}$, derived using the eigenvector corresponding to the largest eigenvalue is given by:

$$\mathbf{A} = .36 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}. \tag{54}$$

Thus, except for scaling, the component may be described as just a short position in the first two individuals and an equal and opposite long position in the other two. This contract is, as we might expect, a swap between the two blocks of individuals. This component is quite different from the first principal component of $\mathbf{\Sigma}$. That matrix has two first principal components, both with the same eigenvalue. These components are proportional to the vectors $[1\ 1\ 0\ 0]'$ and $[0\ 0\ 1\ 1]'$; if we created a contract in either of these, then we would not provide any means for the two groups of individuals to swap their risks. The vector $\mathbf{q}$ corresponding to the eigenvector in equation (54) is given by:

$$\mathbf{q} = .69 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}. \tag{55}$$

The first two individuals are short the component, the second two are long the component. Note also that in this case, where there is symmetry between the two blocks of individuals, the risk premium, $\bar{D}$, is zero.

It is instructive to look at the $I \times I$ matrix $\mathbf{I} + \mathbf{Aq}'$, whose ith column gives the after-hedging exposure of individual i to the endowment risks of the four individuals:

$$\mathbf{I} + \mathbf{A}\mathbf{q}' = \begin{bmatrix} .75 & -.25 & .25 & .25 \\ -.25 & .75 & .25 & .25 \\ .25 & .25 & .75 & -.25 \\ .25 & .25 & -.25 & .75 \end{bmatrix}. \tag{56}$$

Not all elements of this matrix equal .25, as would be the case if we had included all three possible contracts and thereby spanned the world risk-sharing opportunities, resulting in each individual holding one quarter of the world. Since we have only one contract for trading income, it is not possible for each individual to hold the world (market) portfolio, but the holdings shown in expression (56) do nearly as well for risk reduction, given the covariance matrix $\mathbf{\Sigma}$ that was assumed. For example, for individual 1 the holding of .75 times its own income minus .25 times individual two's income is almost as good as the holding of .25 times its own income and .25 times individual two's income, given the high correlation between the two.

If we were to create the next two WICs, then each of these WICs would entail a swap between the pairs of individuals within each block; again the risk premium of the contract will be zero. The risk reduction afforded by such swaps is much smaller because the individuals are so highly correlated within each pair.

4.3 Example C

We show the importance of the population of groups of individuals on contract design where individuals within a group have the identical endowment process. These groups are much the same as the countries in our empirical section below. We use a two group case, case 1, and a three group case, case 2 to demonstrate the importance of differing populations. In these cases we assume all individuals have unit variance and the covariances across groups are zero. For case 1 let population in group 2 be much larger than population in group 1. The variance matrix for this case will be as follows. Let pop_k be the population of individuals in group k. Let $\boldsymbol{\iota}_1$ be a $\text{pop}_1 \times 1$ vector of ones and similarly let $\boldsymbol{\iota}_2$ be a $\text{pop}_2 \times 1$ vector of ones. Note that $I = \text{pop}_1 + \text{pop}_2$. Then we can represent the variance matrix of endowments as:

$$\mathbf{\Sigma} = \left[\begin{array}{c|c} \boldsymbol{\iota}_1\boldsymbol{\iota}_1' & 0 \\ \hline 0 & \boldsymbol{\iota}_2\boldsymbol{\iota}_2' \end{array} \right] \tag{57}$$

so that all elements of the diagonal blocks equal one. Despite the unequal total endowment risk in the two groups, the WIC is an equal swap of *per capita* endowment risk, normalized so that the WIC variance is one. Each individual in the world would like to trade off some of her endowment risk for some of the endowment risk individuals in the other group have. Since we normalize the risk premium $\bar{D}$ of the WIC to be positive, it must be that the dividend covaries positively with the world endowment and thus group 1 weights negatively in the WIC while group 2 weights positively in the WIC. If we were to give a weight to all individuals in the world in this WIC, the weight for individuals in country 1 would be $\frac{-1}{\sqrt{2}\mathrm{pop}_1}$ and the weight for individuals in country 2 are $\frac{1}{\sqrt{2}\mathrm{pop}_2}$. This must be the case since both groups have equal but opposite weights in the contract and group 2 is larger. Individuals in group 1 will buy the contract, and receive a risk premium, while individuals in group 2 will sell the contract and pay a risk premium. Individuals in group 1 benefit enormously by both reducing their risk and receiving a premium for doing so. Individuals in group 2 have much less risk reduction and also pay a premium to achieve the reduction.

In case 2, we add in a third group, group 3. Let her population equal the population of group 2. Then the first best contract is an equal swap of demeaned (at time zero) *per capita* endowment between group 2 and group 3 with group 1 having a weight of virtually zero. The price of this contract is virtually zero and both groups, 2 and 3, benefit equally by this symmetric swap of risk. The small group hardly benefits at all. As the number of groups in the exercise are increased, the less populous countries tend to have less benefit in the first few contracts. The only way less populous countries can benefit in the first few contracts, is if they happen to correlate well with some more populous country.

5 An Application: The G-7 Countries

As an example of our methods we apply it to the G-7 countries, Canada, United States, Japan, France, Germany, Italy and the United Kingdom. These are representative of the developed world where innovative risk management contracts are most likely to succeed today. In order to apply our methods we need estimates of the variance matrix, $\mathbf{\Sigma}$, the real risk free interest rate, r, the coefficient of absolute risk aversion, γ, and the rate of time preference, ρ.

A few points are very important in the understanding of the results below. First the variance matrix is the only parameter that will affect the WIC securities and the optimal demands for the WIC securities. The choice of the interest rate, the coefficient of absolute risk aversion and the rate of time preference will have no effect on these, though they will affect the welfare gains. Since we are able to formulate the contract design problem as in effect a one-period problem, discounting does not matter in the estimates of contracts but clearly discounting will affect the contracts' worth to individuals.

5.1 A Three-Level Income Model of Individual Income

We are able to estimate the matrix Σ for individuals in these countries even though its dimensions are very large (the combined population of these countries in 1992 was 644,594,000) by assuming that all individuals within a country have the same income (endowment) process and using a model of the changes in incomes of individuals that implies that the entire matrix is determined by four parameters. Our three-level income model represents the change in individual i's income from $t-1$ to t is taken to be the sum of three components representing three levels of income comovement:

$$\Delta y_{ti} = \epsilon_{ti} = u_{tw} + u_{tc} + u_{ts} \tag{58}$$

where u_{tw} is a world shock, common to all people in the world, $u_{tc}, c = 1, .., 7$ is a country shock, common to everyone in country c but uncorrelated with the country shock of individuals in other countries, and $u_{ts}, s = 1, .., 7$ is a spatial shock that is common to everyone in country s but correlated with the spatial shock of individuals in other countries according to a spatial model.[14] Each of the three shocks is assumed to be normally distributed with constant variance through time, serially uncorrelated, and uncorrelated with the other two shocks.

With this method of estimating $\mathbf{\Sigma}$, we impose prior restrictions that all individuals have the same mean (zero) and variance of changes in real per capita income (endowment), and that covariances are determined by a spatial component of risk between countries and a common component of risk. These prior restrictions make it possible to estimate a sensible variance matrix with limited data, one that represents world,

[14] If we were to expand this model to a four-level income model by adding an individual specific shock, that is a shock specific to each individual in each country, the results of this section would be unchanged. From equation (47) and using the fact that $\mathbf{MA} = \mathbf{A}$, one can see that if we add an identity matrix times a constant to $\mathbf{\Sigma}$, then the equations are solved by the same $\mathbf{A}$ matrix, while the eigenvalues are each increased by an amount equal to the constant.

idiosyncratic and spatial shocks, but that otherwise represents countries symmetrically.

Our prior assumptions for estimating the variance matrix of changes in real per capita national incomes are represented by the following formulas for the elements of $\mathbf{\Sigma}$:

$$\begin{aligned} \sigma_{i,i} &= \exp(\alpha^w) + \exp(\alpha^c) + \exp(\alpha^s) \quad i = 1, ..., I \\ \sigma_{i,j} &= \exp(\alpha^w) + \exp(\alpha^s)\exp(-\delta d_{i,j}) \quad i \neq j \end{aligned} \tag{59}$$

where d_{ij} is the distance between countries (individuals) i and j, measured as air miles between the major city in the respective countries. We used the air mile distances between the major cities Montreal, New York, Tokyo, Paris, Berlin, Rome and London. The parameters to be estimated are α^w, α^c and α^s which are associated with the world, country and spatial components, and δ. The correlation between the spatial shock of individuals in country i with the spatial shock of individuals in country j is $\exp(-\delta d_{i,j})$. Since δ is positive, the further away the major city of two countries, the less is the covariance of per capita income among the two countries. This formula corresponds to a valid (i. e, the variance matrix is nonnegative definite for any placement of cities) isotropic (i. e., the model is invariant to rotations of the coordinate system) spatial model where the cities lie in $\mathbb{R}^2$, see Cressie [1991, p. 86]. The formula also corresponds to a valid isotropic spatial model where the cities lie on the surface of a sphere and distances are measured along great circles, as in our application to the earth. Moreover, the variance matrix is strictly positive definite unless two cities coincide.

This formulation restricts all covariances to be positive. The prior restriction that all covariances are positive may seem strong, but it is maintained here as a sort of common sense prior notion that there is really no reason in general for any pairs of countries to tend to move opposite each other. This restriction may serve to reduce the possibilities for diversification, by eliminating the negative correlations that diversifiers seek. The effect of the restriction will tend to be to make it more difficult to make a case for the WIC securities.

We estimate all parameters in the variance matrix with data on *per capita* gross domestic products in 1985 US dollars from the Penn World Table, see Summers and Heston [1991] updated to 1992 from http://www.nber.org. A Maximum likelihood estimate is taken using the 42 observations on *per capita* income changes for each country. Future work will examine similar models of the variance matrix with other “common sense” priors.

5.2 Calibration of r, γ and ρ

In this section we specify the real risk free interest rate, r, and the coefficient of absolute risk aversion, γ. Once we specify these two parameters, the subjective rate of discount, ρ, is determined by equation (35). Empirical studies have found wildly different estimates of the coefficient of relative risk aversion parameter (and the discount rate), depending on the kind of circumstances that generate the data, see Thaler [1990]. Values of the coefficient of relative risk aversion have been estimated in the 100s, but these may be regarded as implausibly high; we choose it equal to three as representing a sort of consensus by many who work in this literature as a reasonable value to assume. In order to obtain the coefficient of absolute risk aversion, γ, we take the average income in 1992, $\bar{y} = 14783.43$, see table 1, and setting $\gamma = \frac{3}{\bar{y}}$, we obtain $\gamma = 0.000203$.

To obtain an estimate of r, we use the data from McCulloch and Kwon [1993] for one month maturity zero coupon bonds for the whole sample, 1946:12-1991:2 for the nominal interest rate. To obtain an inflation series we use the CPI-U not seasonally adjusted from the BLS. We take the geometric average of the real interest rates and obtain $r = 0.49\%$ as the annualized real interest rate. With these values of r and γ we obtain from equation (35) that $\rho = 0.0077$.

5.3 Results

The estimated parameters of (59) are $\hat{\alpha}^w = 11.05$, $\hat{\alpha}^c = 9.88$, $\hat{\alpha}^s = 10.82$ and $\hat{\delta} = 0.00017$. These imply a standard deviation of annual *per capita* income change of \$364.60. Of the three components, the most variable is the world component, and this component has no effect on our WICs since the world component cannot be hedged away. The spatial component is somewhat more important than the country-specific component: its standard deviation is 1.6 times larger. The estimate of δ implies that the correlation between the spatial components of the USA and Canada (320 miles) is 0.95, while the correlation between the spatial components of the USA and Japan (6740 miles) is 0.32.

We show, for $N = 2$, the elements of $\mathbf{A}$ for individuals in each country in Table 1. Since each individual within a country gets the same weight, we need not display all 644,594,000 rows of $\mathbf{A}$: we show only one of the rows for each country. Looking at the first column of the $\mathbf{A}$ matrix, in Table 1 we see that the first contract is essentially a USA vs. Japan swap. This contract weights individuals in the USA and individuals in Japan in the opposite direction with the highest weights. In this contract there are also weights on the core European Union (Core-EU) countries which we define as France, Germany, Italy and the UK and these are on the same side as Japan with similar weights but they are about half of Japan's. Canada has a positive weight, as the US, with less than half the weight given to US individuals. As such we expect that the US and Japanese individuals will gain the most benefit from this contract.

¿From the second column of **A** in Table 1 we see that contract 2 is essentially a Japan vs. (Core-EU) swap. The two contracts also span, approximately a US vs. (Core-EU) swap. This can be achieved by taking a long position in the first WIC and about two thirds of a short position in the second.

With these two WICs contracts it is possible for the US, Japan and the Core-EU as a group, to achieve "most" of the risk sharing possible between them. We see in the next two columns of Table 1, the first two columns of the **q** matrix show that the largest investment positions in the first WIC are taken by individuals in the US and in Japan. The US takes a negative position in the contract as it has a positive weight in the contract, so that it sells off some of her own income (endowment) process. Japan on the other hand takes a positive position as she has a negative weight in the WIC. Thus the individuals in the US pay a premium when they take positions in the WIC, since we normalize $\bar{D}$ to be positive, while individuals in Japan receive a premium. The investments by the Core-EU are about half of that of Japan and for Canada about one-third of that of the US. One can see similar results in the second column of **q** with the US and Canada being insignificant in that WIC.

As mentioned earlier, only the variance matrix matters for contract definition and investment positions in the WIC securities. As such there are two effects that are driving these results. One is the differences in populations, while the second is the correlation between countries. One way to investigate this is to look at the WICs if the variance matrix; a) only had a world component and country component and b) only had a world component and a spatial component. In case a, given the model for the variance matrix above, what would drive the results is the differences in populations across countries. In general the first eigenvector will pick up, roughly speaking, a swap between the country with the largest population versus all other countries; the second will be a swap, roughly speaking, between the country with the second largest population and the remaining countries with smaller populations and so on. Thus in contract one, the US and Canada would enter with opposite signs since Canada has the smallest population and the US the largest. In case b, where spatial correlations drive the results, one finds that the US and Canada in the first and second contracts have very similar (positive) weights with the same sign. The other countries have similar (negative) weights as in the contracts in case a. Thus one may ask why the US and Canada in table 1 have such different weights; it is because when comparing them according to spatial correlations they are most similar countries but comparing them according to populations they are the most different countries.

We could extend our estimated variance matrix to include some of the smaller EU countries, Belgium or Denmark for example, and using their distances to fill out the variance matrix with the estimated parameters. Then the incomes of individuals in these countries would receive smaller weights than do the Core-EU countries in the columns of the **A** matrix, just as individuals in Canada get smaller weights than do individuals in the United States. As such these smaller EU countries would not be significant in the first few contracts.

The welfare gain for the countries involved are given in Table 2. The first three columns of numbers report the endowment increase needed each period to make individuals as well off without risk sharing as they would be with risk sharing using N contracts. The last three columns report the first three columns as a percent of consumption that would have been realized at time zero if no risk sharing occurred.[15]

The welfare gains are quite substantial. Notably, creating two WICs has a welfare increase in Japan of nearly a $1000 per capita each year forever. Japan achieves a higher welfare gain because it is estimated to be relatively less correlated with the other countries, attributed to its great distance from the others, and because it is given a lot of weight in the contracts, owing to its substantial population. But welfare gains for the other countries from the first two contracts are still substantial, around $400 for each country except Canada. Canada achieves less benefit than the other countries from the first two contracts. Because of her relatively small population, the first two WICs give little weight to Canada, and thus offer little opportunity for Canadians to hedge their country-specific shocks.

We believe that these preliminary numbers might understate the potential welfare gain from creating the new markets. Our variance matrix for income changes, estimated using data from the G-7 countries 1950-1992, may understate risks to incomes as this was a quiet period in their history. Moreover, there are some studies which find very high coefficients of relative risk aversion, higher than the value of 3 implicitly assumed

[15]Note that adding securities need not increase the welfare for each country: for Japan, F_{i6} is less than F_{i2} in Table 2. Adding WIC securities does not always raise social welfare in each country. Adding securities changes interest rates and risk premia in existing markets, which may have an adverse effect on individuals in some countries even though risk-sharing opportunities are increased.

here; see Barsky Juster, Kimball and Shapiro [1997]. Raising either the variances or the coefficient of risk aversion would substantially raise the welfare gain. On the other hand, if we used a higher real interest rate which increases the subjective rate of time preference, ρ, as some calibrations do, the welfare benefits would be lower.

There are many studies in the international risk sharing literature that have evaluated the gains from international risk sharing. They measure the percent increase in consumption per period needed to make an individual as well off without full risk sharing as she would be with full risk sharing. In comparison to the analysis above, we compare the welfare gain when all six WICs are constructed, i.e. full risk sharing. There are five studies in international risk sharing which Athanasoulis and van Wincoop [2000] compare to their own new method. They are Cole and Obstfeld [1991], van Wincoop [1994], Tesar [1995], Lewis [1996] and van Wincoop [1999].[16] While there are differences in results of the welfare gains, the important result in this paper is that the **A** matrix depends only on the estimated variance matrix and not on the parameters of the utility function that vary across studies.

6 Summary and Conclusion

We have presented a constant absolute risk premium model of the world economy that takes as given preferences and income processes, and shows how risk premia and the world real interest rate are determined in equilibrium. We have shown a correspondence between the world real interest rate and social welfare, and we have shown which risk-management contracts, WIC contracts, should be created to achieve the risk-optimal interest rate.

The application, using the three-level income model, to per capita income data on countries and derivation of WIC contracts illustrates some very important risk-management contracts that might be considered to be traded on new markets. We believe that the derivation of the WIC contracts, though based on limited data, are suggestive enough that we can consider creating something approximating the first two

[16] Athanasoulis and van Wincoop [2000] analyze the reasons for the differences in the welfare gains across these studies. One important cause of the differences across studies is the assumed interest rate. If we take an unweighted average of the welfare gains in the last column in table 2, we obtain a welfare gain of 4.93%. If we increase the interest rate to 1.6% which is similar to the parameterization in Athanasoulis and van Wincoop [2000], we obtain an unweighted average welfare gain of 1.59% which is much more in accord with the other studies.

WIC contracts. In application it would be a good idea to simplify them such as equalizing the weights which are similar and setting the small weights to zero. That is, we should consider a US vs. Japan and a US vs. (Core-EU) *per capita* income swap. Our analysis indicates that it would be better *not* to lump other countries into these first contracts, not to do a swap involving the entire EU on one side, for example. Further econometric work should be conducted to confirm or reject the conclusions from the simple econometric analysis here.

Our theoretical framework has the potential to provide the foundation for econometric work that will suggest other, and better, definitions of WICs. The variance matrix, estimated from the three-level income model, may be further refined and account may be taken of other factors in income risks besides country factors. Further work may also generalize the assumptions about the stochastic process of income, may move to an overlapping generation framework, or may incorporate endogenous investments in physical capital.

Appendix

The No-Ponzi-Game Condition

In this section we give a simple interpretation of the no-ponzi-game condition itself by solving the budget constraint forward.[17] We begin with our budget constraint at time t;

$$c_{ti} = y_{ti} + (1 + r_{t-1})B_{t-1i} - B_{ti} + \mathbf{q}_{t-1i}(\mathbf{P}_t + \mathbf{D}_t)' - \mathbf{q}_{ti}\mathbf{P}'_t. \tag{60}$$

Let's define $c^*_{ti} = c_{ti}\prod_{j=1}^{t}\left(\frac{1}{1+r_{t-j}}\right)$ where $\{r\}_0^\infty$ is a bounded strictly positive process and $c^*_{0i} = c_{0i}$. Define y^*_{ti}, P^*_t, D^*_t and B^*_{ti} in a similar fashion. Assume for this example that $B_{-1i} = 0$ and $\mathbf{q}_{-1i} = 0$. Then substituting into (60), and summing over T periods we obtain;

$$\sum_{t=0}^{T}(c^*_{ti} - y^*_{ti}) = \sum_{t=0}^{T-1}\theta_{ti}(P^*_{t+1} + D^*_{t+1} - P^*_t) - \theta_{Ti}P^*_T - B^*_{Ti}. \tag{61}$$

Define $Z^*_t = \sum_{j=0}^{t} D^*_t$. Then we can rewrite (61) as;

$$\sum_{t=0}^{T}(c^*_{ti} - y^*_{ti}) = \sum_{t=0}^{T-1}\theta_{ti}(P^*_{t+1} + Z^*_{t+1} - (P^*_t + Z^*_t)) - \theta_{Ti}P^*_T - B^*_{Ti}. \tag{62}$$

We know that under individual i's equivalent probability measure, $P^*_t + Z^*_t$ follows a martingale, see for example Huang and Litzenberger [1988]. Taking expectations under the equivalent probability measure, E^*_i, of equation (62), we obtain;

$$\mathrm{E}^*_{0i}[\sum_{t=0}^{T}(c^*_{ti} - y^*_{ti})] = -\mathrm{E}^*_{0i}[\theta_{Ti}P^*_T - B^*_{Ti}]. \tag{63}$$

The term on the right hand side of (63) is the present value of individual i's wealth at time T. In a finite period economy the assumption is made that $P_T = 0$ and $B_T = 0$ so that wealth at the final date is zero. If we let T go to infinity, then the no-ponzi-game condition is that the present value of wealth at time T as T goes to infinity must be non-negative. Since the agent uses E^*_i to evaluate the present value of future income streams, the budget constraint states that the present value of consumption equals the present value of labor income.

[17] Much of this, except for the no-ponzi-game condition itself, can be found in Huang and Litzenberger [1988].

Derivation of Equation (32)

To obtain equation (32) we begin by solving equation (26) backward to time zero to obtain:

$$B_{ti} = \sum_{\tau=0}^{t-1}(y_{t-\tau i} + \mathbf{q}_i\mathbf{X}'_{t-\tau} + \mathbf{q}_i\bar{\mathbf{D}}' - c_{t-\tau i})(1+r)^{\tau} + (1+r)^t B_{0i}. \tag{64}$$

We can write the consumption and income processes as;

$$c_{t-\tau i} = \mu_i(t-\tau) + c_{0i} + \sum_{k=1}^{t-\tau} v_{ki}, \tag{65}$$

$$y_{t-\tau i} = y_{0i} + \sum_{k=1}^{t-\tau} \epsilon_{ki} \tag{66}$$

and

$$\mathbf{X}'_{t-\tau} = \sum_{k=1}^{t-\tau} \mathbf{A}'\boldsymbol{\epsilon}'_k. \tag{67}$$

Substituting equations (65-67) into equation (64) we obtain;

$$B_{ti} = \sum_{\tau=0}^{t-1}(y_{0i} + \mathbf{q}_i\bar{\mathbf{D}}' - \mu_i(t-\tau) - c_{0i})(1+r)^{\tau} + (1+r)^t B_{0i}. \tag{68}$$

Now substitute equations (29) and (31) into (68) to obtain;

$$\begin{aligned} B_{ti} = \textstyle\sum_{\tau=0}^{t-1}(y_{0i} + \mathbf{q}_i\bar{\mathbf{D}}' - \mu_i(t-\tau) - y_{0i} - \frac{1}{1+r}\mathbf{q}_i\bar{\mathbf{D}}' + \frac{1}{r}\mu_i - \frac{r}{1+r}B^-_{0i})(1+r)^{\tau} + \\ (1+r)^t(-\tfrac{1}{1+r}\mathbf{q}_i\bar{\mathbf{D}}' + \tfrac{1}{r}\mu_i + \tfrac{1}{1+r}B^-_{0i}). \end{aligned} \tag{69}$$

One can rewrite this expression as;

$$\begin{aligned} B_{ti} = -\tfrac{1}{1+r}\mathbf{q}_i\bar{\mathbf{D}}' - \mu_i(1+r)^t \textstyle\sum_{\tau=0}^{t}(t-\tau)(1+r)^{-(t-\tau)} + \\ \tfrac{1}{r}(1+r)^t\mu_i \textstyle\sum_{\tau=0}^{t}(1+r)^{-\tau} + \tfrac{1}{1+r}B^-_{oi}. \end{aligned} \tag{70}$$

We can rewrite this expression as;

$$B_{ti} = -\frac{1}{1+r}\mathbf{q}'_i\bar{\mathbf{D}} + \mu_i(1+r)^t\left[\frac{1}{r}\sum_{\tau=0}^{t}(1+r)^{-\tau} - \sum_{\tau=0}^{t}\tau(1+r)^{-\tau}\right] + \frac{1}{1+r}B^-_{oi}. \tag{71}$$

Noting that $\mu_i(1+r)^t\left[\frac{1}{r}\sum_{\tau=0}^{t}(1+r)^{-\tau} - \sum_{\tau=0}^{t}\tau(1+r)^{-\tau}\right] = \frac{1}{r}\mu_i(t+1)$ gives us the result.

Derivation of Equation (39)

To obtain equation (39) we begin with, from equation (5), expected lifetime utility at time zero as;

$$U_{0i} = \mathrm{E}_0[\sum_{\tau=0}^{\infty} \mathrm{u}_0(\mathrm{c}_{\tau \mathrm{i}})/(1+\rho)^{\tau}] \tag{72}$$

where

$$u_0(c_{\tau i}) = -\exp(-\gamma c_{\tau i}). \tag{73}$$

We also know from the consumption process that;

$$c_{ti} = \mu_i t + c_{0i} + \sum_{k=1}^{t} v_{ki}. \tag{74}$$

If we substitute (74) and (73) into (72) we obtain;

$$U_{0i} = -\exp(-\gamma c_{0i}) - \mathrm{E}_0[\sum_{\tau=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{\tau} \exp(-\gamma(\mu_i \tau + c_{0i} + \sum_{k=1}^{\tau} v_{ki}))]. \tag{75}$$

Noting that v_{ki} is normally distributed with mean zero variance $\sigma_{v_i}^2$ for all k, one can rewrite equation (75) as;

$$U_{0i} = -\exp(-\gamma c_{0i})[\sum_{\tau=0}^{\infty} \left(\frac{1}{1+\rho}\right)^{\tau} \exp(-\gamma \mu_i \tau + \frac{1}{2}\gamma^2 \sigma_{v_i}^2 \tau)]. \tag{76}$$

We can see from equation (18) that;

$$\gamma \mu_i = \ln(1+r) - \ln(1+\rho) + \frac{1}{2}\gamma^2 \sigma_{v_i}^2. \tag{77}$$

Note also that;

$$(1+\rho)^{-\tau} = \exp(-\tau \ln(1+\rho)). \tag{78}$$

substituting equations (78) and (77) into (76) we obtain;

$$U_{0i} = -\exp(-\gamma c_{0i}) \sum_{\tau=0}^{\infty} (1+r)^{-\tau} \tag{79}$$

and evaluating the infinite sum gives us the desired result.

Derivation of Equation (41)

Since we are maximizing $\ln(1+r)$ we begin with the expression for $\ln(1+r)$, equation (35), as follows;

$$\ln(1+r) = \ln(1+\rho) - \frac{1}{2}\gamma^2 \frac{1}{I}\sum_{i=1}^{I}\sigma_{v_i}^2. \tag{80}$$

We also know, from equation (22), that;

$$\sigma_{v_i}^2 = \Sigma_{ii} + \mathbf{q}_i\mathbf{q}_i' + 2\mathbf{\Sigma}_i\mathbf{A}\mathbf{q}_i'. \tag{81}$$

Since we are maximizing $\ln(1+r)$ with respect to $\mathbf{A}$ and ρ is unaffected by the choice of $\mathbf{A}$, we can see from (80) and (81) that we will be minimizing the one period average variance of consumption in the economy. Note now that we can rewrite the average variance of consumption in the economy as;

$$\frac{1}{I}\sum_{i=1}^{I}\sigma_{v_i}^2 = \frac{1}{I}tr[\mathbf{\Sigma} + \mathbf{q}\mathbf{q}' + 2\mathbf{q}\mathbf{A}'\mathbf{\Sigma}]. \tag{82}$$

Now using our optimal demands for individuals, $\mathbf{q} = -\mathbf{M}\mathbf{\Sigma}\mathbf{A}$, and noting that $\mathbf{M}$ is idempotent and symmetric, we obtain;

$$\frac{1}{I}tr[\mathbf{\Sigma} + \mathbf{M}\mathbf{\Sigma}\mathbf{A}\mathbf{A}'\mathbf{\Sigma}\mathbf{M} - 2\mathbf{M}\mathbf{\Sigma}\mathbf{A}\mathbf{A}'\mathbf{\Sigma}]. \tag{83}$$

where tr denotes the trace. Using the result that $tr(\mathbf{AB}) = tr(\mathbf{BA})$ when the dimensions of $\mathbf{A}$ and $\mathbf{B}$ allow it, we can rewrite equation (83) as

$$\frac{1}{I}(tr[\mathbf{\Sigma}] + tr[\mathbf{A}'\mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}] - 2tr[\mathbf{A}'\mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}]). \tag{84}$$

which can be written as

$$\frac{1}{I}(tr[\mathbf{\Sigma}] - tr[\mathbf{A}'\mathbf{\Sigma}\mathbf{M}\mathbf{\Sigma}\mathbf{A}]). \tag{85}$$

Since $\mathbf{\Sigma}$ is not affected by the choice of the contract designer, the result follows.

Derivation of Equation (50)

To derive this equation we use equation (39) to obtain the following two equations;

$$U_{0i} = -\frac{1+r}{r}\exp(-\gamma c_{0i}) \tag{86}$$

and

$$U_{0i}^{-} = -\frac{1+r_{-}}{r_{-}}\exp(-\gamma c_{0i}^{-})\exp(-\gamma F_{iN}). \tag{87}$$

When we equate these two expressions, as in equations (49), $U_{0i} = U_{0i}^{-}$, we obtain;

$$\exp(-\gamma F_{iN}) = \exp(-\gamma(c_{0i} - c_{0i}^{-}))\frac{(1+r)r_{-}}{(1+r_{-})r} \tag{88}$$

Taking the natural logarithm of both sides and dividing by γ gives the desired result.

Country	1992 Population in 000's	1992 GDP per capita	$\mathbf{A}_{i1}$ $\times 10^{-9}$	$\mathbf{A}_{i2}$ $\times 10^{-9}$	$\mathbf{q}_{i1}$	$\mathbf{q}_{i2}$
Canada	27445	$16362	6.04	-0.60	-58.44	3.99
			(1.32)	(0.84)	(17.2)	(5.2)
USA	255000	$17945	14.62	-4.40	-141.48	29.41
			(1.05)	(3.23)	(9.5)	(19.2)
Japan	124000	$15105	-16.33	-24.21	157.97	161.81
			(3.21)	(14.03)	(27.7)	(79.7)
France	57372	$13918	-7.41	17.79	71.67	-118.94
			(2.27)	(9.64)	(22.9)	(54.1)
Germany	65120	$14709	-8.54	17.24	82.59	-115.24
			(1.73)	(9.37)	(18.5)	(52.3)
Italy	57809	$12721	-8.34	17.33	80.71	-115.86
			(1.61)	(9.09)	(17.6)	(51.3)
UK	57848	$12724	-7.03	17.20	68.06	-114.94
			(2.24)	(9.33)	(22.4)	(52.3)

Notes : The $\mathbf{A}_{in}$ columns, $n = 1, 2$ give the ith element of $\mathbf{A}_n$ if individual i is in the country shown in the leftmost column. The $\mathbf{A}_{in}$ elements shown therefore give the weight given to the income of each individual in a country in determining the dividend paid on one contract n. The numbers in the optimal investment columns $\mathbf{q}_{in}, n = 1, 2$ are the numbers of contracts n individual i will buy according to the theory if the individual is in the country shown in the leftmost column. $ denotes 1985 US dollars. The risk premia are $\bar{D}_1 = \$1.76$ when one WIC is constructed and $\bar{D}_1 = \$1.69$ and $\bar{D}_2 = \$.41$ when two WICs are constructed in 1985 dollars. Standard errors, in parentheses, were obtained with Monte Carlo methods.

Table 1: Population, 1992 GDP per capita, Optimal Contracts and Optimal Investments

Country	F_{i1}	F_{i2}	F_{i6}	$\frac{F_{i1}}{C_0^-}$	$\frac{F_{i2}}{C_0^-}$	$\frac{F_{i6}}{C_0^-}$
Canada	\$75.59	\$88.76	\$688.57	0.46%	0.54%	4.21%
	(\$40.26)	(\$41.99)	(\$54.84)	(0.25%)	(0.26%)	(0.34%)
USA	\$398.85	\$415.12	\$422.94	2.22%	2.31%	2.36%
	(\$51.28)	(\$47.53)	(\$48.55)	(0.29%)	(0.26%)	(0.27%)
Japan	\$495.00	\$980.96	\$968.95	3.28%	6.49%	6.41%
	(\$155.9)	(\$127.4)	(\$125.2)	(1.03%)	(0.84%)	(0.83%)
France	\$109.11	\$385.25	\$706.13	0.78%	2.77%	5.07%
	(\$73.09)	(\$67.14)	(\$66.76)	(0.53%)	(0.48%)	(0.48%)
Germany	\$141.91	\$400.53	\$718.99	0.96%	2.72%	4.89%
	(\$64.24)	(\$59.62)	(\$72.15)	(0.44%)	(0.41%)	(0.49%)
Italy	\$135.92	\$397.43	\$775.28	1.07%	3.12%	6.09%
	(\$58.52)	(\$60.04)	(\$75.71)	(0.46%)	(0.47%)	(0.60%)
UK	\$99.28	\$358.27	\$696.95	0.78%	2.82%	5.48%
	(\$68.71)	(\$62.88)	(\$68.89)	(0.54%)	(0.49%)	(0.54%)

Notes : F_{i1}, F_{i2} and F_{i6} are welfare gains in 1985 US dollars each year if 1, 2 or 6 WICs are created. Under our assumptions, with 6 WICs risk sharing is complete, and so F_{i6} represents the total possible welfare gain from risk sharing. $\frac{F_{i1}}{C_0^-}$ and $\frac{F_{i2}}{C_0^-}$ and $\frac{F_{i6}}{C_0^-}$ are welfare gains in 1992 as a percent of 1992 consumption. For these calculations we used $r_- = 0.49\%$, $\gamma = 0.0002029$ and consequently $\rho = 0.0077$. The equilibrium interest rate when one, two and six WICs are constructed are 0.52%, 0.54% and 0.56% respectively. Standard errors, in parentheses, were obtained with Monte Carlo methods.

Table 2: Welfare Gains in Dollars and as Percent of 1992 Consumption

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