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The link approach to measuring consumer surplus in transport networks

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Abstract

Should one calculate user benefits from changes in door-to-door journeys or from changes in the use of separate links of the network? The second approach is often discarded for its perceived inability to deal with new links and the OD-matrix approach is favoured. Differences arise when the set of used routes changes. A consumer model containing a general static transportation network with explicit non-negativity constraints serves as a basis for welfare measures expressed in shadow prices. The approximation error when applying the link approach need not be too severe. A rehabilitation of the link approach may be in order.

Keywords: Cost-benefit analysis, transport networks, consumer surplus, corner solutions

JEL codes: D61, H54, R42

Samenvatting

Moet men in een kosten-batenanalyse welvaartsbaten afmeten aan veranderingen in deur-tot-deur verplaatsingen of aan veranderingen in het gebruik van de afzonderlijke links van het netwerk? De tweede benadering wordt vaak verworpen vanwege een vermeend onvermogen van deze methode om met nieuwe verbindingen te kunnen omgaan. De zogenaamde HB-benadering (Herkomst-Bestemming) verdient dan de voorkeur. Verschillen in uitkomsten ontstaan inderdaad wanneer de verzameling van gebruikte routes verandert. Dit blijkt uit een consumentenbeslismodel dat een algemeen transportnetwerk bevat, waarbij expliciet niet-negativiteitsrestricties meegenomen worden. Uit het model volgt een uitdrukking voor welvaartsbaten in termen van schaduwprijzen. Cruciaal in de vergelijking van de verschillende methodes voor berekening van welvaartsbaten blijkt de Wardrop-aanname: zijn alternatieve routes op een gegeven HB-paar perfecte substituten of niet? Eventuele schattingsfouten van de linkbenadering zullen vaak niet groot zijn. De link-benadering verdient een rehabilitatie.

Steekwoorden: Kosten-batenanalyse, transportnetwerken, consumentensurplus

1 Introduction

Network projects raise many questions for the cost-benefit analysis practitioner. One puzzling issue regards the unit of analysis: should one calculate user benefits on the basis of the change in door-to-door journeys or derive them from the change in the use of each and every separate link of the network affected? The first approach may be favoured since consumers think in terms of door-to-door journeys. However, as policy measures typically involve a change in the user costs of parts of the networks, one is tempted to use the second approach. The journey, or route approach and the link approach will be shown to be theoretically equivalent under certain conditions. Given these, whether one derives the change in welfare from a change in the demand for routes or from a change in the demand for links, the outcome will be the same.

Sugden (1979) pointed already to the link approach as a way of making welfare assessments of transport networks more down to earth, but in day-to-day practice his suggestion was not followed. The approach most encountered is the measurement of user benefits from changes in the total travel demand, and average travel costs, between nodes of origin and nodes of destination. This OD-matrix approach is a special case of the route approach, demonstrated by, for instance, Kidokoro (2006). Only when the alternative routes serving an OD-pair are perfect substitutes this method is sure to yield the correct results.

Textbook models invariably focus on the OD-matrix approach. This applies to both the older literature, for example Jones (1977), and the more recent, for example Jara-Díaz (2007). The same is true for CBA guidelines. Mackie et al (2003, Note 6, p. 7) even write explicitly: “User benefits should be calculated on a matrix basis and not on a link basis”. Unfortunately, they do not explain why. Their recommendation might stem from practical considerations. It is often suggested that the OD-matrix approach must be favoured as a way to deal with problems that arise if the project at hand introduces a new link in the network. Also Eliasson (2009) is adamant: “it is well known that the consumer surplus from a change in travel times and/or travel costs should in general be calculated at the level of origin-destination pairs.” Surprisingly, Eliasson does perform an ex post evaluation of the Stockholm congestion charging system with the link approach.

To examine the ground for the apparent dismissal of the link approach, or the justification for its application, this paper presents a consumer model containing a transportation network. It serves as a rigorous micro-economic framework to discuss the different approaches to measuring user benefits. The model has three distinguishing features. First, it has door-to-door journeys, or routes, determining the preferences of the consumer. Second, the difference between link use and route use is made explicit in the representation of the network. And third, there is explicit recognition of the possibility of zero demand for link and route use. Such corner solutions, of which a new link is an example, are at the heart of the problems of the general route approach and the link approach. The concept of 'virtual prices', developed by Neary and Roberts (1980), proves helpful to deal with the corners, they are the marginal value of used and unused routes. The virtual prices correspond to shadow prices.

This paper is related to Kidokoro (2004) and (2006). In the 2006 article he shows that the Wardrop and the logit model are special cases of a general model. These two models are specific functional forms of a utility function with route use as argument. His 2004 article gives a general equilibrium model with congestion on the network and indirect taxes. Welfare analysis can then be performed in first-best and second-best cases, and useful decompositions of welfare change are derived. In addition it contains a nice example of a wrong application of the OD-matrix approach. This example is re-mentioned in section 4. Extension of Kidokoro's network in 'routes' to a general static network is however not straightforward, simply because it lacks the distinction between links and routes. Moreover, his analysis concerns interior solutions only, even though new transport routes are discussed. It is exactly in these two areas that this paper attempts to contribute. Besseling and van 't Riet (2009) present a general equilibrium model based on Kidokoro and introduce links but they do not consider corner solutions.

Section 2 proceeds with developing notation for a general static transportation network. The three approaches for measuring user benefits are then defined as application of 'the rule of a half' based on routes, links or OD-pairs. An example illustrates why the OD-matrix and link approach do not yield the same result, even under the assumption of perfect substitutes.

Next, in section 3, a consumer model is presented where links and a network constraint are added and non-negativity is made explicit. The consequences of binding constraints on a marginal cost-benefit rule are shown. Virtual prices come to the rescue in the sense that Roy's identity is salvaged, relevant for measures of welfare change. The conditions for equivalence between the link and OD-matrix approach are given.

Section 4 makes the switch from theoretical models to more practical issues and approximation of consumer surplus. The bridge with theory is the rule of a half expressed in virtual prices. This however seems to defeat the whole purpose of the practicality of the rule because virtual prices are not observable for unused routes. The assumption of perfect substitutes is shown to circumvent this problem. This fact explains, at least partially, the success of the OD-matrix approach. Alternatively, the problem of unobservability can also be ignored. An expression is derived for the approximation error when actual route or link prices are used. Finally, the error may be estimated to demonstrate that the link approach is justified. This is what Eliasson (2009) does, making use of the characteristics of the transportation network around Stockholm. His expression for the approximation error is a special case of the general one derived in this paper. A final section summarizes the arguments.

Appreciation of this paper requires a certain predisposition to accept that two alternative ways of travelling from A to B may not be perfect substitutes, even when the modes of transportation are the same.

2 Equivalence of OD-matrix and link approach?

In this section notation is developed and the approaches for measuring user benefits are defined. The network concepts are illustrated using an example and equivalence of the approaches is investigated.

2.1 Network, flows and costs

The network considered in this paper is a general static transportation network allowing for different modes of transportation. This abstract network consists of a set of nodes and a set of links which are the direct connections between a pair of nodes. A link is typically a road segment but it may also be the service of a bus company between two successive stops of a given line. A route is a set of connected links, and thus also a set of adjacent nodes. Links themselves are routes too. Every route connects a unique origin-destination (OD-) pair. And any pair of distinct nodes can be an OD-pair: the network is assumed to be connected. Even when cyclical routes are excluded, the number of routes of a network will be enormous. A network of ten nodes, with one link between each pair of nodes, has some ten million non-cyclical routes. Clearly, in an economic context with cost minimization, the bulk of these routes will never be eligible for use.

The economic goods in relation to the network are trips, or journeys, over this network: travel from one node to another via routes *r*. Let x_r^R be the continuous number of trips over such a specific route *r* of the network. Use of a route implies the use of one or more links. Flow variable x_l^L represents the number of trips over a specific link l. Superscripts *R* and *L* refer to variables pertaining to routes and links respectively. Let M be the matrix indicating which links a route is composed of. Element M_{lr} is 1 when link *l* is part of route *r* and is 0 otherwise. Total link use is equal to the sum of the use of those routes it is part of.

$$x_l^L = \sum_r M_{lr} x_r^R \tag{2.1}$$

It is important to note that the route-link incidence matrix is here taken to be given and is not dependent on the set of routes actually used. For a similar way of modeling see Van Dender (2004).[1]

Next consider the cost of using the network. It is assumed that the consumer only pays for link use, which has given 'prices' p_l^L. These prices are the generalised user costs of a link, including time costs. In the context of a general equilibrium model these costs would, of course, be endogenous variables most likely increasing with the total use of the link because of congestion. Many transport models nowadays do have an equilibrium mechanism iterating between travel demand and travel costs, the latter being supplied by the network after an assignment. Here however they are considered to be fixed, as the focus is on the consumer model for the purpose of welfare analysis.

For some components of the generalised costs addition over links may be straightforward, such as distance, or time. This additivity is less obvious for the valuation of time travelled, as the valuation will depend on the total trip distance or journey duration. Nevertheless, such a linear relation is often assumed, see for instance Wohl en Hendrickson (1985, p.49). The cost of using a route r, 'price' p_r^R, is then the sum of the cost of using the composing links. This is referred to as the linear cost assumption.

$$p_r^R = \sum_l p_l^L M_{lr} \tag{2.2}$$

In vector notation equation (2.2) implies a matrix pre-multiplication, making use of the convention among economists that prices and costs are represented by row-vectors. Substitution of equations (2.1) and (2.2) neatly gives the equality of expenditure on links and on routes.[2]

$$p^L x^L = p^L M x^R = p^R x^R \tag{2.3}$$

[1] Eliasson (2009) refers to changing route-link incidence matrices and uses an OD-pair-link incidence dependent on actual use. Verhoef (2002) employs fixed route-link incidence but an endogenous route-OD-pair incidence.

[2] Such a 'translation' of routes into links, or vice versa, is repeated below.

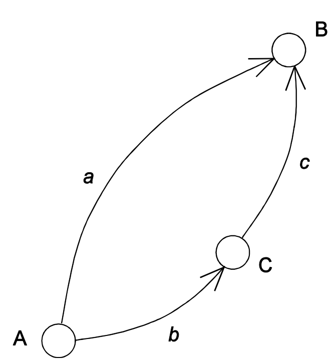
As an extension of the notation consider x_{ijk}^{R} , the number of trips from node *i*, an origin, to node *j* , a destination, over a certain route *k*. OD-pairs play an important role in traffic models and the practice of consumer surplus measurement. Hence the *ijk* notation is required, instead of the more general index *r* for routes, to define the OD-flow, x_{ij}^{OD}, and OD-prices p_{ij}^{OD}. Summation is over the set of all alternative routes that connect the given OD-pair (*i*,*j*), irrespective of being used or not.

$$x_{ij}^{OD} = \sum_k x_{ijk}^{R} \tag{2.4}$$

$$p_{ij}^{OD} = \sum_k p_{ijk}^{R} x_{ijk}^{R} / x_{ij}^{OD} \tag{2.5}$$

Finally consider as an example the following minimal network of 3 nodes, 3 links and 4 routes in figure 1.

Figure 1: Example network



The nodes are A, B and C. The links are the one-way connections *a*, *b* and *c*. The four routes are *a*, *b*, *bc* (=*b* +*c*) and *c*. The one-way-traffic makes that there are just three OD-pairs: (A,B), (A,C) and (C,B). The OD-pair (A,B) has two alternative routes, which are *a* and *bc*. The third route, *bc*, is composed of two connecting links, as reflected in the third column of the route-link incidence matrix, see (2.6).

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{2.6}$$

2.2 The ‘rule of a half’ and the three approaches

With the flow and cost variables introduced above a first pass at measuring welfare change can be made. Widely used in practice, since Neuburger (1971), is the so-called ‘rule-of-a-half’ (RoH). Consider an initial and a new situation with given vectors of prices and quantities: (p^0, x^0) and (p^1, x^1). These values may be obtained from a (traffic) simulation model or could be actual observations, before and after a change. The corresponding change in consumer surplus, ΔCS, can be measured with the rule of a half. The RoH is an approximation of Marshallian CS with a linearization of demand.

$$\Delta CS = -\tfrac{1}{2}\sum_i (x_i^1 + x_i^0)(p_i^1 - p_i^0) = -\tfrac{1}{2}\sum_i (x_i^1 + x_i^0)\Delta p_i \tag{2.7}$$

The RoH can be computed based on routes, links or OD-pairs and the different ‘approaches’ can be thus defined: ΔCS^R, ΔCS^L and ΔCS^{OD}. Consider the first two. Using the ‘translation of routes into links’, shown before, it is evident that under the linear cost assumption the route and link approach are identical: $\Delta CS^R = \Delta CS^L$.
This cannot, in general, be expected from the OD-matrix approach because of the average OD-pair price, see (2.5). However, a common assumption is that the alternative routes serving an OD-pair are perfect substitutes: only getting there matters. Then the Wardrop principle applies (Wardrop, 1952): in equilibrium the generalized costs are equal for all used routes.

$$p_{ij}^{OD} = p_{ijk}^R \quad \text{for all } k \text{ with } \quad x_{ijk}^R > 0 \tag{2.8}$$

Under the Wardrop assumption of perfect substitutes, the route and OD-matrix approach are identical too: $\Delta CS^R = \Delta CS^{OD}$.
Therefore also the link and OD-matrix approach give exactly the same approximation of consumer surplus. This seems to bode good news for the CBA practitioners in the field of transport investment: alas, the example below shows that the linear cost assumption is not sufficient for equivalence of the approaches.

2.3 Corner solutions and virtual prices

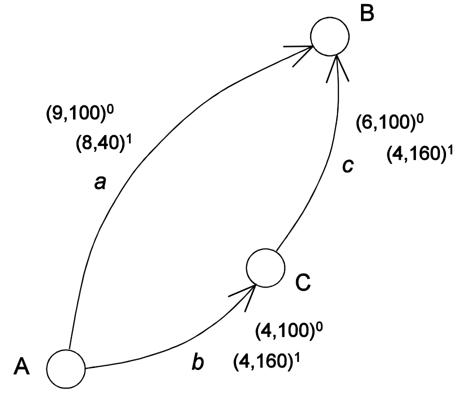
The three approaches are illustrated with an example based on the network introduced earlier. There are a hundred trips on each OD-pair, before an investment and after. The investment reduces the generalized travel costs of link *c* which makes travelers from A to B switch routes from the direct route *a* to route *bc*. The alternative routes are perfect substitutes, so the Wardrop principle applies. Reduced traffic on link *a* makes its cost go down too and a new user equilibrium is reached, see table 1 and figure 2.

Observe that route *bc* has cost 10 (=4+6) in the 'before situation', where the cost of link *a* are only 9. This is consistent with the Wardrop assumption as all travel from A to B uses the cheaper route, being link *a*.

Table 1: Example with link data before and after

		link *a*	**link *b***	**link *c***	**total**
before	**cost**	9	4	6	
	travel	100	100	100	
after	**cost**	8	4	4	
	travel	40	160	160	
diff	**Δ cost**	-1	0	-2	
	Δ CS-links	70	0	260	330

Figure 2: Example with link data before $(p,x)^0$ and after $(p,x)^1$



The user link and route approach yield the same result which differs from the outcome of the OD-matrix approach, despite the Wardrop assumption. Apparently corner solutions wreck the equivalence between the approaches. The corner is found in the initial situation where one of the two alternative AB-routes is not used.

$$\Delta CS^R = \Delta CS_a^R + \Delta CS_b^R + \Delta CS_{bc}^R + \Delta CS_c^R \qquad \text{(4 routes, 1 zero)}$$

= ½(100+40)(9-8) + 0 + ½(0+60)(10-8) + ½(100+100)(6-4) = 70+0+60+200 = 330

$$\Delta CS^L = \Delta CS_a^L + \Delta CS_b^L + \Delta CS_c^L \qquad \text{(3 links, 1 zero)}$$

= ½(100+40)(9-8) + 0 + ½(100+160)(6-4) = 70+0+260 = 330

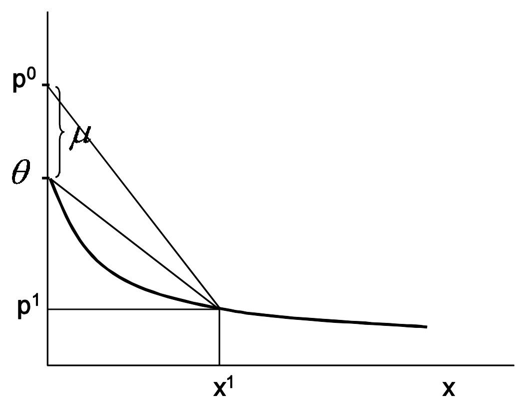
$$\Delta CS^{OD} = \Delta CS_{AB}^{OD} + \Delta CS_{AC}^{OD} + \Delta CS_{CB}^{OD} \qquad \text{(3 OD-pairs, 1 zero)}$$

= ½(100+100)(9-8) + 0+ ½(100+100)(6-4) = 100+0+200 = 300

To understand why the equivalence fails, consider demand for a route. A given market price for it may be just too costly for a consumer to purchase a single unit. Routes are non-essential goods and hence its demand may be zero. When this occurs the given price no longer represents the value of consumption of an additional unit. In fact, the given price will exceed that value, unless it is exactly the one where demand just turns zero.

Consequently, the RoH applied to a situation where demand is zero in the initial situation will overestimate the benefits. This is depicted in figure 3. When demand becomes zero in the new situation, the RoH will overestimate the loss of surplus. The price at which demand just turns zero, θ, is the virtual price. The wedge between the virtual and actual price is μ.

Figure 3: Virtual price and overestimation by RoH



Zero demand matters since the set of all possible routes over a network is enormous. Of course, most routes will not be economically relevant, not before and not after an investment. Thus, the vast amount of these routes can and will be ignored. Still, some routes may not be used in one situation but will be so in another. The set of used, or relevant, routes can change, corresponding with different corner solutions of the consumer problem. Link use can be zero as well. This is especially relevant for modelling investment in a new link. This would typically involve assuming that the link already did exist but that the cost of using it were totally prohibitive: you don't drive your car through an agricultural field even if it would shorten the distance of your trip. Investment then would reduce the cost of using the link so that it would be used. Also longer routes that contain the 'new' link may switch regime from dormancy to activity. Such regime switches are endogenous to the following consumer model.

3 A consumer model with a network and corners

A consumer model with explicit non-negativity and a transportation network is presented in this section. Virtual prices are defined. Different cases of the full model are examined.

3.1 The full model with links

At the heart of the deliberations of a consumer are routes: will I take the car to work or go by train, will I make a nice tour on my motorcycle this weekend or visit, long overdue, my aunt by car? Point of departure is therefore a consumer who derives utility from door-to-door trips. Vector x^R of route use is argument in the utility function next to consumption z of a composite good: $U = U(z, x^R)$.

All substitutability and complementarity of trips is contained in the utility function. Perfect substitutability of alternative routes on a given OD-pair is a special case of the above general utility function. Route use is non-essential and recall that it is taken to be a continuous variable. The composite good is essential and is the numeraire.

Links are mere intermediate goods, necessary for the production of routes. Link use is final consumption when the link represents a simple route. All that is needed to cover a static general transportation network is contained in the fixed route-link incidence matrix M.

It is assumed that the consumer only pays for link use, referred to as the linear cost assumption. The consumer has income y to spend on the composite good and on travel, facing prices $p_z = 1$ and p^L.

The consumer problem is an optimization with constraints. The usual constraint is the budget which has marginal utility of income λ as its multiplier.[3] The network constraint of link use is added and has multiplier ρ. The non-negativity constraints are made explicit. Route use has η^R as its Kuhn-Tucker multipliers and for links these are η^L. It is these added constraints, and their interaction, which make the model of interest.

[3] A separate time budget or constraint is not required here as time is covered by the generalized costs.

$$z+\sum_{l} p_l^L x_l^L = y \tag{3.1}$$

$$\sum_{r} M_{lr} x_r^R = x_l^L \tag{3.2}$$

$$x_r^R \geq 0 \tag{3.3}$$

$$x_l^L \geq 0 \tag{3.4}$$

Non-negativity constrained indirect utility, $\tilde{V}$, as a function of link prices and income, is as follows.

$$\tilde{V}(p^L, y) = \quad \max_{z\geq 0, x_r^R, x_l^L} \quad U(z,..,x_r^R,..) \qquad \text{subject to (3.1) up to (3.4)} \tag{3.5}$$

The composite good z is essential, so $z > 0$ and $\partial U / \partial z = \lambda$ matters only for normalisation. The first-order conditions for the choice of route and link use are given below.

$$\partial U / \partial x_r^R - \sum_l \rho_l M_{lr} + \eta_r = 0 \tag{3.6}$$

$$-\lambda p_l^L + \rho_l + \eta_l^L = 0 \tag{3.7}$$

Next there are the complementarities for routes and links. Of each pair of multiplier and use variable at least one must be zero: $x \geq 0$, $\eta \geq 0$ and $\eta x = 0$.

Define now μ as the normalised values of the non-negativity multipliers η . And define virtual prices θ of network use equal to their normalised marginal values.

From (3.7) the normalised marginal value of link use is obtained: the virtual price θ_l^L .

$$\theta_l^L = \rho_l^L / \lambda = p_l^L - \eta_l^L / \lambda = p_l^L - \mu_l^L \tag{3.8}$$

From (3.6) the virtual price of route use is found. Non-negativity of this virtual price follows from assumptions on the utility function[4] and it cannot exceed its actual price.

$$\theta_r^R = \frac{1}{\lambda}\frac{\partial U}{\partial x_r^R} = \sum_l \frac{\rho_l^L}{\lambda} M_{lr} - \frac{\eta_r^R}{\lambda} = \sum_l \theta_l^L M_{lr} - \mu_r^R \tag{3.9}$$

Most important to observe here, see (3.9), is that the virtual prices of routes are not fully determined by the virtual prices of links. The sum of the virtual link prices may overvalue the route when it is not used and this excess value cannot be assigned to individual links. This is in contrast with the definition of the actual route prices which are fully determined by the link prices, see (2.2). Thus the linear cost assumption does not hold in virtual prices.

The virtual price of route use can be expressed in the given 'normal' link prices or in the route prices, complemented with the non-negativity rents. Symbol $\hat{\mu}_r^R$ represents the combined rents for route *r*.

$$\theta_r^R = \sum_l (p_l^L - \mu_l^L) M_{lr} - \mu_r^R = p_r^R - \sum_l \mu_l^L M_{lr} - \mu_r^R = p_r^R - \hat{\mu}_r^R \tag{3.10}$$

When a certain route is being used its virtual price equals its 'normal' price. Virtual prices can be the thought of as reservation prices: goods are not consumed unless their reservation price equals their market price. This idea can be captured in 'regime switching conditions' (see Lee and Pitt, 1986): $\theta \le p$, $x \ge 0$ and $x(\theta - p) = 0$.

3.2 Equivalence of models

Different cases of the full model are now examined, with and without links and the network, with and without binding non-negativity constraints, evaluated at market or virtual prices.

[4] Lee & Pitt (1986): "the preference function is strictly quasi-concave, continuous, and strictly monotonic".

First observe the equivalence between the full model with links and the network, $\tilde{V}(p^L, y)$ and a model in routes only, without the network constraint, $\tilde{V}(p^R, y)$. The objective function is obviously the same and so is the budget equation given the definition of route prices, in (2.2). And the relation between route and link use, (2.1), must hold outside or inside a consumer model. Non-negativity does not impact on this. Thus the non-negativity constrained full model is equivalent to the constrained model in routes only.

$$\tilde{V}(p^L, y) = \tilde{V}(p^R, y) \tag{3.11}$$

Next consider the unconstrained case of the model in routes: $V(p_r^R, y)$. The approach to welfare change proceeds with a total differential of the indirect utility function and application of Roy's identity.[5] For discrete changes, an integral of the expression below is taken.

$$dV(p_r^R, y) = \sum_r \frac{\partial V}{\partial p_r^R} dp_r^R + \frac{\partial V}{\partial y} dy = -\lambda \sum_r x_r^R dp_r^R + \lambda dy \tag{3.12}$$

However, with binding non-negativity constraints application of Roy's identity is inappropriate. The concept of virtual prices turns out to be helpful here. Neary and Roberts (1980) introduce virtual prices for consumers under rationing as those prices that would support the very same consumption vector for unconstrained consumers as the outcome for the rationed consumers. The non-negativity constraints are a special case of a 'ration' (Lee and Pitt, 1986). They are even a simple case because rations equal to zero have no impact on the budget. The Neary and Roberts result implies that the constrained indirect utility equals unconstrained indirect utility evaluated at the virtual prices.

$$\tilde{V}(p^R, y) = V(\theta^R, y) \tag{3.13}$$

[5] Marshallian demand is the negative value of the partial derivative of indirect utility divided by the marginal utility of income; see any microeconomic textbook, for instance Varian (1984).

This in its turn implies the following first-order expression of utility change, where Roy's identity is applied to the unconstrained case for virtual prices. Alas, just as utility is unobservable, so are the virtual prices for unused routes. This matter is taken up in section 4.

$$d\tilde{V}(p^R, y) = dV(\theta^R, y) = -\lambda \sum_r x_r^R d\theta_r^R + \lambda dy \tag{3.14}$$

The fact that the virtual route prices are not simply the sum of the virtual prices of the links they consist of, see (3.9), ruins the following equivalence: constrained indirect utility of the full model in actual link prices is not equal to unconstrained indirect utility evaluated at virtual link prices. The reason is that the latter prices do not contain sufficient information to determine the marginal value of unused routes.

$$\tilde{V}(p^L, y) = V(\theta^L, \mu^R, y) \neq V(\theta^L, y) \quad (3.15)$$

What does hold is that the non-negativity constrained full model is equivalent to the unconstrained model in routes only, evaluated at virtual route prices.[6] This equivalence will be the basis for welfare measures.

$$\tilde{V}(p^L, y) = \tilde{V}(p^R, y) = V(\theta^R, y) \tag{3.16}$$

Finally, consider an interior solution to the full model. The unconstrained model is a special case of the constrained model. The former can now be evaluated at actual prices instead of virtual prices.

$$V(p^L, y) = V(p^R, y) \tag{3.17}$$

Moreover, Roy's identity can be shown to hold in terms of link prices, which means we have the following.

$$dV = -\lambda \sum_r x_r^R dp_r^R + \lambda dy = -\lambda \sum_l x_l^L dp_l^L + \lambda dy \tag{3.18}$$

[6] Since all links are simple routes, they have a route and a virtual route price too.

For interior solutions, and given the linear cost assumption, the link and route approach are equivalent. And because the OD-matrix approach is a special case of the route approach, under these conditions, the link approach is equivalent to the OD-matrix approach.

For corner solutions the equivalence of the link and route approach does not hold. This is because the marginal value of unused routes cannot be derived from the marginal values of its links.

4 Measuring consumer surplus and approximation errors

From theoretical models we move to more practical issues and approximation of consumer surplus. The bridge with theory is the rule of a half expressed in virtual prices. But virtual prices are not observable for unused routes. The assumption of perfect substitutes is shown to circumvent this problem. Alternatively, the problem of unobservability of the marginal value of unused routes can also be ignored. An expression is derived for the approximation error when actual route or link prices are used. Finally, the approximation error may be estimated. This is what Eliasson (2009) does.

4.1 The rule of a half in virtual prices

To move from the formal model to practical implications and to the different approaches to measure user benefits the rule of a half is derived in terms of virtual prices. Consider (3.14) as a basis for a monetarized measure of change in Marshallian consumer surplus for discrete changes.

$$\Delta CS = -\int \sum_r x_r^R d\theta_r^R \tag{4.1}$$

Approximation of demand with linear functions leads to the rule of a half: it reads in route use and virtual prices.

$$\Delta CS^R(\theta) = -\tfrac{1}{2} \sum_r ((x_r^R)^1 + (x_r^R)^0) \Delta\theta_r^R \tag{4.2}$$

First notice that only used routes matter. The relevant routes are those with a positive number of trips in at least one of the situations, before and after the change. A wrong idea is that only those routes matter for which there is a change in the actual price. Consumption of a commodity may become zero when an alternative product becomes cheaper. Even when the market price of the first commodity does not change there still will be a welfare effect.

Secondly, virtual prices are unobservable for non-used routes. This defeats the whole purpose of the rule of a half. It conventionally reads in observable quantities and prices, which of course still holds for all used routes because for them the virtual price equals the actual market price. There are a number of ways around the unobservability: assuming it away already at the level of preferences, ignoring the possible error, or performing additional estimations. These ways are now discussed where approximation errors are considered additional to the error involved by the approximation with linear demand.

4.2 The Wardrop assumption

The assumption of perfect substitutability of all alternative routes on an OD-pair has an interesting consequence. The marginal value of all unused routes is known and is equal to the price (generalised costs) of the used routes on the pair. A slight reformulation of the Wardrop principle is then: the generalized virtual costs are equal for all routes, used or not, on the same OD-pair.

$$\theta_{ijk}^{R} = p_{ij}^{OD} \quad \text{for all } k \tag{4.3}$$

This feature has undoubtedly contributed to the dominance of the OD-matrix approach. Although it is just a special case of the route approach, it solves the problem of unobservability of the marginal value of unused routes. It does so, if you wish, by assumption.

In fact, the OD-matrix approach is so common that it also applied when the condition of perfect substitutes clearly is not met. Besseling and van ’t Riet (2010) give an example encountered in a CBA of the high speed rail track Amsterdam (Van Hasselen and van Schijndel-Pronk, 1994): travel by road, air, conventional train and high speed rail were considered perfect substitutes. This resulted in a significant overestimation of user benefits.

Kidokoro (2004) points to a manual of the Japanese Institute for Transport Policy Studies. This manual dictates the use of a weighted average method for cost-benefit analysis of railway projects in Japan. This method amounts to the OD-matrix approach. Kidokoro then illustrates the error made using an numerical example from the manual itself. The example involves again different modes of transportation. He finds an overestimation of 24 percent. The lesson for CBA-practitioners is to be aware of the assumption of perfect substitutability. Admittedly, for a given mode the assumption may not be too strong.

However, even applying the OD-matrix approach when perfect substitutability is not ascertained need not result in large errors. When the alternatives are close substitutes the variation in prices will be small which leads to a small error. This does not seem to be unreasonable to expect of most alternative routes. The two practical examples above, however, do have considerable variation in prices because they concern different modes of transportation, thus explaining the possibility of a sizable error. Obviously there will be no error when the OD-matrix approach is applied to outcomes of a traffic simulation model based on the Wardrop principle.

4.3 The link approach?

The question can be turned around: what if perfect substitutability holds and the link approach is applied in the presence of a change in used routes? We derive an expression for the approximation error when actual link prices are used. Thus, the RoH based on OD-pairs is taken to be the correct estimate. This is equal to the RoH based on routes but with virtual route prices. The latter are known because of the reformulated Wardrop principle for all routes, see (4.3). Then the virtual route prices are broken down into actual market prices and non-negativity rents, see (3.10). The RoH based on links and in actual link prices appears but the error term is in non-negativity rents of routes.

$$\Delta CS^{OD} = \Delta CS^{R}(\theta) = \Delta CS^{L}(p) + \tfrac{1}{2}\sum_{r}((x_r^R)^1 + (x_r^R)^0)\Delta\hat{\mu}_r^R \tag{4.4}$$

$$\Delta CS^{L}(p) - \Delta CS^{OD} = \tfrac{1}{2}\sum_{r}((x_r^R)^1(p_r^R - \theta_r^R)^0 - (x_r^R)^0(p_r^R - \theta_r^R)^1) \tag{4.5}$$

The latter equation is a general expression for the approximation error, on top of the linearization of demand, for unused routes. Assuming perfect substitution makes the expression measurable. The additional approximation error arises only for those routes used in one situation and not used in the other situation. A switch from non-use to use of a route leads to an overestimation and a switch from use to non-use means an underestimation by the link approach compared to the OD-matrix approach. In general nothing can be said about the sign of the approximation error. The size of the error will be reduced by the two opposing effects. Also, when the number of routes changing regime is small, the error is also likely to be small.

In a CBA of the Stockholm congestion charging system Eliasson (2009) estimates this approximation error. He uses observed rather than model-forecasted data on travel time and flows. These observations were before and during the so-called Stockholm trial in 2006, when a congestion charging system was in place. The results are used to determine the period the system would have to be in place to break-even. The travel time benefits are thus that in just around 4 years investment and operations costs are recovered. This amounts to convincing empirical support for the "theoretically irrefutable" road pricing.

Eliasson states that: "it is well known that the consumer surplus from a change in travel times and/or travel costs should in general be calculated at the level of origin-destination pairs." To justify his use of the link approach he derives an expression for the error made compared to an OD-matrix approach.[7] In this derivation he applies the linear cost assumption. Though link costs are used, his final error expression also reads in OD-flows. This is comparable to the general error expression derived in this paper being based on routes.

Next Eliasson uses the structure of the network and the fact that the charging cordon is a closed circle to pinpoint a single OD-pair on which there is a switch of used routes. He finds an error of 0.6 million Swedish Kroner per year. This must be related to the 536 million Kroner per year on monetarised benefits of shorter travel times: an error just over 0.1 percent.

[7] Eliasson uses a route-link incidence matrix depending on actual use. By contrast, the one used in this paper is fixed.

No explanation however has been given why, ultimately, the comparison must be with the OD-matrix approach. Are travel through the city of Stockholm or over a bypass necessarily perfect substitutes?

Finally, the link approach has often been discarded in the past, sometimes motivated by its perceived inability to cope with a new link. Certainly, some estimate of the marginal value of the new link in the situation before the investment is then needed. This estimate could very well be found by considering the new link as a perfect alternative to the other routes actually used connecting the same origin and destination. For consistency the estimate should not be less. We see no serious obstacle, theoretical or practical, to the extra step of adding such estimates to the link approach when required. The link approach should have advantages too, in order to make it worthwhile to apply it. Some advantages are mentioned in the concluding section.

5 Conclusion

The link approach and the OD-matrix approach will yield identical outcomes of user benefits when the set of used routes remains unchanged and route costs are the direct sum of the link costs. This conditional identity was established at the level of the underlying consumer model, and is therefore not a mere artefact of the rule of a half. A change in the set of used routes, e.g. as a result of an infrastructure project or pricing measures, ruins the equivalence: the marginal value of an unused route is not the sum of the marginal values of its links. The linear cost assumption does not hold for virtual prices.

Moreover, the marginal value of unused routes cannot be observed. This problem can be solved by the assumption of perfect substitutability of the alternative routes on an OD-pair: the marginal value of the unused routes equals those of the used ones. This explains, at least partially, the success of the OD-matrix approach. Its application on imperfect substitutes need not give a serious approximation error. It will do so when the alternatives are not close substitutes. Unfortunately, this is encountered in practice.

Applying the link approach with actual link costs will lead to overestimation of user benefits for routes which are used after the change but were not used before. Routes no longer used will lead to underestimation of benefits. As these effects work in opposing directions nothing can be said in general about the sign of the overall error. The size of the error may be small because of the opposing effects, or because of a small set of routes with regime switches. Eliasson (2009) estimates this effect in a CBA of the Stockholm congestion charging system. He finds an error of only 0,1 percent of travel time benefits. This approximation error could only computed by contrasting the link approach with the OD-matrix approach, since the latter supplies, by assumption, the marginal value of non-used routes. To keep things in perspective, note that the rule of a half itself is already a linear approximation of consumer surplus.

The additivity of link costs is important for results derived in this paper and its applicability to actual CBA’s. Increasingly CBA’s include reliability of travel time as a possible source of benefits. Measures of reliability at link level do not easily translate to route level. This is because link travel times, and variances, will not be independent in the case of congestion. Then evaluation at route level must be preferred.

Still, the link approach has some notable advantages. It allows, for example, to show exactly where on the network gains and losses materialise. In addition it makes ex post evaluation possible by measurement of traffic flows and time on individual links of the network, without having to question travelers about their origin and destination. Also the number of calculations may remain limited as travel time benefits on individual links are likely to occur mainly in the area where the investment takes place.

The link approach has often been discarded in the past, sometimes motivated by its perceived inability to cope with a new link. Certainly, a shortcut is then needed to provide an estimate of the marginal value of the new link in the situation before the investment. All taken together it seems that the link approach deserves more serious consideration.

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Mathematical elaborations

with

The link approach to measuring consumer surplus in transport networks

Maarten van ’t Riet

Subsection 2.2, after (2.7)

Under the linear cost assumption the route and link approach are identical: $\Delta CS^R = \Delta CS^L$. This is the ‘translation of routes into links’, see (2.3).

$$\Delta CS^R = -\tfrac{1}{2}\sum_r ((x_r^R)^1 + (x_r^R)^0)\Delta p_r^R = -\tfrac{1}{2}\sum_r\sum_l \Delta p_l^L M_{lr}((x_r^R)^1 + (x_r^R)^0)$$

$$= -\tfrac{1}{2}\sum_l\sum_r ((M_{lr}x_r^R)^1 + (M_{lr}x_r^R)^0)\Delta p_l^L = -\tfrac{1}{2}\sum_l ((x_l^L)^1 + (x_l^L)^0)\Delta p_l^L = \Delta CS^L$$

Subsection 2.2. after (2.8)

Under the Wardrop assumption, the route and OD-matrix approach are identical too: $\Delta CS^R = \Delta CS^L$. Use is made of (2.8) to bring the price changes inside the accolades.

$$\Delta CS^{OD} = -\tfrac{1}{2}\sum_i\sum_j ((x_{ij}^{OD})^1 + (x_{ij}^{OD})^0)\Delta p_{ij}^{OD} = -\tfrac{1}{2}\sum_i\sum_j \{\sum_k ((x_{ijk}^R)^1 + (x_{ijk}^R)^0)\}\Delta p_{ij}^{OD}$$

$$= -\tfrac{1}{2}\sum_i\sum_j \{\sum_k ((x_{ijk}^R)^1 + (x_{ijk}^R)^0)\Delta p_{ijk}^R\} = -\tfrac{1}{2}\sum_r ((x_r^R)^1 + (x_r^R)^0)\Delta p_r^R = \Delta CS^R$$

Subsection 3.1, after (3.10)

When a certain route is being used its virtual price should equal its ‘normal’ price. This is verified as follows. A route being used, $x_r^R > 0$, implies $\eta_r^R = 0$ and $\mu_r^R = \eta_r^R / \lambda = 0$. And a route being used also means that all the links it is composed of are being used. Thus $x_l^L > 0$ for the relevant set of links. And thus $\eta_l^L = 0$ and $\mu_l^L = 0$. Together $\hat{\mu}_r^R = 0$ is obtained. Substitution in (3.10) gives the required: $\theta_r^R = p_r^R$.

Subsection 3.2, equation (3.14)

Roy's identity applied to the unconstrained case evaluated at virtual prices.

The trick here is that the budget equation holds in virtual prices.

In vector notation and omitting superscripts: $y = px = \theta x$ because $\hat{\mu}x = 0$.

Partially differentiate this budget equation and $V(\theta, y) = U(x(\theta, y))$.

$$\frac{\partial V}{\partial \theta_i} = \sum_k \frac{\partial U}{\partial x_k}\frac{\partial x_k}{\partial \theta_i} = \lambda \sum_k \theta_k \frac{\partial x_k}{\partial \theta_i}, \qquad \text{and} \qquad \sum_k \theta_k \frac{\partial x_k}{\partial \theta_i} + \sum_k x_k \frac{\partial \theta_k}{\partial \theta_i} = \sum_k \theta_k \frac{\partial x_k}{\partial \theta_i} + x_i = 0 \ .$$

Then $\partial V(\theta, y)/\partial \theta_i = -\lambda x_i$ is obtained.

Subsection 3.2, equation (3.18)

Moreover, Roy's identity then holds in terms of link prices, applied in (3.18). This is a combination of the proof of Roy's identity and the translation from routes to links.

First partially differentiate $V(p^L, y) = U(x^R(p^L, y))$.

$$\frac{\partial V(p^L, y)}{\partial p_l^L} = \sum_r \frac{\partial U}{\partial x_r^R}\frac{\partial x_r^R}{\partial p_l^L} = \lambda \sum_r p_r^R \frac{\partial x_r^R}{\partial p_l^L}$$

Then partially differentiate budget equation $p^R x^R = y$.

$$\sum_r p_r^R \frac{\partial x_r^R}{\partial p_l^L} + \sum_r x_r^R \frac{\partial p_r^R}{\partial p_l^L} = \sum_r p_r^R \frac{\partial x_r^R}{\partial p_l^L} + \sum_r \frac{\partial p_r^R}{\partial p_l^L} x_r^R = 0$$

And use the linear cost assumption.

$$\sum_r p_r^R \frac{\partial x_r^R}{\partial p_l^L} + \sum_r \frac{\partial \sum_m p_m^L M_{mr}}{\partial p_l^L} x_r^R = \sum_r p_r^R \frac{\partial x_r^R}{\partial p_l^L} + \sum_r M_{lr} x_r^R = \sum_r p_r^R \frac{\partial x_r^R}{\partial p_l^L} + x_l^L = 0$$

So $\sum_r p_r^R \frac{\partial x_r^R}{\partial p_l^L} = -x_l^L$ and $\frac{\partial V(p^L, y)}{\partial p_l^L} = -\lambda x_l^L$ is found.

Subsection 4.1, with (4.1) and (4.2)

Alternatively, consider (3.14) as a basis for a second-order approximation of welfare change. Equation (4.2) can be seen to be derived from this expression too.

$$dW = dV / \lambda = -\sum_r (x_r^R + \tfrac{1}{2} dx_r^R) d\theta_r^R + \lambda dy$$

Subsection 4.2, on the error of the OD-matrix approach

However, applying the OD-matrix approach when perfect substitutability is not ascertained need not result in large errors. Consider the rule of a half, see (2.7), expressed in terms of average prices $\bar{p}$ and total quantities, or flows, X .

$$\Delta CS = -\tfrac{1}{2}\{ (\bar{p}^1 - \bar{p}^0)(X^1 + X^0) + \sum_i (p_i^1 - \bar{p}^1) x_i^0 - \sum_i (p_i^0 - \bar{p}^0) x_i^1 \}$$

When the alternatives *i* are perfect substitutes, and we have an interior solution, the second and third term on the right hand side drop out: all individual prices equal the average price. These same two terms therefore also represent the error in applying the OD-matrix approach. With no extra information, not even the sign of the error is known. When the variation in prices is small, which does not seem to be unreasonable to expect of most alternative routes, the error is likely to be small too.

Subsection 4.3, equation (4.4)

$$\Delta CS^{OD} = \Delta CS^R(\theta) = -\tfrac{1}{2}\sum_r ((x_r^R)^1 + (x_r^R)^0)\Delta\theta_r^R$$

$$= -\tfrac{1}{2}\sum_r ((x_r^R)^1 + (x_r^R)^0)(\Delta p_r^R - \Delta\hat{\mu}_r^R) = \Delta CS^L(p) + \tfrac{1}{2}\sum_r ((x_r^R)^1 + (x_r^R)^0)\Delta\hat{\mu}_r^R$$

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