Intertemporal Preference for Flexibility

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Abstract

Following Kreps (1979), we consider a decision maker who is uncertain about her future taste for immediate consumption. This uncertainty leaves the decision maker with a preference for flexibility: When choosing among menus containing alternatives for future choice, she weakly prefers menus with additional alternatives. Existing representations accommodating this choice pattern cannot address dynamic decision situations like a consumption savings problem. We provide representations of choice over continuation problems that are recursive and take the form of Bellman equations. Two specific models are axiomatized. They feature stationary and Markovian beliefs over future tastes, respectively. The parameters of the representations, which are relative intensities of tastes, beliefs over those tastes and the discount factor, are uniquely identified from behavior. We characterize a natural notion of 'greater preference for flexibility' in terms of a stochastic order on beliefs and give an example of a Lucas tree economy, where a representative agent with greater preference for flexibility corresponds to larger price volatility in the sense of second order stochastic dominance.

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*Preliminary. Comments most welcome.

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1. Introduction

A decision maker, DM, who is uncertain about her future taste prefers not to commit to a course of action today, and so has a preference for flexibility. While this intuition is inherently dynamic, standard models that accommodate preference for flexibility, most prominently Kreps (1979) and Dekel, Lipman and Rustichini (2001) (henceforth DLR; a relevant corrigendum is Dekel at al (2007), henceforth DLRS), are static.

We provide a dynamic model of preference for flexibility. Following Gul and Pesendorfer (2004), DM chooses between infinite horizon consumption problems (IHCPs), which provide lotteries over pairs of consumption and continuation problems. Our representations are recursive and take the form of standard Bellman equations.

The domain of IHCPs is extremely rich and allows for very complicated choice behavior. For example, choice might depend on time, past consumption and past tastes. In order to attain the recursive structure of a Bellman equation we make the following simplifying assumptions. Preferences over IHCPs are linear, that is, a form of *Independence* holds. They are also *separable* with respect to present consumption and continuation problems. Finally, we consider two ways to link present consumption and the value of continuation problems.

Firstly, we assume that DM has *stationary preferences* and is *strategically rational* with respect to continuation problems. The corresponding Bellman equation features stationary beliefs over tastes for instantaneous consumption.

Second, to allow for more interesting dynamics, we replace the assumptions of stationary and strategically rational preferences by assuming *consumption contingent strategic rationality*. This axiom roughly states that DM is strategically rational, contingent on choosing a particular alternative from a sufficiently large menu. The corresponding Bellman equation features beliefs that follow a Markov process, that is, period t consumption preferences are a sufficient statistic for beliefs about tastes in period $t + 1$.

Our models could be analyzed using standard dynamic programming techniques. Therefore, we hope that they will be applied to incorporate decisions under uncertainty without observable states of the world in macroeconomic modelling and in asset pricing models. For example, an agent might be uncertain about her future preference for leisure, or her future risk aversion in a portfolio choice problem.

Preference for flexibility is the preference for non degenerate menus over singletons. Intuitively, one DM has *greater preference for flexibility* than another, if she has a stronger preference for menus over singletons. We formalize this notion. Since preference for flexibility is the behavioral manifestation of uncertainty about tastes, it is ideally characterized in terms of beliefs about those tastes. All the parameters in our representations can be uniquely identified from behavior. For a large class of stationary preferences, this allows us to compare decision makers who disagree only in terms of their beliefs and to characterize 'greater preference for flexibility' in terms of the increasing convex order of those beliefs.

We apply this characterization to describe the price volatility in a version of the Lucas tree economy with three possible output levels. Two different representative agents, A and B, who are uncertain about their degree of future risk aversion, behave as in our model. We show that the distribution of the price of the dividends[1] in an economy inhabited by representative agent A second order stochastically dominates the price distribution in an economy inhabited by B if, and only if, B has more preference for flexibility than A. This result suggests that price volatility might be the consequence of uncertainty about future risk aversion.[2]

The unique identification of the parameters of the representation from choice in the stationary as well as the Markovian case contrasts with the non-uniqueness in the models of Kreps and DLR, where the separation of beliefs and utilities is not behaviorally meaningful. It is worthwhile to demonstrate why it is not possible to disentangle beliefs and probabilities in the static problem analyzed by DLR. Suppose there is a representation of the form

$$V(c) = \int_{S_K} \max_{\alpha \in c} \; u_s(\alpha) \;\; d\mu(s)$$

where S_K is a set of subjective states, and each u_s is a vNM utility, and μ is a probability measure. Now, consider a function $f : S_K \to \mathbb{R}$ that is measurable and bounded and also has $\frac{1}{f}$ bounded (μ-a.s.). Define the state dependent utility function $\tilde{u}_s = \big(\int \frac{1}{f}\, d\mu\big) f(s) u_s$, and the measure $\tilde{\mu} := \big[\frac{1}{f}\mu\big] / \big(\int \frac{1}{f}\, d\mu\big)$. It is clear that $\tilde{\mu}$ is also a probability measure and we also have the representation

$$V(c) = \int_{S_K} \max_{\alpha \in c} \; \tilde{u}_s(\alpha) \;\; d\tilde{\mu}(s)$$

It is in this sense that utilities and beliefs can not be separately identified in the static model. As we will see, DLR's model can be viewed as an extreme case of our representation, where there is no intertemporal tradeoff. In the general

[1]Usually, one would normalize the prize of the dividend, and consider changes in the prize of the asset. For our results it is more instructive to normalize the prize of the asset and allow the pice of the dividend to adjust.

[2]Uncertainty about future risk aversion should also imply a propensity to delay investment, or to underinvest. In ongoing research we formally explore this connection between underinvestment and price volatility.

case, there is at least some intertemporal tradeoff. We show that this tradeoff allows the separate identification of beliefs and utilities.

Sadowski (2010) instead considers DLR's model as the extreme case of a situation where observable states of the world contain some relevant information and shows that generically beliefs over tastes become uniquely identified. Shenone (2010) introduces a numeraire with state independent evaluation to identify beliefs uniquely in the same sense they are uniquely identified in Anscombe and Aumann (1963).

Other models of dynamic preference for flexibility impose strong constraints on either the domain or the preferences. Rustichini (2002) considers deterministic sequences of choice problems. This domain can not accommodate a basic intertemporal consumption problem. Higashi et al. (2009) provide a stationary representation, where the preference for flexibility stems exclusively from a random discount factor. The representation does not generalize static models of uncertain tastes for immediate consumption and can not accommodate dynamic evolution of uncertainty.

The remainder of the paper is structured as follows. Section 1.2 gives a preview of our results. Section 2 lays out the model. Section 3 provides a general representation of preference for flexibility with a non finite prize space. Section 4 provides the foundations for a Bellman equation with stationary beliefs. Section 5 concerns the case of Markovian beliefs. A characterization of 'greater preference for flexibility' can be found in section 6 and section 7 applies the result to compare two (discretized) Lucas tree economies. This is an early first draft. Consequently the body of the paper is lacking comments and interpretations and most proofs follow inline. However, some proofs are relegated to an appendix.

1.1. Preview of Results

Choice from an infinite horizon consumption problem (IHCP), determines (a lottery over) consumption in the present period and a continuation problem, which is a new infinite horizon consumption problem starting next period.

We explicitly model only today's choice between consumption problems for tomorrow. We consider *continuous* preferences over such problems that are *non-trivial* and satisfy *independence*.

In DLR, menus contain lotteries over some finite prize space. The set of all continuation problems, in contrast, is not finite. We first provide a representation result after DLR for infinite prize spaces. In the case of a finite price space, the collection of future tastes that are relevant is essentially unique. This uniqueness may fail when the prize space is infinite. We proceed to put additional structure on preferences to obtain representations with a meaningful relevant taste space. In particular, preferences are assumed to be *separable* in consumption and the continuation problem. In addition we assume *monotonicity,* the central assumption in Kreps (1979), which states that more choice is always (weakly) preferable.

Our first model is based on stationary beliefs. It requires two more assumptions: First, preferences are *stationary*, that is, the ranking of any two IHCPs is the same as the ranking of two IHCPs that agree on consumption and provide the respective original IHCPs as continuation problems. Second, preferences satisfy *continuation strategic rationality*, which requires that there is no preference for flexibility with respect to continuation problems. Let $x, z \in Z$ be IHCPs and let $k \in K$ be consumption; $p \in x$ denotes a lottery over $K \times Z$. The axioms listed above are the behavioral content of a representation of Stationary Preference for Flexibility (SPF),

$$U(x) = \int_{S_K} \max_{p \in x} \left\{ \int_{K \times Z} \left[u_s(k) + \delta U(z) \right] dp(k, z) \right\} d\mu(s)$$

where $\delta < 1$ is the discount factor, S_K is the space of tastes for consumption, a collection of vNM rankings, u_s represents taste $s \in S_K$ and μ is a probability measure on S_K. All the parameters of the representation are unique in the appropriate sense. This representation provides a truly dynamic theory of preference for flexibility: It takes the recursive form of a Bellman equation and features uncertain tastes for immediate consumption in a setting where consumption choices can affect what is available in the future. However, it can not accommodate a dynamic evolution of uncertainty.

In our second representation beliefs follow a Markov process. We first simplify the problem by posing an axiom that limits the number of relevant states. The combination of stationarity and continuation strategic rationality is very strong. Up to the intertemporal tradeoff it allows eliciting the ranking of IHCPs in a static setting. We drop stationarity and replace continuation strategic rationality with *consumption contingent strategic rationality.* This axiom roughly requires that, for a large enough set of alternatives, the ranking of continuation problems when paired with a particular consumption alternative satisfies strategic rationality. We call the corresponding representation a representation of Markovian Preference for Flexibility (MPF): There is a measure μ_0 on S_K, such that $V(x, \mu_0)$ represents the preference, where

$$V(x, \mu) = \int_{S_K} \max_{p \in x} \left\{ \int_{K \times Z} \left[u_s(k) + \delta V(z, \mu_s) \right] dp(k, z) \right\} d\mu(s)$$

and where there is a transition operator $\mathcal{A}$ with $\mathcal{A}(s) = \mu_s$. The MPF representation is similar to the SPF representation, but here beliefs about future tastes for consumption follow a Markov process. Again the discount factor δ is unique. Furthermore, the relative intensities of tastes and beliefs are unique for any set of states that is ergodic with respect to the Markov process. Across ergodic sets of states, beliefs and utilities are only jointly identified, as in DLR.

Comparing two preferences that each have a SPF representation, we say that one has more preference for flexibility than the other, if they agree on the ranking of singletons, and the first prefers a nondegenerate menu over a singleton, whenever the second does. In order to characterize this property in terms of the beliefs in the SPF representations, we compare two preferences whose representations differ only in terms of beliefs. In the context of the corresponding axiom, called Numeraire, one preference has a greater preference for flexibility than another if, and only if, it is dominated in the increasing convex order on the relevant part of the taste space. For the case of only three prizes, the relevant part of the taste space turns out to be one dimensional, and the condition on beliefs amounts to second order stochastic dominance.

2. The Model

Let K be a finite set of prizes with typical member k and $\mathcal{P}(K)$ the set of lotteries over K, with typical lotteries being α, β, γ. We follow Gul and Pesendorfer (2004) in defining infinite horizon consumption problems (IHCPs). A consumption problem yields a consumption (lottery) α in the present period and a new infinite horizon problem starting in the next period. Let $Z = \mathcal{K}\big(\mathcal{P}(K \times Z)\big)$ be the collection of all Infinite Horizon Continuation problems (IHCP).[3]

Typical elements $x, y, z \in Z$ are interpreted as menus of lotteries over consumption and continuation problems. Typical lotteries in $\mathcal{P}(K \times Z)$ are p, q, r. We will be interested in menus of lotteries over present consumption and continuation problems. Menus of consumption lotteries will be represented by closed subsets of $\mathcal{P}(K)$, and denoted by $\mathcal{K}\big(\mathcal{P}(K)\big)$, with typical members being a, b, c. By the recursive nature of Z, continuation problems are members of Z. As above, $\mathcal{P}(Z)$ is the space of lotteries over Z, and a menu is a closed subset of lotteries, denoted by $\mathcal{K}\big(\mathcal{P}(Z)\big)$; typical members are A, B, C. To ease the notational burden, we will often write $\mathcal{K}$ for $\mathcal{K}\big(\mathcal{P}(K \times Z)\big)$, $\mathcal{K}_K$ for $\mathcal{K}\big(\mathcal{P}(K)\big)$ and $\mathcal{K}_Z$ for $\mathcal{K}\big(\mathcal{P}(Z)\big)$.

Technical Note: Even though K is finite, Z is infinite and infinite dimensional. Gul and Pesendorfer (2004) show that Z is metrisable and compact, hence $\mathcal{P}(K \times Z)$ when endowed with the topology of weak convergence is metrisable, so $\mathcal{K}\big(\mathcal{P}(K \times Z)\big)$ is also metrisable with the Hausdorff metric.

Each probability measure p over $K \times Z$ induces marginal distributions p_k and p_z over K and Z respectively. For $c \in \mathcal{K}_K$ and $A \in \mathcal{K}_Z$, we write (c, A) to denote the *rectangular menu* $\{(p_k, p_z) : p_k \in c, p_z \in A\}$. When there is no risk of confusion, we identify prizes and continuation problems with degenerate lotteries. That is, we will denote the lottery over Z that gives $z \in Z$ with probability 1 by z.

A preference relation $\succcurlyeq \subset Z \times Z$ is a complete, transitive binary relation. We will assume without further comment that $\succcurlyeq$ is nontrivial. Continuity of the

[3]See Gul and Pesendorfer (2004) for the recursive construction of Z. Notice that even if C is finite, Z is necessarily infinite dimensional (and infinite).

preference relations requires that the sets $\{x : x \succcurlyeq y\}$ and $\{x : y \succcurlyeq x\}$ are always closed. We consider a continuous preference relation $\succcurlyeq$ on Z and this too shall be a standing assumption. Therefore, a preference for us will be a nontrivial, complete, transitive and continuous binary relation.

In what follows, we will take the convex sum of sets to be the Minkowski sum, namely $tx + (1-t)y := \{tp + (1-t)q : p \in x, q \in y\}$ whenever $t \in [0,1]$. We are now ready to state our main structural axiom.

Axiom 1 (Independence)**.** $x \succ y$ implies $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$ for all $\alpha \in (0,1]$.

Independence is a familiar axiom from the theory of choice under risk. But it is not immediately obvious what is ruled in and what is ruled out by Independence in this dynamic setting. Nevertheless, to focus on other issues that are more compelling at the present, we shall, in what follows, assume that Independence always holds.

Dekel, Lipman and Rustichini (2001) (henceforth DLR) studied the domain $\mathcal{K}\big(\mathcal{P}(F)\big)$ where F is a finite set. It is easy to see that our domain is one where F is infinite. We therefore need an additive EU representation after DLR for infinite prize sets. We obtain such an abstract representation next.

3. A General Representation

Let Z be a compact metric space, $C(Z)$ the Banach space of uniformly continuous functions on Z, and let $\mathcal{M}(Z)$ be the space of all finite, signed, regular Borel measures on Z (with the associated sigma algebra). Then, $\langle C(Z), \mathcal{M}(Z)\rangle$ is a dual pair.

Let $\mathcal{P}(Z) \subset \mathcal{M}(Z)$ represent the space of probability measures on Z and fix $p_0 \in \mathcal{P}(Z)$. Define $X' := \text{span}\big(\mathcal{P}(Z) - p_0\big)$ which is a subspace of $\mathcal{M}(Z)$. Let $X'^{\perp} := \{x \in C(Z) : \langle x, x'\rangle = 0 \text{ for all } x' \in X'\}$ be the annihilator of X', so that $X := \{x \in C(Z) : \langle x, x'\rangle \neq 0 \text{ for some } x' \in X'\}$. It is easy to see that $X'^{\perp} = \{\alpha \mathbf{1} : \alpha \in \mathbb{R}\}$ is the space of constant functions. This verifies that $\dim(X'^{\perp}) = 1 = \text{codim}(X)$, and $X \oplus X'^{\perp} = C(Z)$. Moreover, $\langle X, X'\rangle$ is a dual pair.

Let U' be the closed unit ball of X' (assuming X' has the total variation norm), and U the closed unit ball of X. Let $\mathcal{K}$ denote the space of weak* compact, convex subsets of $\mathcal{P}(Z)$. Let $\varphi : \mathcal{K} \to \mathbb{R}$ be Lipschitz continuous and linear. For ease of notation, we shall write ∂U as $\mathcal{U}$. This will serve as our *universal subjective state space*. Each $x \in \mathcal{U}$ is a continuous function on Z, and hence a vNM function. Moreover, as established earlier, no $x \in \mathcal{U}$ is constant on Z. Notice that $\mathcal{U}$ is metrisable (by the norm), hence normal and Hausdorff. Let ba_n denote the set of bounded, normal, finitely additive (signed) measures, ie charges on $\mathcal{A}_{\mathcal{U}}$, the algebra generated by the open sets in $\mathcal{U}$. We can now state the following.

Theorem 3.1. *The function $\varphi : \mathcal{K} \to \mathbb{R}$ is linear and Lipschitz if and only if there exists a finite charge $\mu \in ba_n(\mathcal{A}_{\mathcal{U}})$ such that*

$$\begin{aligned}\varphi(A) &= \int_{\mathcal{U}} h_A \, d\mu \\ &= \int_{\mathcal{U}} \sup_{x' \in A} \langle x, x'\rangle \, d\mu(x)\end{aligned}$$

for all $A \in \mathcal{K}$.

Proof. See appendix. □

Preferences that neither value a preference for flexibility nor a preference for commitment are special, and shall be said to possess ***strategic rationality.***

More formally, a preference $\succcurlyeq$ is strategically rational if for all menus A and B, it is the case that $A \succcurlyeq B$ implies $A \sim A \cup B$. We shall see next that a preference is strategically rational if, and only if, the charge that represents the preference $\succcurlyeq$ is carried by a singleton. We begin with a lemma.

Lemma 3.2. Let $S \subset \mathcal{U}$. Then, $|S| = 1$ if, and only if, for all $p, q \in \mathcal{P}(Z)$ and for all $u_1, u_2 \in S$, $u_1(p) \geqslant u_1(q)$ if and only if $u_2(p) \geqslant u_2(q)$.

Proof. The 'only if' part is easily seen. To see that 'if' part, notice that by definition of $\mathcal{U}$, there exists $p_0 \in \mathcal{P}(Z)$ such that $u(p_0) = 0$ for all $u \in \mathcal{U}$. Moreover, $\|u\|_\infty = 1$ for all $u \in \mathcal{U}$. Therefore, no $u_1 \in \mathcal{U}$ is a positive affine transformation of some other u_2 in $\mathcal{U}$.

Now suppose $|S| > 1$, and let $u_1, u_2 \in \mathcal{U}$ be distinct. But by definition of S, u_1 and u_2 represent the same expected utility preference $\succcurlyeq^*$ on $\mathcal{P}(Z)$. Therefore, by the expected utility theorem and since $u_1(p_0) = u_2(p_0) = 0$, $u_1 = \alpha u_2$ for some $\alpha > 0$, which contradicts the definition of $\mathcal{U}$, wherein $\|u\|_\infty = 1$ for all $u \in \mathcal{U}$. This proves our claim. □

Recall that the ***carrier*** of the charge μ is defined as

$$S_\mu := \bigcap \{N : N \text{ is closed, and } \mu(N^c) = 0\}.$$

The carrier of the charge always exists, and is clearly well defined. For any $p, q \in \mathcal{P}(Z)$, define $S_{p,q} := \{u \in \mathcal{U} : u(p) > u(q)\}$ and $S^\circ_{p,q} := \{u \in \mathcal{U} : u(p) = u(q)\}$. Notice that $S_{p,q}$ is always open and $S^\circ_{p,q}$ is always closed, since p is a continuous (linear) functional on $\mathcal{U}$, which is a closed set. We can now prove

Lemma 3.3. Suppose $\succcurlyeq$ on $\mathcal{K}$ is strategically rational. Then, for all $p, q \in \mathcal{P}$, $\min\{\mu(S_{p,q}), \mu(S_{q,p})\} = 0$.

Proof. We shall prove the contrapositive. Suppose then that there exist p, q such that $\min\{|\mu(S_{p,q})|, |\mu(S_{q,p})|\} = |\mu(S_{p,q})| > 0$. It is clear then that $\{p, q\} \not\sim \{p\}, \{q\}$, which proves our claim. □

Lemma 3.4. Suppose for all $p, q \in \mathcal{P}(Z)$, $\min\{|\mu(S_{p,q})|, |\mu(S_{q,p})|\} = 0$. Then, $|S| = 1$.

Proof. If for all $p, q \in \mathcal{P}$, $\min\{\mu(S_{p,q}), \mu(S_{q,p})\} = 0$, then either (i) $S_\mu \subset S_{p,q} \cup S^\circ_{p,q}$, or (ii) $S_\mu \subset S_{q,p} \cup S^\circ_{p,q}$, but not both (by the definition of S_μ and since $S_{p,q}$ and $S_{q,p}$ are open).

In other words, for all $p, q \in \mathcal{P}(Z)$, and for all $u_1, u_2 \in \mathcal{U}$, $u_1(p) \geqslant u_1(q)$ if, and only if, $u_2(p) \geqslant u_2(q)$. Lemma 3.2 now implies that $|S| = 1$, as required. □

We may now put all this together, as follows.

Theorem 3.5. *Let $\succcurlyeq$ be a preference on $\mathcal{K}$, represented by the charge μ. Then $\succcurlyeq$ is strategically rational if, and only if, the carrier of the charge S_μ is a singleton, ie $|S_\mu| = 1$.*

Proof. The 'if' part of the claim is clear. We shall prove the 'only if' part. By Lemma3.4, strategic rationality implies

$$\min\{|\mu(S_{p,q})|, |\mu(S_{q,p})|\} = 0$$

for all $p, q \in \mathcal{P}(Z)$. By Lemma 3.4, we can then conclude that $|S_\mu| = 1$, as desired. □

4. Stationary Intertemporal Choice

We now consider the behavioural axioms for stationary intertemporal choice.

Axiom 2 (Separability)**.** $\frac{1}{2}\big(c,\{x\}\big)+\frac{1}{2}\big(c',\{x'\}\big)\sim\frac{1}{2}\big(c',\{x\}\big)+\frac{1}{2}\big(c,\{x'\}\big)$

Axiom 3 (Stationarity)**.** $\{(\alpha,x)\}\succ\{(\alpha,y)\}$ if, and only if, $x\succ y$.

Axiom 4 (Monotonicity)**.** $x\cup y\succcurlyeq x$

Axiom 5 (Continuation Strategic Rationality)**.** $(\alpha,x)\succ(\alpha,y)$ implies $(\alpha,x)\sim\{(\alpha,x),(\alpha,y)\}$.

To state our first result, let us define $S_K:=\big\{s\in\mathbb{R}^{|K|}:\sum s_i=0;\ \|s\|_2=1\big\}$ to be the set of all twice-normalized vNM utility functions over instantaneous consumption. For every lottery $\alpha\in\mathcal{P}(K)$, $s(\alpha)=\langle s,\alpha\rangle$. S_K is the canonical model of *subjective states*. In this subjective setting, *tastes* and subjective states will be treated as synonyms. Similarly, in what follows, all probability measures will interpreted as subjective beliefs, and the two terms will be used interchangeably. We can now define the central representation of this section.

Definition 4.1. Let (u_s) be a collection of vNM utilities of the form $u_s=s\lambda_s$ where $\lambda_s\geqslant 0$ for each $s\in S_K$, and λ_s is μ-integrable. Let μ be a probability measure on S_K, and let $\delta\in(0,1)$. A triple $\big((u_s),\mu,\delta\big)$ is a ***Stationary Preference for Flexibility*** (SPF) since each $\big((u_s),\mu,\delta\big)$ induces a function $U\in\mathcal{C}(Z)$ wherein

$$U(x)=\int_{S_K}\max_{p\in x}\left\{\int_{K\times Z}\big[u_s(k)+\delta U(z)\big]\,dp(k,z)\right\}d\mu(s)\tag{4.1}$$

We shall now show that the triple $\big((u_s),\mu,\delta\big)$ induces a unique $U\in\mathcal{C}(Z)$ that satisfies equation (4.1) above.

Proposition 4.2. For each triple $\big((u_s),\mu,\delta\big)$, there is a unique continuous function $U\in\mathcal{C}(Z)$ that satisfies the Bellman equation (4.1) above.

Proof. Let $W\in\mathcal{C}(Z)$, and consider the Bellman operator $\Phi:\mathcal{C}(Z)\to\mathcal{C}(Z)$, given by

$$(\Phi W)(x):=\int_{S_K}\max_{p\in x}\left\{\int_{K\times Z}\big[u_s(k)+\delta W(z)\big]\,dp(k,z)\right\}d\mu(s)$$

It is easy to see that Φ is monotone, that is $W \leqslant W'$ implies $\Phi(W) \leqslant \Phi(W')$, and satisfies discounting, ie $\Phi(W+\rho) \leqslant \Phi(W) + \delta\rho$ where $\rho \geqslant 0$. If we assume that $\Phi(W) \in \mathcal{C}(Z)$ for all $W \in \mathcal{C}(Z)$, it follows that Φ is a contraction mapping (with modulus δ), and has a unique fixed point, which establishes the proposition. All that remains is to establish that Φ is an operator on $\mathcal{C}(Z)$.

For each $x \in \mathcal{K}_Z$, $s \in S_K$ and $W \in \mathcal{C}(Z)$, define

$$f(x,s) = \max_{p \in x} \int_{K \times Z} \left[u_s(k) + \delta W(z)\right] dp(k,z)$$

For each $s \in S_K$, it is clear that

$$\begin{aligned} |f(x,s)| &\leqslant \max_{p \in x} \int_{K \times Z} |u_s(k) + \delta W(z)| \, dp(k,z) \\ &\leqslant \max_{a \in \mathcal{K}_K} \left|\max_{\alpha \in a} u_s(\alpha)\right| + \max_{A \in \mathcal{K}_Z} \left|\max_{y \in A} \delta W(y)\right| \\ &= \lambda_s \max_{a \in \mathcal{K}_K} |\sigma_a(s)| + \max_{A \in \mathcal{K}_Z} \left|\max_{y \in A} \delta W(y)\right| \\ &\leqslant \lambda_s M_1 + M_2 \end{aligned}$$

where $M_1, M_2 > 0$ and σ_a is the support function of the menu $a \in \mathcal{K}_K$.[4] Recall that λ_s is μ-integrable, and $\sigma_a(s)$ is continuous in a for each s (indeed the continuity is uniform), hence the bounds above.

By definition of u_s and W, $u_s + \delta W \in \mathcal{C}(K \times Z)$, and is a continuous, linear functional on $\mathcal{P}(K \times Z)$, where the latter is endowed with the topology of weak convergence. Therefore, for each s, by the Maximum Theorem, $f(x,s)$ is continuous in x.

Consider any sequence (x_n) that converges to x. By the bounds established above, $|f(x_n,s)| \leqslant \lambda_s M_1 + M_2$, and $\lambda_s M_1 + M_2$ is μ-integrable. Moreover,

$$\begin{aligned} \lim_{n \to \infty} (\Phi W)(x_n) &= \lim_{n \to \infty} \int_{S_K} f(x_n, s) \, d\mu(s) \\ &= \int_{S_K} \lim_{n \to \infty} f(x_n, s) \, d\mu(s) \\ &= \int_{S_K} f(x,s) \, d\mu(s) \\ &= (\Phi W)(x) \end{aligned}$$

[4]For each $a \in \mathcal{K}_K$, $\sigma_a : S_K \to \mathbb{R}$ is defined as $\sigma_a(s) := \max_{\alpha \in a} \langle s, \alpha \rangle$. See DLR or Chatterjee and Krishna (2009) for properties of the support function.

Since x and (x_n) are arbitrary, we conclude that $\Phi W \in \mathcal{C}(Z)$ whenever $W \in \mathcal{C}(Z)$. In the equalities above, we have used the Dominated Convergence Theorem to interchange the order of limits and integration, and the continuity of $f(\cdot, s)$ for each s to establish the pointwise limit. This completes the proof. □

We shall say that a SPF represents a preference $\succcurlyeq$ if the induced utility function U represents $\succcurlyeq$. We can now state our main representation theorem.

Theorem 4.3. *For any preference $\succcurlyeq$, the following are equivalent:*

(a) *$\succcurlyeq$ satisfies axioms 1–5.*

(b) *There is a SPF that represents $\succcurlyeq$.*

Moreover, the SPF $\big((u_s), \mu, \delta\big)$ above is unique up to a common scaling of the collection (u_s).

Some remarks about the result above are in order. As demonstrated abpve, in the additive EU representation of DLR, μ and (u_s) are jointly identified and cannot be disentangled. Nevertheless, in the theorem above, beliefs μ are identified uniquely. Roughly, this is because the continuation problem z is valued equally in every subjective state $s \in S_K$. In DLR the subjective utilities are state dependent, which makes it impossible to disentangle probabilities and utilities. But the existence of continuation problems that are valued equally in all states can now serve as a numeraire, so that we can, in fact, disentangle utilities (u_s) and probabilities, as in the representation theorem. DLR suggest and Shenone (2010) carries out identification of beliefs in a static model of preference for flexibility by introducing a numeraire, like money. In our dynamic model the numeraire naturally appears in the form of continuation problems and, as we will see in the next section, this does not even depend on the assumption of stationarity. Of course, our actual argument is more involved than the rough intuition above, because changing the SPF also changes the function U that values continuation problems.

We will now prove Theorem 4.3 via a sequence of lemmas, and define some auxiliary concepts in the process. Notice that $\mathcal{K}_K \times \mathcal{K}_Z \subset \mathcal{K}\big(\mathcal{P}(K \times Z)\big)$. This allows us to introduce two induced preference relations.

Definition 4.4. Fix $A^* \in \mathcal{K}_Z$ and $a^* \in \mathcal{K}_K$. Define the induced preference relations $\succcurlyeq_K$ and $\succcurlyeq_Z$ on $\mathcal{K}\big(\mathcal{P}(C)\big)$ and $\mathcal{K}_Z$ respectively as follows:

(a) for all $a, b \in \mathcal{K}_K$, $a \succcurlyeq_K b$ if, and only if, $(a, A^*) \succcurlyeq (b, A^*)$, and

(b) for all $A, B \in \mathcal{K}_Z$ $A \succcurlyeq_Z B$ if, and only if, $(a^*, A) \succcurlyeq (a^*, B)$.

Our first lemma states that the definitions of $\succcurlyeq_K$ and $\succcurlyeq_Z$ do not depend on the choice of a^* and A^* respectively.

Lemma 4.5. The preferences $\succcurlyeq_K$ and $\succcurlyeq_Z$ are independent of a^* and A^* respectively.

Proof. Fix any $\tilde{A} \in \mathcal{K}_Z$. Applying in turn, Independence, Separability and again Independence, we see that $(a, \tilde{A}) \succ (b, \tilde{A})$ iff $\frac{1}{2}(a, \tilde{A}) + \frac{1}{2}(b, A^*) \succ \frac{1}{2}(b, \tilde{A}) + \frac{1}{2}(b, A^*)$ iff $\frac{1}{2}(b, \tilde{A}) + \frac{1}{2}(a, A^*) \succ \frac{1}{2}(b, \tilde{A}) + \frac{1}{2}(b, A^*)$ iff $(a, A^*) \succ (b, A^*)$ iff $a \succ_C b$. The other part of the lemma is proved analogously. □

Notice that since $\succcurlyeq$ is nontrivial, it follows from Stationarity that $\succcurlyeq_K$ and $\succcurlyeq_Z$ are also nontrivial. (If $\succcurlyeq_K$ is trivial, Stationarity would imply that $\succcurlyeq$ is also trivial.) Now that we have a preference relation $\succcurlyeq_K$, we want to represent it with a utility function, as in DLR. This is next.

Lemma 4.6. The preference $\succcurlyeq_K$ is represented by

$$U_K(c) = \int_{S_K} \max_{\alpha \in c} \langle s, \alpha \rangle \, d\mu_K(s)$$

where S_K is as defined above and μ_K is a probability measure, defined on S_K (with the Borel σ-algebra).

Proof. In the lemma above, $s(\alpha) = \langle s, \alpha \rangle$ is just extension by linearity. The proof is just the representation theorem of DLR. □

We also have a representation for the preference $\succcurlyeq$ in Theorem 4.3 above.

Lemma 4.7. The preference $\succcurlyeq$ above is represented by

$$U(x) = \int_{S'} \max_{p \in x} \left\{ \int_{K \times Z} U'_s(k, z)\, dp(k, z) \right\} d\mu(s)$$

where $S' = \{U' \in \mathcal{C}(Z) : \|U'\|_\infty = 1,\ U' \text{ not constant}\}$ is as defined above and μ is a probability charge, defined on $\mathcal{U}$ (with the Borel σ-algebra), and for each $s \in \mathcal{U}$, $U'_s \in \mathcal{C}(K \times Z)$, the space of uniformly continuous functions on $K \times Z$.

Proof. This is merely a restatement of Theorem 3.1 above. □

Finally, we have a representation of $\succcurlyeq_Z$ as follows.

Lemma 4.8. The preference $\succcurlyeq_Z$ can be represented by

$$U_Z(A) = \max_{p_z \in A} \int_Z U(z)\ dp_z(z)$$

where $U \in \mathcal{C}(Z)$ is the utility representation of $\succcurlyeq$ obtained in lemma 4.7 above.

Proof. Lemma 4.7 says that $\succcurlyeq_Z$ can be represented by a charge on $S' = \{U' \in \mathcal{C}(Z) : \|U'\|_\infty = 1,\ U' \text{ not constant}\}$. By Theorem 3.5 above, Continuation Strategic Rationality implies that the carrier of the charge is a singleton, so

$$U_Z(A) = \max_{p_z \in A} \int_Z U'(z)\ dp_z(z)$$

where U' is unique up to positive affine transformation. Stationarity implies that U' and U are equivalent, modulo a positive affine transformation. □

Before moving to the proof of Theorem 4.3, we state some more auxiliary results. For each $A \in \mathcal{K}_Z$, we know that by the representation of $\succcurlyeq_K$

$$(\bigstar) \qquad U(c, A) = \eta_1(A) \int_{S_K} \max_{p_k \in c}\ \langle s, p_k \rangle\ d\mu(s) + \theta_1(A)$$

and for each $a \in \mathcal{K}\big(\mathcal{P}(C)\big)$,

$$(\clubsuit) \qquad U(a, A) = \eta_0(a) \max_{p_z \in A} \int U(z)\ dp(z) + \theta_0(a)$$

We want to show that $\eta_1(A)$ and $\eta_0(a)$ are independent of A and a respectively.

Lemma 4.9. In equations (★) and (♣) above, $\eta_1(A)$ and $\eta_0(a)$ are independent of A and a respectively.

Proof. Suppose first that η_0 is not constant, so there exist $a, b \in \mathcal{K}\big(\mathcal{P}(C)\big)$ such that $\eta_0(a) > \eta_0(b)$. Since $\succcurlyeq$ is nondegenerate, there exist $x, y \in Z$ such that $U(x) > U(y)$. Therefore, by Stationarity, $(a, x) \succ (a, y)$ and similarly when b is offered as consumption in the present. We then see that $\big[\eta_0(a) - \eta_0(b)\big]U(x) > \big[\eta_0(a) - \eta_0(b)\big]U(y)$. We can rewrite this as $\eta_0(a)U(x) + \eta_0(b)U(y) > \eta_0(b)U(x) + \eta_0(a)U(y)$. Collecting all these inequalities, we see that

$$\begin{aligned}
&U\big(\tfrac{1}{2}(a, x) + \tfrac{1}{2}(b, y)\big) \\
&\quad = \tfrac{1}{2}\big[\eta_0(a)U(x) + \theta_0(a)\big] + \tfrac{1}{2}\big[\eta_0(b)U(y) + \theta_0(b)\big] \\
&\quad > \tfrac{1}{2}\big[\eta_0(b)U(x) + \theta_0(b)\big] + \tfrac{1}{2}\big[\eta_0(a)U(y) + \theta_0(a)\big] \\
&\quad = U\big(\tfrac{1}{2}(b, x) + \tfrac{1}{2}(a, y)\big)
\end{aligned}$$

which violates Separability. Notice that the inequality was established above.

The proof that η_1 is constant is similar, once we recognise that by Stationarity and nontriviality of $\succcurlyeq$, there always exist a and b such that $U_C(a) > U_C(b)$. We now follow the first part of the proof, to establish that η_1 must be constant. □

We now move to the proof of Theorem 4.3.

Proof of Theorem 4.3. Notice that each rectangular menu (c, A) has two representations, as in equations (★) and (♣) above. Therefore, it follows by matching coefficients and constants that U can be written as

$$U(c, A) = \int_{S_K} \max_{p \in (c, A)} \left\{ \int_{K \times Z} \big[u_s(k) + \delta U(z)\big] \, dp_k(k) dp_z(z) \right\} d\mu(s)$$

It follows immediately from this that $\delta < 1$. To see this, consider constant consumption menus, where $c = \alpha$ and $x = (\alpha, x)$. This sum converges if, and only if, $\delta < 1$. Moreover, the measure μ has marginal μ_C on S_K and a Dirac point measure on the marginal S_Z. Therefore, we identify μ with μ_C and ignore S_Z while integrating over possible states.

We now address the issue of uniqueness. We claim that since U does not depend on s, μ can be determined uniquely. To see this, notice first that for any SPF $\big((u_s), \mu, \delta\big)$ that induces the utility function U over IHCPs, the SPF $\big((\rho u_s), \mu, \delta\big)$ induces the utility function $U' = \rho U$ over IHCP's, whenever $\rho > 0$.

Suppose now, to the contrary, that there is another representation

$$\tilde{U}(c, A) = \int_{S_K} \max_{p \in (c,A)} \left\{ \int_{K \times Z} \left[\tilde{u}_s(k) + \tilde{\delta}\tilde{U}(z) \right] dp_k(k) dp_z(z) \right\} d\tilde{\mu}(s)$$

of the preference $\succcurlyeq$. By the expected utility theorem, U must be a positive affine transformation of $\tilde{U}$. Therefore, scaling $(\tilde{u}_s)$ by a positive number means we can assume, without loss of generality, that $U(x) = \tilde{U}(x)$ for all $x \in \mathcal{K}$, that is U and $\tilde{U}$ are identical on $\mathcal{K}$.

We now claim that $u_s = \tilde{u}_s$ for all s, $\delta = \tilde{\delta}$, and $\mu = \tilde{\mu}$. To see this, notice that for any fixed A, $U(\cdot, A)$ still represents $\succcurlyeq_K$. Therefore, μ and $\tilde{\mu}$ are mutually absolutely continuous. That is, we have $f = \frac{d\tilde{\mu}}{d\mu}$ as the Radon-Nikodym derivative, so we can write $\tilde{\mu} = f\mu$.

Now, fix some $c \in \mathcal{K}_K$, and consider $x, y \in \mathcal{K}_Z$. Clearly,

$$\begin{aligned}
\delta\big[U(x) - U(y)\big] \int_{S_K} f \, d\mu(s) &= U(c, x) - U(c, y) \\
&= \tilde{U}(c, x) - \tilde{U}(c, y) \\
&= \tilde{\delta}\big[U(x) - U(y)\big] \int_{S_K} d\mu(s)
\end{aligned}$$

By choosing x and y so that $U(x) \neq U(y)$, this implies $\int f \, d\mu = \delta/\tilde{\delta}$. But we know that $\int d\tilde{\mu} = \int f \, d\mu = 1$ since $\tilde{\mu}$ is a probability measure, so it must be that $\delta = \tilde{\delta}$. Therefore, all that is left to prove is that $u_s = f_s \tilde{u}_s$, μ-a.s., and that $f = 1$ μ-a.s. We begin with some notation and a useful fact.

Let $p^* := \left(\frac{1}{|K|}, \dots, \frac{1}{|K|}\right)$ be the uniform measure on the set of consumption outcomes that also gives 0 utility in each subjective state, and let $B_r := \{p \in \Delta^{|K|-1} : \|p - p^*\|_2 \leqslant r\}$ be a menu for $r > 0$. For any state $s \in S_K$ and $\varepsilon > 0$, define $N_\varepsilon(s) = \{s' \in S_K : \|s - s'\|_2 < \varepsilon\}$ to be the ε-neighbourhood of s in S_K. The following follows immediately from the definition of subjective states in DLR.

Fact. Fix B_r. For any $s^* \in \operatorname{supp}\mu$ and $\varepsilon > 0$, there exists $p \notin B_r$ such that $u_s(p) \geqslant \max_{q \in B_r} u_s(q)$ iff $s \in N_\varepsilon(s^*)$.

Suppose now that $u_s \neq f\tilde{u}_s$ μ-a.s. Then, fixing B_r, $s^* \in S_K$ and $\varepsilon > 0$, there exists p such that $u_s(p) \geqslant \max_{q\in B_\delta} u_s(q)$ iff $s \in N_\varepsilon(s^*)$. Let $z \in \mathcal{K}$ be arbitrary. Recall also that $u_s = s\lambda_s$ and $\tilde{u}_s = s\tilde{\lambda}_s$ for each $s \in S_K$, where $\lambda, \tilde{\lambda} \geqslant 0$, μ-a.s. Then,

$$\begin{aligned}\int_{S_K} \lambda_s\big[\langle s,p\rangle - r\big]1\{N_\varepsilon(s^*)\}\,d\mu(s) &= U\big(B_r \cup \{p\}, z\big) - U(B_r, z)\\ &= \tilde{U}\big(B_r \cup \{p\}, z\big) - \tilde{U}(B_r, z)\\ &= \int_{S_K} \tilde{\lambda}_s\big[\langle s,p\rangle - r\big]1\{N_\varepsilon(s^*)\}f(s)\,d\mu(s)\end{aligned}$$

But since s^* and ε were arbitrary, it follows that $\lambda_s = f_s\tilde{\lambda}_s$, μ-a.s. All that remains is to show that $f = 1$ μ-a.s.

Suppose $f \neq 1$, μ-a.s. Then, there exists m^* such that the set $E := \{s \in S_K : f(s) < 1 - \frac{1}{m^*}\}$ has positive measure, that is $\mu(E) > 0$. By proposition 2.26 of Lee (2003), for each $\varepsilon > 0$, there exist $F_\varepsilon \subset E \subset O\varepsilon$ where F_ε is closed and O_ε is open, such that $\mu(O_\varepsilon \setminus F_\varepsilon) < \varepsilon$, and a function $\varphi(\cdot; O_\varepsilon, F_\varepsilon, \varepsilon) : S_K \to \mathbb{R}$ such that $\varphi(s; O_\varepsilon, F_\varepsilon, \varepsilon) = 0$ for $s \in O_\varepsilon^c$ and $\varphi(s; O_\varepsilon, F_\varepsilon, \varepsilon) = 1$ for $s \in F_\varepsilon$. Moreover, by the construction of Theorem 6 in Chatterjee and Krishna (2009), there exist menus (of consumption) A_ε and B, where B is independent of ε, such that φ is the difference of the support functions of A_ε and B. In particular, for all $s \in F$,

$$\max_{p\in A_\varepsilon} u_s(p) - \max_{p\in B} u_s(p) = \varphi(s; O_\varepsilon, F_\varepsilon, \varepsilon) = \lambda_s$$

Let θ be the menu that gives the agent p^* in each period. Since $u_s(p^*) = 0$ for all $s \in S_K$, this implies $U(\theta) = 0$. Now consider the menu $x_\varepsilon := \{(A_\varepsilon, \theta), (B, y)\}$ where $y \in \mathcal{K}$ is some continuation problem. Notice that

$$U(x_\varepsilon) - U(B, y) = \int_{O_\varepsilon \setminus F_\varepsilon} D(\varepsilon)\,d\mu + \int_{F_\varepsilon} \max\{\lambda_s - \delta U(y), 0\}\,d\mu$$

and

$$\tilde{U}(x) - \tilde{U}(B, y) = \int_{O_\varepsilon \setminus F_\varepsilon} \tilde{D}(\varepsilon)f(s)\,d\mu + \int_{F_\varepsilon} \max\{\lambda_s - f_s\delta U(y), 0\}\,d\mu$$

where $D(\varepsilon)$ and $\tilde{D}(\varepsilon)$ are functions such that

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{O_\varepsilon \setminus F_\varepsilon} D(\varepsilon)\,d\mu &= 0\\ &= \lim_{\varepsilon\to 0} \int_{O_\varepsilon \setminus F_\varepsilon} \tilde{D}(\varepsilon)f(s)\,d\mu\end{aligned}$$

By definition of U and $\tilde{U}$, it must be that the above displays are equal, ie $U(x_\varepsilon) - U(B, y) = \tilde{U}(x_\varepsilon) - \tilde{U}(B, y)$.

But notice the following facts: (i) $f_s < 1 - \frac{1}{m}^*$ on F, (ii) we can always choose y such that $0 < \mu\big(\{\lambda_s > \delta U(y)\}\big) < \mu\big(\{\lambda_s > f_s \delta U(y)\}\big)$ and the set $\{s \in F : \lambda_s - \delta U(y)\}$ has positive measure, and (iii) $\max\{\lambda_s - \delta U(y), 0\} \leqslant \max\{\lambda_s - f_s \delta U(y), 0\}$, where the inequality is strict on a set of positive measure due to the appropriate choice of y.

Therefore, $\int_{F_\varepsilon} \max\{\lambda_s - f_s \delta U(y), 0\}\, d\mu - \int_{F_\varepsilon} \max\{\lambda_s - f_s \delta U(y), 0\}\, d\mu > 0$, and is nonincreasing in ε. Also, $\int_{O_\varepsilon \setminus F_\varepsilon} \tilde{D}(\varepsilon) f(s)\, d\mu - \int_{O_\varepsilon \setminus F_\varepsilon} D(\varepsilon)\, d\mu$ is strictly decreasing in ε, so that by the regularity of μ and the definition of $D(\varepsilon)$ and $\tilde{D}(\varepsilon)$, there is an $\varepsilon^* > 0$ such that $\tilde{U}(x_{\varepsilon^*}) - \tilde{U}(B, y) > U(x_{\varepsilon^*}) - U(B, y)$, which is a contradiction. Thus, $f = 1$, μ-a.s., which proves our claim. □

5. Nonstationary Intertemporal Choice

An important feature of stationary dynamic choice is that regardless of what subjective state the agent is in at any moment of consumption, the distribution of subjective states in the future remains the same. In this section, we enrich this behaviour. This requires some new definitions, and strengthening of some axioms seen before.

For a fixed collection $a^{t-1} := (a_0, a_1, \dots, a_{t-1}) \in \mathcal{K}_Z^{t-1}$ for $t \geqslant 1$ (where $a^{-1} = \varnothing$), define, for all $t \geqslant 0$, $\succcurlyeq^t \subset Z \times Z$ as follows: $x \succcurlyeq^t y$ if, and only if, $(a^{t-1}, x) \succcurlyeq (a^{t-1}, y)$. Each preference $\succcurlyeq^t$ also induces the binary relations $\succcurlyeq^t_K$ and $\succcurlyeq^t_Z$ as in section 4 above. Separability implies that this definition is independent of the choice of a^{t-1}. The nonstationary context requires a strengthening of our nontriviality assumption on $\succcurlyeq$. We call this Dynamic Nontriviality.

Axiom 6 (Dynamic Nontriviality)**.** For all $t \geqslant 0$, $\succcurlyeq^t$ and therefore $\succcurlyeq^t_K$ and $\succcurlyeq^t_Z$ are nontrivial.

Dynamic nontriviality says that after any consumption history, the agent will always value consumption and continuation problems nontrivially, which means that he values IHCPs nontrivially. In the context of stationary choice, this follows immediately from Stationarity.

5.1. Axioms for Nonstationary Choice

To keep the main new axiom as well as the following theorem simple, we confine attention to preferences that require only a finite relevant state space to describe $\succcurlyeq^t_K$ at any time $t \geqslant 0$.

Axiom 7 (Finiteness)**.** For all $a \in \mathcal{K}_K$ and $t \geqslant 0$, there is a finite set $b \subset a$ with $b \sim^t_K a$.

Dekel, Lipman and Rustichini (2009) show this axiom indeed implies that the representation of $\succcurlyeq^t_K$ requires only finitely many relevant states. The intuition is clear: Consider a menu a that is supported by a constant support function. This menu provides a different strictly best alternative in every state s of the universal state space S. If any finite subset b of a is as good as a itself, then only a finite subset $S' \subset S$ of states can be relevant. The following is the main behavioural axiom of this section.

Axiom 8 (Choice Contingent Strategic Rationality)**.** If $a \cup b \succ_K^t b$, then there is c such that $a \cup b \cup c \succ_K^t b \cup c$ and

$$(a, A) \cup (b \cup c, A \cup B) \succcurlyeq^t (a, B) \cup (b \cup c, A \cup B)$$

implies

$$(a, A) \cup (b \cup c, A \cup B) \sim^t (a \cup b \cup c, A \cup B).$$

The implications of the axiom are most easily understood in the context of finitely many relevant consumption states. Every such state must be relevant in some period t. Further, for each relevant state s, there are menus $a \cup b \succcurlyeq_K^t b$, such that a delivers greater (ex post) utility than b only in state s. Moreover, $a \cup b \cup c \succcurlyeq_K^t b \cup c$ implies that a also outperforms c in that state. Contingent on a providing a better consumption choice than $b \cup c$ in period t, the preference satisfies strategic rationality with respect to the continuation problems. Since the axiom holds for all $a \cup b \succcurlyeq_K^t b$, we can establish strategic rationality for any one of the relevant states. The menu c in the qualifier of the menu implies that the axiom does not generically imply the stronger continuation strategic rationality axiom assumed in the previous section. The axiom implies that the consumption state in period t is a sufficient statistic for period $t+1$ preferences over continuation problems. We can now define the class of representations that are central to this section.

Definition 5.1. Consider a family (u_s) of vNM utilities, where $u_s = s\lambda_s$ for some $\lambda_s > 0$ and $s \in S \subset S_K$ where S is countable. Let $\mu \in \mathcal{P}(S)$ be a probability measure on S, and let $\mathcal{A} : S \to P(S)$ be a transition operator from states to probability measures on S, so that $\mathcal{A}(s) = \mu_s$, and let $\delta \in (0, 1)$. A tuple $\big((u_s), \mu_0, \mathcal{A}, \delta\big)$ is a representation of ***Markovian Preference for Flexibility*** (MPF), and induces a utility over IHCPs $V(x, \mu_0)$, where V is defined recursively as

$$V(x, \mu) = \int_{S_K} \max_{p \in x} \left\{ \int_{K \times Z} \left[u_s(k) + \delta V(z, \mu_s)\right] dp(k, z) \right\} d\mu(s)$$

where $\mathcal{A}(s) = \mu_s$. We refer to (u_s) as the relevant set of subjective tastes, μ_0 as initial beliefs, μ_s as state contingent beliefs over future subjective states, $(\mu_s) \cup \mu_0$ as the collection of all beliefs, and δ as the discount factor.

Let S^* denote the set of relevant states, those states that can be reached, given μ_0 and $\mathcal{A}$. A set of states $E \subset S^*$ is ***ergodic*** if (i) conditional on being in the set E of states, the probability of transitioning to another state outside E is zero, and (ii) there is no proper subset of E with this property. We can now state the main theorem of this section.

Theorem 5.2. *A preference $\succcurlyeq$ satisfies Axioms 1, 2, 4, 6–8 if, and only if, it has a SPF representation $((u_s), \mu_0, \mathcal{A}, \delta)$ where μ_0 and all $\mathcal{A}(s) = \mu_s$ have finite support. In this representation, δ is unique, and for any ergodic set of states $S' \subset S^*$, the relative intensities of tastes, $\lambda_{s'}/\lambda_{s''}$ as well as relative beliefs $\mu_0(s')/\mu_0(s'')$ and $\mu_s(s')/\mu_s(s'')$ are unique for all $s', s'' \in S'$ and any $s \in S^*$. Across sets of ergodic states, (u_s) and (μ_s) are at least jointly identified as in DLR.*

The proof consists of showing that a representation exists, where the discount factor may also depend on the state, and then showing that any such representation, with a state dependent discount factor, can be renormalized to give an SPF with a unique discount factor, and then establishing the uniqueness of the parameters of the SPF proves the theorem. It is more instructive to first assume the existence of a representation with a state dependent discount factor, and show that there is also a unique representation with a constant discount factor, and defer showing the existence of such a representation.

5.2. Representation with Constant Discounting

We now show this via a number of steps.

§ 5.2.1. *Existence of Representation with Constant Discount Factor*

Suppose

$$V(x, \mu_0) = \int_S \max_{p \in x} \left\{ \int_{C \times Z} [u_s(c) + \delta(s) V(z, \mu_s)] \, dp(c, z) \right\} d\mu_0(s)$$

represents $\succcurlyeq$. We now prove that a representation with constant discount factor exists. Note that S^* is countable. Consider a MPF representation, which features a constant discount factor,

$$\hat{V}(x, \hat{\mu}_0) = \int_{\hat{S}} \max_{p \in x} \left\{ \int_{C \times Z} \left[\hat{u}_s(c) + \hat{\delta} \hat{V}(z, \hat{\mu}_s) \right] dp(c, z) \right\} d\hat{\mu}(s)$$

Let ξ be the collection of scaling factors for the relevant tastes, that is $\xi(s) = \lambda_s / \hat{\lambda}_s$. Abusing notation, we write $\langle \mu, \xi \rangle$ to denote $\sum_{s' \in S'} \mu(s')\xi(s')$. The uniqueness result in DLR applies to the representation of $\succcurlyeq_K^t$ for all t. Hence, $\hat{S}^* \equiv S^*$ and $\hat{\mu}(s) = \frac{\mu(s)\xi(s)}{\langle \mu, \xi \rangle}$ must hold for all $\mu \in (\mu_s) \cup \mu_0$. Then clearly

$$\int_{S^*} \max_{p \in x} \left\{ \int_{C \times Z} \left[\hat{u}_s(c) + \frac{\delta(s)}{\xi(s)} V(z, \mu_s) \right] dp(c, z) \right\} d\hat{\mu}_0(s)$$

represents $\succcurlyeq$, as it is a renormalisation of $V(x, \mu_0)$. Therefore, $\hat{\delta}\hat{V}(z, \hat{\mu}_s) = \frac{\delta(s)}{\xi(s)} V(z, \mu_s)$ must hold for all $s \in S^*$. At the same time $\hat{V}(z, \hat{\mu}) = \frac{V(x,\mu)}{\langle \mu, \xi \rangle}$. Hence, $\hat{\delta} = \frac{\delta(s)}{\xi(s)} \langle \mu_s, x \rangle$. To show that there is such a representation, we have to show that there is ξ such that $\frac{\delta(s)}{\xi(s)} \langle \mu_s, \xi \rangle$ is constant for all $s \in S^*$.

Consider a situation where $S^* = \{s_1, s_2, ...\}$. Since a common rescaling of all tastes is always possible, we can confine attention to ξ with $\sum_{i \geqslant 1} \xi(s_i) = 1$. We proceed by induction. Consider the following induction hypothesis:

Suppose the following is true for $n-1$: For any $\kappa < 1$ such that $\sum_{i \geqslant n} \xi(s_i) = \kappa$, there exist $\xi(s_i; \kappa, n-1)$ such that (i) $\sum_{i \leqslant n-1} \xi(s_i; \kappa, n-1) = 1 - \kappa$, and (ii) for all $i \leqslant n-1$, $\langle \mu_{s_i}, \xi \rangle \frac{\delta(s_i)}{\xi(s_i)}$ is a constant.

We now establish that the proposition is also true for n, if it is true for $n-1$. Let $\sum_{i > n} \xi(s_i) = 1 - \kappa$ and suppose $\xi(s_n) = r \geqslant 0$. For each $r \in [0, 1-\kappa]$, by the induction hypothesis, we can find $\xi(s_i; \kappa, n-1)$ such that (i) $\sum_{i \leqslant n-1} \xi(s_i; \kappa, n-1) = 1 - \kappa - r$, and (ii) for all $i \leqslant n-1$, $\langle \mu_{s_i}, \xi \rangle \frac{\delta(s_i)}{\xi(s_i)}$ is a constant. Clearly, when $r \approx 1 - \kappa$, this constant is strictly bigger than $\delta(s_n) \langle \mu_{s_n}, \xi \rangle / \xi(s_n)$, and when $r \approx 0$, this constant is some positive number, that is strictly less than $\delta(s_n) \langle \mu_{s_n}, \xi \rangle / \xi(s_n)$. Therefore, by the intermediate value theorem, there exists an r^* such that the hypothesis holds for n. It is easy to see that the induction hypothesis holds for $n - 1 = 1$, which proves our claim.

§ 5.2.2. *Uniqueness of* δ

We now claim that δ is unique in any MPF representation. Suppose not, then there would be a MPF representation $\left((\hat{u}_s), \hat{\mu}_0, \hat{\mathcal{A}}, \hat{\delta}\right)$ with $\hat{\delta} = \frac{\delta}{\xi(s)} \langle \mu_s, \xi \rangle$. This is only possible for $\frac{\xi(s)}{\langle \mu_s, \xi \rangle}$ constant. This requires $\xi(s)$ to be constant on the support of μ_s, and therefore $\hat{\delta} = \delta$.

§ 5.2.3. *Uniqueness Properties of* $\left((u_s), \mu_0, \mathcal{A}\right)$

The intertemporal trade off between consumption and continuation problems in the additively separable MPF representation implies that rescaling u_s by $\frac{1}{\xi(s)}$ also must lead to a rescaling of $\delta V(\cdot, \mu_s)$ by $\frac{1}{\xi(s)}$. As noted above, $\hat{V}(\cdot, \hat{\mu}) = \frac{V(\cdot,\mu)}{\langle \mu, \xi \rangle}$.

It is useful to recall some facts from linear algebra. For an $n \times n$ matrix Q, $\rho_R \in \mathbb{R}$ is a *right eigenvalue* if there exists a vector x_R such that $Qx_R = \rho_R x_R$. The vector x_R is referred to as the *eigenvector* corresponding to the eigenvalue ρ_R. Also, $\rho_L \in \mathbb{R}$ is a *left eigenvalue* if there exists a vector x_L such that $x_L^\top Q = \rho_L x_L$ (where $x_L^\top$ is the transpose of x_L), and the vector x_L is referred to as the *eigenvector* corresponding to the eigenvalue ρ_L. Notice that any scalar multiple of an eigenvector is also an eigenvector. Therefore, when speaking about the uniqueness of an eigenvector, we identify all eigenvectors that are scalar multiples of each other. It can be shown that the set of left and right eigenvalues is the same, so we may refer to *eigenvalues* without further qualification. However, the left and right eigenvectors are, in general, not identical. The following lemma allows us to connect the number of left and right eigenvectors.

Proposition 5.3. Let Q be a $n \times n$ stochastic matrix (that is, all the rows are nonnegative and sum to 1). Suppose $\rho = 1$ is an eigenvalue of Q and suppose also that there is a unique left eigenvector corresponding to the eigenvalue $\rho = 1$. Then, there is a unique right eigenvector.

Proof. Notice that if Q is ergodic, there is a unique left eigenvector corresponding to $\rho = 1$. Let this left eigenvector be x_L. Then, x_L solves $x_L^\top\left[Q - I\right] = \mathbf{0}$, where

I is the identity matrix. We know that the solutions from the space orthogonal to the column vectors of the matrix $Q - I$. By the Rank-Nullity Theorem, Lang (1987, Chapter 3, Theorem 3.2), we know that

$$\text{column rank} + \text{dim space of solutions} = n$$
$$\text{row rank} + \text{dim space of solutions} = n$$

But the row rank and the column rank are equal; see, for instance Lang (1987, Chapter 5, Theorem 3.2). Therefore, the dimension of the set of left and right eigenvectors is the same, and equal to 1. This is equivalent to saying that there is a unique right eigenvalue (up to scalar multiples). $\square$

Given that δ is constant, it must be true that $\xi(s) = \langle \mu_s, \xi \rangle$ for all s in the support of μ_s. Clearly, $\mathbf{1}$ is a solution to this equation. More generally, any constant vector ξ is a solution to this equation. By the proposition above, every solution must be a scalar multiple of $\mathbf{1}$. The observation above helps us conclude that $\xi(s')/\xi(s'') = 1$ for all $s', s'' \in S'$.

At any time t there only finitely many states that are or have been relevant for consumption. Consider any ergodic subset of these states, for $\mathcal{A}$. Each such set has, by ergodicity of the set, a unique stationary distribution, which by definition must be ξ. The trade off of gains in different ergodic sets of states is, from the ex-ante perspective, analogous to trade of across states in DLR's original model, and the same indeterminacy of beliefs results.

In "steady state", that is at a point in time where subjective tastes have reached an ergodic set, all the parameters of the representation are unique.

5.3. Existence of Representation for Nonstationary Choice

As before, $\succcurlyeq$ is a preference relation on Z. We know that a preference $\succcurlyeq$ satisfies Independence and Monotonicity if and only if there exists a finite charge μ such that the function

$$V(x) := \int_{S'} \max_{p \in x} \langle p, s \rangle \; d\mu(s)$$

for all $x \in \mathcal{K}$, represents $\succcurlyeq$.

This theorem is proved above. In particular, $S_{K\times Z}$ is the unit ball of a certain hyperplane in the space $\mathcal{C}(K \times Z)$, the space of uniformly continuous functions on $K \times Z$. We also have the following proposition.

Proposition 5.4. A preference $\succcurlyeq$ satisfies Independence, Monotonicity and Separability if and only if there exists a finite measure μ_K and a finite charge μ_Z such that the function

$$V(x) := \int_{S_K \times S_Z} \max_{p \in x} \left[u(p_k, s_k) + \nu(p_z, s_z)\right] \, d\mu_K(s_k) \, d\mu_z(s_z)$$

for all $x \in \mathcal{K}$, represents $\succcurlyeq$.

Proof. The proof is the same as the proof of separability in the stationary case, in lemma 4.9. In that proof, we only used Separability, Independence and Monotonicity, and stationarity to show that $\succcurlyeq_K$ is not trivial. Instead, here we use the axiom Dynamic Nontriviality which says that $\succcurlyeq_K^t$ is nontrivial for all $t \geqslant 0$. The proof then follows that of lemma 4.9. □

This is an important observation, since it allows us to separate the valuation of the present and the future. This is useful since it is much easier to analyze $S_K \times S_Z$ than $S_{K\times Z}$, since $\mathcal{C}(K) \times \mathcal{C}(Z)$ is much simpler than $\mathcal{C}(K \times Z)$. (We note that $S_K \times S_Z \subset S_{K\times Z}$.)

First some notation. For each $s \in S_K$, define $\Pi(s) := \left\{t \in S_Z : (s, t) \in S_K \times S_Z\right\}$, as the fiber over s. Moreover, for each $s \in S_K$, $\mu_K(s) := \int_{\Pi(s)} d\mu(s, t) = \mu(\Pi(s))$. That is, μ_K is the measure induced by μ on S_K.

We can now state our next proposition, that places a lot of structure on how the future is valued.

Proposition 5.5. Consider a preference functional of the form

$$V(x) := \int_{S_K \times S_Z} \max_{p \in x} \left[u(p_k, s_k) + \nu(p_z, s_z)\right] \, d\mu(s_k, s_z)$$

representing a preference $\succcurlyeq$, where $x \in \mathcal{K}$, such that $\text{proj}_{S_K}(\text{supp}\, \mu_K) < \infty$. The following are equivalent.

(a) The set $\Pi(s)$ is a singleton for each $s \in S^*$.

(b) The preference $\succcurlyeq$ satisfies Axioms 6, 7 and 8.

Proof. Recall that $p^* := \left(\frac{1}{|K|}, \ldots, \frac{1}{|K|}\right)$ is the uniform measure on the set of consumption outcomes that also gives 0 utility in each subjective state, and $B_\delta := \{p \in \Delta^{|K|-1} : \|p - p^*\|_2 \leqslant \delta\}$. For a consumption menu, A, denote it's ε neighbourhood by $N(A; \varepsilon)$.

Since the support of μ_K is finite, there is an $\varepsilon_1 > 0$ such that for any $s \in \operatorname{supp} \mu_K$, $p \in N(B_\delta; \varepsilon_1)$ is chosen only in state s and something else from B_δ is chosen in all the other states. So, fix the state s and the corresponding p, and notice that by construction, $B_\delta \cup \{p\} \succ_K B_\delta$. Therefore, by Consumption Continuation Strategic Rationality, there is $c \in \mathcal{K}_K$ such that (i) $B_\delta \cup \{p\} \cup c \succ_K B_\delta \cup c$, and (ii) $\left(B_\delta \cup \{p\} \cup c, \{x, y\}\right) \sim (p, y) \cup (B_\delta \cup c, \{y, z\})$ for all $x, y \in \mathcal{K}_Z$. Then, it must be the case that

$$\int_{\Pi(s)} \max_{\substack{(q,\tilde{y}) \in \\ \{p,y\} \cup (B_\delta \cup c, \{y,z\})}} \left[u(q, s) + \nu(\tilde{y}, t)\right] \, d\mu_Z(t|s) = \int_{\Pi(s)} \left[u(p, s) + \max_{\tilde{y} \in \{y,z\}} \nu(\tilde{y}, t)\right] \, d\mu_Z(t|s)$$

where the left hand side is the utility from the menu $(p, y) \cup (B_\delta \cup c, \{y, z\})$ contingent on being in state s, and the right hand side is the utility from the menu $\left(B_\delta \cup \{p\} \cup c, \{x, y\}\right)$ also contingent on being in state s. (In any other state $s' \neq s$, because there is a preference for flexibility, the agent will always pick from $B_\delta \times \{y, z\}$, which is available to both menus.)

But notice that, by definition, $u(p, s) > u(q, s)$ for all $q \in B_\delta$, which means that for almost all $t \in \operatorname{supp} \mu_Z(t|s)$,

$$\max_{\substack{(q,\tilde{y}) \in \\ \{p,y\} \cup (B_\delta \cup c, \{y,z\})}} \left[u(q, s) + \nu(\tilde{y}, t)\right] \leqslant u(p, s) + \max_{\tilde{y} \in \{y,z\}} \nu(\tilde{y}, t).$$

But since the integrals above are equal, this actually means that for almost all $t \in \operatorname{supp} \mu_Z(t|s)$, we actually have

$$\max_{\substack{(q,\tilde{y}) \in \\ \{p,y\} \cup (B_\delta \cup c, \{y,z\})}} \left[u(q, s) + \nu(\tilde{y}, t)\right] = u(p, s) + \max_{\tilde{y} \in \{y,z\}} \nu(\tilde{y}, t).$$

Let us assume, without loss of generality, that $y \in \arg\max_{\{y,z\}} \nu(\tilde{y}, t)$. Recalling that $u(p, s) > u(q, s)$, we conclude that almost all $t \in \operatorname{supp} \mu_Z(t|s)$, $\nu(y, t) \succcurlyeq \nu(z, t)$. But this is equivalent to saying that there is a singleton in $\Pi(s)$ that is assigned full measure by $\mu_Z(t|s)$. □

Notice that the lemma says that for every way of valuing present consumption, there is a unique way of valuing future consumption. This is how we exploit recursivity. From here, the Markovian representation follows relatively easily.

Another comment is in order. Without the above proposition, the representation is both unwieldy and difficult to interpret. In particular, it would suggest that there are things quite apart from consumption utility that determines how the agent values streams of consumption. While this may be interesting (and natural) in some contexts, we prefer to stay as close as possible to standard infinite horizon consumption models and the subjective state space of DLR. We can now finish the proof of the theorem.

§ 5.3.1. *Existence of a Recursive Representation*

Note that $\succcurlyeq^t$ satisfies separability, monotonicity, independence and non-trviality. Consequently, it can be represented by a DLR representation. On the one hand, according to Proposition 5.2, $\succcurlyeq^1$ is represented by

$$V_t(x) = \int_{S_K} \max_{p \in x} \left[\hat{u}(p_k, s) + \hat{\nu}(p_z, s)\right] d\hat{\mu}(s)$$

for some $\hat{\mu}$. On the other hand, recalling the definition of $\succcurlyeq^t$, consider a representation of $\succcurlyeq^1$:

$$\begin{aligned} V_1(x) \propto V(a, x) &= \int_{S_K} \max_{p \in (a,x)} \left[u_s(p_k) + \nu(x, s)\right] d\mu(s) \\ &= \int_{S_K} \max_{p_K \in a} u_s(p_k)\, d\mu(s) + \int_{S_K} \nu(x, s)\ d\mu(s) \end{aligned}$$

or

$$V_1(x) \propto \int_{S_K} \nu(x, s)\, d\mu(s)$$

where μ has the same support as $\hat{\mu}$.

Hence,

$$v(x,s) = \int_{S_K} \max_{p \in x} \left[\hat{u}(p_k, s) + \hat{v}(p_z, s)\right] d\tilde{\mu}(s)$$

for some $\tilde{\mu}$ with the same support as $\hat{\mu}$. Now rescale $\hat{u}(\cdot, s')$ to become $u_{s'}(\cdot)$ for all $s' \in S_K$. Let $\xi(s') = \frac{\hat{u}(\cdot,s')}{u(\cdot,s')}$. Find the corresponding measure $\mu_s(s') = \frac{\tilde{\mu}(s')\xi(s')}{\int_{S_K} \xi(s')\, d\tilde{\mu}(s')}$. Finally, let $\delta(s) := \int_{S_K} \xi(s')\, d\tilde{\mu}(s')$ to arrive at the representation

$$V(x, \mu_0) = \int_{S_K} \max_{p \in x} \left\{ \int_{K \times Z} \left[u_s(k) + \delta(s) \int_{S_K} \max_{p' \in z} \int_{K \times Z} \left[u_{s'}(k') + \right.\right.\right.$$
$$\left.\left.\left. + v(z', s')\, dp'(k', z')\right] d\mu_s(s') \right] dp(k, z) \right\} d\mu(s)$$

Induction over t gives the recursive representation

$$V(x, \mu_0) = \int_{S_K} \max_{p \in x} \left\{ \int_{C \times Z} \left[u_s(c) + \delta\,(s)\, V(z, \mu_s)\right] dp(c, z) \right\} d\mu(s).$$

6. Comparative Statics

Preference for flexibility is the preference for non degenerate menus over singletons. Intuitively, one DM has more preference for flexibility than another, if she has a stronger preference for menus over singletons. To formalize this notion, compare two preference rankings that have an SPF representation, and agree on singletons.

Definition 6.1. If $\succ$ and $\succ^*$ have an SPF representation, $\succ^*$ has a ***greater preference for flexibility*** than $\succ$ if

$$\alpha \succ_K \beta \quad \text{if, and only if,} \quad \alpha \succ_K^* \beta$$

and

$$a \succ_K \beta \quad \text{implies} \quad a \succ_K^* \beta$$

for all $\alpha, \beta \in \mathcal{P}(K)$, $a \in \mathcal{K}_K$.

Since preference for flexibility is the behavioral manifestation of uncertainty about tastes, it is ideally characterized in terms of beliefs. In our dynamic context the intertemporal tradeoff separately identifies beliefs and intensities. This allows us to compare decision makers that agree in terms of the intensities of their tastes but may differ in their beliefs. To do so, suppose there are two prizes, $m, M \in K$, that can play the role of numeraires across tastes: M is unequivocally better than m, and the strength of the preference for M over m is independent of the relevant taste s. Intuitively, this requires that the cost of giving up an ε amount of M (in probability) for an extra unit of continuation consumption is the same for each taste. In order to capture the behavioural content of this requirement in an axiom, let $\alpha, \beta \in \mathcal{P}(K)$ be such that $\alpha(m) > \beta(m) > \varepsilon$. Define α' such that $\alpha'(m) := \alpha(m) - \varepsilon$, $\alpha'(M) := \alpha(M) + \varepsilon$ and $\alpha'(k) := \alpha(k)$ for all other $k \notin \{M, m\}$. Analogously define β'. Therefore, α' differs from α in that α' has an ε lesser mass on m than α, but an ε greater mass on M than α, while agreeing on all other prizes. Moreover, β' differs from β in the same way. The following axiom now ensures the existence of numeraires.

Axiom 9 (Numeraire)**.** $x \cup (\alpha, p_z) \sim x \cup (\alpha', 0)$ implies $z \cup (\beta, p_z) \sim z \cup (\beta', 0)$ for all z, $p_z \in \mathcal{K}(Z)$.

Recall that the normalization of the taste space S in the representation SPF above was arbitrary. Consider instead a renormalization that takes into account intensities.

Definition 6.2. Let $\big((u_s), \mu, \delta\big)$ be the SPF representation of $\succ$. For $u_s = s\lambda_s$, renormalize the taste space S_K according to the intensities (λ_s), and include only relevant tastes, that is, tastes in the support of μ:

$$S_\lambda := \left\{ s\lambda_s : s \in \operatorname{supp}(\mu) \subset S \right\}$$

The unique SPF representation of $\succ$ can be rewritten in terms of S_λ as

$$U(x) = \int_{S_\lambda} \max_{p \in x} \left\{ \int_{K \times Z} \left[\langle s, p_k \rangle + \delta U(z) \right] \, dp(k, z) \right\} \, d\mu(s)$$

and denoted by (S_λ, μ, δ), where μ now denotes a probability measure over S_λ with full support. Let $s(k)$ denote the component of s that corresponds to prize $k \in K$. It can be verified that Axiom Numeraire indeed implies that S_λ can be normalized such that $s(M) = 1$ and $s(m) = 0$ for all $s \in S_\lambda$. Consequently, $S_\lambda \subset \mathbb{R}^{|K|-2}$ and if $\succ$ and $\succ^*$ both satisfy Axiom Numeraire, then $S^*_\lambda = S_\lambda$. In order to characterise a notion of greater preference for flexibility, we need the following definition.

Definition 6.3 (Increasing convex order)**.** Let μ and μ^* be probability measures with support in S_λ. Then, μ dominates μ^* in the ***increasing convex order*** if for every increasing convex function $\varphi : \mathbb{R}^{|K|-2} \to \mathbb{R}$, $\int \varphi \, d\mu \leqslant \int \varphi \, d\mu^*$.

We can now characterise a greater preference for flexibility in terms of the measure μ.

Proposition 6.4. If $\succ$ and $\succ^*$ have an SPF representation and satisfy Axiom Numeraire, then $\succ^*$ has a greater preference for flexibility than $\succ$ if, and only if, μ dominates μ^* in the increasing convex order.

Sketch of Proof. Since S_λ is bounded, we can restrict attention, without loss of generality, to convex functions u defined on the closed convex hull $\overline{\text{conv}}(S_\lambda)$ of S_λ. Moreover, we can restrict attention to increasing convex functions that are Lipschitz, with rank less than 1. Call this set Φ. But the set of increasing, piecewise linear, convex functions with Lipschitz rank less than 1, Φ_0, is dense in Φ. Therefore, we will be done if we can show that for all nonnegative functions $\varphi \in \Phi_0$, $\int \varphi \, d\mu \leqslant \int \varphi \, d\mu^*$.

But notice that the utility of a lottery p is a linear function of the state $s \in S_\lambda$. Therefore, the utility of any finite menu is a piecewise linear convex function. Indeed, for any nonegative function $\varphi \in \Phi_0$ with $\varphi(0) = 0$, we can construct a finite menu x such that the utility of the menu x is $U(x) = \int \varphi(s) \, d\mu(s)$. Thus, a greater preference for flexibility corresponds exactly to dominance in the increasing convex order. □

To illustrate this result, consider the example of two decision makers, DM and DM*, both of whom have monotone preferences over $\{0, 1/2, 1\}$ and are uncertain about their future risk aversion. Further, suppose their preferences, $\succ$ and $\succ^*$ respectively, have an SPF representation and satisfy Axiom Numeraire for $m = 0$ and $M = 1$.[5] Axiom Numeraire suggests that we may take $u(0, s) = 0$, and $u(1, s) = 1$ for all $s \in S_\lambda$. What is uncertain is the utility of 1/2, which is $s \in S_\lambda = [0, 1]$.

Proposition 6.5. DM* has a greater preference for flexibility than agent DM if, and only if, μ second order stochastically dominates μ^*.

This follows easily from proposition 6.4, since in one dimension, the increasing convex order corresponds to second order stochastic dominance. Nevertheless, we provide a proof, both because it is simple, and because it is illustrative.

Proof. A lottery is $\alpha = (\alpha_0, \alpha_{1/2}, \alpha_1)$. The utility of a lottery α to DM is

$$U(\alpha) = \alpha_1 + \alpha_{1/2} \int s \, d\mu(s) \tag{6.1}$$

DM* has a greater preference for flexibility than DM. This means that they rank singletons the same. Given the normalisations, this corresponds to requiring $U^*(\alpha) + U(\alpha)$ for all lotteries $\alpha \in \mathcal{P}(K)$. Equation (6.1) then implies that their taste measures μ and μ^* must have the same mean, that is, they must satisfy

$$\int s \, d\mu(s) = \int s \, d\mu^*(s) \tag{6.2}$$

Also, that DM* has a greater preference for flexibility than DM means that for each menu $\{\alpha, \beta\}$, $U^*(\alpha, \beta) \geqslant U(\alpha, \beta)$.

[5]For example the choice of DM who is uncertain about the parameter of risk aversion ρ in the CRRA utility $u_\rho(k) = k^{1-\rho}$ satisfies this assumption, as $u_\rho(1) = 1$ and $u_\rho(0) = 0$ for all $\rho > 0$, $\rho \neq 1$. Note that this is not the isoelastic CRRA utility often used in standard models. Isoelasticity would imply that there is no 'numeraire' that would facilitate the cardinal comparison of utilities across different levels of risk aversion. This cardinal comparison is central in a model of uncertainty about future risk aversion, but meaningless in standard models.

Consider a menu $\{\alpha, \beta\}$ such that $r := (\alpha_1 - \beta_1)/(\beta_{1/2} - \alpha_{1/2}) \in [0,1]$. Clearly, α is preferred by DM to β in state $s \in S_\lambda$ if, and only if, $s \leqslant r$. Therefore, the (expected) utility of the menu $\{\alpha, \beta\}$ to DM is

$$\begin{aligned} U(\alpha, \beta) &= \int_0^r \left[\alpha_1 + s\alpha_{1/2}\right] d\mu(s) + \int_r^1 \left[\beta_1 + s\beta_{1/2}\right] d\mu(s) \\ &= U(\alpha) + \left[\beta_{1/2} - \alpha_{1/2}\right] \int_r^1 (s - r)\, d\mu(s) \end{aligned}$$

with a similar expression for DM*.

Notice that by choosing the lotteries p and q appropriately, we can force r to take every value in $[0,1]$. Therefore, DM* has a greater preference for flexibility than DM if, and only if, for each $r \in [0,1]$, $\int_r^1 (s-r)\, d\mu^*(s) \geqslant \int_r^1 (s - r)\, d\mu(s)$.

Since μ and μ^* have the same mean, this is equivalent to requiring that

$$\int_0^r (s - r)\, d\mu(s) \geqslant \int_0^r (s - r)\, d\mu^*(s) \quad \text{for all } r \in [0,1] \tag{6.3}$$

Therefore, DM* has a greater preference for flexibility than DM if, and only if, (6.2) and (6.3) hold. But (6.2) and (6.3) correspond exactly to requiring that μ second order stochastically dominates μ^* (see, for instance, p 33 of Laffont (1989)), which completes the proof. □

The propositions may appear to hold even in a static context, since they are valid for every period t. However, the comparison of decision makers who agree in terms of the intensities of their tastes but may differ in their beliefs is only possible, because the intertemporal tradeoff separately identifies beliefs and intensities. In contrast, the notion of 'greater preference for flexibility' proposed in DLR for the static context cannot rely on beliefs, μ, but only on the (uniquely identified) support of μ. In terms of behavior this implies that, instead of characterizing whether one DM has a stronger preference for flexibility than another, DLR can only characterize whether one DM has any preference for flexibility, whenever the other does. Note that neither ranking is complete. While ours can only compare preferences that agree on the ranking of singletons, the ranking in DLR can only compare preferences with representations for which the support of the measure is ordered by set inclusion.

Next we connect our behavioral notion of 'greater preference for flexibility' to price behavior in a simple Lucas tree economy.

7. A Lucas Tree Economy

We shall consider a discretised version of the Lucas tree after Lucas (1978). There is an economy with a representative agent, and one productive asset. The asset produces $y \geqslant 0$ units of output, or dividends in each period. Output varies over time, according to the Markov process $F(y', y)$, with stationary distribution φ.

The agent has $z \in [0, 1]$ shares in the asset, which gives him a proportional right to the output. Specifically, with probability z, he gets all the output, and with complementary probability, he gets none of the output. He is in (taste) state $s \in S$ in each period, and his tastes evolve according to the Markov measure $\mu(s', s)$.

There is a market where the agent can trade for the probability q of getting all of the output of output and shares for the next period's output. The price of a unit of q is normalized to 1 in each state (y, s), while the price of a share is $p(y, s)$. Therefore, the agent's value function, when he owns z shares in the asset is

$$v(z, y, s) = \max_{q,x} \Big[u\big([q; y], s\big) + \delta \iint v(x, y', s') \, dF(y', y) \, d\mu(s', s) \Big]$$

subject to

$$q + p(y, s)x \leqslant z + p(y, s)z$$

where $[q; y]$ is the the lottery that gives y with probability q and 0 with probability $1 - q$. Then, $u\big([q; y], s\big) = qu(y, s) + (1 - q)u(0, s)$.

As in proposition 6.5 above, we shall assume that $u(0, s) = 0$ for all $s \in S$. By following the arguments in Lucas, we can show that for each continuous $p(y, s)$, there exists a unique continuous, bounded, nonnegative function $v(z, y, s)$ that satisfies the Bellman equation above, and is concave in z.

We know that in equilibrium, we must have $q = z + p(y,s)z - p(y,s)x$, so that $u\big([q;y],s\big) = u(y,s)\big[z + p(y,s)z - p(y,s)x\big]$. Following the arguments in Lucas (1978), we can show that the pricing function $p(y,s)$ is the unique solution to the functional equation

$$p(y,s) = \frac{g(y,s)}{u(y,s)} + \delta \iint \frac{u(y',s')}{u(y,s)} p(y',s')\, dF(y',y)\, d\mu(s',s) \tag{7.1}$$

where

$$g(y,s) = \delta \iint u(y',s')\, dF(y',y)\, d\mu(s',s)$$

We remark that we could just as easily have normalized the price of a share to be 1 in each state (y,s), in which case the price of a unit of q (the probability for immediate consumption) becomes $\psi(y,s) = 1/p(y,s)$. In what follows, it will be more natural to work with the price $\psi(y,s)$.

7.1. Stationary Distributions

Consider the special case where the Markov chains F and μ are iid over time. Then, $g(y,s)$ is a constant (albeit one that clearly depends on F and μ), and the pricing equation (7.1) can be rewritten as

$$f(y,s) = g + \delta \iint f(y',s')\, dF(y)\, d\mu(s)$$

where $f(y,s) = p(y,s)u(y,s)$. Since g is constant, the unique solution is given by

$$\begin{aligned} f(y,s) &= \frac{g}{1-\delta} \\ &= \frac{\delta}{1-\delta} \iint u(y',s')\, dF(y')\, d\mu(s') \\ &= \frac{\delta}{1-\delta} \mathbf{E}[u;F,\mu] =: \Lambda(F,\mu) \end{aligned}$$

which can be rewritten as

$$\psi(y,s) = \frac{u(y,s)}{\Lambda}$$

7.2. Preference for Flexibility and Price Volatility

Suppose, as before, that there are only three levels of output, 0, 1/2 and 1. We are now in a position to relate the distribution of prices with the distribution of tastes. Consider two exchange economies, with representative agents A and B respectively. (We shall refer to economies as A and B respectively.) We shall assume, as above, that both agents have preferences which have an SPF representation and satisfy axiom Numeraire with $M = 1$ and $m = 0$. We shall also assume that $\succ^A$ and $\succ^B$ agree on the intertemporal tradeoff for getting 1 instead of 0, which implies $\delta_A = \delta_B$.

Recall that $\Lambda_i = \frac{\delta}{1-\delta}\mathbf{E}\left[u; F, \mu_i\right]$ for $i = A, B$ and in state (y, s), the price of a unit of probability of consumption q is

$$\psi_i(y,s) = \frac{u(y,s)}{\Lambda_i}$$

for the economy inhabited by agent i. Notice first that $\mathbf{E}\left[u; F, \mu_i\right] = \int \left[0 \cdot f_0 + s \cdot f_{1/2} + 1 \cdot f_1\right] d\mu_i(s) = f_1 + f_{1/2} \int s \, d\mu_i(s)$ is independent of i, where f_j is the probability that output is $j \in \{0, 1, 2\}$. This implies $\Lambda_A = \Lambda_B = \Lambda$. Hence,

$$\begin{aligned} \psi_i(0,s) &= 0 \\ \psi_i(1,s) &= \frac{1}{\Lambda} \\ \psi_i(1/2,s) &= \frac{s}{\Lambda} \end{aligned}$$

for $i = A, B$ and $s \in [0, 1]$. In both economies the price of a unit (in probability) of consumption is constant across tastes, if output is either 0 or 1. Let $\psi_i(0)$ and $\psi_i(1)$ denote these prices. However, since tastes are stochastic, we can say something about the distribution of prices in the two economies for the case where output is $y = 1/2$. We let $H_i(\lambda) = \mathbf{P}\left(\psi(1/2, s) \leqslant \lambda\right)$ denote this distribution in economy i.

Proposition 7.1. In the two economies above, $\psi_A(0) = \psi_B(0) = 0$, $\psi_A(1) = \psi_B(1) = \frac{1}{\Lambda}$, and H_A second order stochastically dominates H_B if, and only if, agent B has a greater preference for flexibility than agent A.

Proof. Recall that the distribution of tastes s in economy i is given by the measure μ_i. By the arguments above, we have shown that the distribution of $\psi_A(1,s)$, H_A, second order stochastically dominates the distribution of prices H_B, if, and only if, μ_A second order stochastically dominates μ_B. But by proposition 6.5 above, this happens exactly when B has a greater preference for flexibility than A, which completes the proof. □

8. Proof of Theorem 3.1

We collect here proofs that could not find a home in the paper.

8.1. A General Representation

Let us recollect the notation from the paper. Let Z be a compact metric space, $C(Z)$ the Banach space of uniformly continuous functions on Z, and let $\mathcal{M}(Z)$ be the space of all finite, signed, regular Borel measures on Z (with the associated sigma algebra). Then, $\langle C(Z), \mathcal{M}(Z)\rangle$ is a dual pair.

Let $\mathcal{P}(Z) \subset \mathcal{M}(Z)$ represent the space of probability measures on Z and fix $p_0 \in \mathcal{P}(Z)$. Define $X' := \operatorname{span}\big(\mathcal{P}(Z) - p_0\big)$ which is a subspace of $\mathcal{M}(Z)$. Let $X'^{\perp} := \{x \in C(Z) : \langle x, x'\rangle = 0 \text{ for all } x' \in X'\}$ be the annihilator of X', so that $X := \{x \in C(Z) : \langle x, x'\rangle \neq 0 \text{ for some } x' \in X'\}$. It is easy to see that $X'^{\perp} = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$ is the space of constant functions. This verifies that $\dim(X'^{\perp}) = 1 = \operatorname{codim}(X)$, and $X \oplus X'^{\perp} = C(Z)$. Moreover, $\langle X, X'\rangle$ is a dual pair.

Let U' be the closed unit ball of X' (assuming X' has the total variation norm), and U the closed unit ball of X. By Alaoglu's theorem, U' (the unit ball in X') is weak* compact and metrisable. Let d be a metric that induces the weak* topology on U'. Let $\mathcal{F}$ be the space of all weak* closed, convex subsets of U'. For any weak* compact, convex subset A of X', let $\overline{h}_A : X \to \mathbb{R}$ be its support function, ie $\overline{h}_A(x) = \sup_{x' \in A} \langle x, x'\rangle$. The support function is sublinear and Mackey continuous (Theorem 5.102, Aliprantis and Border, 1999), and hence is also norm continuous (Corollary 6.27, Aliprantis and Border, 1999), since the Mackey and norm topology coincide on normed spaces.

Notice that a support function is completely defined by the values it takes on ∂U, the boundary of U. Therefore, for any sublinear and norm continuous function $\overline{h} : X \to \mathbb{R}$, we shall consider its restriction to ∂U, denoted by h. We shall call a function $h : \partial U \to \mathbb{R}$ a *support function* if its unique extension to X by positive homogeneity is sublinear and norm continuous, ie is a support function in the sense described above. (Thus, we have support functions defined on both X and ∂U. Which function we are referring to will be clear from the context, and also from the notation — support functions defined on X will have a 'bar' on top, while support functions defined on ∂U will not.)

For any support function $h : \partial U \to \mathbb{R}$, define $A_h = \{x' \in X' : \langle x, x' \rangle \leqslant h(x),\ \forall x \in \partial U\}$. Support functions have the following duality. For any weak* compact subset A of X', $A_{h_A} = A$ and for any support function h, $h = h_{A_h}$. The support function has the following useful properties: (i) $A \subset B$ if and only if $h_A \leqslant h_B$, (ii) $h_{A+\lambda B} = h_A + \lambda h_B$ whenever $\lambda \geqslant 0$, and (iii) $h_{A \cap B} = h_A \wedge h_B$ and $h_{\mathrm{conv}(A \cup B)} = h_A \vee h_B$.

Recall that the *polar* of the set U is $U^\circ := \{x' \in X' : |\langle x, x' \rangle| \leqslant 1\}$, so that U'°. Moreover, $(U'^\circ = U^{\circ\circ} = U$. Notice that with this definition, $\sup_{x \in \partial U} h_{U'}(x) = 1$.

Let $\lambda > 0$. The space $\mathcal{F}_\lambda$ of compact, convex subsets of $\lambda U'$ can be metrised by the Hausdorff metric ρ_d. We shall write $\mathcal{F}_1$ as $\mathcal{F}$. Let us now show that the Hausdorff distance between two sets can be measured by the uniform distance between their support functions. The following result is essentially Lemma 6.41 in Aliprantis and Border (1999).

Proposition 8.1. Let $A, B \in \mathcal{F}$. Then,

$$\begin{aligned}\rho_d(A, B) &= \sup\left\{\left|\bar{h}_A(x) - \bar{h}_B(x)\right| : x \in U\right\} \\ &= \sup\left\{\left|h_A(x) - h_B(x)\right| : x \in \partial U\right\}\end{aligned}$$

where $\partial U = \{x \in U : \|x\| = 1\}$ is the boundary of U.

Proof. Recall that $A \subset B$ if and only if $h_A \leqslant h_B$. Also, a characterisation of the Hausdorff metric is

$$\rho_d(A, B) = \inf\{\varepsilon > 0 : B \subset A + \varepsilon U' \text{ and } A \subset B + \varepsilon U'\}.$$

But $h_{A+\varepsilon U'} = h_A + \varepsilon h_{U'}$. Therefore, $B \subset A + \varepsilon U'$ if and only if $h_B(x) - h_A(x) \leqslant \varepsilon h_{U'}(x) \leqslant \varepsilon$ for all $x \in \partial U$. Thus, $|h_A(x) - h_B(x)| \leqslant \varepsilon$ for all $x \in \partial U$ if and only if $\rho_d(A, B) \leqslant \varepsilon$. In light of the characterisation of the Hausdorff metric, the claim is proved.

The second equality displayed above follows immediately from the positive homogeneity of support functions. □

An easy corollary of the above proposition is the following. For $\lambda > 0$, and $\lambda A, \lambda B \in \mathcal{F}_\lambda$, $\rho_d(\lambda A, \lambda B) = \lambda \rho_d(A, B)$. This follows from the fact that for a support function $h_{\lambda A} = \lambda h_A$.

Let $\mathcal{K}$ denote the space of weak* compact, convex subsets of $\mathcal{P}(Z) - p_0$, and notice that $\mathcal{K} \subset \mathcal{F}_\lambda$ for some $\lambda > 0$. Define $K_0 := \{h \in \mathbb{R}^X : h = h_A \text{ for some } A \in \mathcal{K}\}$. Let $\varphi : \mathcal{K} \to \mathbb{R}$ be Lipschitz continuous and linear. This induces a Lipschitz continuous linear functional $\overline{\Phi} : K_0 \to \mathbb{R}$, so that $\varphi(A) = \overline{\Phi}(A)$ for all $A \in \mathcal{K}$. Proposition 8.1 ensures that the two functions preserve the appropriate limits, in the sense that for any sequence (A_n) in K_0, $A_n \to_{w^*} A$ if and only if $\sup_{x \in \partial U} |h_{A_n}(x) - h_A(x)| \to 0$.

For ease of notation, we shall write ∂U as $\mathcal{U}$. This shall serve as our *universal subjective state space*. Each $x \in \mathcal{U}$ is a continuous function on Z, and hence a vNM function. Moreover, as established earlier, no $x \in \mathcal{U}$ is constant on Z.

Notice that $\mathcal{U}$ is metrisable (by the norm), hence normal and Hausdorff. Also, $K_0 \subset C_b(\mathcal{U})$, the space of all bounded, continuous functions on $\mathcal{U}$. Since $\overline{\Phi} : K_0 \to \mathbb{R}$ is Lipschitz continuous and linear, by the Hahn-Banach theorem, it has a linear extension to $C_b(\mathcal{U})$ that we shall denote as Φ. (First extend $\overline{\Phi}$ to the span(K_0) by linearity, and then extend to $C_b(\mathcal{U})$ by the Hahn-Banach theorem.) Moreover, Φ has the same Lipschitz constant as $\overline{\Phi}$.

> When Z is finite, it is prudent to let p_0 be the uniform measure. Then, $\mathcal{U} := \{x \in \mathbb{R}^{|Z|} : \sum_z x(z) = 0 \text{ and } \|x\|_2 = 1\}$ is compact. Moreover, span(K_0) is then dense in $C_b(\mathcal{U}) = C(\mathcal{U})$ (since $\mathcal{U}$ is compact). This means there is a unique Φ that extends $\overline{\Phi} : K_0 \to \mathbb{R}$ to $C_b(\mathcal{U})$. Unfortunately, the density result doesn't seem to be true when Z is infinite. At any rate, we aren't able to settle this one way or the other. This means we can't ensure the uniqueness of Φ.

Let ba_n denote the set of bounded, normal, finitely additive (signed) measures, ie charges on $\mathcal{A}_\mathcal{U}$, the algebra generated by the open sets in $\mathcal{U}$. We can now state the following.

Theorem 8.2. *The function $\varphi : \mathcal{K} \to \mathbb{R}$ is linear and Lipschitz if and only if there* *exists a finite charge* $\mu \in ba_n(\mathcal{A}_\mathcal{U})$ *such that*

$$\begin{aligned}\varphi(A) &= \int_\mathcal{U} h_A \, d\mu \\ &= \int_\mathcal{U} \sup_{x' \in A} \langle x, x' \rangle \, d\mu(x)\end{aligned}$$

for all $A \in \mathcal{K}$.

Proof. The 'if' part is clear. We shall only prove the 'only if' part.

By the construction above, we see that there exists $\Phi : C_b(\mathcal{U}) \to \mathbb{R}$ that is linear and continuous (hence Lipschitz continuous) such that $\Phi|_{K_0} = \varphi$. By a Riesz representation theorem, for instance Theorem 13.10 of Aliprantis and Border (1999), we see that there exists a finite normal charge μ such that $\Phi(f) = \int f \, d\mu$ for all $f \in C_b(\mathcal{U})$. Using the definition of the support function then proves our theorem. □

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