ACADEMY OF ECONOMIC STUDIES

DOCTORAL SCHOOL OF FINANCE –DOFIN

ANALYZING ASYMETRIC DEPENDENCE IN EXCHANGE RATES USING COPULA

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Bucharest

2010

In this paper I aimed to analyze the use of copulas in financial application, namely to investigate the assumption of asymmetric dependence and to compute some measures of risk. For this purpose I used a portfolio consisting in four currencies from Central and Eastern Europe. Due to some stylized facts observed in exchange rate series I filter the data with an ARMA×GJR model. The marginal distributions of filtered residuals are fitted with a semi-parametric CDF, using a Gaussian kernel for the interior of distribution and Generalized Pareto Distribution for tails. To obtain a better view of the dependence among the four currencies I proposed a decomposition of large portfolio in other three bivariate sub-portfolios. For each of them I compute Value-at-Risk and Conditional Value-at-Risk and then backtest the results.

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1. Introduction

A series of far-back observation have been reported the non-normality of distribution in the case of almost economic and financial variables. In this sense Mandelbrot (1963) highlighted for the first time existence of *leptokurtosis* effect, he indicating the fact that large changes tend to be also followed by several large changes of either sign (*volatility clustering* effect). Later in 1976, Black underlined the *leverage* effect as tends of assets prices correlates negatively with volatility movements. Furthermore in 1998 Ramchand and Susmel emphasized the evidence of common volatility tends across markets that could lead to contagion effects.

Taking into account for the existence of all these *stylized facts* in financial markets, we can conclude about complex behaviour of the financial instruments. Over the past 20 years it have be seen an enormous interest for obtaining better knowledge of financial markets. This process occurs along the rapid development of financial instruments. In parallel there raised the interest in protecting against some turbulent motions of the markets. Thus the risk management became one of the most important concerns in both private and academic environment.

The first step in developing quantitative tools designed to measure the risk of random events was made by the risk department of J.P Morgan. In 1994, the CEO of J.P Morgan, Dennis Weatherstone asked to the risk department that every day at 4.30 P.M to submit a report relating to the bank risk measure and a corresponding risk measure. Thus it takes birth the Risk Metrics Department managed by Till Guildman that elaborated the *Value-at-Risk* (VaR) model. Value-at-Risk is a statistic model which is designed to express the risk of an exposure by a single number. More exactly Value-at-Risk model estimates the worst potential loss for a financial instruments portfolio over a given time horizon and confidence level. Despite the simple assumptions, the original Value-at-Risk provided satisfactory results from the beginning. For this reason and also because it is relative easily to implement, this model became the main risk instrument used in banking and financialsystem. Furthermore due to the increased importance given by the Group of 30' Report in 1993 and especially the introduction to Basel I amendment Value-at-Risk became standard measurement in risk management modelling

But Value-at-Risk have received a lot of criticism over time due to theirs simplistic assumptions that made this model to have many limitations in quantifying the risks. In 2001 Dembo and Freeman proved that Value-at-Risk models, like volatility, don't provide a satisfactory distinction between "good" risks and "bad" risks. In 1997, Artzner called an axiomatic approach and set some conditions to certify a satisfactory risk measure. Thus Artzner called the risk measures which satisfy the formulated axioms as "*coherent*". He proved that Value-at-Risk model is not a coherent risk measure because it doesn't satisfy one of axiomatic condition, namely the sub-additivity one. Other important criticism was that Value-at-Risk model only provides a limit of the losses but tell nothing about the potential loss when the limit is exceeded.

Meanwhile several recent resounding failures as LTCM, Barings Bank, or more recent as Enron, Bear Sterns or Lehman Brothers which have brought in discussion that used risk management models do not take enough into account for potential occurrence of extreme events. This also applies to the Value at Risk model assumptions, which is actually considered one of the main catalysts for the current financial crisis because it is widely used in banking and financial system. But returning about first example of LTCM a very interesting fact was that the worst scenario provided by their models predicted a loss of only 20% as compared to 60% which was recorded when the situation began to deteriorate. In an interview accorded to the Wall Street Journal in 2000, John Meriweather that with Nobel Prize winners Robert Merton and Myron Scholes led LTCM to the time of its bankruptcy, he said that globalization will lead to increasing occurrence of multiple crises and therefore he encouraged consider extreme events. Instead few years later, Professor Paul Embrechts who is called Mr. Extreme Values due to its important contribution in this area declared in a Swiss paper that models with normality assumption (such as Black & Scholes or VaR) perform poorly in practice, one possible explanation being the existence of heterosckedasticity. Also Professor Embrechts added that for this purpose mathematics should be used to adjust some kind of models such that to be consistent with reality.

Thus in recent years have been proposed many alternatives to the original assumption of Value-at-Risk models. The flash points were to incorporate in Value-at-Risk the heteroskedasticity and some distributions that take into account the existence of extreme events. For the first task there appeared several volatility models from GARCH family as asymmetric GARCH (GJR) proposed by Glosten, Jagannathan and Runkle (1993) or exponential GARCH (EGARCH) proposed by Nelson (1991) that includes a Boolean function wich accounts for negative impact of bad news (*leverage* effect). Also stochastic volatility models, *Fuzzy*-GARCH or Markov-Switching GARCH models are other appropriate choices to model the heteroskedasticity. For the second task I quote the Bollerslev's conclusion (1987) that Student distribution provide suitable fits for univariate distribution, but performs poorly in the multivariate case. Thus a better choice to model the multivariate distribution is the use of copulas which permits decomposition of joint distribution in dependence structure and marginal distributions. Of course in the univariate case Extreme Value distributions are more appropriate in financial modelling than Student distribution. But a more detailed description of these concepts I will do later.

So in this paper I aimed to analyze the use of copulas in financial application, namely to investigate the assumption of asymmetric dependence and to compute measures of risk. For this purpose I used a portfolio containing four currencies from Central and Eastern Europe. Due to some stylized facts observed in exchange rate series I filter the data with an ARMA×GJR model. The marginal distributions of filtered residuals are fitted with a semi-parametric CDF, using a Gaussian kernel for the interior of distribution and Generalized Pareto Distribution for tails. To capture a better view of the dependence among the four currencies I propose a decomposition of large portfolio in other three bivariate sub-portfolios. For each of them I compute Value-at-Risk and Conditional Value-at-Risk and then backtest the results.

2.Literature Review

The use of copula in modelling economic and financial processes has recorded a fast growth in recent years, even though the first applications of copulas date back to late 70s.

Copula concept was firstly introduced in mathematics by Sklar (1959) who defined a theorem according to which any multivariate joint distribution can be decomposed into a dependence structure and its *n* marginal distributions. 1959 actually refers only to the appearance of this theorem for decomposition of multivariate distributions. Sklar explicitly calls *copula* concept in 1996 as a function that satisfies the theorem formulated by him in 1959. Epistemology of *copula* word comes from Latin and means connection or link. But until 1996, the functions that fulfils the Sklar's theorem from 1959 circulated under different names as: *dependence function* (Deheuvels, 1978), *standard form* (Cook and Jonson, 1981) or uniform representation (Hutchinson and Lai, 1990).

Copula was used for the first time in the joint-life models by of Joe Clayton (1978), studying the bivariate life tables of sons and fathers. Others important contributions to the Clayton's models have been made by Cook and Johnson (1981) and Oakes (1982).

Hougaard (1992) studied the join-survival of twins born in Denmark between 1881 and 1930 using a Gumbel copula (1960). Frees (1995) used Frank copula to investigate mortality of annuitants in joint- and last- survivor annuity contracts. Also using Frank copula, Shih and Louis (1995) studied the joint-survival of a series of patients infected with HIV.

After 2000 a wave of copula applications works in finance came due the growing interest for risk management. Rockinger and Jondeau (2001) used Plackett copula to analyze the dependence among S&P500, Nikkei 225 and some European stock indices. Patton (2002) computed the first conditional copula in order to allow first and second order moments of distribution function to vary over time. Patton (2004) used conditional copulas to analyze the asymmetric distribution between Deutsche Mark and Yen against Dollar. Jondeau and Rockinger (2006) use time-varying Gaussian and Student copula to model the bivariate dependence between countries, while for univariate marginal distributions propose Skewed-t GARCH models.

Frey and McNeil (2003) and Goorbergh, Genest and Werker (2005) have used copula functions to account for dependence in option pricing. Hotta, Lucas and Palaro (2006) estimates Value-at-Risk using ARMA×GARCH model to filter returns, while the marginal distributions were modelled by an GPD approach and dependence structure by Gumbel copula. ARMA×GARCH model was used previously to filter the returns series by Embrechts and Dias (2004) and Patton (2006). Hotta and Palaro (2006) use conditional copula to estimate Value-at-Risk for an bivariate indices portfolio.

Chollete, Heineny and Valdesogo (2008) use Gaussian and Canonical Vine copulas to model the asymmetric dependence between financial returns. Heineny and Valdesogo (2009) introduce a Canonical Vine autoregressive copula to model dynamic dependence between more than 30 assets. .

3. Methodology

3.1. Extreme Value Theory

Extreme Value Theory (EVT) represents a domain of the probability theory that deals with the study of extreme events. Such events are characterized by extreme deviations from the normal median of their probability distributions. More exactly, the EVT studies and models the behaviour of distributions in their extreme tails. These rare events are described by a thickening of the tails that determines an excess of the *kurtosis* above the characteristic value for of the Gaussian distribution. Therefore the apparitions of the so-called *fat tails* are also known as the *leptokurtic* distributions. An important remark about the modelling of extreme events is that it is not necessary to make a prior specification or assumption about the shape of the studied distribution. In literature exists two main theories that provided two approaches for applying the EVT theory.

3.1.1Generalized Extreme Value distributions

Thus the first method is known as *Block maxima* approach and it is based on the theorems which were introduced independently by Fisher[1] and Tippett[2] *et al.* (1928) and Gnedenko *et al.* (1943). This technique supposes that a sample should be divided into blocks and then the maximum or minimum value of each block is treated as extreme event. In 1958, Emil Julius Gumbel[3] showed that depending if the samples of maximum or minimum are bounded below or above, the extreme value distribution can be modelled as a few known limiting distributions. More exactly Gumbel demonstrated that if a distribution has a continuous repartition function and also has an inverse, then the asymptotic distribution of the maximum or minimum sample will converge to Gumbel, Fréchet or Weibull distributions. Therefore in a standard form these three types of distributions are considered as *Generalized Extreme Value* (GEV) distributions. It also should be noted that the principle of the limiting distributions is very close to the *Central Limit Theorem*whicht limits the normal distribution to a sample of averages. Firstly we will define the three distributions of rare events:

Gumbel:

[1] Sir Ronald Aylmer Fisher (1890 –1962) was an English statistician, biologist , geneticist and eugenicist. He introduced the maximum likelihood approach.

[2] Leonard Tippett (1902 - 1985) was an English statistician and physicist who studied under Professor Karl Pearson.

[3] Gumbel (1891-1661) was a German mathematician and political writer who founded the Extreme Value Theory together with Tippett and Fisher.

$$(1)\ \Psi^{Gumbel}(x) = -e^{-e^{x}},\ \text{where}\ x \in \mathbb{R}$$

Fréchet:

$$(2)\ \Psi_{\alpha}^{Fréchet}(x) = \begin{cases} 0, \text{ for } x \leq 0 \\ e^{-x^{-\alpha}}, \text{ for } x > 0, \alpha > 0 \end{cases}$$

Weibull:

$$(3)\ \Psi_{\alpha}^{Weibull}(x) = \begin{cases} e^{-(-x^{\alpha})}, \text{for } x < 0, \alpha < 0 \\ 1, \text{for } x > 0 \end{cases}.$$

For a given random vector $X = (X_1, \dots, X_n)$ in which the random variables are *independently and identically distributed* (i.i.d.) and $\vartheta = \max(X_1, \dots, X_n)$, then $P(\vartheta \leq \lambda) = \Phi^{n}(\lambda)$ is the distribution function of Ψ. Thus for an appropriate choice of constants θ_n and τ_n such that $\lambda = \theta_n x + \tau_n$ it will be fulfilled a convergence of the maxima's distribution functions to the following continuous distribution function:

$$(4)\ \Phi^{n}(\theta_n x + \tau_n) \to \Psi(x),$$

where $n \to \infty$ and $\Psi(x)$ is a non-degenerate distribution function that belongs to one of the three extreme distributions families.

Considering $\xi = 1/\alpha$, von Mises[4] *et al.* (1936) and Jenkinson *et al.* (1955) proposed a parametization for the GEV distribution to encompasses the three family of extreme distribution defined above:

$$(5)\ \Psi_\xi(x) = \begin{cases} e^{-(1+\xi x)^{-\frac{1}{\xi}}}, \text{for } 1+\xi x > 0, \xi \neq 0 \\ e^{-e^{x}}, \xi = 0 \end{cases},$$

where $\xi = 1/\alpha$ denotes the tail index and indicates the degree of thickness for the tails of the distribution. More detailed, the tail index reflects the velocity with which the probability decays in the extreme of the tail and also approaches to zero. Thus the heavier the tail, it will result a slower speed of decreasing probability and also a higher tail index. A very important property of the tail index $\xi = 1/\alpha$ is that this one indicates the number of moments which exist in a distribution. Therefore for $\xi = 4$ results that in the studied distribution it exists the first four moments: mean, variance, skewness and kurtosis, but the higher moments have infinite values. The chosen family of the extreme distribution to model the EVT is determined by the value of the tail index such that:

- If $\xi < 0$, then it will be chosen a Weibull distribution;
- If $\xi > 0$, then it corresponds a Fréchet distribution;
- And if $\xi = 0$, it results as appropriate a Gumbel distribution.

From the three families of extreme distributions it has been found that Fréchet distribution it is the most appropriated to the fat-tailed financial data due to the particularity of its tail index $\xi > 0$, because we know the smaller ξ corresponds to heavier tails.

In 1943, Gnedenko showed the necessary and sufficient conditions for each parametric distribution to belong to one of the three families of extreme distributions. Therefore he demonstrated that normal or log-normal distributions converges to a Gumbel distribution,; when the parameter α denotes degree of freedom, a Student distribution lead to a Fréchet distribution for its extremes; or a uniform distribution belongs to the attraction domain of a Weibull distribution.

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[4] Richard von Mises (1883– 1953) was an austrian athematcian. His brother was the economist Ludwig von Mises.

A very important advantage of the GEV approach is that for a given unknown initial distribution, the modelling of asymptotic distribution doesn't suppose any assumption about the particularities of the initial distribution of the sample. An exception of this observation is shaping by the modelling of a parametric *Value-at-Risk* (VaR).

3.1.2 Generalized Pareto Distribution

This second approach supposes to set a threshold value such that all the realizations over this limit are considered and also modelled as extreme events. The main idea behind this method called also *peak-over-threshold* is that difference between the realized extreme events and the set threshold are considered as excesses. Therefore peak-over-threshold approach involves the estimating of a conditional distribution of the excesses situated above a given set threshold. For a random vector $X = (X_1, \ldots, X_n)$ with a distribution function φ, let consider the threshold V as $V < x_{\varphi}$. Thus φ_v denotes the distribution function of excesses over the threshold V:

$$(6)\varphi_v(x) = \mathrm{P}(X - v \le x | X > v), x \ge 0.$$

Independently Balkema and de Haan *et al.* (1974) and Pickands *et al.* (1975) provided theorems that demonstrated since the threshold v was estimated and for a sufficiently high v to satisfy $v \to \infty$, the conditional distribution φ_v can be fit using *Generalized Pareto Distribution* (GPD). Therefore we will define the following relationship regarding the fitting of conditional distribution function φ_v using GPD:

$$(7)\ \varphi_v(x) \cong G_{\xi,\gamma,\beta}(x), v \to \infty, x \ge 0,$$

where

$$(8) G_{\xi,\sigma,\beta}(x) = \begin{cases} 1 - \left(1 + \xi \dfrac{x-\beta}{\gamma}\right)^{-\frac{1}{\xi}}, for\ \xi \neq 0 \\ 1 - e^{-\frac{(x-\beta)}{\gamma}}, \qquad for\ \xi = 0 \end{cases}$$

and

$$(9)\ x \in \begin{cases} [\beta, \infty], for\ \xi \geq 0 \\ \left[\beta, \beta - \frac{\gamma}{\xi}\right], for\ \xi < 0 \end{cases}$$

In the above relations, the parameter γ denotes the scale parameter, while β represents the location parameter. An important observation is that in the case when $\beta = 0$ and $\gamma = 1$, then the relations (8) and (9) constitute a standard GPD.

The relationship between GEV and GDP approaches can be expressed as following:

$$(10)\ G_{\xi_i}(x) = 1 + \log \Psi_{\xi}(x), for \log \Psi_{\xi}(x) > -1.$$

3.2 What means dependence?

In the probability theory, two random variables are independence if a part of the information's genesis of one variable does not found in the other variable. More exactly two random variables presents independence if and only if it is respected the following inequality:

$$(11)\ \Pr[X \leq x; Y \leq y] = \Pr[X \leq x] * \Pr[Y \leq y].$$

Instead, the concept of dependence between two or more random variables has to be described more in detail, owing to the high complexity of the concept. Another very important concept is the *mutual complete dependence* that states in the case of two random variables X and Y, the information' genesis of X implies the knowledge of Y, and inversely. Thus the predictability of one random variable to other can be defined as following:

$$(12) Y = f(X)$$

where f is either strictly increasing or strictly decreasing mapping, implying that the two random variables are *co-monotonic*. For a better understanding of the dependence concept, we will note the most important families of dependence measures.

3.2.1 Linear Correlations

This method for the measurement of dependence between random variables is the most used in the finance and insurance areas. Thus the linear correlation coefficient is found in the structure of many popular models like CAPM or *Value-at-Risk* (VaR). The main advantage of the linear correlation coefficient is the easiness of the estimation:

$$(13)\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{Var[X] * Var[Y]}}$$

where $Cov[X,Y]$ represents the covariance between X and Y, while $Var[X]$ and $Var[Y]$ are the variances of X, respectively of Y.

The properties of the linear correlation coefficient are the following:

i) $\rho(X,Y) \epsilon [-1,1]$;

ii) If X and Y are independent, then $\rho(X,Y) = 0$;

iii) $\rho(X,Y) = \rho(Y,X)$.

When between X and Y exists a perfect linear dependence defined as : $Y = \alpha + \beta X$, then $\rho(X,Y)$ equals ± 1, depending on whether β is positive or negative. The main disadvantage of the linear correlation coefficient is that it supposes a normal distribution of the analyzed series; otherwise it provides the so-called *spurious correlation.*

The linear correlation coefficient estimates the overall correlation between two random variables, basing on the assumption that $\rho(X,Y)$ is invariant only under linear changes. Instead, Docksum *et al.* (1994) developed a coefficient that measures the *local correlation* between two random variables, allowing to analyze when the correlation remains constant or not, in order to the random variables' realizations. Given the relationship between X and Y: $Y = \alpha + \beta X + \varepsilon$, $\varepsilon \sim iid$, then the correlation coefficient admits the representation:

$$(14)\rho = \frac{\beta * \sigma_X}{\sqrt{\beta^2 * \sigma_X^2 + \sigma_\varepsilon^2}}$$

where σ_X^2 and σ_ε^2 are the variances of X, respectively of the error term. More exactly the local correlation allows the analysis of the changes in the correlation strength using a function of the random variables' realizations. This approache is very useful in the study of systemic risks, contagions of crisis or to analysis the *flight-to-quality*[5] phenomenon.

3.2.2 Concordance Measures

A very important limit of the linear correlation coefficient is that this one isn't a robust estimator of the correlation. Instead the financial risk management aim to analysis the joint behaviour of the assets, investigating the propensity of the assets to move together. In these conditions, the linear correlation coefficient can provides misspecificated results according with most of the empirical studies' conclusions that testified the non-normal distributions of the assets. Therefore have been developed other measures of correlation to avoid the potentially misalignments. These new approaches, named *concordance measure*, are based on the idea that „large" values from one series corresponds to those „large" from other series and also the principle is validating for the „lower" ones. Given two independent random variables X and Y with the realizations (X_1, Y_1) and (X_2, Y_2), we say that the two pairs of realizations are concordance if $(X_1 - X_2) * (Y_1 - Y_2) > 0$, elsewhere these ones are disconcordance.

The *concordance measures* fulfils the following properties:

i) $\rho(X, Y) \in [-1, 1]$;

ii) $\rho(X, Y) = \rho(Y, X)$;

iii) If X and Y are independent, then $\rho(X, Y) = \rho(Y, X) = 0$;

iv) If we denote f and g as two linear or non-linear increasing functions, then: $\rho(f(X), g(Y)) = \rho(X, Y)$.

Kendall's Tau:

$$(15)\ \rho_\tau^{Kendall}(X, Y) = \frac{2}{n(n-1)} \sum_{i<j} sign[(X_i - X_j)(Y_i - Y_j)]$$

Spearman's Rho:

[5] See Malevergne and Sornette (2006) for more details.

$$(16)\ \rho_{\rho}^{Spearman}(X,Y) = \frac{12}{n(n^2-1)}\sum_{i=1}^{n}\left(rank(X_i) - \frac{n+1}{2}\right)\left(rank(Y_i) - \frac{n+1}{2}\right)$$

3.2.3 Dependence Metric

Granger *et al.* (2004) defined the *Dependence metrics* as any dependence measure that fulfils the following properties:

i) Given a measure $V[X,Y]$ of the dependence between two random variables, V is defined for both continuous and discrete random variables;

ii) V represents a distance;

iii) V is invariant under the continuous changes of the realizations of X and Y;

iv) V is ranging between 0 and 1;

v) If X and Y are independent, then V equals 0;

vi) If the relationship between X and Y can be defined as a measurable mapping: $X = f(Y)$, then V equals 1.

The concept of dependence metric has the role to test the complicated serial correlations, being used in the forecasting of financial time series and to analyze the *goodness-of-fit* in financial modelling. Therefore we will define two of the most important dependence metrics.

Bhattacharaya, Matusita and Hellinger measure:

$$(17) S = \frac{1}{2}\int_{R^2}\left[1 - \left(\frac{f(x)g(y)}{h(x,y)}\right)^{\frac{1}{2}}\right]^2 dH(x,y),$$

where f and g are the marginal densities of X and Y, h is the joint density function and H represents the distribution function of the two random variables. More in detail, the dependence metric defined above relates the entropy between the bivariate density function h and product of marginal densities f and g. The relation (17) measures ¼ of the symmetric relative entropy between h and $f * g$ for the ½ -class entropy.

Kullback-Leibler distance:

Given the ***k***-class entropy family named as well as Tsallis entropy, we define the *Kullback-Leibler distance* as:

$$(18) H_k(f) = \frac{1}{k-1}(1 - \mathrm{E}[f^{k-1}]), k \neq 1$$

$$= -\mathrm{E}[\ln f], k = 1,$$

where f is the density function of a random variable or of a vector. Also the *Kullback-Leibler distance* is commonly used in *Bayesian econometrics* to measure the relative entropy in the moving process from prior distribution to posterior distribution.

3.2.4 Tail Dependence

The concept of *Tail dependence* tests the probability of two random variables to posts extreme movements in the same timeframe. Thus the upper tail dependence can be defined as:

$$(19) \gamma_U = \lim_{u \to 1} \Pr[X > F_X^{-1}(u) | Y > F_Y^{-1}(u)] ,$$

This formula measures the probability to records a large value of X, given that Y is itself large, at level of probability u.

The coefficient of lower tail dependence admits the following representation:

$$(20) \gamma_L = \lim_{u \to 0^+} \Pr[X < F_X^{-1}(u) | Y < F_Y^{-1}(u)].$$

If $\gamma = 0$, then random variables X and Y are *asymptotically independent*, otherwise when $\gamma > 0$ the large events post common movements. In other words two random variables are independents if it is fulfilling the following conditions:

$$(21) \frac{F(x,y)}{F_x(x)F_y(y)} = 1$$

where the numerator denotes the repartitions function of the joint distribution, while the $F_x(x)$ and $F_y(y)$ represents the margins distribution of X and Y.

In real world, two random variables X and Y can denotes volatilities of two stocks, while the coefficient γ represents the probability that both stocks post simultaneously extreme volatilities. For example we can consider the *Value-at-Risk* (*VaR)* model because it constitutes the subject of this paper. Therefore considering a VaR, we define the two random variables $\mathbf{X}$ and $\mathbf{Y}$ as assets or portfolios, u is the confidence level, while $F_X^{-1}(u)$ and $F_Y^{-1}(u)$ represents the two quantiles. Given the same level of confidence u, the probability that $\mathbf{X}$ and $\mathbf{Y}$ exhibit tail dependence and exceed their VaR[6] equals:

$$(22)\gamma_U * (1-u), \text{ where } u \to 1.$$

3.2.5 Quadrant and Orthant Dependence

The main disadvantage of the concordance measures is the fact that they are difficult to estimate for portfolios with financial assets. Other limit of the generalization of concordance measures for more than two random variables is related to the "*frustration*" phenomenon, which was introduced in statistical physics. The concept of "*frustration*" says that constraints tend to determine opposite and divergent states in two interacting variables that can't be incorporated in systems of three or more random variables. More exactly the "*frustration*" phenomenon generally leads to existence of multiple equilibria. A way to remove this limit is the *quadrant dependence approach* (*PQD*). Therefore we say two random variables are positive quadrant dependent if admit the following representation:

$$(23)\Pr[X \le x, Y \le y] \ge \Pr[X \le x] * \Pr[Y \le y], \forall x, y.$$

The relation (23) states that probability of the two random variables to be in the same timeframe small is at least equal with probability of the case when these ones are independent. A very important property is that the PQD random variables post a positive correlation coefficient. In real world an example of the PQD variables are the assets preferred by risk-averse investors, having a concave utility function.

But to generalize the concept of PQD for cases with more than two random variables we will define the concept of *positive orthant dependence* (*POD*). Thus for *N* random variables $X_1, X_2, \dots, X_N$ we define the *positive lower orthant dependence* (*PLOD*) and the *positive upper orthant dependence* (*PUOD*) as:

$$(24)\ \Pr[X_1 \le x_1, \dots, X_N \le x_N] \ge \Pr[X_1 \le x_1] * \dots * \Pr[X_N \le x_N]$$

[6] See Malevergne and Sornette (2006) for more details.

$$(25)\ \Pr[X_1 > x_1, \dots, X_N > x_N] \geq \Pr[X_1 > x_1] * \dots * \Pr[X_N > x_N].$$

Therefore if *N* random variables $X_1, X_2, \dots, X_N$ are PLOD or PUOD, then it results that they are POD. The POD random variables have the same interpretation as the PQD ones, only that the principle is generalized for the multivariate case. In financial world, the concept of POD is commonly used to analyze and to adopt some strategies of trading that are market neutral. More exactly the portfolio managers are using the concept of POD to reduce and to remove the impact of market movements on portfolios evolution, therefore aiming to minimize the propagation of negative effects produced by some potential extreme events.

3.2.6 Conditional Correlation Coefficient

The concept of *conditional correlation coefficient* represents a very useful tool for researchers that study the contagion phenomenon among different markets or economies and to detect the propagation of the systemic risks. Also the notion of conditional correlation coefficient offers a better overview of the correlation between two portfolio's assets when the volatility posts different movements of different magnitudes.

Given two random variables X and Y, their conditional correlation coefficient admits the following representation:

$$(26)\ \rho_F = \frac{\mathrm{Cov}(X, Y | X \epsilon F)}{\sqrt{\mathrm{Var}(X | X \epsilon F)\mathrm{Var}(Y | X \epsilon F)}},$$

where X and Y are conditioned upon $X \epsilon F$, while *F* represents a subset of **R** that fulfils the condition $\Pr\{X \epsilon F\} > 0$. Thus the above relation permits to analyze the impact on the underlying model and of the conditioning set of information on the evolution of ρ_F . It has to be mentioned that relation (26) represents a standard form of the conditional correlation coefficient that can be generalized for different types of distribution or model.

For example if we consider that X and Y have a multivariate *Gaussian* distribution with ρ as unconditional correlation coefficient, then the conditional correlation coefficient has the form:

$$(27)\rho_F = \frac{\rho}{\sqrt{\rho^2 + (1-\rho^2)\frac{\mathrm{Var}(X)}{\mathrm{Var}(X|X\epsilon F)}}}.$$

Relation (27) implies that $\mathrm{Var}(Y)$ has not a direct influence on ρ_F, mentioning also that ρ_F can be either greater or smaller than ρ because $\mathrm{Var}(X|X\epsilon F)$ can be either greater or smaller than $\mathrm{Var}(X)$. Also a very important remark states that ρ_F can changes without ρ to change or at least ρ don't posts the same manner of the change.

In the case when X and Y are conditioned upon $X\epsilon F$ and $Y\epsilon G$, where F and G are subsets of **R** that fulfils the condition $\Pr\{X\epsilon F, Y\epsilon G\} > 0$, then we define the conditional correlation coefficient as:

$$(28)\ \rho_{F,G} = \frac{\mathrm{Cov}(X,Y|X\epsilon F, Y\epsilon G)}{\sqrt{\mathrm{Var}(X|X\epsilon F, Y\epsilon G)\mathrm{Var}(Y|X\epsilon F, Y\epsilon G)}}.$$

The conditional correlation coefficient on both variables presents a higher grade of difficulty regarding the transformation into closed formula for several types of models or distributions as compared with the conditional correlation coefficient on a single variable. Furthermore $\rho_{F,G}$ does not add any special improvement versus the correlation coefficient on a single variable.

Many researchers have studied the efficiency of the conditional correlation coefficient to detect the contagion phenomenon in the case of emerging markets from Latin America during the 1994 Mexican crises[7]. In accordance with the conclusions of Calvo, Garcia, Lizondo, Reinhart or Rose regarding the contagion occurred in Latin America economies, the conditional correlation coefficient didn't yield very clear information about the contagion effects. In addition the conditional correlation coefficient provided artificial changes, while the unconditional correlation coefficient remained constant. Therefore the conditional correlation coefficient does not constitute the a very useful tool to study the behaviour of extreme events.

[7] See Meerschaert and Scheffler *et al.* (2001)

3.2.7 Conditional Concordance Measures

The main idea behind the *conditional concordance measures* is to condition the random variables on values that are larger than a given threshold and also let this threshold to converge to infinity.

Noting $A = F_X(X)$ and $B = F_Y(Y)$, it results that the *Spearman's rho* $\rho_\rho^{Spearman}$ is the linear correlation coefficient of the uniform variables A and B that in fact represents nothing but the *correlation coefficient of the rank*:

$$(29)\rho_\rho^{Spearman} = \frac{\mathrm{Cov}(A,B)}{\sqrt{\mathrm{Var}(A)\mathrm{Var}(B)}},$$

A very important advantage of using the correlation coefficient of the rank is that this one analyzes only the dependence structure of the random variables, as compared with the linear correlation coefficient that aggregates in addition the marginal distributions of the studied variables. Given a threshold, we define the *conditional correlation coefficient of the rank* as:

$$(30)\ \rho_\rho^{Spearman}(\varepsilon) = \frac{\mathrm{Cov}(A,B|A \geq \varepsilon)}{\sqrt{\mathrm{Var}(A|A \geq \varepsilon)\mathrm{Var}(B|A \geq \varepsilon)}},$$

where the random variables A and B are conditioned on A, which is larger than ε.

Unlike the conditional correlation coefficient, the transformation of conditional rank correlation coefficient into closed formula for Gaussian or Student distribution presents a greater grade of difficulty. Instead the researches made by Meerschaert and Scheffler *et al.* (2001) and Edwards and Susmel *et al.*(2001) in analyizing the contagion across the Latin America markets during the 1994 Mexican crises concluded that the conditional Spearman's rho provides a higher accuracy than the conditional correlation coefficient.

3.2.8 Lagged time-varying dependence

A very disputed topic in the fields as economics, econometrics or finance is so well-known concept of *causality* between two time series $X(t)$ and $Y(t)$. The concept of causality used in the mentioned domains doesn't represents a causality in a strictly sense. Therefore the concept of causality used in economics, econometrics or finance aims to analyze which economic variable might influence and determines other economic process. Causality is widespread used to study the interactions between GDP and inflation, unemployment and inflation, interest rate and exchange rate, bond yields and stock prices, a.s.o. Thus a naive measure of causality is the *lagged time-varying dependence*:

$$(3.1)\ \rho_{X,Y(\vartheta)} = \frac{\mathrm{Cov}[(X(t)Y(t+\vartheta)]}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}},$$

where ϑ represents the time lag.

Given a positive ϑ, the lagged cross-correlation coefficient $\rho_{X,Y(\vartheta)}$ states that the knowledge of X at t provides information on the future evolution of Y at a later moment $t+\vartheta$. But it have to be noted that the lagged cross-correlation coefficient doesn't imply unguarded the existence of the causality between the two time series. This phenomenon is owed to the fact that correlation between two time series is provided by a common source of influence. The main deficiency of the lagged cross-correlation coefficient $\rho_{X,Y(\vartheta)}$is that this one represent a linear measure of dependence and could omits important properties of the non-linear dependence.

The most used approach to test the causality is the so-called *Granger causality* that states between two time series there exist causality if the knowledge of $X(t)$ and of its past values improves the forecasting of $Y(t+\vartheta)$, for a positive ϑ. Also it have to be mentioned that the Granger causality is just consistent related with the real causality, being in accordance with Hume's principle that the effect has to succeed the cause over time

3.3 Copula models

In probability field, a joint distribution can be decomposed in a dependence structure that represent a copula and into marginal distributions related to the number of random variables. So the copulas describe the dependence between two or more random variables, with different marginal distributions. The main advantage of using copulas is that this procedure allows the modelling of both parametric and non-parametric marginal distributions into a joint risk distribution. Also the dependence structure of these joint risk distributions created by copula models are characterized more in detail as compared with a simple correlation matrix.

Mathematically speaking, in order to notations used by Nelsen (1999), the notion of copula can be described as following:

Definition. A function $\boldsymbol{C}$ **:** $[0,1]^n \rightarrow [0,1]$ is a copula with *n* dimensions only if it follows the properties:

i) $\forall u \in [0,1]$, $C\ (1, \dots, 1, u, 1, \dots, 1) = u$;

ii) $\forall u_i \in [0,1]$, $C\ (u_1, \dots, u_n) = u$ if at least one of the u_i's equals zero;

iii) *C* is *n*-increasing and grounded, therefore the *C*- volume of every box is positive only if its vertices are ranging in $[0,1]^n$.

Also there have to be mentioned that if a function fulfils the property *i*) then respective function is grounded. The name of "copula" attributed to the function ***C*** results from the following theorem.

Sklar's Theorem (1959). *If F is a n-dimensional joint distribution function with the continuous marginal distributions* $F_1, \dots, F_n$*, then there exist a unique n-copula C* $[0,1]^n \rightarrow [0,1]$, *such that*:

(32) $F(x_1, \dots, x_n) = C\big(F_1(x_1), \dots, F_n(x_n)\big)$,

for every $x_1, \dots, x_n \in \bar{R}$. A very important remark about Sklar's theorem is that *C* is unique only if the $F_1, \dots, F_n$ are continuous. In conclusion, the theorem mentioned above shows that any joint distribution can be dimensioned in a copula and into marginal distribution functions. In 1996, Sklar defined copula like "a function that links a multidimensional distribution to its one dimensional margins".

Inversely, if there are known the density functions for the *n*-dimensional joint distribution and marginal distributions, then the copula is given by the following formula:

$$(33)\ C(u_1, \dots, u_n) = F\big(F^{-1}{}_1(u_1), \dots, F^{-1}{}_n(x_n)\big),$$

as Nelsen (1999) mentioned that above relation hold only if the $F_1, \dots, F_n$ are continuous. Also Nelsen (1999) shown that for a bidimensional distribution function, the two margins F_1 and F_2 are given by $F_1(x_1) = F(x_1, +\infty)$, respectively $F_2(x_2) = F(+\infty, x_2)$.

Other powerful property registered by all the copulas is referring to theirs invariance:

Invariance Theorem. *Let define n continuous random variables* $Y_1, \ldots, Y_n$ *that have a C copula. So, if* $g_1(Y_1), \ldots, g_n(Y_n)$ *are increasing functions on the range of* $Y_1, \ldots, Y_n$ *, then the random variables* $X_1 = g_1(Y_1)$ *, … ,* $X_n = g_n(Y_n)$ *have also the same copula C.*

More exactly, the above theorem underlines one of the most important advantages of the modelling using copulas, namely that the dependence structure is insensitive to the monotonically changes of random variables.

In accordance with the Lipschitz's condition of continuity on $[0,1] \times [0,1]$, we will define the following property of copulas:

Theorem. *Let consider an n-copula C. Then for all* $u_1, \ldots, u_n \in [0,1]$ *and all* $v_1, \ldots, v_n \in [0,1]$*:*

$$(34)\ |C(v_1, \ldots, v_n) - C(u_1, \ldots, u_n)| \leq |v_1 - u_1| + \ldots + |v_n - u_n|.$$

The above relation is given by the property that copulas are *n*-increasing. Roughly speaking, the theorem states that every copula ***C*** is uniformly continuous on its domain.

Other important property of these dependence structures refers to the partial derivatives of a copula with respect to its variables:

Theorem. *Given a n-dimensional copula C, for every* $\in [0,1]$*, the partial derivative* $\partial C / \partial v$ *exists for every* $v \in [0,1]$*, such that:*

$$(35)\ 0 \leq \frac{\partial C}{\partial v}(u, v) \leq 1\ .$$

Also it have to been mentioned that the analogous is true for $/\partial v$. Additionally the functions $u \rightarrow C_v(u) = \partial C(u, v) / \partial v$, respectively $v \rightarrow C_u(v) = \partial C(u, v) / \partial u$ are defined and non-decreasing almost everywhere on [0,1].

3.3.1 Examples of Copula Families

Furthermore we will present a few examples of copula families.

Product Copula

Definition. *Let denote* H_1 *and* H_2 *as two random variables. These ones are independent if and only if the product of their distribution functions* F_1 *and* F_2 *equals their joint distribution* *F***:**

$$(36)\ F(r_1, r_2) = F_1(r_1) \times F_2(r_2), \text{ for all } r_1, r_2 \in \bar{R}.$$

Theorem. Given *two random variables* H_1 *and* H_2 *with continuous distribution functions* F_1 *and* F_2 *and joint distribution F, then* H_1 *and* H_2 *are independent if and only if* $C_{H_1 H_2} = \prod$.

Therefore it will result the independence copula ***C*** = $\prod$ from **:**

$$(37)\ \prod(u_1, \dots, u_2) = \prod_{i=1}^{n} u_i.$$

Also the relation (36) becomes obvious from the Sklar's theorem that states as there exists a unique copula *C* **:**

$$(38) P(H_1 \leq r_1, H_2 \leq r_2) = F(r_1, r_2) = C(F_1(r_1), F_2(r_2)).$$

Elliptical[8] Copulas

The most important examples of elliptical copulas are the Gaussian and Student copulas. In fact, from technical viewpoint, these two copulas are very close to each other. Furthermore the two copulas become closer and closer in their tail only when the number of freedom degrees of Student copula increases.

Gaussian (Normal) Copula

According to the notations used by Yannick Malevergne and Didier Sornette (2005), a *Gaussian n*-copula ***C*** can be defined as following:

$$(39) C_{\rho,n}^{Gauss}(u_1, \dots, u_n) = \varphi_{\rho,n}(\varphi^{-1}(u_1), \dots, \varphi^{-1}(u_n)),$$

[8] The name come from the fact that for each iso-density locus represents an ellipse.

where φ denotes the standard *Gaussian* distribution, $\varphi_{\rho,n}$ is the *n*-dimensional Normal distribution with correlation matrix ρ. The *Gaussian* copulas are derivate from the multivariate Gaussian distributions.

So the density function of the Normal copula is given by:

$$(40)\, c_{\rho,n}^{Gauss}(u_1, \dots, u_n) = \frac{\partial C_{\rho,n}(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n}.$$

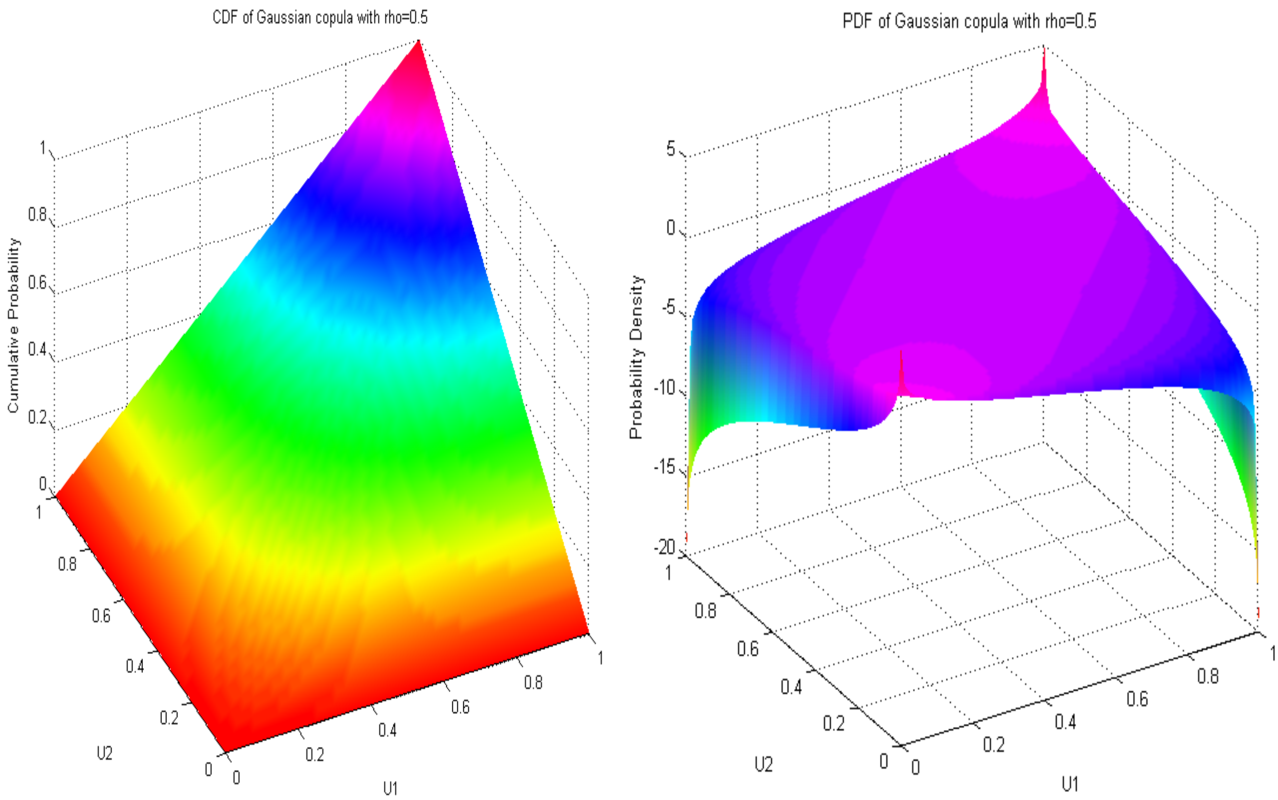

Figure 1. CDF and PDF of Gaussian copula

Noticing with $y^t(u) = (\varphi^{-1}(u_1), \dots, \varphi^{-1}(u_n))$, then it will result:

$$(41)\, c_{\rho,n}^{Gauss}(u_1, \dots, u_n) = \frac{1}{\sqrt{\det \rho}} exp(-\frac{1}{2} y^t(u)(\rho^{-1} - I)y(t).$$

Student *t*-Copula

Also the Student ***t***-copulas are derived from the Student multivariate distributions. Likewise the *Gaussian* copulas, ***t***-copulas are found in the form of *meta-elliptical* distributions, providing a generalization of the multivariate distributions. More exactly, the *meta-elliptical* distributions have the same dependence structure like ***n***-dimensional distributions, but differ in their marginal distributions.

Let denote $T_{n,\rho,v}$ as a multivariate Student distribution with v degrees of freedom and correlation matrix :

$$(42) T_{n,\rho,v}(x) = \frac{1}{\sqrt{\det\rho}} \frac{\vartheta\left(\frac{v+n}{2}\right)}{\vartheta\left(\frac{v}{2}\right)(\pi v)^{n/2}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{dx}{\left(1+\frac{x^t \rho^{-1} x}{v}\right)^{\frac{v+n}{2}}}$$

then the Student copula is:

$$(43) C_{n,\rho,v}^{Student}(u_1, \dots, u_n) = T_{n,\rho,v}(T_v^{-1}(u_1), \dots, T_v^{-1}(u_n)),$$

where T_v is the univariate ***t*** distribution with v degrees of freedom.

Therefore the density function of the ***t***-copula is defined as following:

$$(44) c_{n,\rho,v}^{Student}(u_1, \dots, u_n) = \frac{1}{\sqrt{\det\rho}} \frac{\vartheta\left(\frac{v+n}{2}\right)\left[\vartheta\left(\frac{v}{2}\right)\right]^{n-1}}{\left[\vartheta\left(\frac{v+1}{2}\right)\right]^{n}} \frac{\prod_{k=1}^{n}\left(1+\frac{y_k^2}{v}\right)^{\frac{v+1}{2}}}{\left(1+\frac{y\rho^{-1}y}{v}\right)^{\frac{v+n}{2}}}$$

where $y^t(u) = (T_v^{-1}(u_1), \dots, T_v^{-1}(u_n))$.

The Student copulas are characterized by two parameters: the shape matrix ρ, which also appears in the Normal copulas, and the number of freedom's degrees v that supposes a high level of accuracy for its value's estimation. Thus the ***t***-copula presents higher degree of difficulty to use and to calibrate than the *Gaussian* copula.

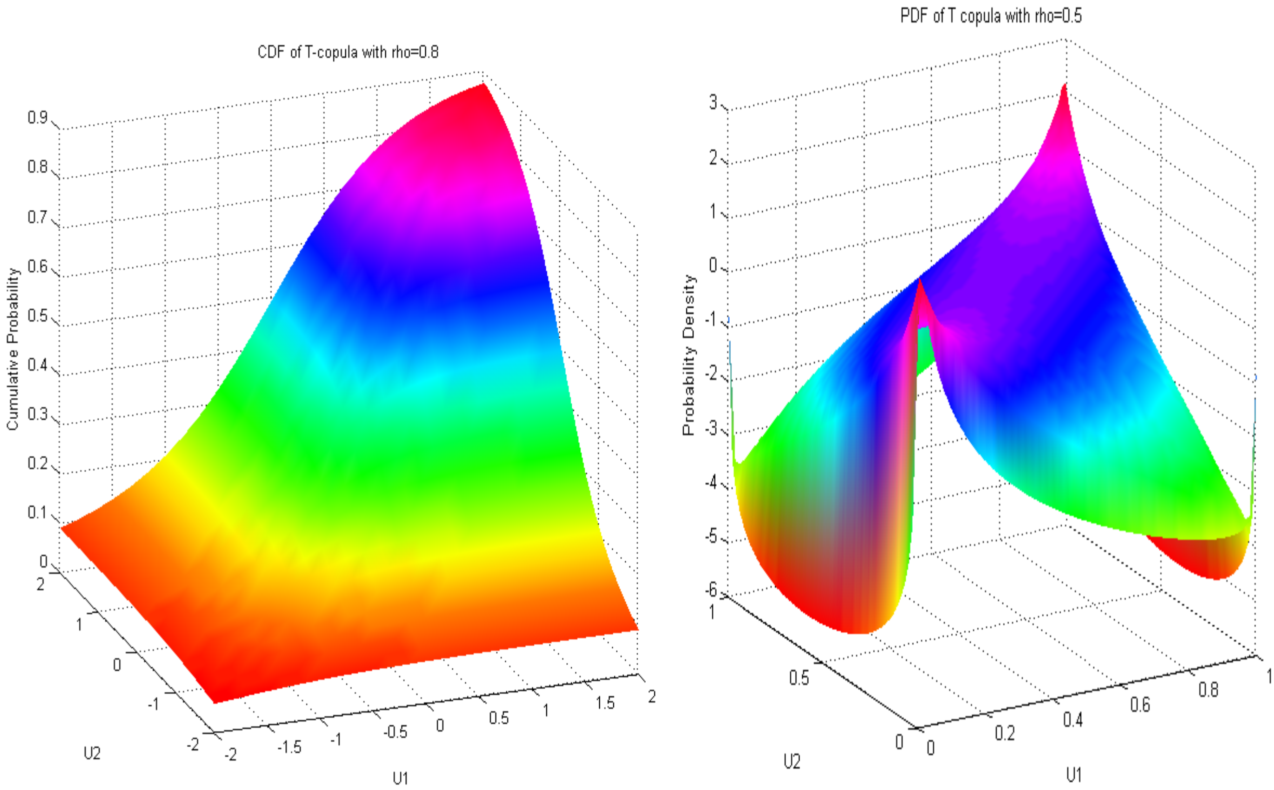

Figure 2.. CDF and PDF of Student copula

From the principle of *Large Numbers' Law* it results that when the number of freedom's degrees v incline to infinity, then the Student copula tends to Normal copula:

$$(45)\, v \to +\infty, \sup_{u \in [0,1]^n} \left| C_{n,\rho,v}(u) - C_{n,\rho}(u) \right| \to 0.$$

<u>Archimedean Copulas</u>

Unlike the *meta-elliptical* copulas, Archimedean copulas are not derived from the multivariate distributions through the use of Sklar's theorem. In addition the Archimedean copulas can be defined as the closed-form solutions. A copula belongs to Archimedean family if it fulfils the properties:

Definition. *Given* φ *as a continuous function from* $[0,1]$ *onto* $[0,\infty]$, *strictly decreasing and convex, such that* $\varphi(1) = 0$ *and* $\varphi^{[-1]}$*is a pseudo-inverse of* φ*:*

$$(46)\varphi^{[-1]}(t) = \begin{cases} \varphi^{[-1]}(t), if\ 0 \le t \le \varphi(0) \\ 0, \quad if\ t \ge \varphi(0) \end{cases},$$

then the function

$$(47)\ C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$$

is an Archimedean copula with generator φ *.*

A strict condition for ***C*** to be an Archimedean copula is that :

$(-1)^k \frac{d^k \varphi^{[-1]}(t)}{dt^k} \ge 0$, as $\forall\ k = 0,1,\dots,n$, or more exactly if $\varphi^{[-1]}$ is monotonic.

Thus we can generalize relation (15) for ***n***-Archimedean copulas:

$$(48) C_n(u_1,\dots,u_2) = \varphi^{[-1]}\big(\varphi(u_1) + \ \dots + \varphi(u_n)\big).$$

The main idea behind Archimedean copulas is that the dependence structure among *n* variables is represented by a function of a single variable, which is the generator φ.

From the large Archimedean family of copulas, we will mention the most known of these ones:

Clayton Copula

Joe Clayton (1978) has used for the first time the concept of copula in the joint-life models, studying the bivariate life tables of sons and fathers. Others important contributions to the Clayton's models were developed by Cook and Johnson (1981) and Oakes (1982). A Clayton copula can be defined as following:

$$(49) C_\theta^{Clayton}(u,v) = \max\left(\left[u^{-\theta} + v^{-\theta} - 1\right]^{-\frac{1}{\theta}}, 0\right), \theta \in [-1,\infty),$$

having the role of a limit copula, with the generator $\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1)$, whose *Laplace* transformation is a *Gamma* distribution.

Thus the density function of the Clayton copula is:

$$(50) c_{\theta}^{Clayton}(u,v) = (1+\theta)[uv]^{-\theta-1}(u^{-\theta} + v^{-\theta} - 1)^{-2-\frac{1}{\theta}}.$$

Gumbel-Hougaard Copula

This copula developed independently by Gumbel (1960) and Hougaard (1986) admits the following representation:

$$(51) C_{\theta}^{Gumbel\&Hougaard}(u,v) = \exp(-[(-\ln u)^{\theta} + (-\ln v)^{\theta}]^{\frac{1}{\theta}}), \theta \in [1,\infty),$$

with the generator $\varphi(t) = (-\ln t)^{\theta}$.

Using this kind of copula, Hougaard (1992) studied the join-survival of twins born in Denmark between 1881 and 1930.

Frank Copula

In 1979, Frank introduced the following copula:

$$(52) C_{\theta}^{Frank}(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right), \theta \in R,$$

having the generator $\varphi(t) = -\ln\frac{(e^{-\theta t} - 1)}{e^{-\theta} - 1}$.

This type of copula is very suitable for empirical applications, due to its desirable properties. In 1995, Frees used Frank's copula to investigate mortality of annuitants in joint- and last-survivor annuity contracts. Also using Frank's copula, Shih and Louis (1995) studied the joint-survival of a series of patients infected with HIV.

Extreme Value Copula

An Extreme Value Copula can be defined as following:

$$(53) C_{\theta}^{Extreme\ Value}(u_1, \dots, u_n) = \exp\left[-V\left(-\frac{1}{\ln u_1}, \dots, -\frac{1}{\ln u_n}\right)\right]$$

and $(54)\, V(x_1, \dots, x_n) = \int_{\pi_n} \max_i \left(\frac{w_i}{x_i}\right) dH(w),$

where H is any positive finite measure such that $\int_{\pi} w_i \, dH(w) = 1$ and

$\pi_n = \{w \in R_+^n : \sum_{i=1}^{n} w_i = 1\}$.

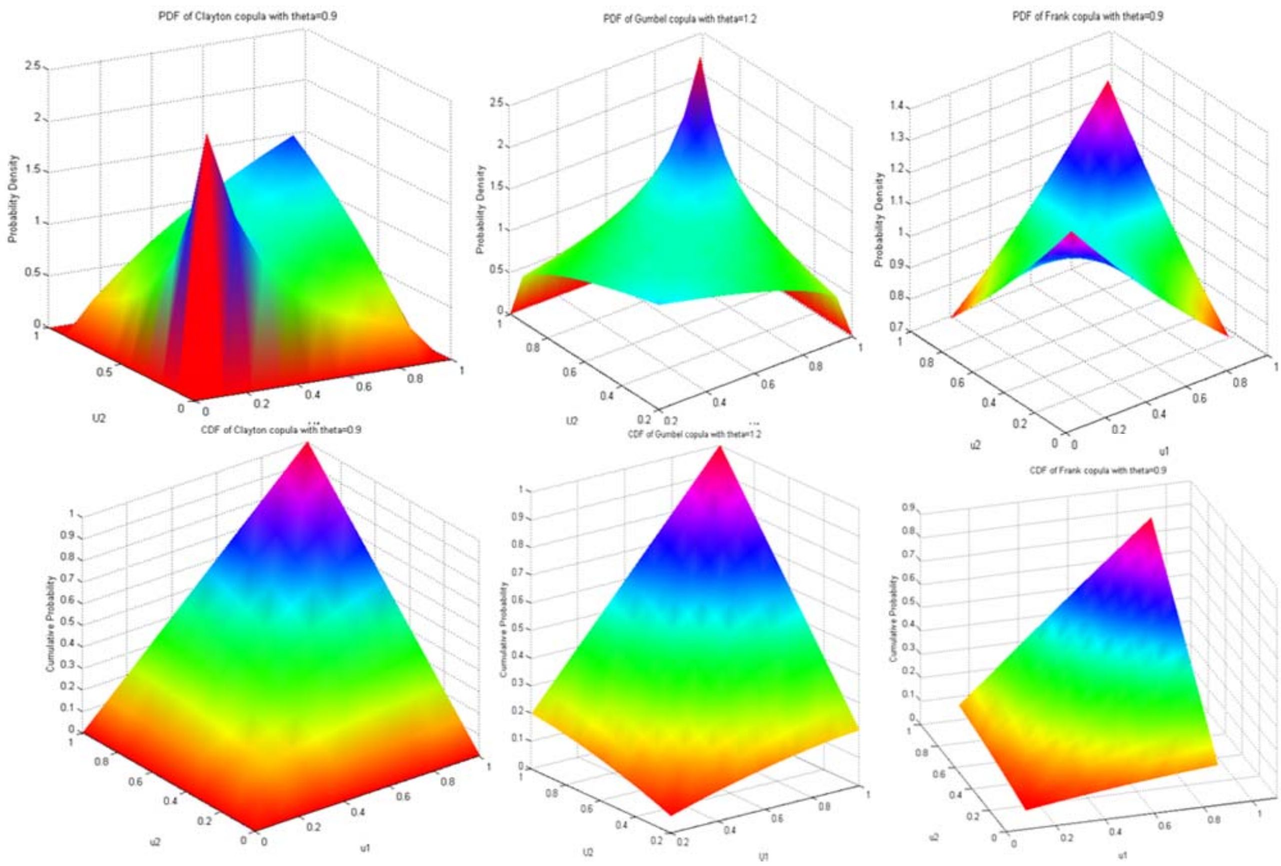

Figure 3. PDF and CDF of Archimedean copulas

Plackett Copula

Plackett copula which was introduced in 1965 after the name of the English statistician Robert Plackett, it is a very useful toll in many application in finance that analyzes the bivariate dependence. Like Student and Gaussian, Plackett copula presents completely symmetry in tail dependence. An important remark is that Plackett copula which is one parameter copula doesn't belong to parametric family, but in applications it is usually nested with Elliptical copulas due to the absence of asymmetry. The Plackett copula is defined as following:

$$(55)\ C_{\theta}^{Plackett} = \begin{cases} \frac{1}{2(\theta-1)}\left[1+(\theta-1)(u+v)-\sqrt{[1+(\theta-1)*(u+v)]^2-4uv\theta(\theta-1)}\right] & if\ \theta \neq 1 \\ uv & if\ \theta = 1 \end{cases}$$

Starting from 'BB7' copula of Joe (1997) or Joe-Clayton as it is also known in literature, Patton (2004) introduced the Symetrised Joe-Clayton(SJC) copula:

$$(56)\ \ C^{SJC}(u,v|\tau^U,\tau^L) = 0.5*(C^{SJC}(u,v|\tau^U,\tau^L) + C^{SJC}(1-u,1-v|\tau^U,\tau^L) + u + v - 1).$$

Unlike originally 'BB7', the Symetrised Joe-Clayton copula may take into account for completely presence or absence of asymmetry in the tail dependence. In fact the SJC copula represents a special case of the Joe-Clayton when $\tau^U = \tau^L$. Empirical facts indicate SJC copula as a more interesting choice to model the dependence in economic and financial processes.

Fréchet-Hoeffding Upper- and –Lower Bounds

In the case of a copula ***C*** with ***n*** dimensions, giving all $u_{,}, \dots, u_n \in [0,1]$, then:

$$(57) \max(u_1 + \dots + u_n - n + 1, 0) \leq C(u_1, \dots, u_n) \leq \min(u_{,}, \dots, u_n).$$

The properties of *Fréchet-Hoeffding bounds* are very important for the study copula science, because the lower bound is an Archimedean copula, while the upper bound apart to the family of Extreme Value copulas. Furthermore the upper bound has the special property that is the strongest form of dependence met at the random variables. In addition, the Fréchet-Hoeffding upper bound represents itself an ***n***-dimensional copula, while the lower bound is a copula only in the bivariate case.

3.3.2 Copula-Garch Model

A major criticism of the copula models in the favour of multivariate GARCH model was That former suppose a static measure of dependence. Even though the separately modelling of the marginal distribution and dependence structure provides a higher degree of robustness over time of the copula parameters, the empirical findings proving that the high frequency data records a continuously switching of the regimes. Thus in 2001, Patton took the first initiative to extend the copula function to conditional case, in order to account the impact of the past information on the state of copula parameters. He introduced for the first time the concept of time varying dependence which does nothing to incorporate the heteroschedasticity in dynamic copula modelling. So to extend the Sklar's theorem to conditional cumulative distribution functions, Patton has defined the following conditional σ-algebra:

$$(58)\sigma - \text{algebra} = \sigma\{y_{1t-1}, y_{2t-1}, \ldots, y_{nt-1}, y_{1t-2}, y_{2t-2}, y_{nt-2}, \ldots\}$$

for $t = 1, \ldots, T$. In fact the above equation tell us that σ-algebra is generated by all the past information up to time t. Therefore the Sklar's theorem cam be expressed as:

$$(59)F(y_{1t}, \ldots, y_{2t}|\sigma - \text{algebra}) = C_t(F_{1t}(y_{1t}|\sigma_t - \text{algebra}), \ldots, F_{nt}(y_{nt}|\sigma_t - \text{algebra})|\sigma_t - \text{algebra})$$

More exactly the main idea behind the equation (58) and (59) is that in modelling of the marginal distributions, the conditional mean follows an autoregressive process, while the conditional variance is modelled as a GARCH(1,1) process.

Further I will define the time-varying equations for Gumbel and SJC copulas which I will use later to model the dependence between exchange rates over the analyzed period. A general form of the conditional dependence can be expressed as:

$$(60)\rho_t = \Lambda\left(\omega + \beta\rho_{t-1} + \alpha\frac{1}{m}\sum_{j=1}^{m}\emptyset^{-1}(u_{t-j})\emptyset^{-1}(v_{t-j})\right),$$

where $\Lambda \equiv \frac{1-e^{-x}}{1+e^{-x}}$ is the modified logistic transformation that holds the dependence parameter ρ_t in the interval (-1,1). The right hand of above equation contains an autoregressive term $\beta\rho_{t-1}$, a forcing variabile and m denotes the window length. Equation (60) was designed for modelling dynamic Elliptical copulas.

For non-Elliptical copulas Patton proposed the following general form to model the evolution of the dependence parameter:

$$(61)\theta_t = \Lambda\left(\omega + \beta\theta_{t-1} + \alpha\frac{1}{m}\sum_{j=1}^{m}\left|u_{t-j} - v_{t-j}\right|\right),$$

where Λ is an appropriate transformation function designed to keep the dependence parameter in its domain. This transformation function can take different forms as: $\frac{1}{1+e^{-x}}$ for tail dependence, e^{x} for Clayton copula or $e^{x}+1$. For SJC copula Patton proposed the following dynamic equations:

$$(62)\tau_t^{U} = \Lambda\left(\omega_U + \beta_U\tau_{t-1}^{U} + \alpha_U\frac{1}{m}\sum_{j=1}^{m}\left|u_{t-j} - v_{t-j}\right|\right)$$

$$(62)\tau_t^{L} = \Lambda\left(\omega_L + \beta_L\tau_{t-1}^{L} + \alpha_L\frac{1}{m}\sum_{j=1}^{m}\left|u_{t-j} - v_{t-j}\right|\right),$$

where τ_t^{U} and τ_t^{L} represents the upper, respectively lower tail dependence and $\left|u_{t-j} - v_{t-j}\right|$ denotes the mean absolute difference over the past observations. Thus the window length can be seen as a switching parameter of the forcing variable. A very important remark is that the Patton's model for conditional dependence supposes the time-varying of parameters according to defined dynamic equation, while the functional form of copula remains constant over horizon. Instead Rodriguez (2003) proposed a Markov switching regime for the functional form of copula.

3.3.3. Estimation of Copula's parameters

In the fields of economics, finance or actuarial risks it exists a lot of aproaches used to estimate the parameters of copulas. Broadly speaking we can devide such techniques of copulas' estimations in three main categories: nonparametric, semi-parametric and parametric.

3.3.3.1 Nonparametric estimation

Empirical Copula

Paul Deheuvels *et al.*(1979,1981) elaborated the first approach for the estimation of copula's parameters, which is based on the generalization of the multivariate distribution's estimator.

Thus for a random vector Y with n-dimensions $Y = (Y_1, \dots, Y_n)$ and for a sample size T, $\{(y_1(1), y_2(1), \dots, y_n(1)), \dots, (y_1(T), y_2(T), \dots, y_n(T))\}$, then there have to estimate the empirical density function F of Y:

$$(63)\ \hat{F}(y) = \frac{1}{T}\sum_{k=1}^{T} 1_{\{y_1(k) \le y_1, \dots, y_n(k) \le y_n\}}$$

and the empirical marginal distributions of Y_i's:

$$(64)\ \hat{F}_i(y_i) = \frac{1}{T}\sum_{k=1}^{T} 1_{\{y_i(k) \le y_i\}}.$$

Given a copula C, applying the *Sklar's theorem* we will obtain a unique nonparametric estimator, which is defined at discrete points $\left(\frac{t_1}{T}, \dots, \frac{t_n}{T}\right)$, having $t_k \in \{1, 2, \dots, T\}$. Operating the inverse of the marginal distribution function, it results the so-called *empirical copula*:

$$(65)\ \hat{C}\left(\frac{t_1}{T}, \dots, \frac{t_n}{T}\right) = \frac{1}{T}\sum_{k=1}^{T} 1_{\{y_1(k) \le y_1(t_1;T), \dots, y_n(k) \le y_n(t_n;T)\}},$$

where $y_g(k;T)$ represents the k^{th} order statistics of the sample $\{y_g(1), \dots, y_g(T)\}$.

There have to be noted that almost surely the empirical distribution function $\hat{F}$ converges to the underlying distribution function F as $T \to \infty$, also resulting that this property holds for the nonparametric estimator:

$$(66)\ \sup_{u \in [0,1]^n} \left|\hat{C}(u) - C(u)\right| \xrightarrow{asymptotically} 0.$$

In the same manner we can estimate the empirical copula density:

$$(67)\ \hat{c}\left(\frac{t_1}{T}, \dots, \frac{t_n}{T}\right) = \begin{cases} \frac{1}{T}, \text{if } \{y_1(t_1;T), \dots, y_n(t_n;T)\} \text{ belongs to the sample,} \\ 0, \quad \text{in rest.} \end{cases}$$

Multivariate Kernel Estimator

Fermanian and Scaillet *et al.*(2003,2005) proposed a kernel approach to estimate the parameters and the derivatives of a copula. The *Gaussian kernel* is probably the most widespread used in economic and financial modeling, being defined as following:

$$(67) \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}.$$

Given a random vector Y with $\boldsymbol{n}$-dimensions $Y = (Y_1, \dots, Y_n)$ and for a sample size $\boldsymbol{T}$, $\{(y_1(1), y_2(1), \dots, y_n(1)), \dots, (y_1(T), y_2(T), \dots, y_n(T))\}$ such that $F(Y_1, \dots, Y_n) = C(F_1(Y_1), \dots, F_n(Y_n))$, the kernel estimator of $F(x)$ is:

$$(68)\ \hat{F}(y) = \frac{1}{T}\sum_{t=1}^{T}\prod_{i=1}^{n}\theta\left(\frac{y_i - y_i(t)}{h_i}\right),$$

and

$$(69)\ \theta(y) = \int_{-\infty}^{y} \varphi(t)\,dt,$$

where $\theta(y)$ is the density function, and $(h_1, \dots, h_n)$ represents the bandwidth, satisfying the conditions:

$$(70)\ h_i(T) > 0, \forall\ T, i \in \{1, \dots, n\}$$

and

$$(71)\ \prod_{i=1}^{n} h_i(T) + \left[T\prod_{i=1}^{n} h_i(T)\right]^{-1} \to 0,\ \text{as}\ T \to \infty.$$

More detailed, in practice the bandwidth is set as $h_i = \hat{\sigma}_i\left(\frac{4}{3T}\right)^{\frac{1}{5}}$, where $\hat{\sigma}_i$ is the standard deviation of the sample $\{y_i(1), \dots, y_i(T)\}$.

Thus the kernel estimator of a copula admits the following representation:

$$(72)\ C(u_1, \dots, u_n) = \frac{1}{T}\sum_{t=1}^{T} K_{u_1,h}\left(\frac{u_1 - \hat{F}_1(y_1^t)}{h}\right) * \dots * K_{u_n,h}\left(\frac{u_n - \hat{F}_n(y_n^t)}{h}\right).$$

where $K_{u_i,h}(\cdot)$ is a kernel with the bandwidth h, while $\hat{F}_1, \dots, \hat{F}_n$ represents the empirical repartition functions of the marginal distributions.

Fermanian and Scaillet have demonstrated that under mild conditions, the kernel defined by relation (72) converges asymptotically to Gaussian distribution as:

$$(73) \left(T\prod_{t=1}^{n} h_t\right)^{\frac{1}{2}} * \left(\hat{C}(u) - C(u)\right) \xrightarrow{asymptotically} N(0, C(u)).$$

3.3.3.2 Semiparametric estimation

Semi-parametric approaches are very useful estimation when the sample is not large enough. This technique supposes the use of a parametric estimation only for the copula and nonparametric one to estimate the univariate marginal distributions.

Parameters' Estimation based on Concordance Measures

This approach supposes a nonparametric estimation of the parameters that depend only on the copula. The main idea behind the approach mentioned before is that once the parameters have been estimated using concordance measures are expressed the parameters of copula as functions of the former ones. Oakes *et al.*(1982) emphasized the relation between the estimated parameter θ of Clayton copula and Kendall's tau as:

$$(74) \theta = \frac{2\tau}{1-\tau}.$$

So that the estimator of the parameter θ is defined as following:

$$(75) \hat{\theta}_T = \frac{2\hat{\tau}_T}{1-\hat{\tau}_T}$$

where $\hat{\tau}_T$ represents version of Kendall's tau for the sample. Given the bivariate sample of size T: $\{(x_1, y_1), \dots, (x_T, y_T)\}$, we define $\hat{\tau}_T$ as:

$$(76) \hat{\tau}_T = 2\frac{C-D}{T(T-1)}$$

where C represents the number of concordant pairs: $(x_i - x_j) * (y_i - y_j) > 0$, while D denotes the number of discordant pairs $(x_i - x_j) * (y_i - y_j) < 0$.

After Lindskog, McNeil and Schmock *et al.*(2003) have demonstrated that for any elliptical copula which provides a dependence structure for any pair of random variables there exists the relation:

$$(77)\tau = \frac{2}{\pi} \arcsin \rho,$$

in the same manner it is obtained ρ for the bivariate sample defined before:

$$(78)\hat{\rho}_T = \sin\left(\frac{\pi}{2}\hat{\tau}_T\right).$$

The main advantage of this approach is represented by its simplicity, but it don't provides a very accurate estimation of the parameters.

Pseudo Maximum Likelihood Estimation

In 1995 Genest, Ghoudi and Rivest provided a more elaborated method for estimating the parameters of copula, based on the maximization of a *pseudo* likelihood function. Given a sample of size T $\{(y_1(1), y_2(1), \ldots, y_n(1)), \ldots, (y_1(T), y_2(T), \ldots, y_n(T))\}$ which is derived from a common distribution F that have a copula C and margin distributions F_i such that for the random vector U it is obtained the following relationship: $U_i = F_i(Y_i)$. Let consider θ as the vector of parameters of the copula C and supposing that $C = C(\cdot\,;\theta)$ belongs to the family of copulas $\{C(u_1, \ldots, u_n; \theta);\ \theta \epsilon R\}$, then we will define the likelihood function of the sequence $\left\{\left(u_1(k) = F_1(y_1(k))\right), \ldots, \left(u_N(k) = F_N(y_N(k))\right)\right\}_{k=1}^{T}$ as following:

$$(79)\ln L = \sum_{t=1}^{T} \ln c\left(F_1(y_1(t)), \ldots, F_n(y_n(t)); \theta\right),$$

where $c(\cdot\,;\theta)$ represents the density function of $C(\cdot\,;\theta)$. Because the sequence $\left\{\left(u_1(k) = F_1(y_1(k))\right), \ldots, \left(u_N(k) = F_N(y_N(k))\right)\right\}_{k=1}^{T}$ is *independently and identically distributed* (*i.i.d.*) it results that also all the $y_i(k)$'s are also *i.i.d.* Another important remark is the fact that since the marginal distributions are unknown then it is desirable to use the empirical marginal distributions $\hat{F}_i$ for the estimating of random vector U:

$$(80)\ \hat{U} = \left(\hat{F}_1(Y_1), \dots, \hat{F}_n(Y_n)\right).$$

Also it has to be mentioned that extracting the *pseudo*-sample $\{(\hat{u}_1(k), \dots, \hat{u}_n(k))\}_{k=1}^{T}$, even all the $y_i(k)$'s are *i.i.d.* the $\hat{u}_i(k) = \hat{F}_i(y_i(k))$ isn't *i.i.d.* Therefore based on the sample of size T $\{(y_1(1), y_2(1), \dots, y_n(1)), \dots, (y_1(T), y_2(T), \dots, y_n(T))\}$ and substituting all $u(k)$'s with $\hat{u}(k)$'s in relation (18), we will define the *pseudo* log-likelihood of the model as:

$$(81)\ \ln \hat{L} = \sum_{t=1}^{T} \ln c\left(\hat{F}_1(y_1(t)), \dots, \hat{F}_n(y_n(t)); \theta\right).$$

In these conditions, maximizing the *pseudo* log-likelihood we obtain the estimation of the parameter vector θ:

$$(82)\ \hat{\theta}_T = \arg\max_{\theta} \ln \hat{L}\ (\{y_1(t), \dots, y_n(t)\}; \theta).$$

In conclusion, the *pseudo* maximum likelihood estimation is more reasonable for the low dimensional sample, while the Kendall's tau is probably the best when we work with large portfolios because the last one requires less time consuming. Genest *et al.*(1995) demonstrated that in the case of Clayton's θ the *pseudo* maximum likelihood estimation provides a smaller variance than the Kendall's tau.

3.3.3.3 Parametric estimation

In literature exists different parametric methods used for the estimation of the copula's parameters. So we will define the most known of them:

Canonical Maximum Likelihood (CML) Estimation

The CML approach supposes an estimation of the copula's parameters, without any assumption about the parametric form of marginal distributions. Thus CML technique uses nonparametric approaches such the *kernel* estimation for the modelling of marginal distributions. Maashal and Zeevi *et al.*(2002) proposed an estimation algorithm based on crossing of the following two steps:

i) Firstly using the empirical marginal distribution, a given dataset $(y_1^t, \dots, y_n^t)$ with $t = 1, \dots, T$ is transformed into uniform variates $(\hat{u}_1^t, \dots, \hat{u}_n^t) = (F_1(y_{1t}), \dots, F_n(y_{nt}))$;

ii) Secondly it is estimated the vector of copula's parameters θ as:

$$(83) \hat{\theta}_{CML} = \arg\max L(\theta), \text{ where}$$

$$(84)\ L(\theta) = \sum_{t=1}^{T} \ln c(\hat{u}_1^t, \dots, \hat{u}_n^t).$$

Therefore the main advantage of the CML approach is the easily of its utilization from the numerical viewpoint.

Exact Maximum Likelihood (EML) Estimation

The EML approach is based on an algorithm which estimates commonly the parameters of both the copula C and the marginal distributions. Thus for a dataset $(y_1^t, \dots, y_n^t)$ with $t = 1, \dots, T$, having the marginal distribution F_i, its univariate density function can be defined as:

$$(85)\ f(y_1, \dots, y_n) = c(F_1(y_1), \dots, F_n(y_n)) \prod_{i=1}^{n} f_i(x_i),$$

where c represent the copula's density function that is resulted from the following relation:

$$(86)\ c(u_1, \dots, u_n) - \frac{\partial C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n}.$$

Also we denote the vector of parameters $\vartheta(\theta, \delta)$ with $\theta = (\theta_1, \dots, \theta_n)$ representing the copula's parameters, while the parameters of marginal distributions are defined as: $\delta = (\delta_1, \dots, \delta_m)$.

Thus given the repartition function of the marginal distribution F_i and their density function f_i, the log-likelihood function admits the following representation:

$$(87)\ L(\vartheta) = \sum_{t=1}^{T} \ln c\left(F_1(y_1^t; \delta_1), \ldots, F_n(y_n^t; \delta_n); \theta\right) + \sum_{t=1}^{T}\sum_{i=1}^{n}\left(f_i(y_i^t; \delta_i)\right).$$

Therefore the EML estimator maximizes the relation (26) such that:

$$(88)\ \hat{\theta}_{EML} = \arg\max L(\theta)$$

From a statistical view the EML method provides the highest degree of accuracy because its properties are the nearest ones front MLE, but the computational difficulty of this approach is much greater than CML.

Inference Functions for Marginals (IFM)

The main idea behind the IFM approach is to estimate separately the parameters of marginal distributions from the copula parameters. Thus the algorithm of this method is compounded by two steps:

i) Given that the relation (26) is equivalent with:

$$(89)\ L(\vartheta) = L_1(\theta, \delta) + L_2(\delta),$$

then the estimation of the marginal distributions' parameters admits the following representation:

$$(90)\ \hat{\delta}_{IFM} = \arg\max L_2(\delta);$$

ii) Secondly, knowing the parameters of marginal distributions, we estimate the copula's parameters θ as:

$$(91)\ \hat{\theta}_{IFM} = \arg\max L_1(\theta, \hat{\delta}_{IFM}).$$

Numerically, the IFM approach provides a better accuracy than the EML method even the algorithms of the two method are very closely.

3.3.4 Goodness-of-Fit Tests

Given that we can choose from a wide range of models in order to estimate parameters, the efficiency of the selected method will be establish by comparing the empirical distribution to the theoretical one. Therefore we define two distances and one information criteria that are designed to amount the efficiency of the chosen approach through a backtesting.

Kolmogorov-Smirnov Distance

The Kolmogorov-Smirnov(K-S) approach determines the maximum local distance among all quantiles, noting that the maximum is most often located in the bulk of distribution. So that the Kolmogorov-Smirnov distance is defined as:

$$(92)D^{K-S} = \max_z \left|F_{z^2}(z^2) - F_{\chi^2}(z^2)\right|,$$

Where $F_{z^2}(z^2)$ and $F_{\chi^2}(z^2)$ denotes the empirical distribution, respectively the χ^2 distribution of the random variable z. The null hypothesis of the Kolmogorov-Smirnov test is that the sample is drawn from the χ^2 distribution. In addition, for a higher accuracy of the measurement, we define the average of the Kolmogorov-Smirnov distance:

$$(93)\ D^{K-S}_{Avg} = \int \left|F_{z^2}(z^2) - F_{\chi^2}(z^2)\right| dF_{\chi^2}(z^2).$$

Unlike D^{K-S} that can present a higher degree of sensitivity to the presence of outliers, the D^{K-S}_{Avg} is less sensitive to an outlier because this one is weighted with the order $1/T$, where represents the size of the sample.

Anderson-Darling Distance

Like the K-S distance, the Anderson-Darling (A-D) distance asses the existing differences between a sample and a specific distribution. Thus under the null hypothesis about a distribution, the Anderson-Darling distance assumes that sample's data arise from the specific distribution and therefore transforms the data into a uniform distribution. So the Anderson-Darling distance admits the following representation:

$$(94)D^{A-D} = \max_z \frac{\left|F_{z^2}(z^2) - F_{\chi^2}(z^2)\right|}{\sqrt{F_{\chi^2}(z^2)\left[1 - F_{\chi^2}(z^2)\right]}},$$

while the average of A-D distance is defined as:

$$(95) D_{Avg}^{A-D} = \int \frac{\left|F_{Z^n}(z^2) - F_{X^n}(z^2)\right|}{\sqrt{F_{X^n}(z^2)\left[1 - F_{X^n}(z^2)\right]}} dF_{X^n}(z^2).$$

Also like D^{K-S}, the point which maximizes the argument of the $\max(\cdot)$ function especially exerts a control on the D^{A-D} also makes it more sensitive to the presence of outliers. Therefore D_{Avg}^{A-D} provides more valuable information about the similitude between the sample and specific distribution.

Also other important approach to check the efficiency of chosen copula is to compute the *Akaike information criterion* (AIC), which is given by:

$$(96)\ \text{AIC} = -2 \ln L\left(\left(y_1(t), \dots, y_n(t)\right); \hat{\theta}\right) + 2 \dim \theta,$$

where $\left(y_1(t), \dots, y_n(t)\right)$ represents a vector of random variable and θ is the vector of copula's parameters. In this paper I use the information criteria to chose the best copula.

3.3.5. Simulation

Once the copula parameters were estimated and marginal distributions were modelled using Generalized Pareto Distribution, the next stage is to simulate the jointly dependent returns of the FX portfolio. Thus simulating the cumulative distribution function for a given horizon of time, we can compute VaR measures.

Firstly we have to generate randomly dependent uniform variates for each series for a given horizon of time. Using the estimated parameters for dependence structure of our portfolio given by each type of used copula, we simulate *n* trials that are uniformly distributed U(0,1). For this first task I have given an example of bivariate copula, using the method of conditional distribution:

$$(97) c_u(v) = \text{P}(V \leq v, U = t)$$

for given random variables U and V with $C(u, v)$, where c_u is the conditional distribution function for the random variable *V* at a given value u of U. From previous relation we can write the conditional distribution as:

$$(98) c_u(v) = \lim_{\Delta u \to 0} \frac{C(u+\Delta u, v) - C(u,v)}{\Delta u} = \frac{\partial}{\partial x} C(u,v) = C_u(v)$$

Given the above result we can generate *n* pairs of (u, v) of pseudo random variable. For reach this goal, the first step is two generate two independent pseudo random variables $u \text{ and } z$, and then we compute the inverse of conditional parameters as:

$$(99) v = c_u^{-1}(z),$$

taking into account the estimated for each copula. Instead for some copulas like Gumbel, this invers can't be calculated analytically and it is numerically computed. Then using the Monte Carlo simulation we obtain a desired pair vector of random variables in order to the estimated dependence parameters:

$$(100)\ (y_1 = \Phi_1^{-1}(u), y_2 = \Phi_2^{-1}(v)\ ).$$

But this procedure using Monte Carlo simulation is performed iteratively *N* times to obtain a sample $Y = (y^{(1)}, y^{(2)}, \dots, y^{(n)}$. The main advantage of simulation with copulas is they allow for a differentiation in the type of dependence structure and marginal distributions. For example we can use Student and Gaussian copula to simulate random vectors in which the marginal distribution follow a Student, respectively Gamma distribution.

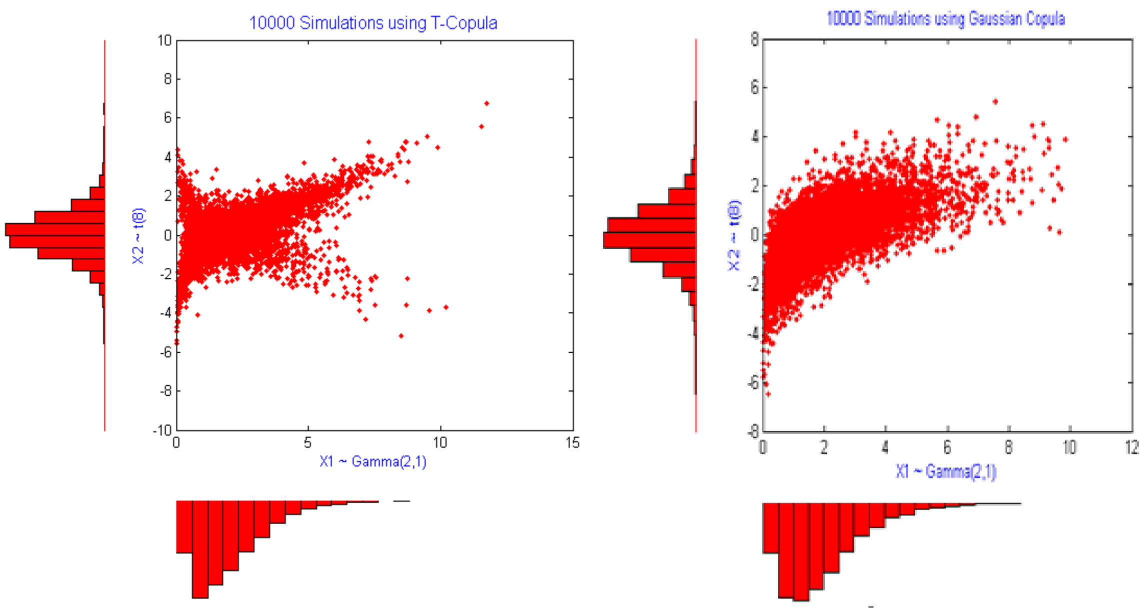

Figure 4. Monte Carlo Simulation eith Copula

The second stage consist in the transformation of uniform variates into standardized residuals using for this purpose the inversion of semi-parametric marginal CDF which in this paper are modelled with GPD. At this step are simulated standardized residuals that have the same features as those resulting from a filtering process with an ARMA× GARCH model.

Using the parameters of dynamic equations estimated with an ARMA× GARCH model, at third stage reintroduce autocorrelation and heteroskedasticity in the simulated standardized residuals to obtain conditional returns.

3.4 Risk Measurements

In finance theory, a metric for market risk represents a measurement of uncertainty related to future evolution of the portfolio's value. In fact a risk metric attends to summarise the potential deviations over time from an expected value of a portfolio, which are defined as profits or losses.

The first step in developing quantitative tools designed to measure the risk of random events was made by the risk department J.P Morgan. In 1994, the CEO of J.P Morgan, Dennis Weatherstone asked to the risk department that every day at 4.30 P.M to submit a report relating to the bank risk measure and a corresponding risk measure. Thus it takes birth the Risk Metrics Department managed by Till Guildman that elaborated the *Value-at-Risk* (VaR) model. VaR is a statistic model which is designed to express the risk of an exposure by a single number. More exactly VaR model estimates the worst loss for a financial instruments portfolio over a given time horizon and for a given confidence level. Thus the first form of the VaR model was defined as:

$$(101)\ VaR_{h,\alpha} = -q_{\alpha}X,$$

where h is the horizon of time, $100(1-\alpha)\%$ denotes de confidence level, q_{α} is the lowest quantile of distribution function and X represents the value of a given assets or portfolio. In the case of normal distribution $N(\mu,\sigma)$, there could be used the normal transformation:

$$(102)\ Z_{\alpha} = \frac{q_{\alpha}-\mu}{\sigma}$$

to define the lowest quartile of distribution as:

$$(103)\ q_{\alpha} = Z_{\alpha}\sigma + \mu$$

where Z_α denotes the lowest quantile α of the standard normal distribution.

Therefore the previous two relations lead to Analytical VaR:

$$(104) VaR_{h,\alpha} = -(Z_\alpha \sigma + \mu)X.$$

Over time there have been developed various methods for calculating VaR , among the most important are Historical VaR, Bootstrapping VaR or Monte Carlo VaR.

But VaR have received more criticism due to theirs simplistic assumptions that made these model to have many limitations in quantifying the risks. In 2001 Dembo and Freeman proved that VaR models, like volatility, don't provide a satisfactory distinction between "good" risks and "bad" risks. In 1959, Markovitz introduced the concept of *semi-variance* as a downside risk metric that measures the variances of returns which fall below than an expected return:

$$(105) Semi - Variance = E\left(\left(min\left(0, R - E(R)\right)\right)^2\right),$$

where *R* is the return and *E* denotes the expectation operator.

Starting from the above relation, Dembo and Freeman (2001) proposed the concept of *Regret* as a downside risk which replaced the expected return with a benchmark return. Thus the *Regret* concept was defined as:

$$(106) Regret = -E\left(min\left(0, R - Benchmark\_Return\right)\right)$$

From the above relation it could be easily observed that *Regret* operator embeds the form of a *Put* option with benchmark return as strike.

In 1997, Artzner called an axiomatic approach and set some conditions to certify the satisfactory risk measure. Thus Artzner called the risk measures which satisfy the formulated axioms as "*coherent*". Given a space of risks Ω, a risk measure function φ, a vector of random variables (loss distribution) X, the invested capital δ and the free-risk interest rate, Artzner defined that risk measure $\varphi(X)$ is *coherent* if the following conditions holds:

***i*) Monotonicity**

$$\forall X, Y \in \Omega \qquad X \geq Y \rightarrow \varphi(X) \geq \varphi(Y)$$

The above relation implies that a higher return corresponds to a higher risk.

***ii*) Homogenity**

$\forall X \in \Omega$ and $\forall \lambda \geq 0$, $\varphi(\lambda * X) = \lambda * \varphi(X)$

This axiom states that for a given position, the associated risk will linearly increase with its size.

***iii*) Translational invariance**

$\forall X \in \Omega$ and $\forall \delta \in R$, $\varphi(X + \delta(1 + \mu)) = \varphi(X) - \delta$

The above axiom means that for an investment of capital δ in risk-free assets, the risk decrease with the amount δ.

***iv*) Sub-additivity**

$\forall (X_1, X_2) \in \Omega \times \Omega$, $\varphi(X_1 + X_2) \leq \varphi(X_1) + \varphi(X_2)$

The last axiom ensure that total risk of portfolio is no more than sum of individuals positions' risk that means this condition encourages the portfolio managers to diversify their overall risk through the aggregate of different positions.

Therefore VaR model is not a coherent risk measure because it doesn't satisfy the sub-additivity condition. Other important criticism was that VaR models only provides a limit of the losses but tell nothing about the potential loss when the limit is exceeded. From this purpose, Artzner (1999) defined the Conditional Value-at-Risk which is a coherent risk measure, satisfying all four axioms mentioned above. Conditional VaR represents the expected loss in the case when VaR limit is violated:

$$(107)\ \text{Conditional VaR}(\alpha) = E(R|R > VaR),$$

where R denotes the return.

3.5 Backtesting

The main objective of VaR models is to minimize the errors resulting from estimating the maximum possible loss for an individual asset or portfolio in a given time horizon and for a given level of confidence. In financial literature there are many approaches designed to quantify the accuracy and performance of VaR models. In order to assess the performance of different copulas in estimation of VaR models I used three main approaches:

i) firstly, I compare the in-sample estimation of VaR and CVaR for the whole distribution with out-of-time empirical returns;

ii) I computed an out-of-sample estimation with 1 day window length for the last three years of sample and then compared the number of empirical violations with confidence levels of VaR;

iii) I computed a Bernoulli backtest to estimate the confidence intervals for the number of excesses and then calibrate results to the *traffic light* approach proposed by Basel amendment;

iii) I computed a Kupiec backtest to analyze the "success" probability of empirical excesses to equal the confidence levels provided by VaR.

Thus we can consider that daily empirical returns follow a Bernoulli process and define an *indicator function* to accounts the exceedance of 100$\alpha\%$ daily VaR:

$$(108) I_{\alpha,t+1} = \begin{cases} 1, if\ R_{t+1} < -VaR_{1,\alpha,t} \\ 0, otherwise \end{cases}$$

where R_{t+1} is the realized return and $VaR_{1,\alpha,t}$ denotes the forecasted value-at-risk.

Given that forecasting sample has *n* observations, expected number of successes in the test sample equals $n\alpha$. Thus denoting $X_{n,\alpha}$ as number of successes which has a binomial distribution, the expected number of successes will be:

$$(109) E(X_{n,\alpha}) = n\alpha$$

and the variance is:

$$(110) V(X_{n,\alpha}) = n\alpha(1-\alpha).$$

The standard error resulted from variance $\sqrt{n\alpha(1-\alpha)}$ represents a measure of uncertainty related to the expected value. So when *n* is large, the distribution of $X_{n,\alpha}$ converges to normal distribution and we can compute two-sided confidence interval:

$$(111)\ \left(n\alpha - \Phi^{-1}(\alpha^{*})\sqrt{n\alpha(1-\alpha)}, n\alpha + \Phi^{-1}(\alpha)\sqrt{n\alpha(1-\alpha)}\right)$$

where $\Phi^{-1}(\alpha^{*})$ is the inverse cumulative distribution function for standard normal distribution in the case of α^{*} confidence interval. Thus the null hypothesis of Bernoulli test is that VaR is an accurate model. In practice is unlikely to obtain the expected number of excesses provided by VaR due to modelling errors, so we use these confidence intervals around to the expected value of successes.

Kupiec introduced in 1995 an unconditional coverage test where the indicator function follows a Bernoulli process. The null hypothesis of Kupiec test is that indicator function is accurate in levelling the significance level of VaR. The statistic of Kupiec test is a likelihood ratio statistic:

$$(112) LR_{Kupiec} = \frac{\theta_{exp}^{n1}(1-\theta_{exp})^{n0}}{\theta_{emp}^{n1}(1-\theta_{emp})^{n0}}$$

where θ_{exp} is the expected proportion of exceedances, θ_{emp} is the empirical proportion of exceedances and *n*1 denotes the number of exceedances from the backtest sample size *n*. Thus θ_{exp} equals α, while θ_{emp} is equal to $\frac{n1}{n0}$ and the asymptotic distribution of $-2lnLR_{Kupiec}$ follows a chi-square distribution with 1 degree of freedom.

4. Data and Results

4.1 Data

To analyze the behaviour of dependence among currencies of Central and Eastern I used daily exchange rate of Czech Koruna (CZK), Hungarian Forint (HUF), Polish Zloty (PLN) and Romanian New Leu (RON) against Euro (EUR) between February 1999 and February 2010. Each data set used in this paper represents the closing rate of analyzed currencies, being provided by Bloomberg.

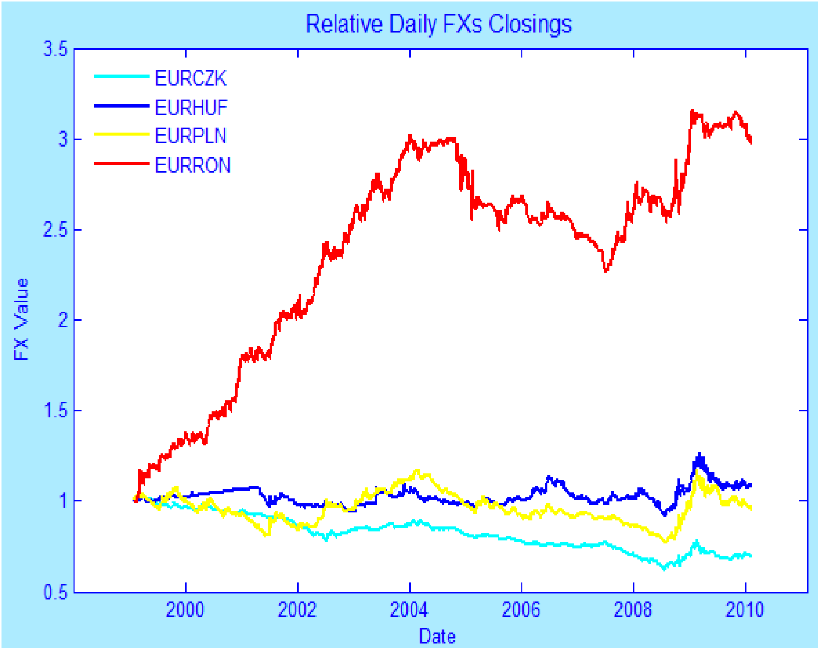

Figure 5. Evolution of analyzed daily exchange rate between February 1999 – February 2010.

From the above plot we can observe that patterns of analyzed exchange rate were quite different. The Czech Koruna was the currency which has recorded de most important appreciation against Euro due to sound structural reforms. In the same time EUR/RON has situated at the opposite pole, recording an upward evolution. All of these countries have addressed different policies to stabilize the nominal exchange rate and prices. The four CEE countries have changed their monetary policy rules over past 15 years, adopting inflation targeting regime. Thus Czech Republic adopted inflation target regime in 1998, Poland in 1999, Hungary in 2001 and Romania in 2005. One important requirement of inflation target regime is to increase the flexibly of exchange rate. In these conditions the four countries opted for different types of exchange rate regime: Czech Republic use

Exchange Rate Regimes	
Czech Republic	Classical administrated floating
Hungary	Target zones against Euro
Poland	Independent floating
Romania	Managed floating

Table 1. Exchange rate regimes for analyzed CEE countries

In order to analyze the switches of exchange rate regimes in the observed period I have computed Markov Switching regressions for each of the four currencies' returns.

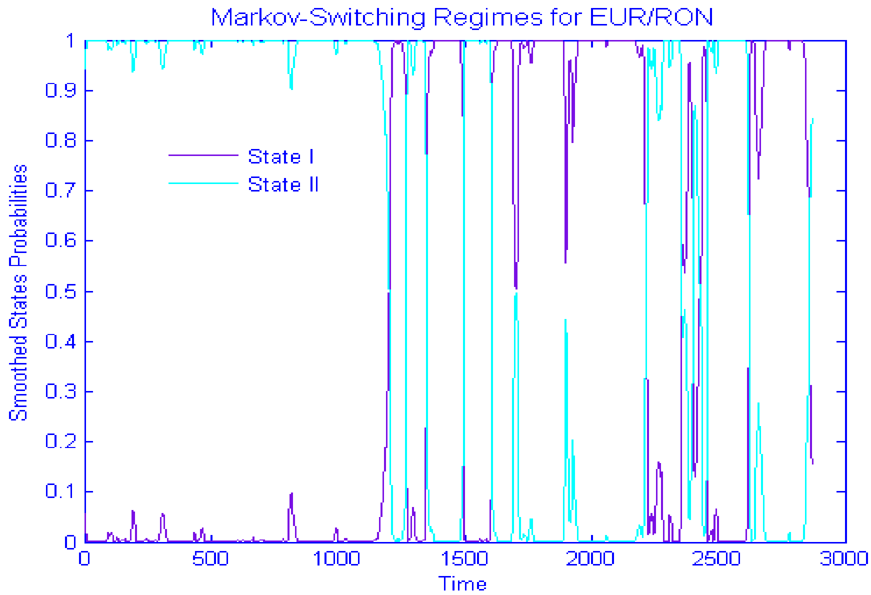

Figure 6. EUR/RON switching regimes in observed period

Thus I modelled the exchange rate returns as one lag autoregressive processes with Student distributed innovations. To capture the transition from one regime to other I switched the AR term and the innovations of regression. We can easily observe from above plot that starting with 2005 the EUR/RON returns transitioned more often between regimes due to the flexibility required by inflation targeting regimes. Instead the EUR/CZK returns recorded far fewer switches due to the exchange rate regime used by Central Bank of Czech Republic(see Appendix I, no.1). The very different behaviour of exchange rates is an important issue in modelling the dependence among the four currencies of portfolio.

The exchange rates have some typical features as compared with other assets traded in financial markets. To capture these stylized facts I analyzed the returns of exchange rates computed using the following formula: $R_{t+1} \approx \ln\left(\frac{S_{t+1}}{S_t}\right)$, where R_{t+1} denotes the return and S_t is daily exchange rate.

Descriptive statistics of returns showed that the four currencies posted quite different evolution in observed period. EUR/RON recorded the highest depreciation and appreciation of four currencies. These extreme values were recorded during 1999-2000 due to high economical, political and social stress at that time. An important remark is that these extreme values recorded by the EUR/RON are very high compared to the minimums and maximums registered by other currencies. This observation is also sustained by the highest standard deviation of EUR/RON. EUR/CZK returns had the fewest extreme variations and also the smallest standard deviation, but the beginning of the financial crisis pushed the Czech Koruna like Hungarian Forint to historical minima and maxima.

On the third and fourth order moments of the distribution we observed that analyzed series recorded very asymmetric evolutions, but all four currencies posted positive skewness and excess kurtosis (the value of kurtosis is higher than 3). EUR/RON recorded by far the highest skewness and kurtosis, while the skewness value of EUR/CZK is very close to the normal distribution, having the most stable distribution. Positive values the third and fourth order moments of the distribution and the test rejecting of Jarque-Bera's null hypothesis indicates that returns are not normally distributed.

Basic Stats	EUR/CZK	EUR/HUF	EUR/PLN	EUR/RON
Mean	-0.000121	0.000033	-0.000008	0.000385
Median	-0.000133	-0.000080	-0.000254	0.000000
Maximum	0.031908	0.065272	0.052507	0.123554
Minimum	-0.032471	-0.029232	-0.038318	-0.072379
Std. Dev.	0.004370	0.005827	0.006967	0.007282
Skewness	0.068426	1.236087	0.400931	1.902772
Kurtosis	8.944810	16.109895	7.309036	42.754610
Jarque-Bera	4228.41	21283.56	2297.29	190724.80
Probability	0.00000	0.00000	0.00000	0.00000
Observations	2870	2870	2870	2870

Table 2. Descriptive statistics of analyzed returns

The evolution of exchange rate returns confirms the existence of some typical stylized facts as the excess kurtosis, heteroskedasticity, volatility clustering and autocorrelation. Also we can observe that as descriptive statistics showed EUR/RON recorded the most unstable evolution, while EUR / CZK located at the opposite pole. In the same time EUR/HUF recorded several clusters of volatility, but EUR/PLN was more stable in the analyzed period. To test the existence of unit roots in the returns series I computed the ADF and KPSS tests (see,4). The null hypothesis of no unit roots for ADF test was rejected in all the cases for three confidence level. Instead the null hypothesis of stationarity in the case of EUR/RON was rejected at 5% and 10% confidence levels that indicate the existence of microstructures noises.

The GPD modelling and use of copulas requires that analyzed time series to be approximately i.i.d. Instead most of financial series and especially the exchange rates post autocorrelation and heteroskedasticity.For this purpose I have computed the autocorrelation function (ACF) for each of the four exchange rate returns. All ACFs exhibit some correlation, with EUR/RON and EUR/HUF having the most coefficients of autocorrelation function significantly different from zero.

Indeed the computed autocorrelation functions for squared returns show some higher persistence of the variance for all four currencies, especially for EUR/CZK returns that reveals the sound pattern of Czech Koruna appreciation against Euro.Instead the Romanian New Leu recorded isolated appreciations against European currency which were due mainly to increase of foreign direct investments starting with 2005.

4.2 GARCH Modelling

However the sample ACF of returns and squared returns indicate the use of GARCH models in order to obtain i.i.d. observation as required by GPD and copulas modelling. For this purpose I have used an asymmetric ARMA$\times$ GARCH to compensate for autocorrelation and heteroskedasticity recorded by exchange rates returns. Thus the conditional mean of each return is fitted using an ARMA process:

$$R_t = c + \sum_{i=1}^{m} \Phi_i R_{t-i} + \sum_{j=1}^{m} \theta_j a_{t-j} + a_t$$

where $a_t \sim \mathrm{i.i.d.}$ The conditional variance is modeled with an asymmetric GARCH, also known as GJR after authors' names:

$$\sigma_t^2 = \omega + \sum_{i=1}^{P} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{Q} \alpha_j a_{t-j}^2 + \sum_{j=1}^{Q} \gamma_j a_{t-j}^2 I_{t-j}$$

where the last term accounts for asymmetry, I_{t-j} being an indicator function that takes 1 if $a_t < 0$ (incorporates the impact of bad news) and 0 when good news arrive. In fact the ARMA$\times$ GJR model acts as a filter to obtain the i.i.d. processes. This approach of filtering time series was firstly used by Embrechts and Dias (2004) and Patton (2006).

To compensate for autocorrelation and heteroskedasticity I engaged an ARMA $\times$ GJR model for each currency. Conditional variance was modelled by an GJR (1,1) all the currencies, while the for conditional mean equation I used an AR (1) for EUR/CZK and EUR/PLN, respectively an ARMA (1,1) process for EUR/HUF and EUR/RON (see Appendix II., nr.1). To test the accuracy of fits for engaged ARMA × GJR models I performed Nyblom and Pearson tests. Additionally I estimated in-sample VaR for each ARMA × GJR and the related backtests (see Appendix II., nr.2). But to check if obtained residual series are i.i.d, I computed Ljung-Box test for filtered residuals. The null of no serial correlation was accepted for all the currencies (see Appendix II., nr.3).

Ljung-Box Test for standardized residuals						Ljung-Box Test for squared standardized residuals				
	EUR/PLN	EUR/HUF	EUR/RON	EUR/CZK			EUR/PLN	EUR/HUF	EUR/RON	EUR/CZK
H	0	0	0	0		H	0	0	0	0
P-value	0.607	0.7633	0.9223	0.8159		P-value	0.8335	0.9738	0.205	0.8972
Q-stat	17.7024	15.2271	11.8153	14.2829		Q-stat	13.9398	4.3487	24.9074	12.5144
Critical Value	31.4104	31.4104	31.4104	31.4104		Critical Value	31.4104	31.4104	31.4104	31.4104

Table 3. Testing for autocorrelation up to lag 25.

4.3 Preliminary statistic analysis

Once the i.i.d. residuals were obtained, the next step is choice an appropriate distribution to fit the data. Even the Student distribution of innovations capture a high degree of the leptokurtosis effect, the unimodal distribution as T or Gaussian are not designed to provide a good fit in the tails. The main reason for this effect is that tails are low density areas and the unimodal distributions are an appropriate choice to fit in the areas where data are most concentrated, namely in mode[9]. Also the exchange rates contain many microstructure noises which Student or Gaussian distributions cannot capture.

[9] See Embrechts (1997) for more details.

But as Embrechts (1997) underlined that before applying some statistical methods, the used data must be well studied. Firstly I have computed the Mean Excess Function (MEF):

$$MEF(t) = \frac{\sum_{i=1}^{n}(Y_i - t)}{\sum_{i=1}^{n} 1_{\{Y_i > t\}}}$$

where t denotes the threshold, $1_{\{Y_i < t\}}$ is an indicator function that accounts for values higher than respective threshold. Ascending ordered sample values are successively chosen as thresholds and it is calculated the average of excesses over the threshold. The threshold was chosen successively in increasing order, then the slope of MEF should have a negative slope converging to zero. If the empirical MEF is positively slope straight line above 0, there is an indication of extreme values and need to use EVT theory.

All four currencies have shown signs of excess kurtosis (Appendix III,1,*i*), with EUR/RON recorded the highest extremes, while the EUR/CZK posted the lower ones as the previous analyzes have indicated. Another important tool in analyzing of extreme behaviour of sample data is the QQ-plot against exponential distribution which is a particular case of GPD when $\xi = 0$. The concave departure of the four filtered residuals series (Appendix III,1,*ii*,) against exponential distributions is an additional argument for the use of EVT in modelling of tail distribution.

Thus taking into account these reasons, the appliance to EVT theory is an appropriate choice. Fitting the data in tails is one of the main concerns in many financial applications, especially in quantile based models as VaR. As compared with GEV or block maxima approach, the GPD method (peak-over-threshold) has the advantage of require a smaller sample and provide a much smaller randomisation of data's distribution in tails. Thus the use of GPD is more appropriate than GEV in VaR application.

A very important concern in modelling the tails of distribution using GPD approach is to chose an appropriate threshold over which are considered the excesses, because various methods for estimating parameters of distributions are very sensitive to the choice of threshold. Embrechts (1997) has suggested the usage of Hill estimator for threshold determination. Hill (1997) proposed the following estimator:

$$\xi = \frac{1}{k-1}\sum_{i=1}^{k-1} \ln Y_{i,N} - \ln Y_{k,N}$$

for $k \geq 2$. In the above relation k are the upper ordered exceedances, N is the sample size and $\xi = \frac{1}{\gamma}$ is the tail index. Computing the Hill-plot for γ as Embrechts (1997) suggested a threshold will be selected from the plot where γ is fairly stable. Therefore I have ordered the highest, respectively lowest 500 i.i.d. observations for each currencies and inferenced the Hill estimator for lower and upper tails (Appendix III,1,*iii*). The lowest thresholds was recorded by EUR/RON for the left tail and EUR/HUF for the left, both accounting for about 10% of sample data over the most stable area of Hill estimator.

In this paper I used a semi-parametric approach to fit the residuals' distribution, namely for tails applied the GPD method, while the interior of distribution was fitted with a Gaussian kernel (Appendix III, no. 2,*i*). Chosen a non-parametric method as Gaussian kernel to fit the interior of distribution is very appropriate because most of the data are found near the mode. Selecting the threshold as 10 % of the residuals in each tail, then parameters of distribution's excesses over this threshold were estimated using a maximum likelihood approach. McNeil and Frey (2000) demonstrated that maximum likelihood estimator is invariant for a selected threshold ranging between 5% and 13% of the sample size.

	Lower tail		**Upper tail**		**Lower tail**		**Upper tail**	
Parameters	ξ	σ	ξ	σ	ξ	σ	ξ	σ
ML estimates	-0.1017 (0.0372)	0.5328 (0.0000)	0.0495 (0,3503)	0.6099 (0.0000)	-0.0941 (0.0845)	0.6138 (0.0000)	0.1599 (0,0192)	0.6562 (0.0000)
Standard Error	0.0488	0.0416	0.0530	0.0486	0.0564	0.0521	0.0683	0.0593
Lower limits of Confidence interval	-0.1974	0.4571	-0.0544	0.5216	-0.2086	0.5198	0.0261	0.5496
Upper limits of Confidence interval	-0.0059	0.6210	0.1534	0.7130	0.0283	0.7248	0.2936	0.7835

Table 4. Estimated GPD parameters for tail distribution. Values under paranthesis are the P-values

An important remark about fitting marginal distributions with GPD method is that size of tail index is determined by the original distribution. Thus Student distribution with tails decreasing as polynomial corresponds to a positive tail index, while Gaussian distribution leads to a zero tail index due to its tails that drop exponentially. When the tail index is negative, the original distribution behaves as beta distribution in its tails.

Estimated parameters for distribution of peaks over the selected threshold indicate quite different behaviour of tails for the filtered residuals (other results in <u>Appendix III</u>, no. 2,*ii*). Different values of tail index emphasizes the asymmetry of innovations due to some stylized fact like leverage effect or volatility clustering. Statistically significant P-values of the tail index for EUR/PLN lower tail shows that the original distribution behaves as beta in the respective tail, while the upper tail index is insignificant differently from zero. Economically speaking the values of tail index recorded for EUR/PLN provides a suitable description for the exchange rate evolution in analyzed period: appreciations against euro were isolated, while depreciations against European currency followed a Gaussian process due to price stabilization. In the same time, significant P-value at 10 % confidence level of EUR/RON lower tail index emphasizes that starting with the explosive growth of FDI in 2005-2006 the Romanian New Leu began a robust appreciation process against euro. Estimated tail indexes for EUR/CZK are insignificant different from zero, while significant shape parameters at 5%, respectively 10% of EUR/HUF indicates the effect of leptokurtosis due to financial crisis that pushed this currency to historical minima and maxima.

A very important concern in tails modelling is that estimated parameters reflect the true behaviour in tails of original distribution such that GPD fit to be close to the empirical distribution. In the sense of parameters estimation' accuracy, the standard errors which are obtained from principal diagonal of the inverse of Fisher's information matrix tells us that if the same estimation could be repeated for a large number of times on sample with the same source, then the parameters estimated with maximum likelihood approach should asymptotically converge to the normal distribution. But if the source of data comes from a very asymmetric distribution as beta or gamma, then the maximum likelihood estimates would not converge to normal distribution. Thus I have tested the asymptotically normal assumption for estimated parameters in order to verify if the negative values for some tail indexes really arrived from beta distribution, while the positive shape parameters were provided by Student distribution. To do this I used a Boostrap[10] approach to extract randomly a number of 10,000 sub-samples of data distributed in each tail. Once the desired numbers of sub-samples were extracted, I computed the maximum likelihood estimation for each sub-samples and then computed the resulted parameters against the quartile of Normal distribution.

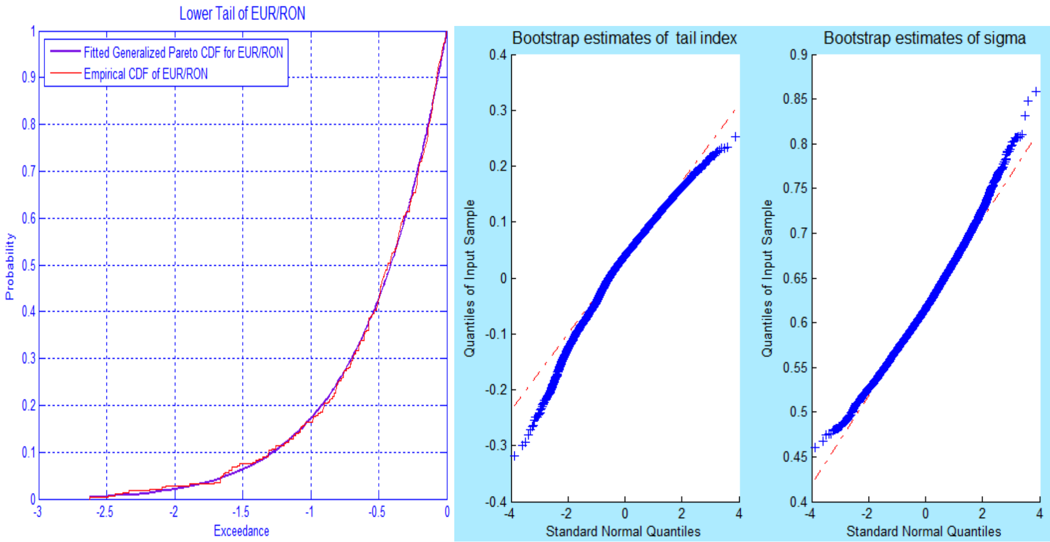

Figure 7. Check for maximum likelihood estimation's accuracy

As we can see the distribution of maximum likelihood estimation for lower tail index doesn't approximate the normal quartile that means the negative value of tail index really indicates the original data behaves as beta in lower distribution (other results in Appendix III, no. 2,*iii*).

4.4 Copula estimation

Once the marginal distributions of filtered residuals were fitted using a GPD approach for tails distribution and a Gaussian kernel for the interior of distribution, the next stage was to estimate the parameters of dependence structure for the analyzed portfolio. Isolating the effects of marginal distribution I have estimated dependence existent among the four currencies. In fact this is the copulas' job: to capture the interaction among the portfolio' assets by isolating the individual behaviour of each asset.

In this paper I used the *Cannonical Maximum Likelihood* (CML) approach to estimate the parameters of copula. Main advantage of CML method is that allows the estimation of

[10] For Bootstraping I recommend the use of *parallel computing* approach provided by Matlab as it leads to a large increase in processing speed.

dependence structure without any assumption about the distribution of marginals. Maashal and Zeevi *et al.*(2002) proposed the following formula for estimation of copulas parameters:

$$L(\theta) = \sum_{t=1}^{T} \ln c(\hat{u}_1^t, \ldots, \hat{u}_n^t)$$

where $\hat{u}_1^t, \ldots, \hat{u}_n^t$ denotes the transformation of semi-parametric CDFs computed for filtered residuals into uniform variates. Once the transformation was made, the following step is to estimate the parameters of copula.

For analyzed portfolio I used two Elliptical copulas to estimate the dependence among exchange rates, namely Student, respectively Gaussian copula. Unlike Gaussian, the estimation of Student copula was made in two steps: firstly given a fixed value for degree of freedom (DoF), the likelihood function is maximized with respect to the dependence parameter; secondly once the results from previous maximization were obtained, DoF parameter is estimated with respect to dependence parameter.

The estimated parameters with both Elliptical copulas revealed positive dependence among the four currencies (Appendix IV, no. 1). Correlations estimated with Student copula are higher than those fitted with Gaussian copula, due to the fact that T copula takes into account for fat tails. Another approach to compute the linear correlation matrix is to first estimate the rank correlation matrix[11]. Then given the previous estimate we can use a robust sine transformation to obtain a linear correlation matrix (Appendix IV, no. 1). We can observe that linear correlation matrix obtained from rank correlation provides higher coefficients of correlation than those estimated with Gaussian copula because the former approach accounts for tail dependence.

By studying the resulted correlation coefficients it can be observed that in all three methods the highest correlation is recorded between EUR/PLN and EUR/HUF, while the lowest one is registered between EUR/CZK and EUR/RON. Another important remark is that each currency is most correlated with the EUR/PLN and at least with the EUR/RON. However the correlation coefficients are smaller than 0.5 that means a low dependence in the evolution of the four currencies. But the empirical events revealed a high dependence among the four

[11] Zeevi and Mashal (2002).

exchange rates on depreciation side as the episode from October 2008 showed, when all these currencies have sharply decreased against European currency. This fact brings the discussion about the existence of both asymmetric dependence and leverage effect as stylized facts and also about the contagion of shocks among these countries. This is very interesting result taking also into account that Poland is the largest country by population and the biggest economy from CEE zone. Thus a shock of Poland on other countries in the region would have the greatest impact on the evolution of portfolio.

But to study this hypothesis is very suitable to engage an analysis of conditional dependence among exchange rates when EUR/PLN plays a pivotal role. From Bayes Law is well-known the fact that a multivariate joint distribution can be decomposed using iterative conditioning as following:

$$f(y_1,\dots,y_n) = f(y_1) * f(y_2|y_1) * f(y_3|y_2,y_1) * \dots * f(y_{n-2}|y_1,\dots,y_{n-1}) * f(y_n|y_1,\dots,y_{n-1})$$

Thus we can decompose the first conditional density in terms of copula:

$$f(y_2|y_1) = c_{12}\left(F_1(y_1),F_2(y_2)\right)f_2(y_2)$$

Further we can continue with the second conditional density as:

$$f(y_3|y_2,y_1) = c_{23|1}\left(F_{2|1}(y_2|y_1),F_{3|1}(y_3|y_1)\right)f(y_3|y_1)$$

where

$$f(y_3|y_1) = c_{13}\left(F_1(y_1),F_3(y_3)\right)f_3(y_3).$$

This model of conditional copula are called *Canonical Vine Copula* and was introduced by Bedford and Cooke (2002) and in financial application were firstly used by Aas (2007) and Berg and Aas (2007). The notations called here are according those used by Aas (2007). A general form of Canonical Vine Copula can be defined as:

$$c(y_1,\dots,y_n) = \prod_{j=1}^{n-1}\prod_{i=1}^{n-j} c_{j,j+i|1,\dots,j-1}\left(F(y_j|y_1,\dots,y_{j-1}),F(y_{j+i}|y_1,\dots,y_{j-1})\right)$$

	Clayton	SJC

Pair	Kendall's tau τ	Upper tail	Lower tail
EUR/PLN-EUR/CZK	**0.1144**	**0.1403**	**0.0742**
EUR/PLN-EUR/HUF	**0.1462**	**0.1735**	**0.1371**
EUR/PLN-EUR/RON	**0.0547**	**0.0219**	**0.0102**
EUR/CZK-EUR/HUF|EUR/PLN	**0.1789**	**0.2774**	**0.1286**
EUR/CZK-EUR/RON|EUR/PLN	**0.0801**	**0.0844**	**0.0183**
EUR/HUF-EUR/RON|EURPLN,EUR/CZK	**0.1072**	**0.1049**	**0.0566**
Log Likelihood	**457.1408**	**726.1495**	

Table 5. Estimation of Cannonical Vine Copula

In the computation of Canonical Vine model for the chosen portfolio I selected the EUR/PLN returns as pivot. Therefore we can observe that highest dependence was recorded between EUR/CZK and EUR/HUF conditioned by EUR/PLN in the lower tail that indicates the existence the spill-over of negative shocks in periods of depreciation against Euro. Also from estimated result it can be observed that conditional dependences in lower tail are twice than those from upper tail. The previous remark underlines the existence of asymmetric dependence among the four currencies and leverage effect

For these reasons we can say that estimated dependence parameters of portfolio with the three methods would provide incomplete information about the really dependence among currencies due to the rigidity in capture asymmetric dependence. Also have to be mentioned that an explanation for much tighter dependency between EURCZK, EUR / HUF and EUR / PLN as compared with EUR/RON is that Czech Republic, Poland and Hungary have adopted in periods very close one of the other inflation targeting regime. All four countries are primarily aimed to accede to ERM II, but they firstly have to satisfy the nominal convergence criteria in order to provide a high stability of exchange rate. Different reactions of Central Banks to changes in prices or interest rates, lead to asymmetric dependences among the analyzed exchange rates' evolution. The other fundamental explanation of asymmetric dependence effect is that all four countries are subject to the same problem: the increasing flows of FDIs from lasts years leads to appreciations of local currencies against Euro, but in the same time this effect coincides with a loss of external competitiveness.

For this reasons we have to take into account the existence of asymmetric dependence because this is one of the main concern in portfolio management. Thus I proposed the decomposition of chosen portfolio in other three bivariate portfolios consisting in EUR/PLN and each of other three currencies. As we have seen from the estimation of dependence parameters with Elliptical copulas each currency records the highest correlation with EUR/PLN, thus supporting the proposal which I made. Another important reason for this choice is that Poland is the greatest country by population and largest economy from CEE and such a shock from this country shows a high probability of having a significant impact on other economies from CEE. Incomplete capture of dependence in the computation of VaR and other risk measures could lead to important misalignments In this case the decomposition of portfolio in other three bivariate sub-portfolios permits a higher flexibility to choice the most suitable copul.s for modelling the dependence in order to obtain a higher accuracy from computation of risk measurements.

In this paper I decomposed the initial portfolio in other three bivariate sub-portfolios consisting in EUR/PLN and each of other three currencies. For each sub-portfolio I used a number of nine copulas for the estimation of dependence parameters and then I computed the information criteria (AIC, BIC). I divided the nine copulas used in two categories: Elliptical copulas plus Plackett copula (due to appropriate properties) and Archimedean family. I made this split in order to select the best copula by goodness-of-fit criteria from each category and then estimate the risks measures and compare the estimation errors with those of the initial portfolio and of other sub-portfolios. Additionally I have computed copula-GARCH models to capture the evolution of dependence over time. On the base of results from information criteria

For bivariate EUR/PLN-EUR/CZK portfolio the lowest negative log likelihood values and the information criteria were recorded by Gumbel and Student copula (see Appendix IV, no. 2, *i*) and *ii*); red inserted values denotes the minimum AIC and BIC). The lower tail dependence between the two currencies was much lower than the upper tail one (see Appendix IV, no. 2, *iii*)).

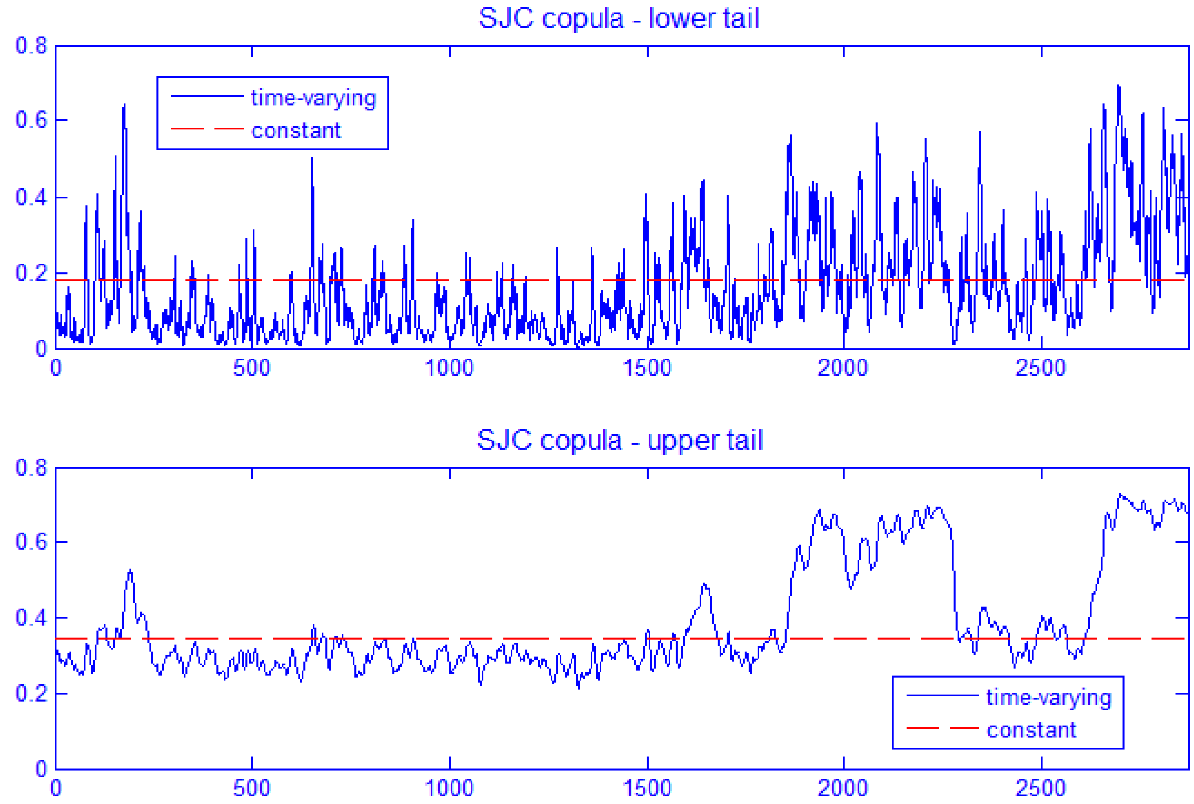

Figure 8. Modelling tyme-varying dependence with SJC-copula

For EUR/PLN-EUR/HUF sub-portfolio it can be observed (see Appendix IV, no. 3, *iii)*) a very strong dependence especially in the upper tail. An interesting remark is that as we can see from tyme-varying SJC copula the dependence between two currencies decreased very sharply in the end of 2006 and the begin of 2007. But from the evolution of Markov-Switching regimes for EUR/HUF (see Appendix IV, no. 1) it can be observed a suddenly rigidity in transition of exchange rate between states, thus sharply fall of EUR/HUF due to intervention of Central Bank is an important explanation for the empirical events emphasized previously. Instead the beginning of financial crisis coincided with a very high dependence for the two currencies. Also for this portfolio the goodness-of-fit analysis indicated the choice of Student and Gumbel copulas. Other important remark is that unlike the case of EUR/PLN-EUR/CZK portfolio, for EUR/PLN-EUR/HUF dependence the tyme-varying Gumbel provided lowers AIC and BIC than those obtain by dynamic SJC copula that reveal a very strong right asymmetry. A very interesting result was obtained for EUR/PLN-EUR/RON sub-portfolio, where the lowest values of information criteria were obtained by Plackett copula for the first category and Frank copula for Archimedean family. The result could be explained by the fact that as we can see from the three dynamic copulas, the (see Appendix IV, no. 4, *iii*)) the upper tail dependence seems very noising, while the lower tail dependence was filled by many peaks. Anyway all three sub-portfolios indicate the presence of a sound right asymmetric dependence.

4.5 Estimates of risk measures

Least but not last I attended the main goal of this paper as to analyze the results of using different types of dependence on the VaR models' accuracy. While the VaR models are not very complicated under their original form, the change of several assumptions creates an important concern in order to check and to compare the accuracy of involved models. Once the best copulas for the three sub-portfolios were selected by information criteria, the next step was to compare the accuracy of the models in order to conclude if my proposal brings certain benefits. For this reason firstly I estimated the in-sample VaR and CVaR and secondly I computed an out-of sample forecasting.

In-sample estimates of VaR and CVaR were computed using the Monte Carlo approach to simulate the cumulative distribution of each portfolio return for a given horizon with respect to the copula parameters and the estimated parameters for whole sample of ARMA×GJR model (results can be found in Appendix V). Then the resulted risk measures were compared with the empirical minimum and maximum returns of portfolio for the respective horizon. Also the estimated out-of-time[12] VaRs and CVaRs were compared with the realized return for each horizon.

From results it can be observed that overall Student copula provides larger measures of potential loss than those computed with Gaussian copula, because the former takes into account for tail dependence (see other estimates in Appendix V, no.1,*i*)). This fact is underlined by simulated CDFs which shows a high right skew for T copula. Estimated CVaRs are very close minima and maxima recorded by portfolio returns, while out-of-time realized returns are within the limits provided by these two risk measures (Appendix V, no.1,*ii)*).

For EUR/PLN-EUR/CZK sub-portfolio, the Student copula provides higher values of VaR for 1% and 5% quantiles and lower ones for 95 % and 99 % because Gumbel is a right asymmetric copula (Appendix V, no.2,*ii)*). This aspect is also observed in the case of EUR/PLN-EUR/CZK. An interesting aspect is that for both EUR/ PLN-EUR /CZK and EUR/ PLN-EUR/CZK portfolios the 99 % quantile VaR for one month horizon simulated with T

[12] Out-of-Time concept denotes here the first period out of sample for a given horizon. As compared with out-of-sample forecasting that use a rolling-window method within the sample, out-of-time method used the parameters estimated for the whole distribution of returns and provides only once estimation of VaR and CVaR.

copula is lower than this one estimated by Gumbel, while CVARs estimated with T copula are higher. In the case of EUR/PLN-EUR/HUF this situation is also available for horizons of 5, respectively 10 days (Appendix V, no.2,*i*)). Thus conclusion we can draw from the previous observations is that simulation with Gumbel just facilities the appearance of some peaks for short horizons and the density of data in tails is smaller than this one simulated with Student copula. Another very interesting remark is that the 99% quantile of Gumbel- CVaR for EUR/PLN-EUR/HUF sub-portfolio is much higher than the maximum 3 months cumulated returns that conclude Gumbel copula provides over conservative risk measures for long horizons. In the case of EUR/PLN-EUR/RON, the Frank copula provided larger potential losses and lower potential gains as compared with those estimated with Plackett copula, because Frank is lighter on the right side (Appendix V, no.3,*i*). This aspect becomes more evident in the simulations on longer horizons.

But these static estimates cannot give full information about the models' forecasting accuracy of maximum potential losses and gains. In this sense I engaged a dynamic process of risk measures' estimation. To do this firstly I split the entire sample in other two: estimation sample and forecasting sample. Then I appealed the rolling-window method, as the estimation sample is used to provide a VaR for the next period of a fixed number of days. Setting constant the size of estimation sample the previous procedure was rolled over the whole period, such that obtained a series of VaR which represents the forecasting sample. This is an out-of-sample forecasting is used to test accuracy of models over time.

Therefore I applied a out-of-sample methodology of forecasting with a horizon of 1 day ahead to large portfolio and to the three sub-portfolios, using for each of them the related copulas in order to compute VaR measures. The forecasting period was between January 2007 and January 2010, containing 808 observations, while the estimation sample accounts for 2062 data. The chosen forecasting horizon is very appropriate to test the accuracy of used Copula-VaR models because contains both a quiet and a turmoil period determined by the financial crisis.

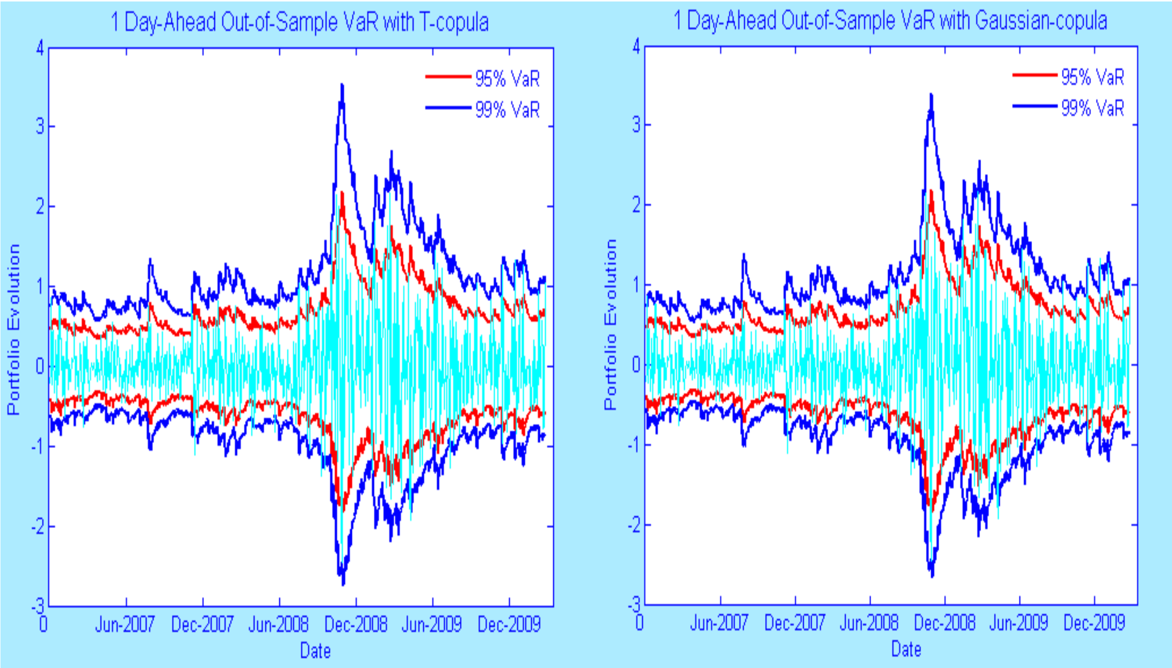

Figure 9. Out-of-Sample forecasts for large portfolio

4.5 Backtesting

Once the out-of-sample forecasts were obtained, I compared the resulted VaRs with empirical returns (see Appendix VI, no. 1). The following stage was to take into account those returns which exceeded the forecasted values of VaR. Thus to check the accuracy of engaged models I compared the number of exceedances with the theoretical levels provided by VaR.

Bernoulli Backtest and Calibration to Basel II *Traffic light*					
	Copula	**95% VaR**	**99% VaR**	**5% VaR**	**1% VaR**
Large Portfolio	T	7.17%	1.36%	7.29%	1.48%
	Gaussian	7.42%	1.61%	7.17%	1.98%
EUR/PLN-EUR/CZK Portfolio	T	5.69%	1.48%	5.69%	2.35%
	Gumbel	5.69%	1.36%	7.05%	1.98%
EUR/PLN-EUR/HUF Portfolio	T	5.32%	1.11%	7.42%	1.48%
	Gumbel	5.81%	1.36%	8.41%	1.24%
EUR/PLN-EUR/RON Portfolio	Frank	**4.45%**	1.48%	**4.94%**	1.11%
	Plackett	**4.57%**	1.36%	5.19%	1.36%

Table 6. Bernoulli Backtest for the number of exceedances

The percent of empirical exceedances over the theoretical levels of VaR resulted from out-of-sample forecasting for each portfolio and related copulas are found in this Table. A firstly indication about the accuracy of Copula-VaR models can be made by comparing the percent of exceeses with VaR confidence levels. We can observe that copula Frank and Plackett recorded lower percentage of excesses (bolded values from table) than theoretical ones provided by VaR.

But this is not a "robust" tool in testing the accuracy of Copula-VaR models involved here. Thus I computed a Bernoulli test under null hypothesis that VaR model is accurate with a 99 % confidence level. Also it has to be mentioned that regulators admit that in period with turmoil the VaR models could produce some misalignments. From this reason I calibrated my results to the errors bands proposed by Basel II *Traffic light* framework. Therefore the green bullets indicate the acceptance of Bernoulli null hypothesis, while the yellow ones indicate some misalignments in predicting maximum potential losses or gains. However the results provided by Copula-VaR models used here gave good results, for which the above table does not contains any red bullets. One important requirement for the power of the Bernoulli test is that number of observation has to be large. Also the existence of positive autocorrelation of exceedances could lead to widening of the confidence interval that affect the power of test (see Appendix VI, no. 2)

From this purposes I have engaged an unconditional test to strengthen and complete the conclusions regarding the comparison of models accuracy. Kupiec test under the null hypothesis that expected number of exceedances equals the number of empirical VaR's violations is based on the sample principle as Bernoulli. Statistic of the Kupiec test is a likelihood ratios statistic, being distributed as χ^2 with one degree of freedom.

	Kupiec test				
	Copula	**95% VaR**	**99% VaR**	**5% VaR**	**1% VaR**
Large Portfolio	T	7.0987	0.9493***	7.8832	1.6597***
	Gaussian	8.7042	2.5394***	7.0987	6.0736*
EUR/PLN-EUR/CZK Portfolio	T	0.7681	1.6597***	0.7681***	10.7608
	Gumbel	0.7681***	0.9493***	6.3514*	6.0736*
EUR/PLN-EUR/HUF Portfolio	T	0.1657***	0.0996***	8.7042	1.6597***
	Gumbel	1.0623***	0.9493***	16.5231	0.4232***
EUR/PLN-EUR/RON Portfolio	Frank	0.3181***	1.6597***	0.0053***	0.0996***
	Plackett	0.3181***	0.9493***	0.0617***	0.9493***

Table 7. Unconditional coverage backtest

***Denotes the acceptance of null at 10%;

**Denotes the acceptance of null at 5%;

*Denotes the acceptance of null at 1%;

Thus Kupiec test as Bernoulli shows that Frank and Plackett copulas provide the best results after both copulas accepted the null of unconditional coverage test at 10 % confidence level for all the quantiles. In the same time Gaussian copula recorded the most poorly accuracy after the null was accepted at 10% was rejected only for 5% quantile of VaR. Also from Kupiec backtest we can observe that Gumbel is not an appropriate choice for sample with data distributed onto the middle of the tail’s range, as the null was rejected for 5% quantile of VaR in the case of EUR/PLN-EUR/HUF portfolio and was accepted only at 1 % confidence level in the case EUR/PLN-EUR/CZK portfolio. Student copula provided similar results as Gumbel, indicating the same bad points: low density of data in middle of the tails.

5. Conclusion

Recent turbulences from financial markets revealed the inflexibility of traditional risk models to capture the observed stylized facts. One of the main concerns in market risk modeling is how to account for common trends of the assets. So in this paper I aimed to analyze the use of copulas in financial application, namely to investigate the assumption of asymmetric dependence and to compute measures of risk. In literature are several methods outside of copulas to analyze the common evolution of financial assets but this paper is not subject to compare such approaches.

The analyze of exchange rate returns computed with a logarithmic formula reveals some typical stylized facts as autocorrelation, heteroskedasticity or volatility clustering. The use of copula requires uniform distributed data so I had to filter the returns series using an ARMA-GARCH model to compensate for autocorrelation and heteroskedasticity. For this purpose I used an asymmetric GARCH (1,1) for conditional variance, called GJR (after the authors’ names), because this model incorporate a Boolean function that takes into account for the impact of bad news. Instead the conditional mean equation was modeled by an AR(1) process for EUR/CZK and EUR/PLN, respectively by an ARMA (1,1) for the other two exchange rates.

Once the filtered residuals were obtained a semi-parametric CDF was fitted for each series. The preliminary statistic analysis revealed the need to use EVT approach for modeling the tails of distribution. To do this firstly I computed the Hill-plot for upper and lower tail of each series in order to select an appropriate threshold. The high density of extreme values in tails of EUR/HUF and EUR/RON indicated the choice of 10%, respectively 90% quantile as thresholds.

The interior CDF for each residual series was fitted by a Gaussian kernel, while GPD method was chosen to model the tails of distribution. The estimated tail parameters showed that EUR/PLN and EUR/RON behave as beta distribution in lower tail. To test the accuracy of parameters estimation I engaged a Bootstrap sampling to check the asymptotic normality. The obtained results indicated largely that estimated parameters consist with the tail behavior of original data.

Given the semi-parametric CDFs for residuals series, the following step was to fit the copula parameters using CML approach. Results indicated a positive dependence among the four currencies, underlining that each currency is most correlated with EUR/PLN and at least with EUR/RON. Estimation of conditional dependence using a Canonical Vine Copula with EUR/PLN as pivot emphasized the asymmetric dependence among the four exchange rates. For this reason I decomposed the large portfolio in three bivariate sub-portfolios consisting in EUR/PLN and each of other currencies.

Student and Gumbel copulas have recorded the lowest values of negative log-likelihood for both EUR/PLN-EUR/CZK and EUR/PLN-EUR/HUF sub-portfolios, while for EUR/PLN-EUR/HUF the information criteria indicated the selection of Frank and Plackett copula as best fit models. In the same time Copula-GARCH models emphasized the evidence of a strong asymetric deopendence in right tail for each of the three sub-portfolio.

In-sample estimation of risk measures for large portfolio and each of the three sub-portfolio with related copulas for different time horizon revealed some interesting remarks about the copula features. However computed CVaRs situated closely to the minimum and maximum of empirical returns, even though the 99% quantile of Gumbel-CVaR for EUR/PLN-EUR/HUF sub-portfolio is much higher than the maximum of 3 months empirical cumulated returns. Out-of-sample forecasting of VaR made possible both an assessment of Copula-VaR models' accuracy and also a comparison between them. Kupiec and Bernoulli backtest shows that Frank copula obtained the best results, followed by Plackett, while the Gaussian copula situated at the opposite pole. Student and Gumbel copulas provided satisfactory results, performing poorly for the 95 % quantile of VaR.

An interesting topic for future research is the use Copula-GARCH models to estimate the risk measures. Also the use of some GARCH models as FIGARCH or HYGARCH that takes into account for the long memory of financial assets could provide consistent improvement of the forecasting results.

6.References

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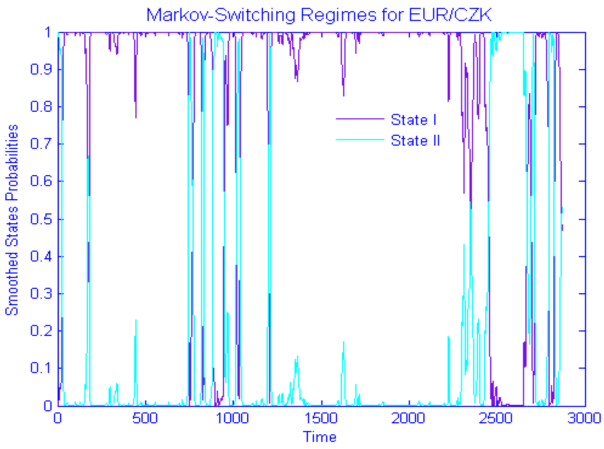
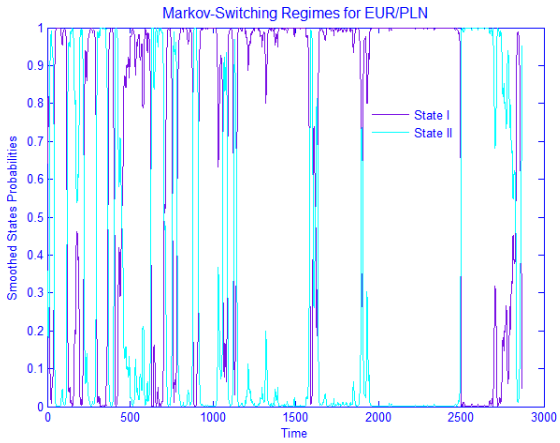
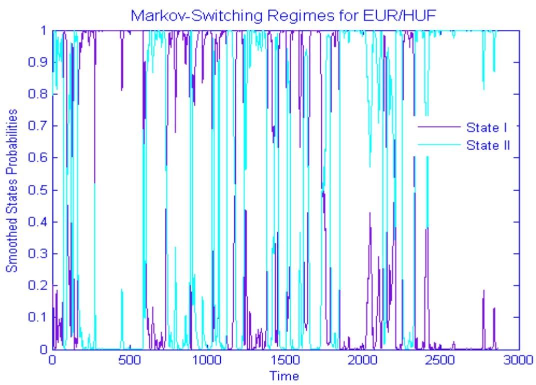
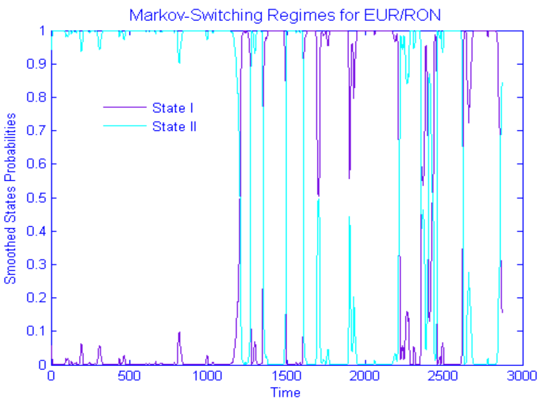
Appendix I

1. Evolution of exchange rate regimes and estimated parameters.

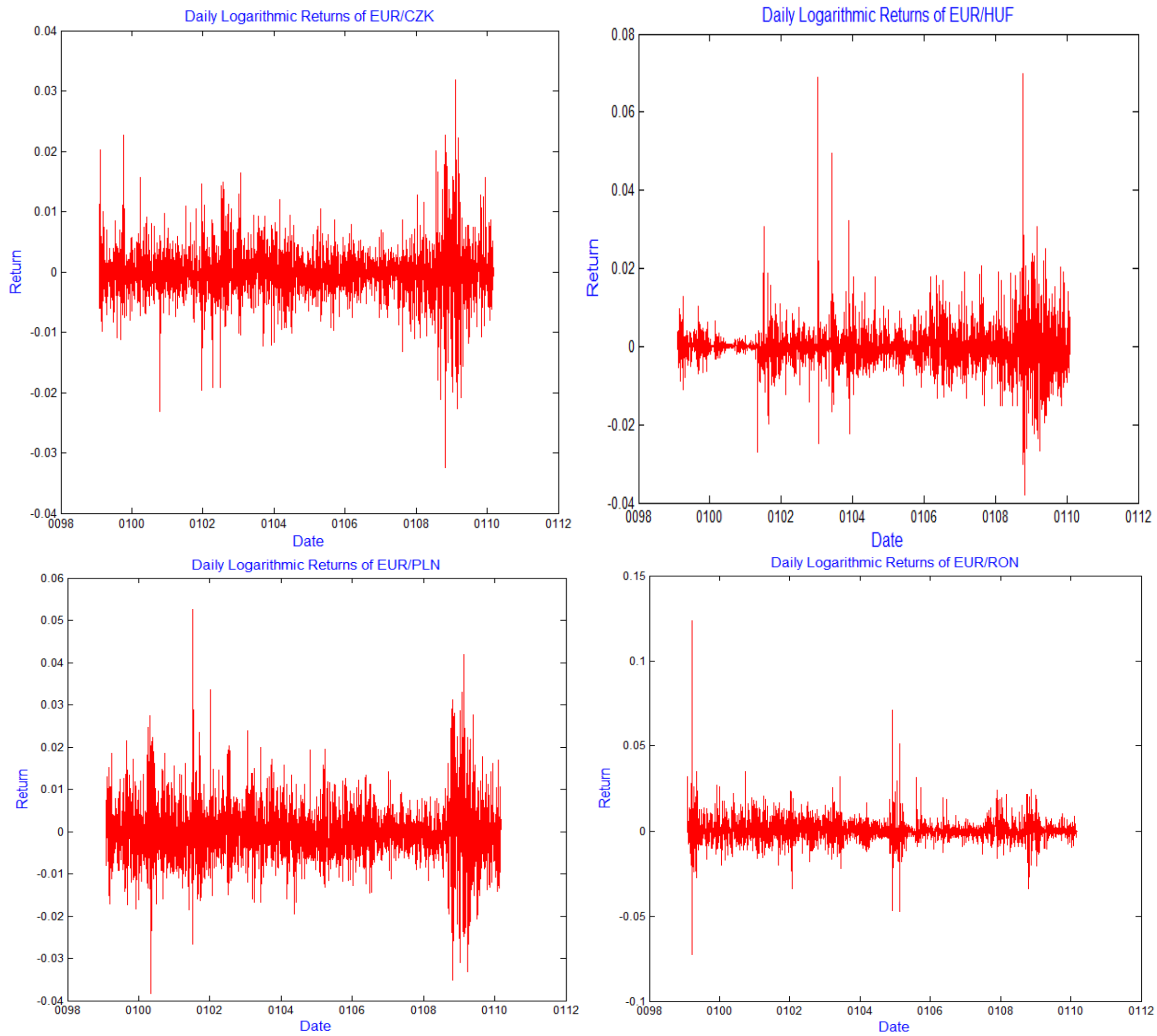




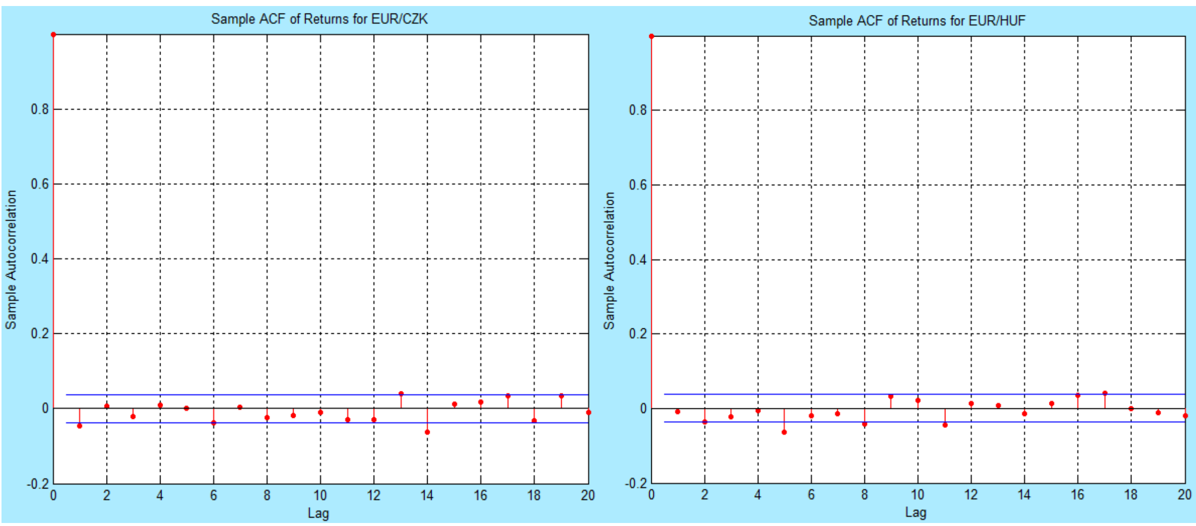


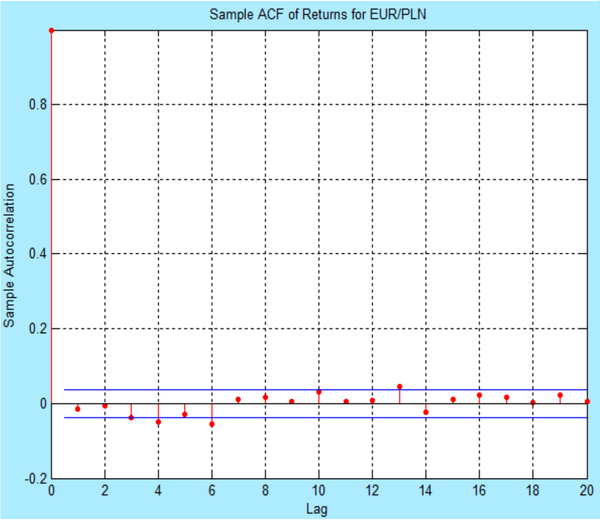


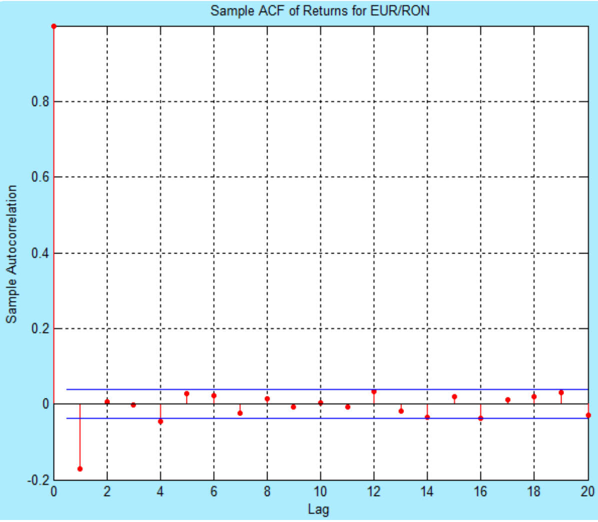
2. Exchange rate returns.

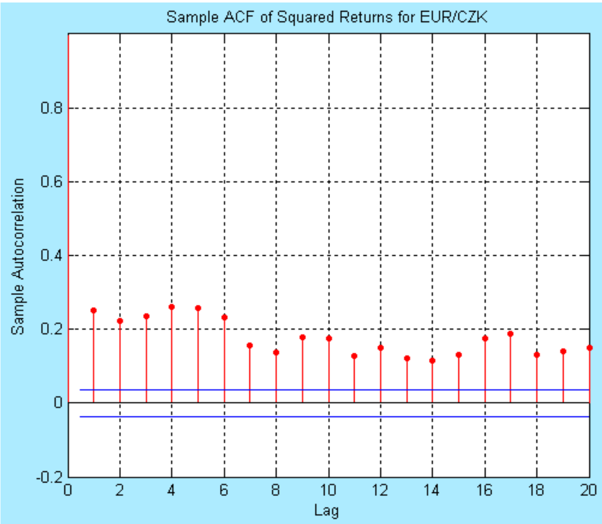


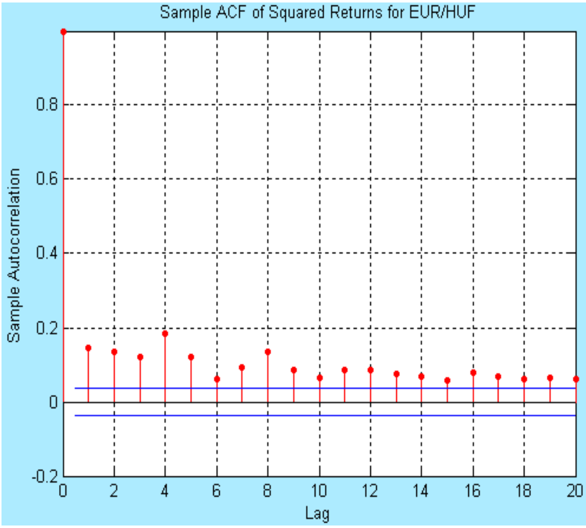
3. Autocorrelation function for exchange rate returns.

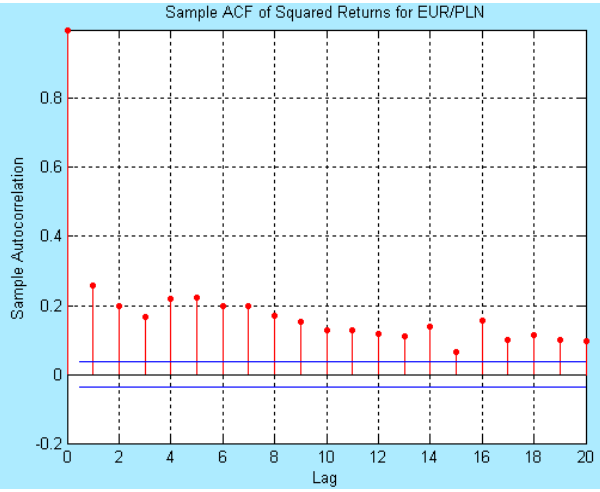


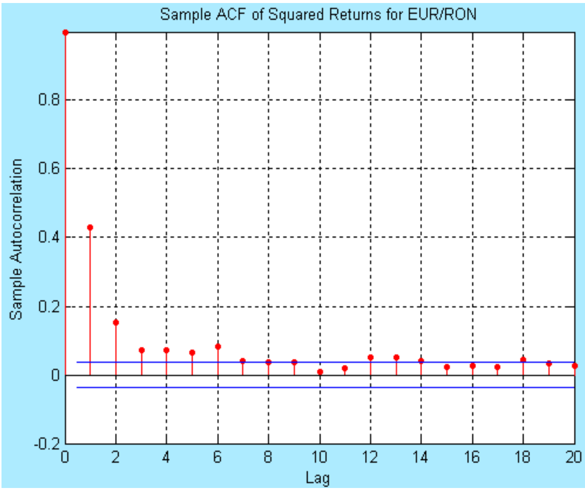

Sample ACF of Returns for EUR/PLN
Sample Autocorrelation
Lag


Sample ACF of Returns for EUR/RON
Sample Autocorrelation
Lag


Sample ACF of Squared Returns for EUR/CZK
Sample Autocorrelation
Lag


Sample ACF of Squared Returns for EUR/HUF
Sample Autocorrelation
Lag


Sample ACF of Squared Returns for EUR/PLN
Sample Autocorrelation
Lag


Sample ACF of Squared Returns for EUR/RON
Sample Autocorrelation
Lag

4. Testing for stationarity.

ADF test

Augmented Dickey-Fuller Unit Root Test on EURCZK_RETURNS

Null Hypothesis: EURCZK_RETURNS has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic - based on SIC, maxlag=27)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-55.99640	0.0000
Test critical values:	1% level	-3.961241	
	5% level	-3.411373	
	10% level	-3.127535	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Unit Root Test on EURHUF_RETURNS

Null Hypothesis: EURHUF_RETURNS has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic - based on SIC, maxlag=27)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-53.93779	0.0000
Test critical values:	1% level	-3.961241	
	5% level	-3.411373	
	10% level	-3.127535	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Unit Root Test on EURPLN_RETURNS

Null Hypothesis: EURPLN_RETURNS has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic - based on SIC, maxlag=27)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-54.28395	0.0000
Test critical values:	1% level	-3.961241	
	5% level	-3.411373	
	10% level	-3.127535	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Unit Root Test on EURRON_RETURNS

Null Hypothesis: EURRON_RETURNS has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic - based on SIC, maxlag=27)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-63.72558	0.0000
Test critical values:	1% level	-3.961241	
	5% level	-3.411373	
	10% level	-3.127535	

*MacKinnon (1996) one-sided p-values.

KPSS test

KPSS Unit Root Test on EURCZK_RETURNS

Null Hypothesis: EURCZK_RETURNS is stationary
Exogenous: Constant, Linear Trend
Bandwidth: 17 (Newey-West automatic) using Bartlett kernel

		LM-Stat.
Kwiatkowski-Phillips-Schmidt-Shin test statistic		0.038160
Asymptotic critical values*:	1% level	0.216000
	5% level	0.146000
	10% level	0.119000

*Kwiatkowski-Phillips-Schmidt-Shin (1992, Table 1)

Residual variance (no correction)	1.91E-05
HAC corrected variance (Bartlett kernel)	1.47E-05

KPSS Unit Root Test on EURHUF_RETURNS

Null Hypothesis: EURHUF_RETURNS is stationary
Exogenous: Constant, Linear Trend
Bandwidth: 38 (Newey-West automatic) using Bartlett kernel

		LM-Stat.
Kwiatkowski-Phillips-Schmidt-Shin test statistic		0.024668
Asymptotic critical values*:	1% level	0.216000
	5% level	0.146000
	10% level	0.119000

*Kwiatkowski-Phillips-Schmidt-Shin (1992, Table 1)

Residual variance (no correction)	3.39E-05
HAC corrected variance (Bartlett kernel)	2.64E-05

KPSS Unit Root Test on EURPLN_RETURNS		
Null Hypothesis: EURPLN_RETURNS is stationary Exogenous: Constant, Linear Trend Bandwidth: 24 (Newey-West automatic) using Bartlett kernel		
		LM-Stat.
Kwiatkowski-Phillips-Schmidt-Shin test statistic		0.062514
Asymptotic critical values*:	1% level	0.216000
	5% level	0.146000
	10% level	0.119000
*Kwiatkowski-Phillips-Schmidt-Shin (1992, Table 1)		
Residual variance (no correction)		4.85E-05
HAC corrected variance (Bartlett kernel)		4.18E-05

KPSS Unit Root Test on EURRON_RETURNS		
Null Hypothesis: EURRON_RETURNS is stationary Exogenous: Constant, Linear Trend Bandwidth: 13 (Newey-West automatic) using Bartlett kernel		
		LM-Stat.
Kwiatkowski-Phillips-Schmidt-Shin test statistic		0.198176
Asymptotic critical values*:	1% level	0.216000
	5% level	0.146000
	10% level	0.119000
*Kwiatkowski-Phillips-Schmidt-Shin (1992, Table 1)		
Residual variance (no correction)		5.29E-05
HAC corrected variance (Bartlett kernel)		3.40E-05

Appendix II.

1) ARMA×GARCH stats; Nyblom and Pearson tests

EUR/CZK

```
Dependent variable : EUR/CZK
Mean Equation : ARMA (1, 0) model.
No regressor in the conditional mean
Variance Equation : GJR (1, 1) model.
No regressor in the conditional variance
Student distribution, with 3.86459 degrees of freedom.
Estimation done using the MaxSA algorithm

Strong convergence using numerical derivatives
Log-likelihood = 12044.4 and temperature 2.42037e-121
Please wait : Computing the Std Errors ...

 Robust Standard Errors (Sandwich formula)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)              -0.0001785.3141e-005   -3.357  0.0008
AR(1)               -0.072902   0.018013   -4.047  0.0001
Cst(V)               0.003384 0.00045757    7.396  0.0000
ARCH(Alpha1)         0.078996  0.0049617    15.92  0.0000
GARCH(Beta1)         0.908996  0.0024765    367.0  0.0000
GJR(Gamma1)          0.015983  0.0055638    2.873  0.0041
Student(DF)          3.864590    0.22347    17.29  0.0000

No. Observations :      2870  No. Parameters   :         7
Mean (Y)         :  -0.00012  Variance (Y)     :   0.00002
Skewness (Y)     :   0.06843  Kurtosis (Y)     :   8.94481
Log Likelihood   : 12044.412
```

```
Joint Statistic of the Nyblom test of stability: 3.08212

Individual Nyblom Statistics:
Cst(M)              0.18878
AR(1)               0.59614
Cst(V)              0.72244
ARCH(Alpha1)        1.50338
GARCH(Beta1)        1.21159
GJR(Gamma1)         0.87239
Student(DF)         0.91688

Rem: Asymptotic 1% critical value for individual statistics = 0.75.
     Asymptotic 5% critical value for individual statistics = 0.47.
```

```
 Adjusted Pearson Chi-square Goodness-of-fit test

 # Cells(g)   Statistic      P-Value(g-1)
     10        16.5923         0.055496
     20        30.2927         0.048190
     30        38.8084         0.105389
     40        43.1150         0.299646
     50        67.7352         0.039246
     60        65.0244         0.275027
 Rem.:  k = 7 = # estimated parameters
```

EUR/HUF

```
Dependent variable : EURHUF
Mean Equation : ARMA (1, 1) model.
No regressor in the conditional mean
Variance Equation : GJR (1, 1) model.
No regressor in the conditional variance
Student distribution, with 2.0181 degrees of freedom.
Estimation done using the MaxSA algorithm

Weak convergence (no improvement in line search) using numeric
Log-likelihood = 11562.1 and temperature 2.35099e-037
Please wait : Computing the Std Errors ...

 Robust Standard Errors (Sandwich formula)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)               0.0000332.8089e-005    1.192  0.2335
AR(1)                0.509308   0.085561    5.953  0.0000
MA(1)               -0.605491   0.084825   -7.138  0.0000
Cst(V)              -0.005780 0.00062141   -9.301  0.0000
ARCH(Alpha1)         0.199153    0.41422    19.20  0.0000
GARCH(Beta1)         0.873435  0.0014679    595.0  0.0000
GJR(Gamma1)         -2.594578     1.0078   -2.574  0.0101
Student(DF)          2.018100  0.0010486    1925.  0.0000

No. Observations :       2870  No. Parameters   :          8
Mean (Y)         :    0.00003  Variance (Y)     :    0.00003
Skewness (Y)     :    1.23609  Kurtosis (Y)     :   16.10989
Log Likelihood   : 11562.107

Warning : To avoid numerical problems, the estimated parameter
Cst(V), and its std.Error have been multiplied by 10^4.
```

Joint Statistic of the Nyblom test of stability: 7.09717

Individual Nyblom Statistics:

Cst(M)	1.09405
AR(1)	1.93688
MA(1)	1.39516
Cst(V)	0.26313
ARCH(Alpha1)	0.70106
GARCH(Beta1)	1.22941
GJR(Gamma1)	0.34467
Student(DF)	0.81756

Rem: Asymptotic 1% critical value for individual statistics = 0.75.
Asymptotic 5% critical value for individual statistics = 0.47.

Adjusted Pearson Chi-square Goodness-of-fit test

# Cells(g)	Statistic	P-Value(g-1)
10	24.1324	0.004097
20	102.1254	0.000000
30	140.0557	0.000000
40	132.3136	0.000000
50	147.9443	0.000000
60	182.1394	0.000000

EUR/PLN

Dependent variable : EURPLN
Mean Equation : ARMA (1, 0) model.
No regressor in the conditional mean
Variance Equation : GJR (1, 1) model.
No regressor in the conditional variance
Student distribution, with 8.06866 degrees of freedom.
Estimation done using the MaxSA algorithm

Strong convergence using numerical derivatives
Log-likelihood = 10614.3 and temperature 3.72529e-008
Please wait : Computing the Std Errors ...

Robust Standard Errors (Sandwich formula)

	Coefficient	Std.Error	t-value	t-prob
Cst(M)	-0.000290	9.2978e-005	-3.115	0.0019
AR(1)	-0.070370	0.019241	-3.657	0.0003
Cst(V)	0.005346	0.0019892	2.688	0.0072
ARCH(Alpha1)	0.087309	0.017575	4.968	0.0000
GARCH(Beta1)	0.918672	0.016096	57.08	0.0000
GJR(Gamma1)	-0.037361	0.017003	-2.197	0.0281
Student(DF)	8.068659	1.2986	6.213	0.0000

No. Observations :	2870	No. Parameters :	7
Mean (Y) :	-0.00001	Variance (Y) :	0.00005
Skewness (Y) :	0.40093	Kurtosis (Y) :	7.30904
Log Likelihood :	10614.335		

```
Joint Statistic of the Nyblom test of stability: 3.08212

Individual Nyblom Statistics:
Cst(M)              0.18878
AR(1)               0.59614
Cst(V)              0.72244
ARCH(Alpha1)        1.50338
GARCH(Beta1)        1.21159
GJR(Gamma1)         0.87239
Student(DF)         0.91688

Rem: Asymptotic 1% critical value for individual statistics = 0.75.
     Asymptotic 5% critical value for individual statistics = 0.47.
         Adjusted Pearson Chi-square Goodness-of-fit test

         # Cells(g)  Statistic      P-Value(g-1)
             10       16.5923         0.055496
             20       30.2927         0.048190
             30       38.8084         0.105389
             40       43.1150         0.299646
             50       67.7352         0.039246
             60       65.0244         0.275027
         Rem.: k = 7 = # estimated parameters
```

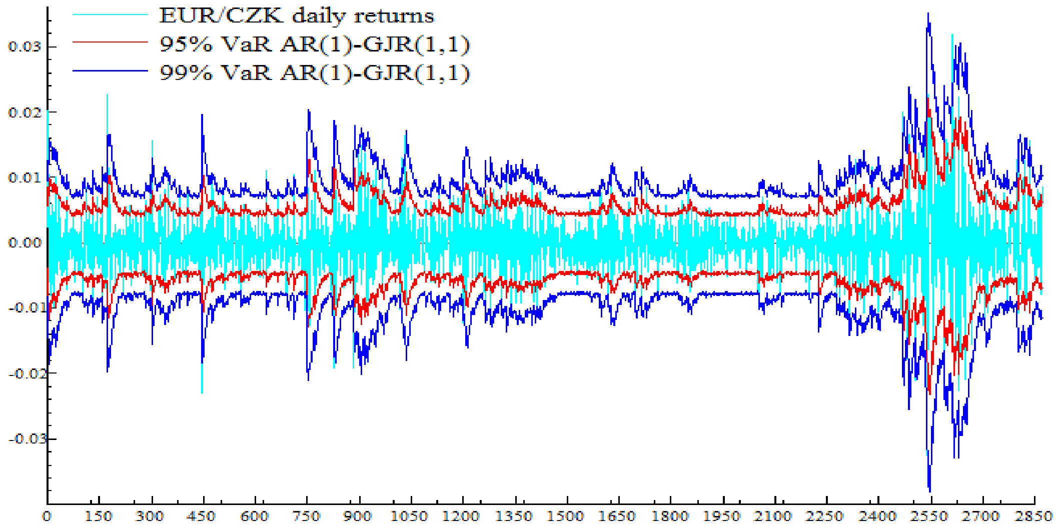
EUR/RON

```
Dependent variable : EURRON
Mean Equation : ARMA (1, 1) model.
No regressor in the conditional mean
Variance Equation : GJR (1, 1) model.
No regressor in the conditional variance
Student distribution, with 3.51438 degrees of freedom.
Estimation done using the MaxSA algorithm

Strong convergence using numerical derivatives
Log-likelihood = 10970.3 and temperature 3.18618e-057
Please wait : Computing the Std Errors ...

 Robust Standard Errors (Sandwich formula)
                  Coefficient  Std.Error  t-value  t-prob
Cst(M)              -0.003541   0.020823  -0.1700  0.8650
AR(1)                0.999776  0.0013028    767.4  0.0000
MA(1)               -0.996381  0.0016558   -601.7  0.0000
Cst(V)               0.006321 0.00076446    8.268  0.0000
ARCH(Alpha1)         0.139777   0.012143    11.51  0.0000
GARCH(Beta1)         0.863213  0.0061180    141.1  0.0000
GJR(Gamma1)          0.050850   0.013263    3.834  0.0001
Student(DF)          3.514377    0.19462    18.06  0.0000

No. Observations :      2870  No. Parameters    :         8
Mean (Y)         :   0.00038  Variance (Y)      :   0.00005
Skewness (Y)     :   1.90277  Kurtosis (Y)      :  42.75461
Log Likelihood   : 10970.292
```

```
Joint Statistic of the Nyblom test of stability: 7.61897

Individual Nyblom Statistics:
Cst(M)              1.46734
AR(1)               0.33566
MA(1)               0.32906
Cst(V)              1.94896
ARCH(Alpha1)        1.33683
GARCH(Beta1)        1.91960
GJR(Gamma1)         1.43161
Student(DF)         2.45990
Rem: Asymptotic 1% critical value for individual statistics = 0.75.
     Asymptotic 5% critical value for individual statistics = 0.47.

         Adjusted Pearson Chi-square Goodness-of-fit test

         # Cells(g)   Statistic       P-Value(g-1)
             10        17.8606         0.036825
             20        24.4530         0.179347
             30        44.3275         0.034159
             40        54.9059         0.046963
             50        62.2648         0.096603
             60        79.8676         0.036569
```

2) In-Sample VaR estimation and backtests

EUR/CZK



In-sample Value-at-Risk Backtesting

Kupiec LR test

- Short positions -

Quantile	Failure rate	Kupiec LRT	P-value	ESF1	ESF2
0.95000	0.95505	1.5941	0.20674	0.0097000	1.4303
0.99000	0.99338	3.7597	0.052503	0.014786	1.2952

- Long positions -

Quantile	Failure rate	Kupiec LRT	P-value	ESF1	ESF2
0.050000	0.044599	1.8259	0.17661	-0.0092412	1.4143
0.010000	0.0066202	3.7597	0.052503	-0.015346	1.2967

Dynamic Quantile Test of Engle and Manganelli (2002)

- Short positions -

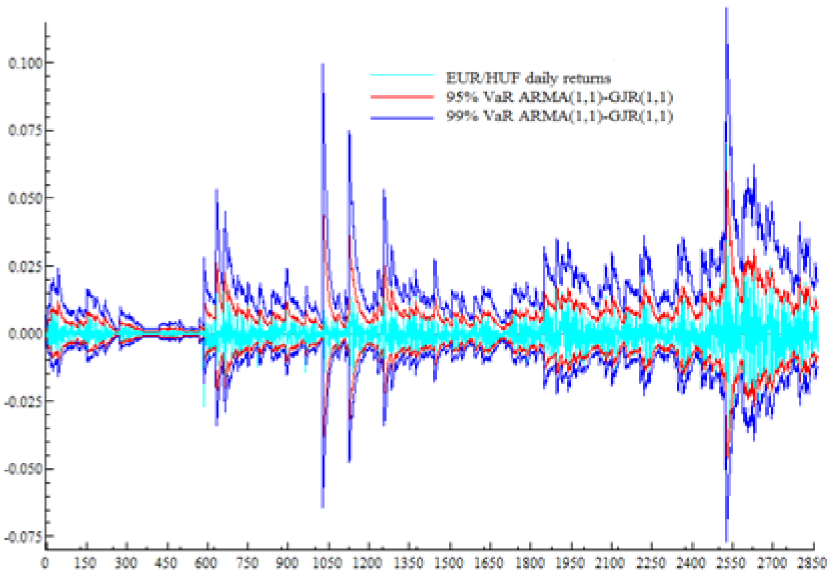
Quantile	Stat.	P-value
0.95000	4.9310	0.55270
0.99000	3.6941	0.71799

- Long positions -

Quantile	Stat.	P-value
0.050000	4.5128	0.60763
0.010000	11.884	0.064611

Remark: In the Dynamic Quantile Regression, p=5.

EUR/HUF



In-sample Value-at-Risk Backtesting

Kupiec LR test

- Short positions -

Quantile	Failure rate	Kupiec LRT	P-value	ESF1	ESF2
0.95000	0.95780	3.8642	0.049326	0.012616	1.7749
0.99000	0.99512	9.3455	0.0022354	0.022211	1.9393

- Long positions -

Quantile	Failure rate	Kupiec LRT	P-value	ESF1	ESF2
0.050000	0.022672	56.115	6.8390e-014	-0.0088022	1.6023
0.010000	0.0013952	33.797	6.1162e-009	-0.014138	2.8745

Dynamic Quantile Test of Engle and Manganelli (2002)

- Short positions -

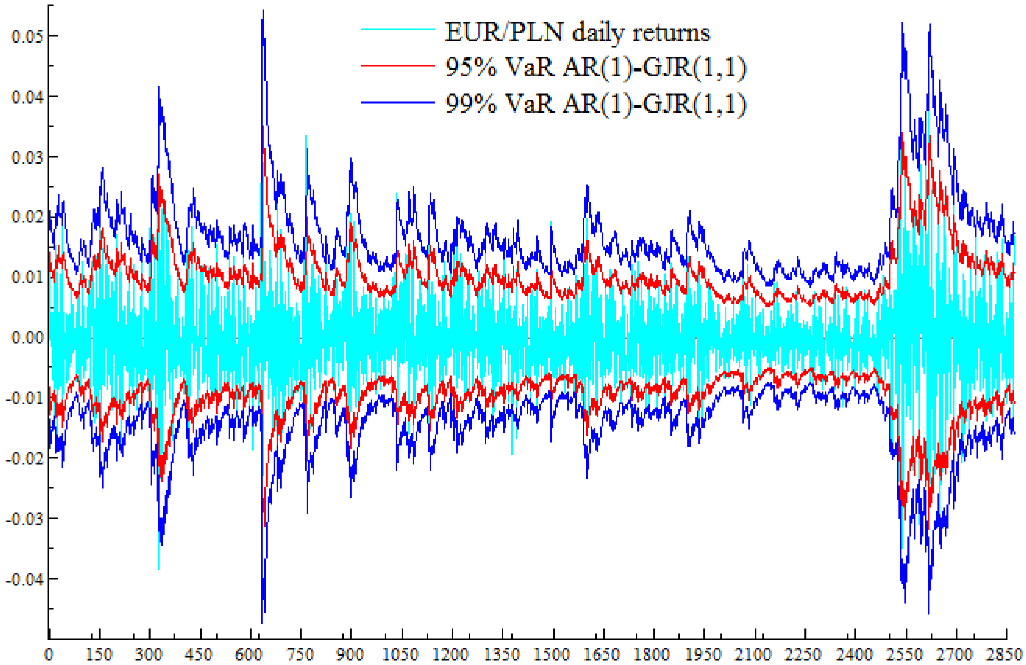
Quantile	Stat.	P-value
0.95000	4.9713	0.54750
0.99000	7.7274	0.25876

- Long positions -

Quantile	Stat.	P-value
0.050000	45.453	3.8033e-008
0.010000	21.447	0.0015246

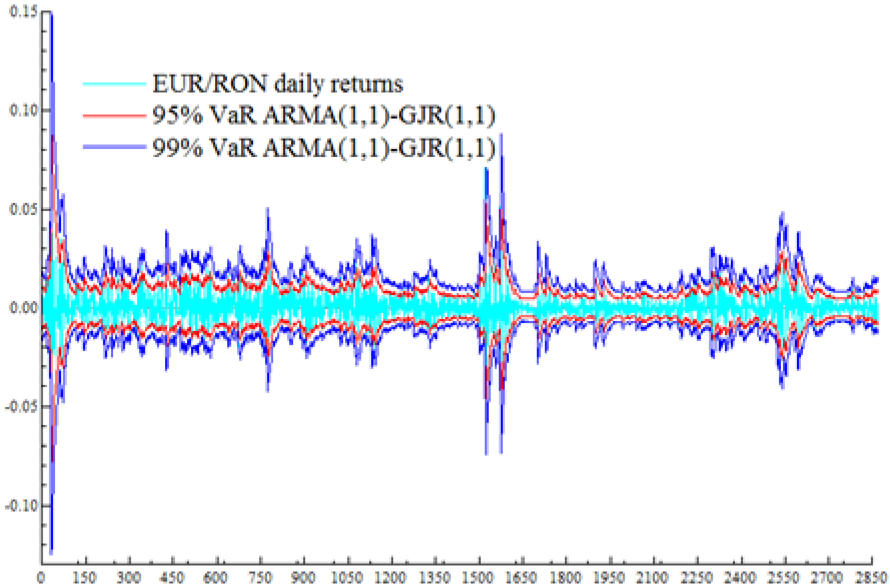
Remark: In the Dynamic Quantile Regression, p=5.

EUR/PLN



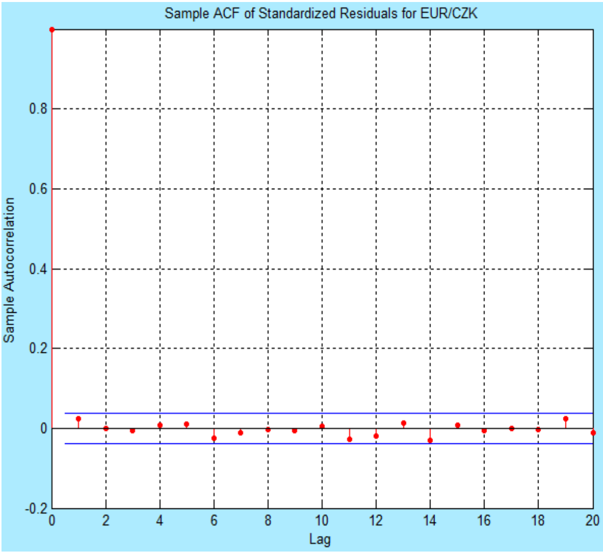
```
In-sample Value-at-Risk Backtesting

Kupiec LR test
--------------
                              - Short positions -
     Quantile    Failure rate   Kupiec LRT     P-value         ESF1          ESF2

      0.95000       0.93763       8.6004     0.0033609     0.013886       1.3962
      0.99000       0.98641       3.3572      0.066911     0.019791       1.3079
                               - Long positions -
     Quantile    Failure rate   Kupiec LRT     P-value         ESF1          ESF2

     0.050000      0.038676       8.3752     0.0038038    -0.013991       1.2749
     0.010000     0.0048780       9.3764     0.0021980    -0.019592       1.1367

Dynamic Quantile Test of Engle and Manganelli (2002)
--------------------------------------------------------------------------------

           - Short positions -
     Quantile          Stat.       P-value

      0.95000         10.410       0.10841
      0.99000         7.1146       0.31038

            - Long positions -
     Quantile          Stat.       P-value

     0.050000         9.9845       0.12531
     0.010000         7.7785       0.25478

Remark: In the Dynamic Quantile Regression, p=5.
```

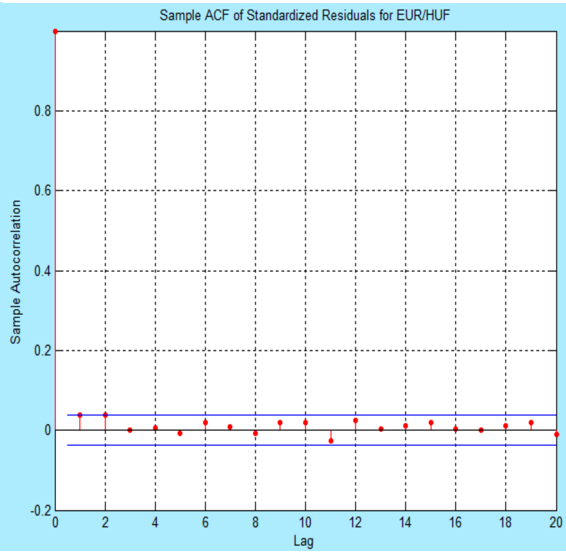
EUR/RON

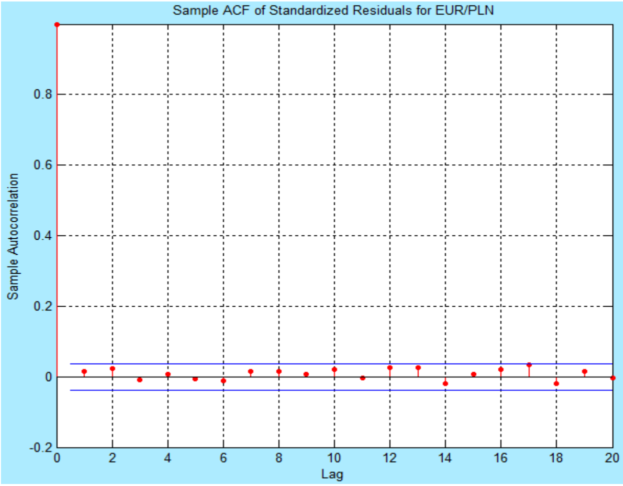


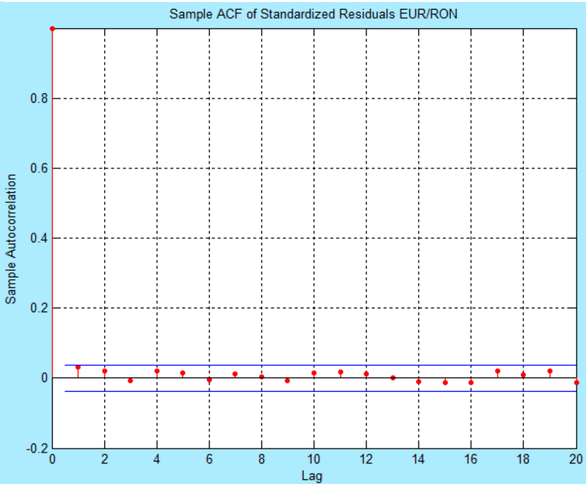
```
In-sample Value-at-Risk Backtesting

Kupiec LR test
--------------
                              - Short positions -
     Quantile   Failure rate   Kupiec LRT    P-value        ESF1          ESF2

      0.95000       0.94070       4.9348     0.026322     0.014622       1.5098
      0.99000       0.99058      0.10020      0.75158     0.028451       1.4844
                               - Long positions -
     Quantile   Failure rate   Kupiec LRT    P-value        ESF1          ESF2

     0.050000      0.044297       2.0380      0.15341    -0.012595       1.3133
     0.010000     0.0031392       18.621  1.5948e-005    -0.020228       1.1598

Dynamic Quantile Test of Engle and Manganelli (2002)
---------------------------------------------------------------------------------


           - Short positions -
     Quantile          Stat.       P-value

      0.95000         7.5895       0.26975
      0.99000         3.1535       0.78934

            - Long positions -
     Quantile          Stat.       P-value

     0.050000         5.3364       0.50145
     0.010000         13.677      0.033455

Remark: In the Dynamic Quantile Regression, p=5.
```

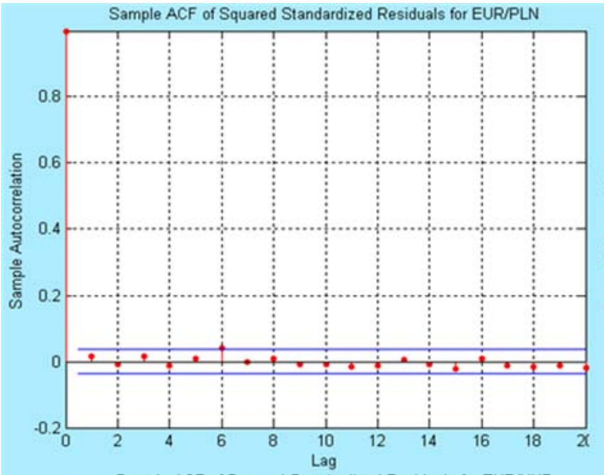
3) Autocorrelation function for filtered residuals.

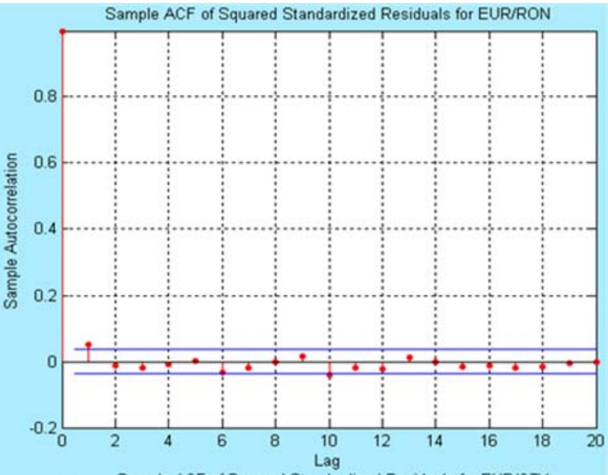


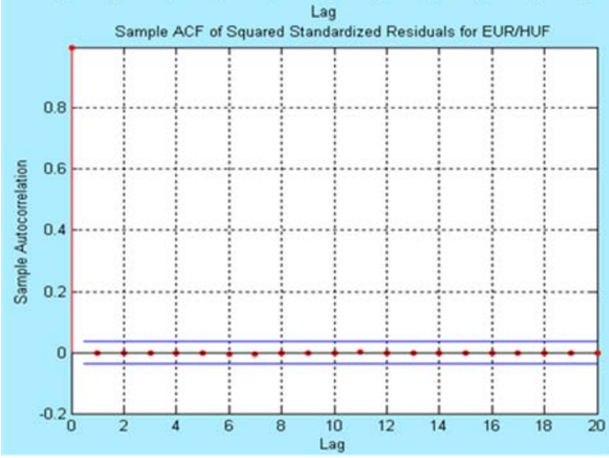


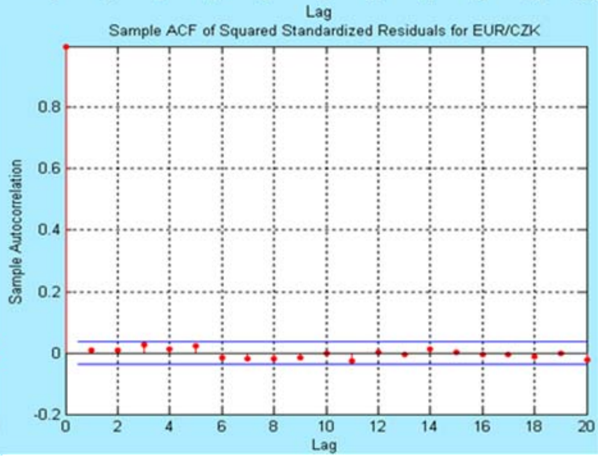

Sample ACF of Standardized Residuals for EUR/PLN
Sample Autocorrelation
Lag


Sample ACF of Standardized Residuals EUR/RON
Sample Autocorrelation
Lag


Sample ACF of Squared Standardized Residuals for EUR/PLN
Sample Autocorrelation
Lag


Sample ACF of Squared Standardized Residuals for EUR/RON
Sample Autocorrelation
Lag

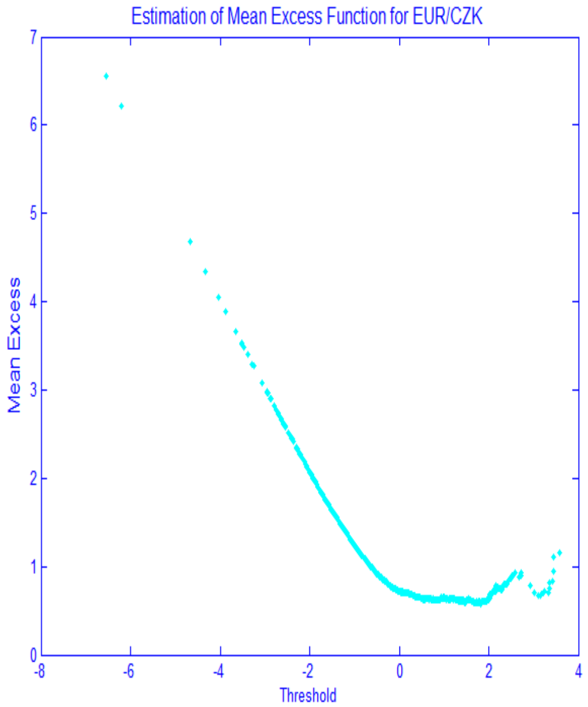

Sample ACF of Squared Standardized Residuals for EUR/HUF
Sample Autocorrelation
Lag

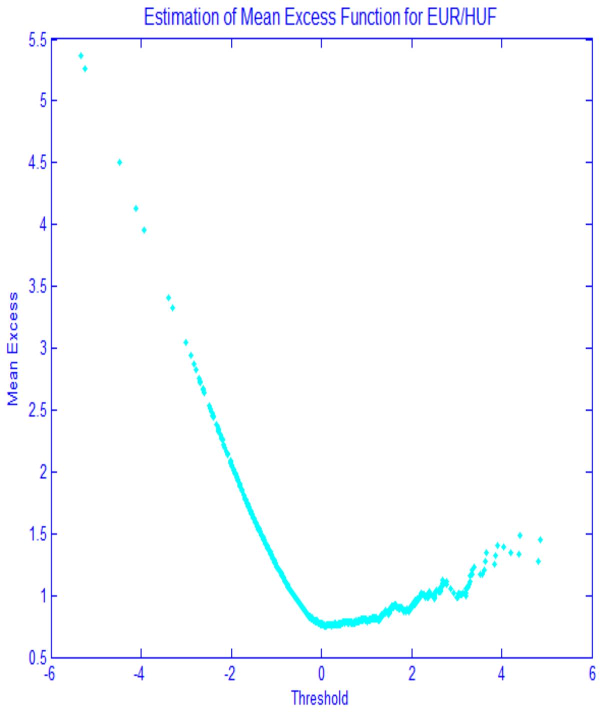

Sample ACF of Squared Standardized Residuals for EUR/CZK
Sample Autocorrelation
Lag

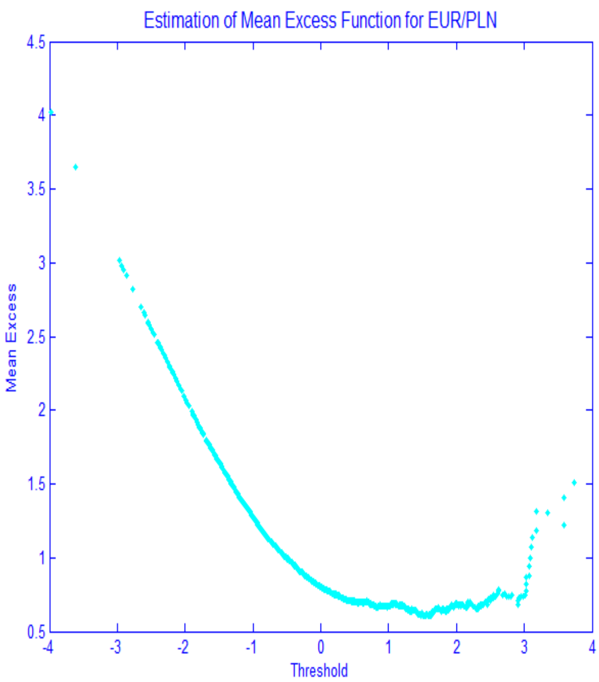
Appendix III

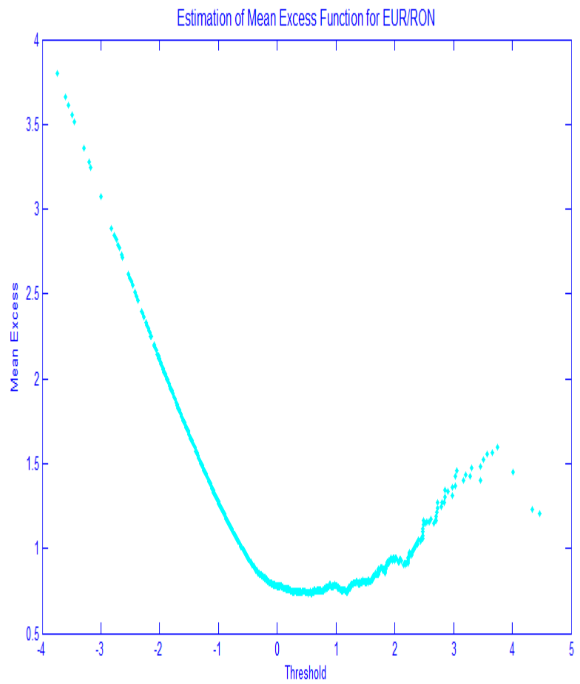
1. Preliminary statistic analyzes of filtered data.

i) Computation of Mean Excess Function

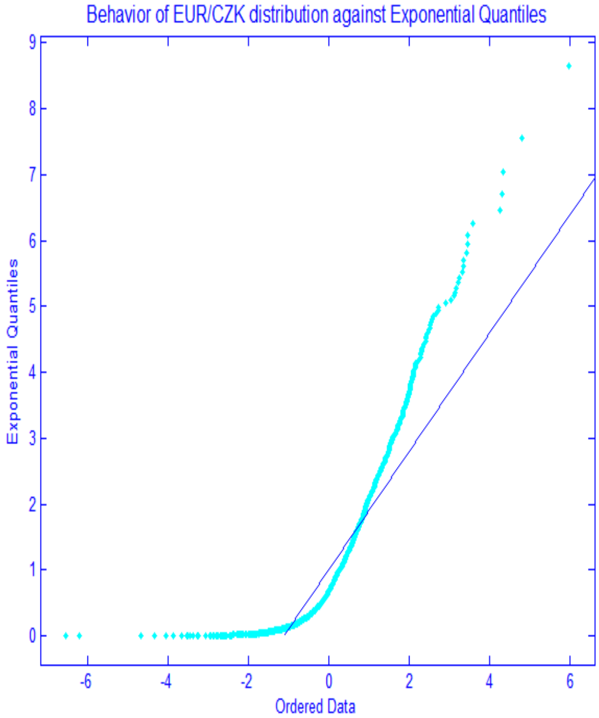


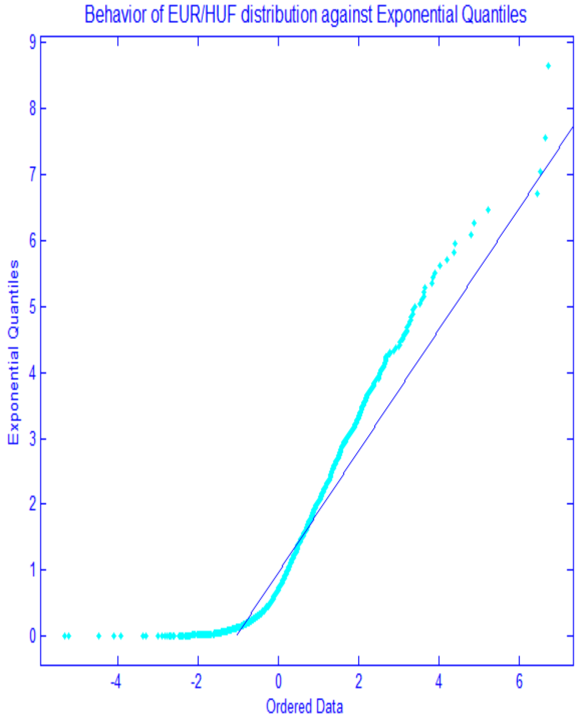


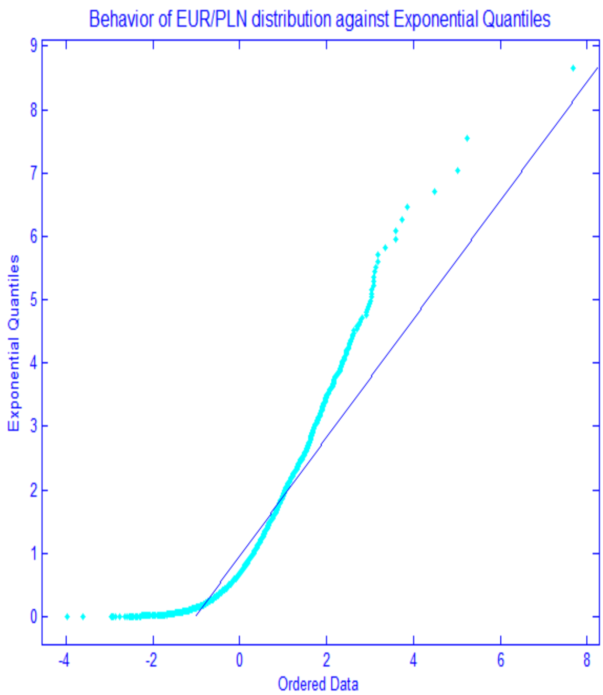


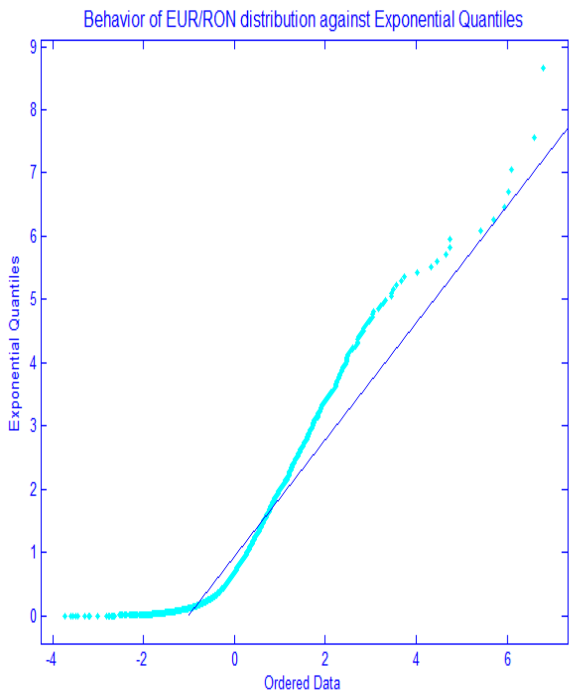


_ii_) QQ-plot against Exponential distribution.



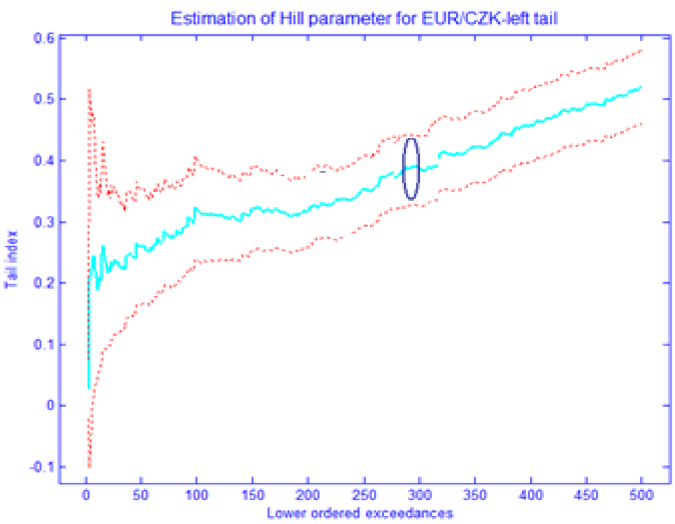


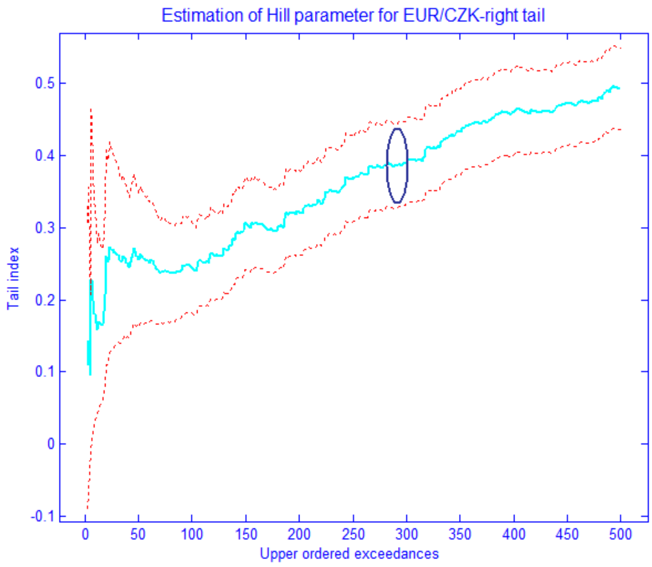




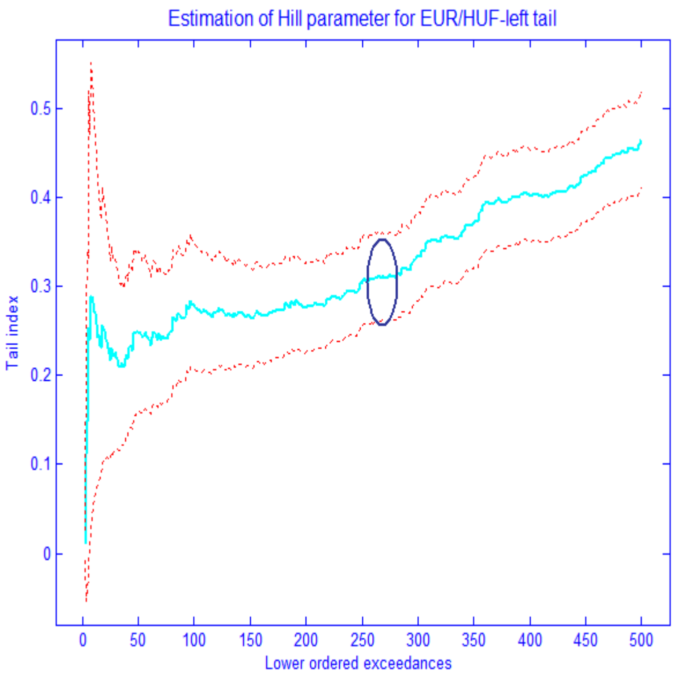
***ii*) Hill-plot inference for lower and upper tails.**

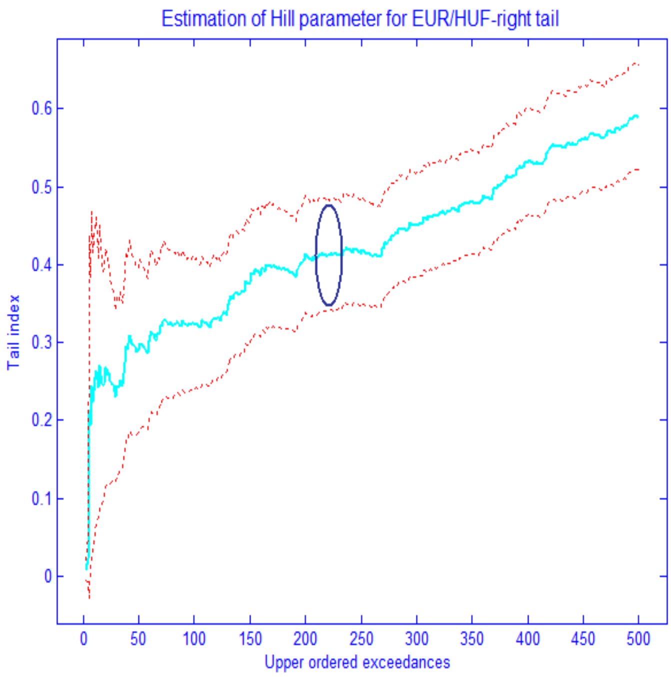
EUR/CZK



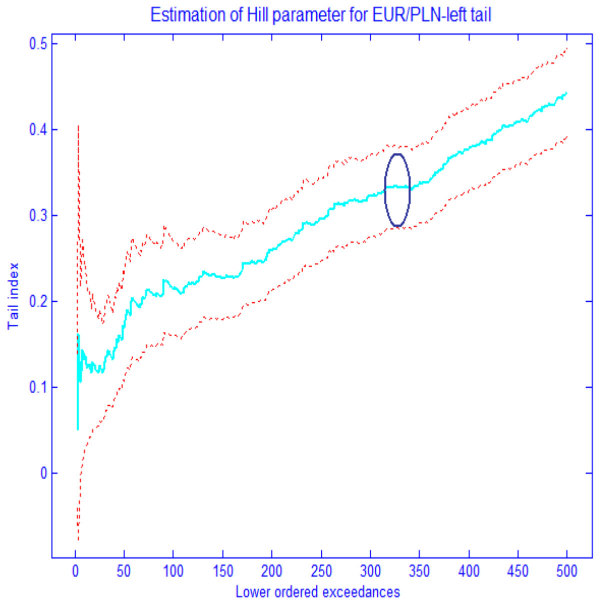


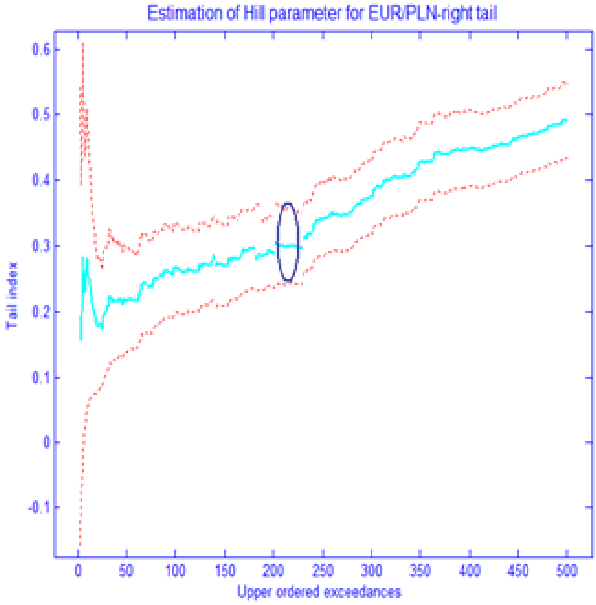
EUR/HUF



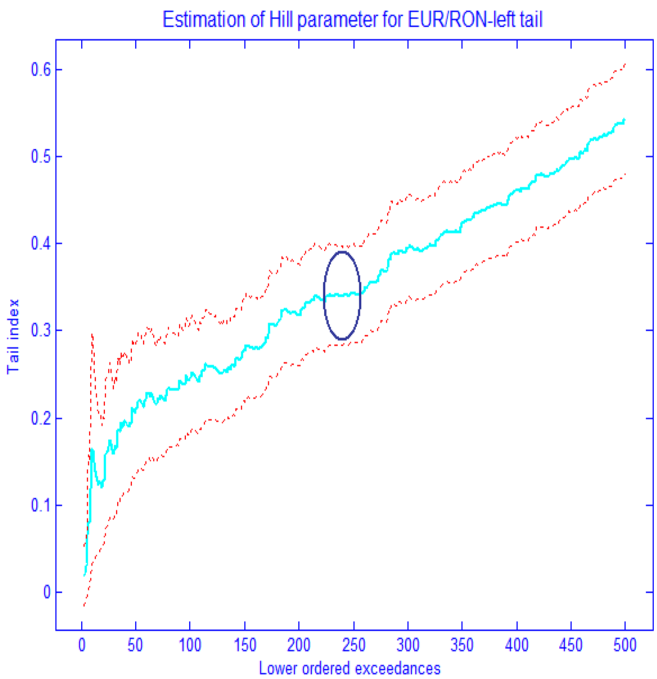


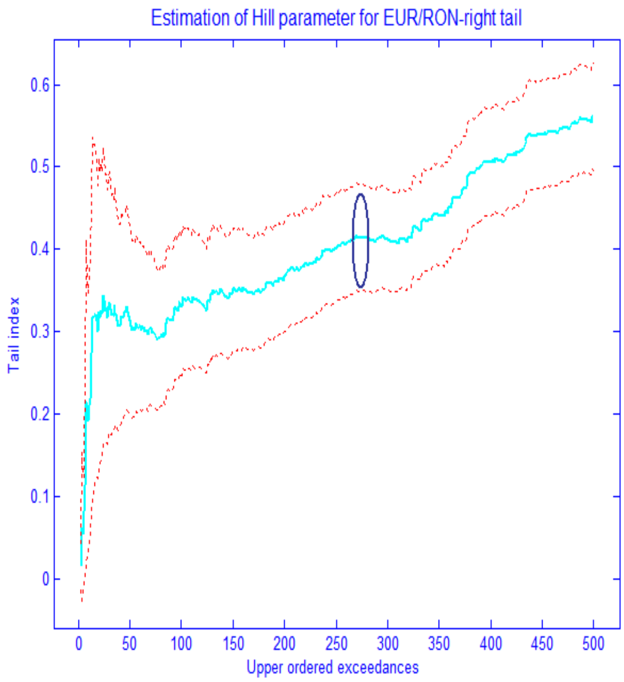
EUR/PLN





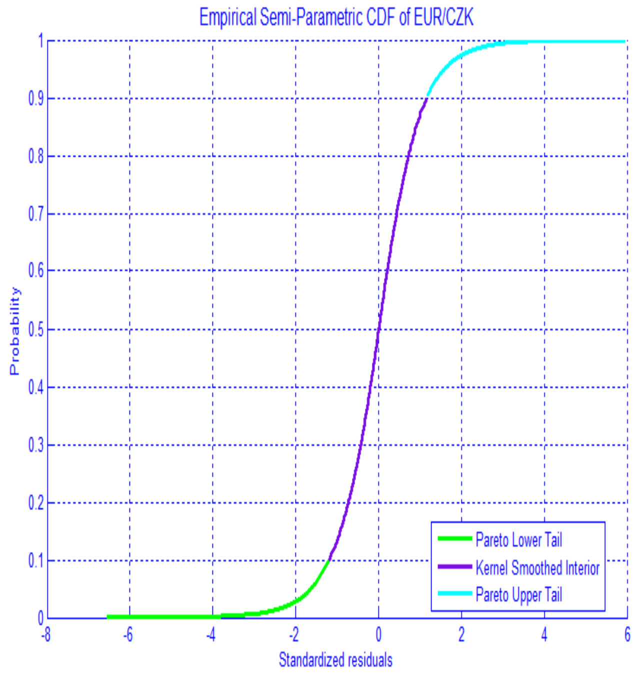
EUR/RON



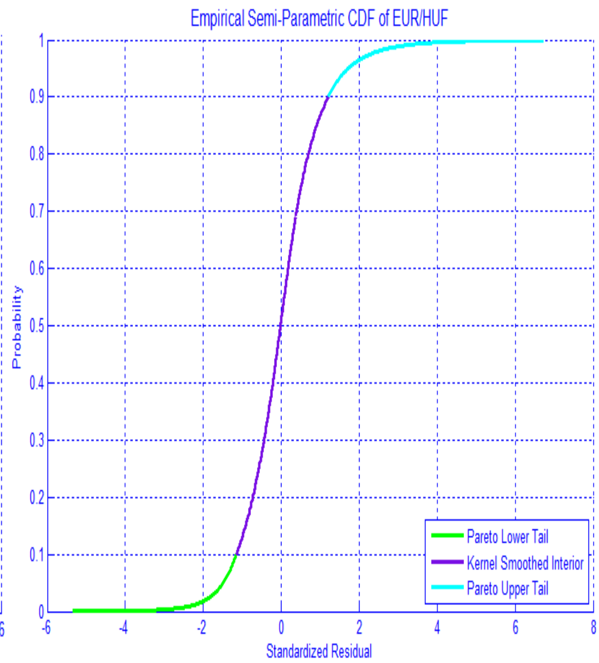


2. GPD modelling.

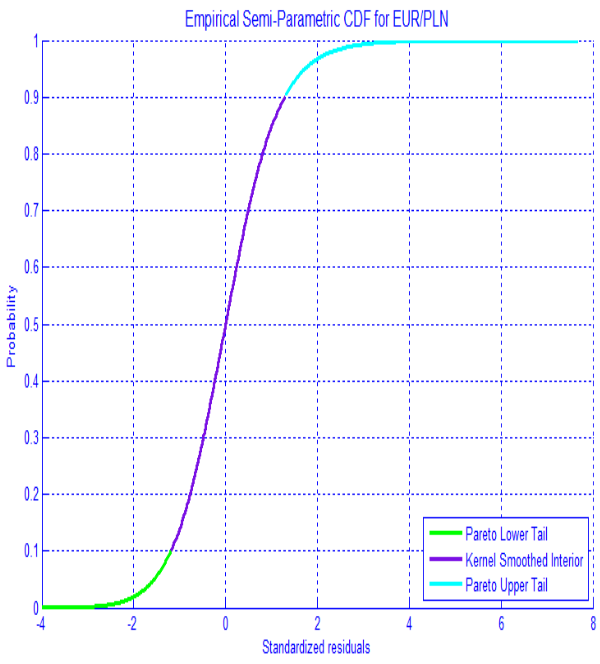
***i*) Semi-parametric CDFs of filtered innovations.**



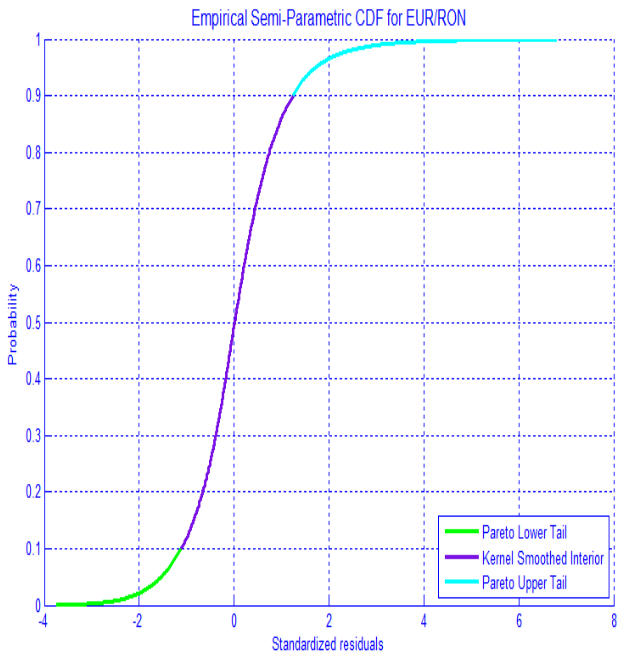
Empirical Semi-Parametric CDF of EUR/CZK
Probability
Standardized residuals
Pareto Lower Tail
Kernel Smoothed Interior
Pareto Upper Tail



Empirical Semi-Parametric CDF of EUR/HUF
Probability
Standardized Residual
Pareto Lower Tail
Kernel Smoothed Interior
Pareto Upper Tail



Empirical Semi-Parametric CDF for EUR/PLN
Probability
Standardized residuals
Pareto Lower Tail
Kernel Smoothed Interior
Pareto Upper Tail

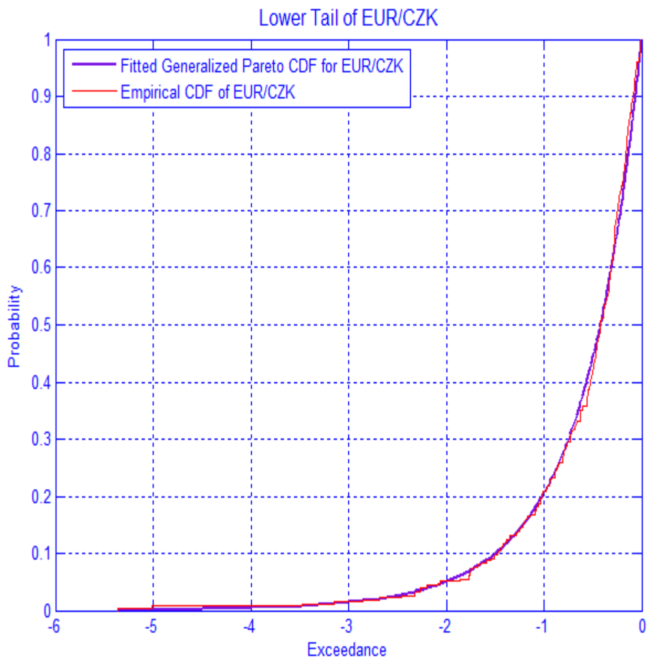


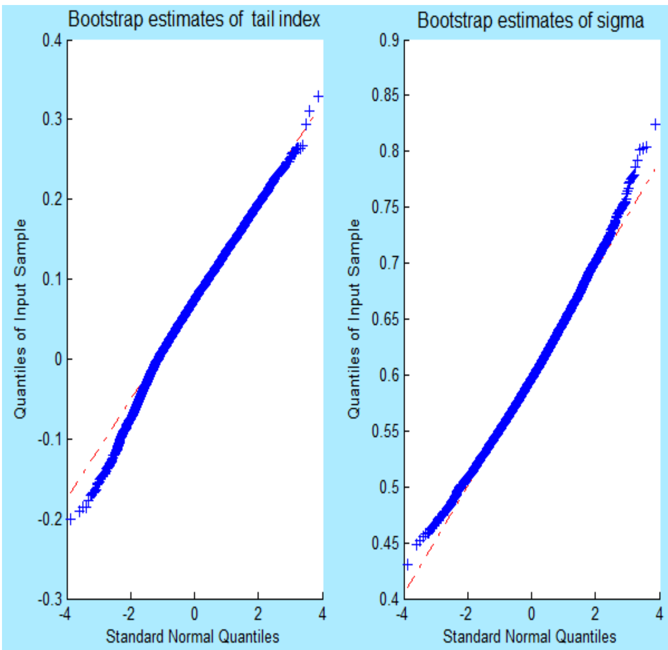
Empirical Semi-Parametric CDF for EUR/RON
Probability
Standardized residuals
Pareto Lower Tail
Kernel Smoothed Interior
Pareto Upper Tail

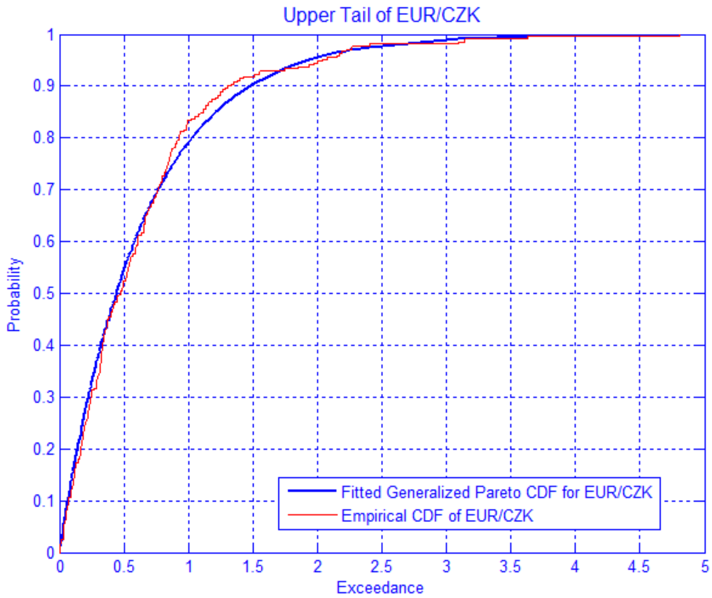
***ii*)Inference for GPD parameters**

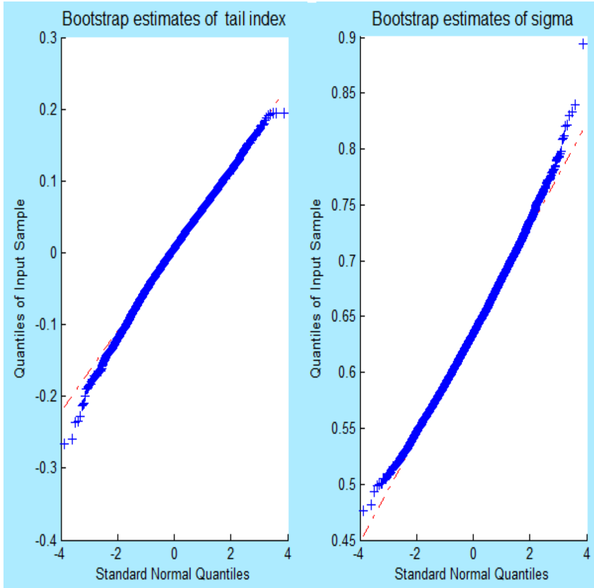
Parameters	EUR/CZK				EUR/HUF			
	Lower tail		Upper tail		Lower tail		Upper tail	
	ξ	σ	Ξ	σ	ξ	σ	ξ	σ
ML estimates	0.0813 (0.1797)	0.5934 (0.0000)	0.0140 (0.7998)	0.6307 (0.0000)	0.1264 (0.0452)	0.4327 (0.0000)	0.1253 (0.0698)	0.7151 (0.0000)
Standard Errors	0.0606	0.0507	0.0552	0.0518	0.0631	0.0373	0.0691	0.0652
Lower limits of Confidence interval	-0.0375	0.5019	-0.0941	0.5370	0.0027	0.3654	-0.0102	0.5981
Upper limits of Confidence interval	0.2000	0.7016	0.1222	0.7407	0.2502	0.5124	0.2607	0.8549

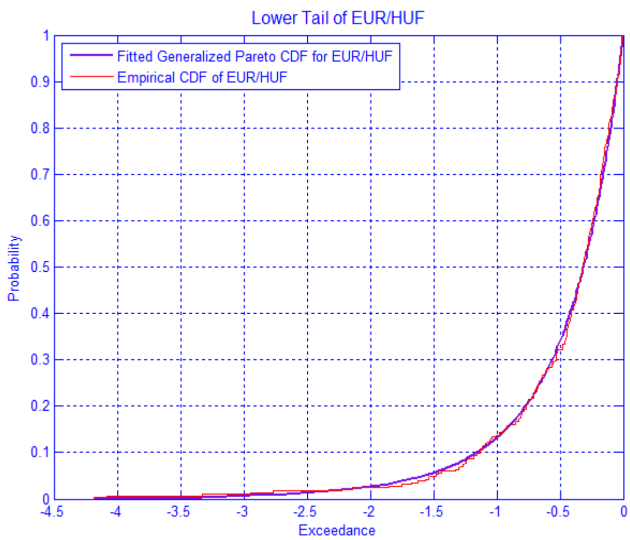
***iii*) Check the Asymptotical Normality**

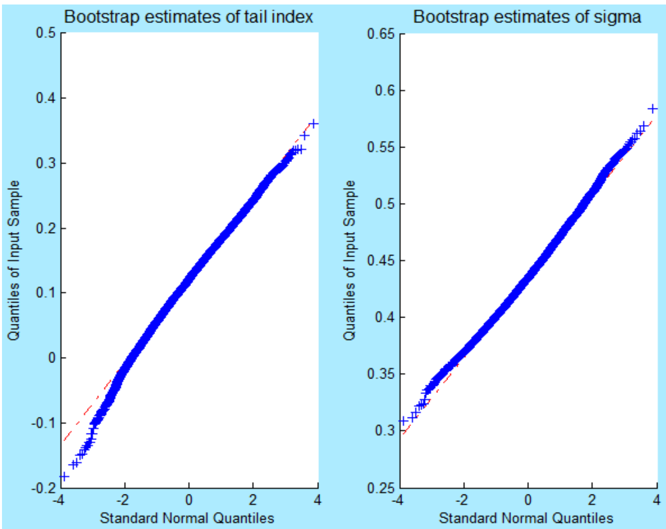


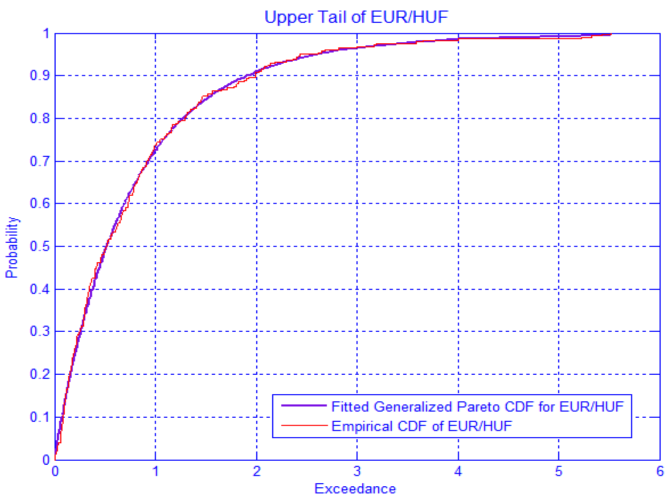


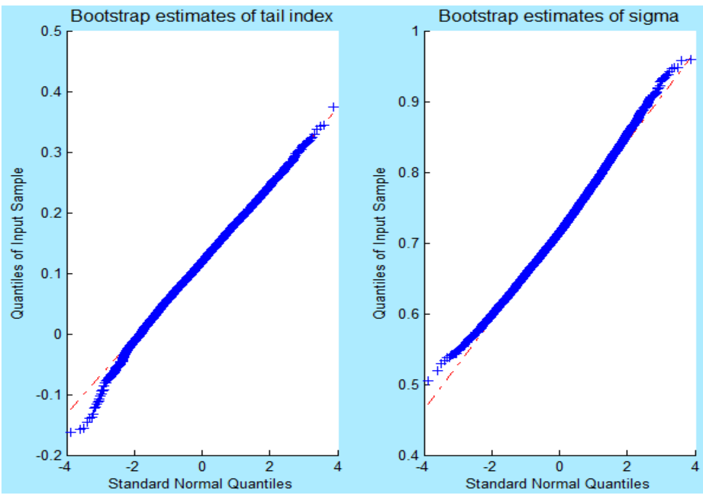

Upper Tail of EUR/CZK
Fitted Generalized Pareto CDF for EUR/CZK
Empirical CDF of EUR/CZK
Probability
Exceedance

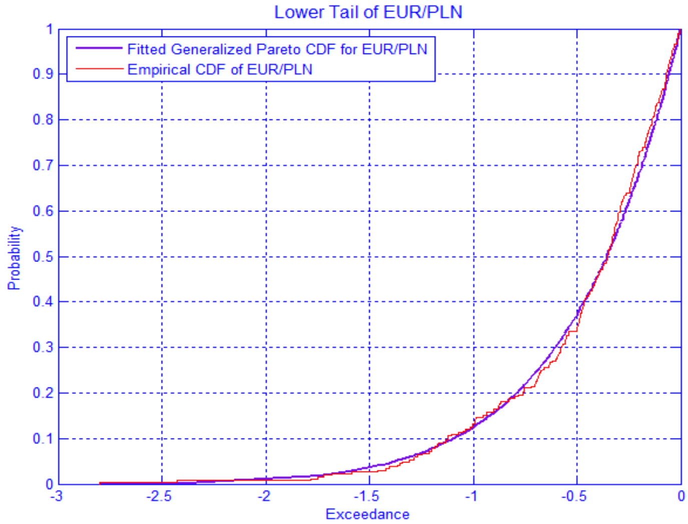

Bootstrap estimates of tail index
Bootstrap estimates of sigma
Quantiles of Input Sample
Standard Normal Quantiles

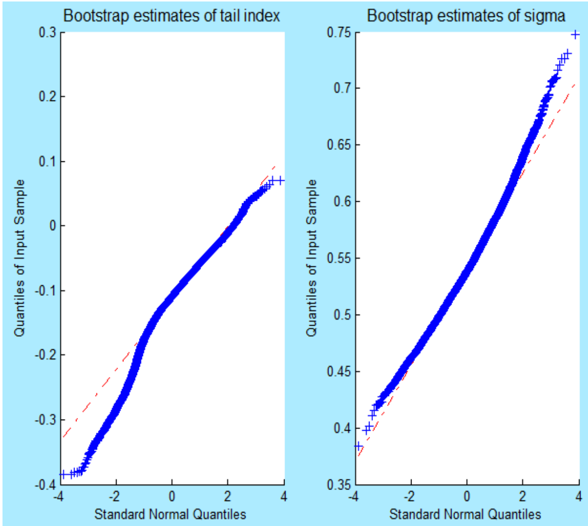

Lower Tail of EUR/HUF
Fitted Generalized Pareto CDF for EUR/HUF
Empirical CDF of EUR/HUF
Probability
Exceedance

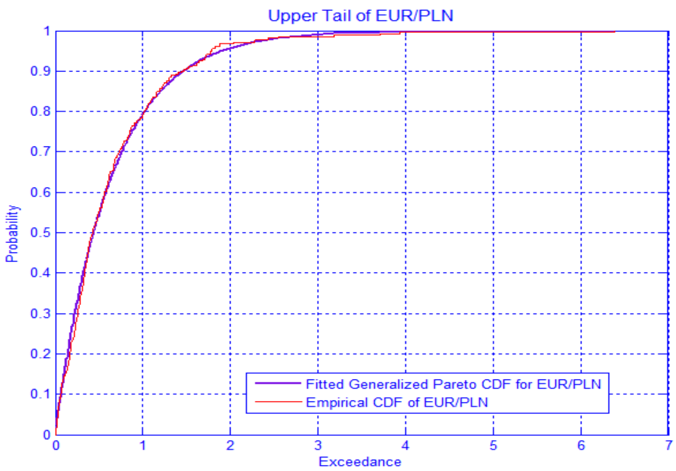

Bootstrap estimates of tail index
Bootstrap estimates of sigma
Quantiles of Input Sample
Standard Normal Quantiles

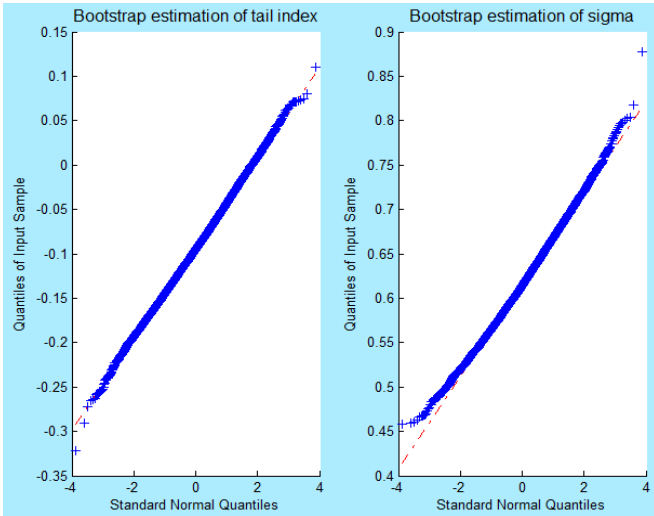

Upper Tail of EUR/HUF
Fitted Generalized Pareto CDF for EUR/HUF
Empirical CDF of EUR/HUF
Probability
Exceedance

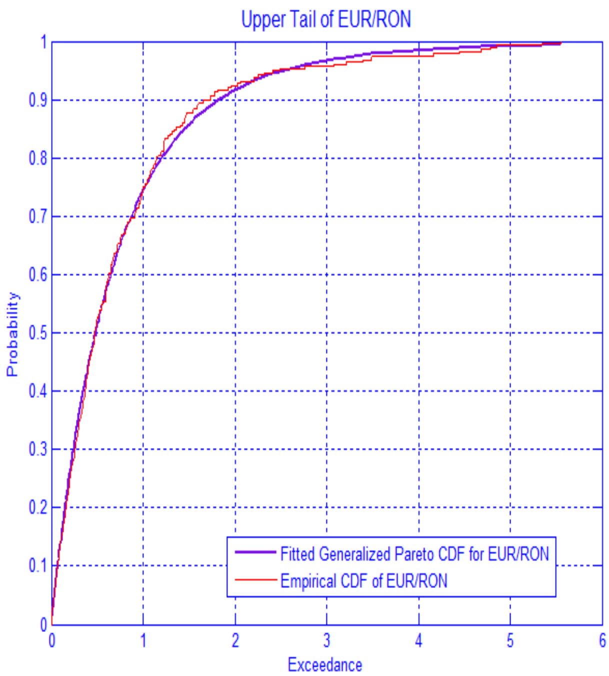

Bootstrap estimates of tail index
Bootstrap estimates of sigma
Quantiles of Input Sample
Standard Normal Quantiles

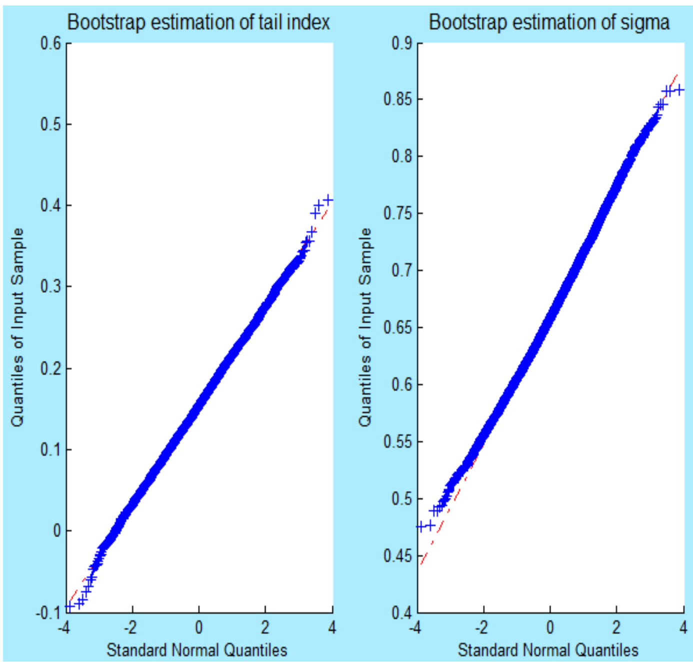

Lower Tail of EUR/PLN
Fitted Generalized Pareto CDF for EUR/PLN
Empirical CDF of EUR/PLN
Probability
Exceedance


Bootstrap estimates of tail index
Bootstrap estimates of sigma
Quantiles of Input Sample
Standard Normal Quantiles


Upper Tail of EUR/PLN
Fitted Generalized Pareto CDF for EUR/PLN
Empirical CDF of EUR/PLN
Probability
Exceedance


Bootstrap estimation of tail index
Bootstrap estimation of sigma
Quantiles of Input Sample
Standard Normal Quantiles


Upper Tail of EUR/RON
Fitted Generalized Pareto CDF for EUR/RON
Empirical CDF of EUR/RON
Probability
Exceedance


Bootstrap estimation of tail index
Bootstrap estimation of sigma
Quantiles of Input Sample
Standard Normal Quantiles

Appendix IV

1. Estimation of copula parameters for the four currencies portfolio.

DoF	DoF CI	
17.3080	12.1811	22.4348

Correlation Matrix for T-copula

	EUR/CZK	EUR/HUF	EUR/PLN	EUR/RON
EUR/CZK	1.0000	0.2954	0.3446	0.1453
EUR/HUF	0.2954	1.0000	0.4764	0.2332
EUR/PLN	0.3446	0.4764	1.0000	0.3388
EUR/RON	0.1453	0.2332	0.3388	1.0000

Correlation Matrix for T-copula

	EUR/CZK	EUR/HUF	EUR/PLN	EUR/RON
EUR/CZK	1.0000	0.2816	0.3303	0.1345
EUR/HUF	0.2816	1.0000	0.4618	0.2240
EUR/PLN	0.3303	0.4618	1.0000	0.3311
EUR/RON	0.1345	0.2240	0.3311	1.0000

Empirical Kendall's *tau*

	EUR/CZK	EUR/HUF	EUR/PLN	EUR/RON
EUR/CZK	1.0000	0.1882	0.2153	0.0896
EUR/HUF	0.1882	1.0000	0.3175	0.1433
EUR/PLN	0.2153	0.3175	1.0000	0.2238
EUR/RON	0.0896	0.1433	0.2238	1.0000

Theoretical R using Kendall's tau

	EUR/CZK	EUR/HUF	EUR/PLN	EUR/RON
EUR/CZK	1.0000	0.2913	0.3317	0.1403
EUR/HUF	0.2913	1.0000	0.4783	0.2232
EUR/PLN	0.3317	0.4783	1.0000	0.3443
EUR/RON	0.1403	0.2232	0.3443	1.0000

Empirical Spearman's *rho*

	EUR/CZK	EUR/HUF	EUR/PLN	EUR/RON
EUR/CZK	1.0000	0.2769	0.3135	0.1336
EUR/HUF	0.2769	1.0000	0.4561	0.2116
EUR/PLN	0.3135	0.4561	1.0000	0.3269
EUR/RON	0.1336	0.2116	0.3269	1.0000

Theoretical R using Spearman'*rho*

	EUR/CZK	EUR/HUF	EUR/PLN	EUR/RON
EUR/CZK	1.0000	0.2890	0.3268	0.1398
EUR/HUF	0.2890	1.0000	0.4731	0.2211
EUR/PLN	0.3268	0.4731	1.0000	0.3407
EUR/RON	0.1398	0.2211	0.3407	1.0000

2. Estimation of copula parameters bivariate EUR/PLN-EUR/CZK sub-portfolio.

i) Copula parameters.

Kendall's tau	Theoretical Rho of the sample	Gaussian	T				Clayton			Frank		
		R	R	DoF	CI		θ	CI		θ	CI	
0.2153	0.3317	0.3302	0.3424	9.5650	5.6044	13.5255	0.3947	0.3448	0.4447	2.1141	1.8906	2.3376

Gumbel			Rotated Clayton			Rotated Gumbel			Plackett			SJC	
θ	CI		θ	CI		θ	CI		θ	CI		τ-Lower	τ-Upper
1.2602	1.2267	1.2937	0.4291	0.3792	0.4791	1.2530	1.2195	1.2866	2.8710	2.6475	3.0945	0.1365	0.1879

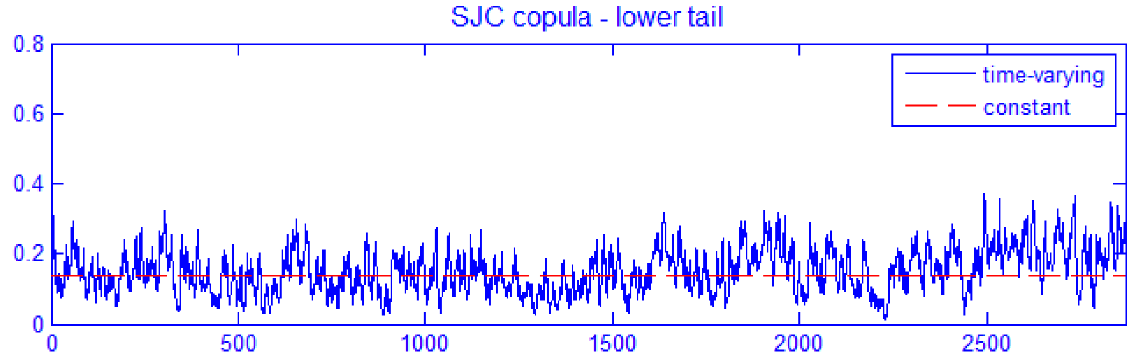
Tyme-varying Rotated Gumbel			Tyme-varying Gumbel			Tyme-varying SJC					
Ω	β	α	Ω	β	α	Ω-Lower	β-Lower	α-Lower	Ω-Upper	β-Upper	α-Upper
0.7420	0.1256	-1.5428	0.2080	0.4210	-0.9072	0.2397	-8.1954	1.2386	0.4134	-7.9968	-1.0793

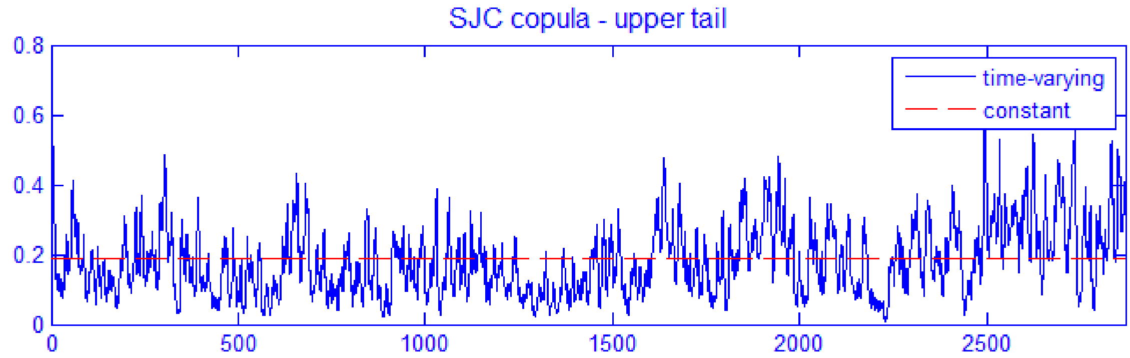
ii) Tail Dependence and Information Criteria.

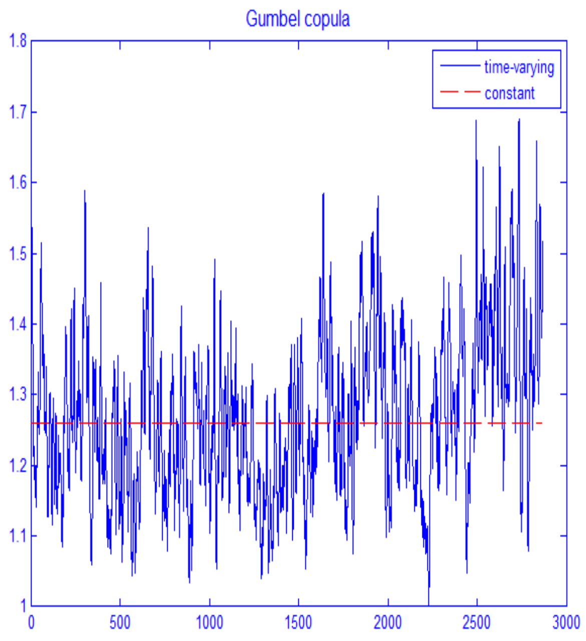
Copula	Tail Dependence Lower	Upper
Gaussian	0	0
Clayton	0.1729	0
Rotated Clayton	0	0.1989
Plackett	0	0
Frank	0	0
Gumbel	0	0.2667
Rotated Gumbel	0.2612	0
T	0.0449	0.0449
SJC	0.1365	0.1879

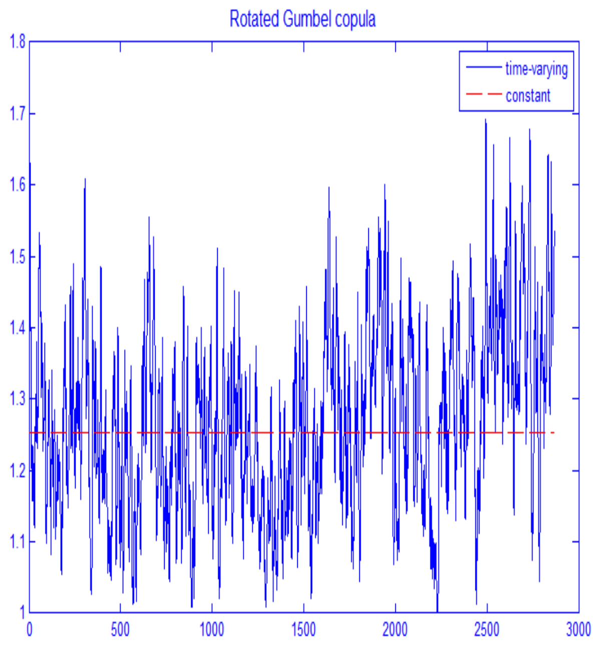
Copula	Information Criteria NLL	AIC	BIC
Gaussian	-165.6123	-331.224	-331.222
Clayton	-117.6286	-235.257	-235.255
Rotated Clayton	-144.4857	-288.971	-288.969
Plackett	-163.1631	-326.326	-326.323
Frank	-154.9511	-309.902	-309.9
Gumbel	-166.6571	-333.314	-333.311
Rotated Gumbel	-150.4854	-300.97	-300.968
T	-179.3969	-358.792	-358.788
SJC	-176.2975	-352.594	-352.59
Copula-GARCH			
Rotated Gumbel	-176.0552	-352.108	-352.102
Gumbel	-188.9844	-377.967	-377.96
Symmetrised Joe-Clayton	-198.7753	-397.546	-397.534

***iii*) Copula-GARCH.**









3. Estimation of copula parameters bivariate EUR/PLN-EUR/HUF sub-portfolio.

i) Copula parameters.

Kendall's tau	Theoretical Rho of the sample	Gaussian	T-copula				Clayton			Frank		
		R	R	DoF	CI		θ	CI		θ	CI	
0.3175	0.4783	0.4618	0.4759	12.6829	6.4495	18.9163	0.5628	0.5085	0.6171	3.2315	2.9968	3.4662

Gumbel			Rotated Clayton			Rotated Gumbel			Plackett			SJC	
θ	CI		θ	CI		θ	CI		θ	CI		τ-Lower	τ-Upper
1.4293	1.3900	1.4687	0.6929	0.6387	0.7472	1.3893	1.3500	1.4286	4.4910	4.2563	4.7257	0.170839	0.345223

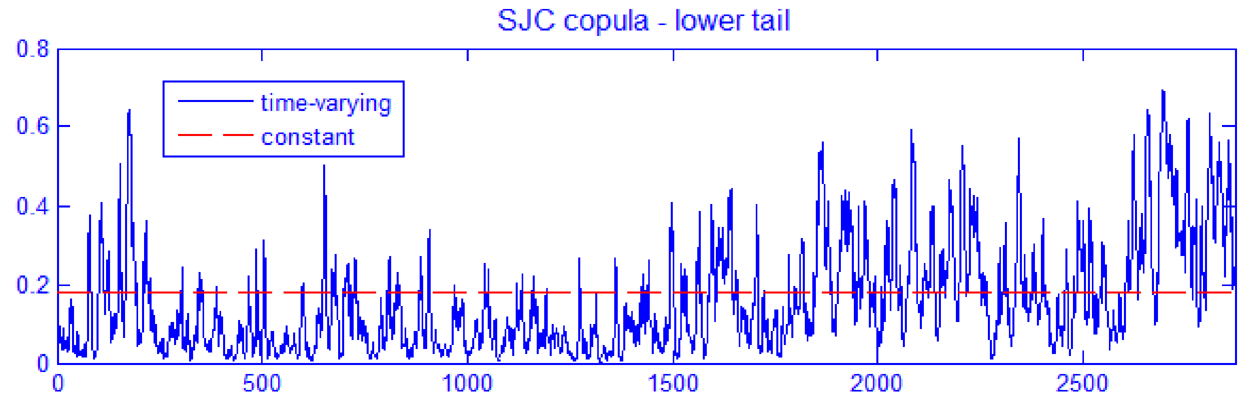
Tyme varying Rotated Gumbel			Tyme varying Rotated Gumbel			Tyme varying SJC					
Ω	β	α	Ω	β	α	Ω-Lower	β-Lower	α-Lower	Ω-Upper	β-Upper	α-Upper
0.336375	0.39426	-1.179	0.357299	0.388526	-1.1701	-1.9619	-0.45904	4.0943	0.946996	-14.2717	0.844943

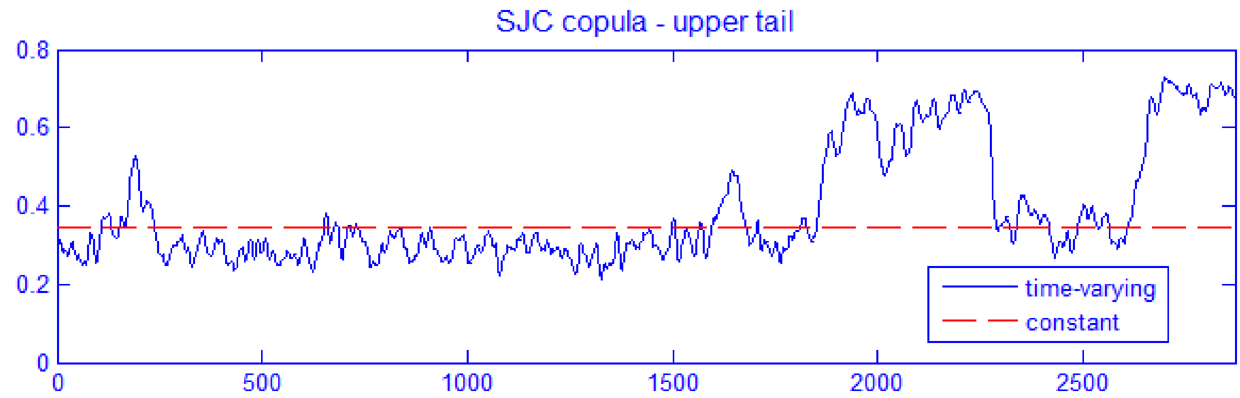
ii) Tail Dependence and Information Criteria.

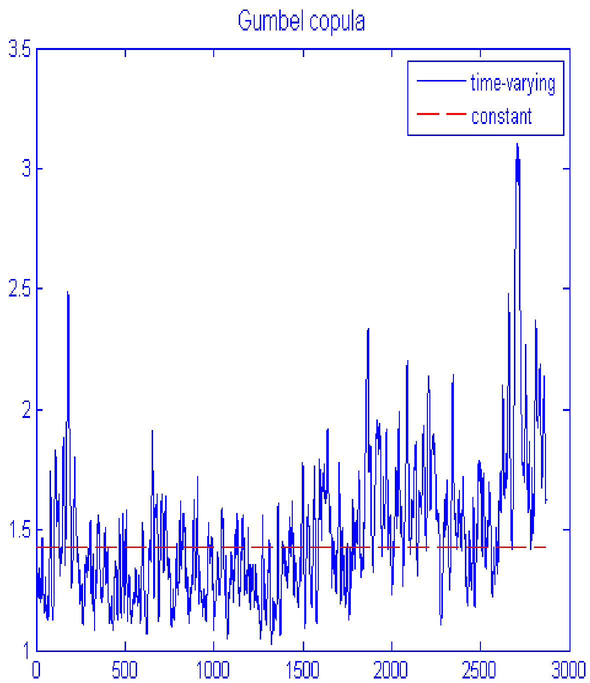
	Tail Dependence	
Copula	Lower	Upper
Gaussian	0	0
Clayton	0.1729	0
Rotated Clayton	0	0.1989
Plackett	0	0
Frank	0	0
Gumbel	0	0.2667
Rotated Gumbel	0.2612	0
T	0.0449	0.0449
SJC	0.1365	0.1879

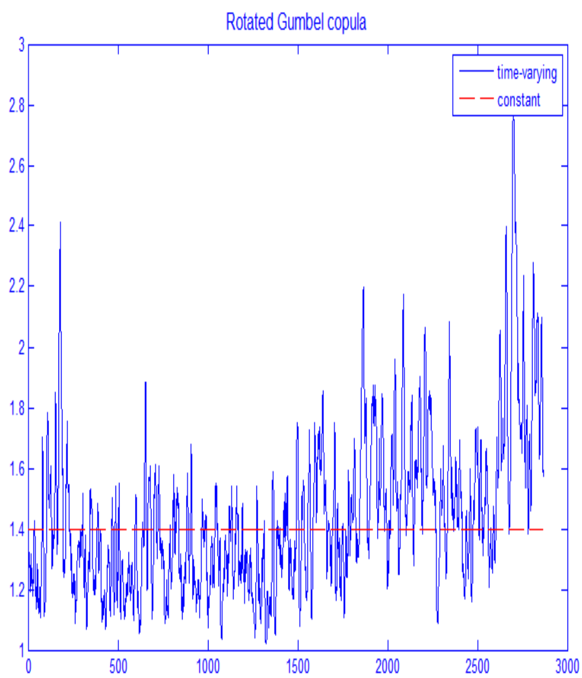
	Information Criteria		
Copula	NLL	AIC	BIC
Gaussian	-344.113	-688.225	-688.223
Clayton	-217.074	-434.146	-434.144
Rotated Clayton	-306.968	-613.936	-613.934
Plackett	-351.701	-703.402	-703.399
Frank	-342.603	-685.205	-685.202
Gumbel	-349.786	-699.570	-699.568
Rotated Gumbel	-280.449	-560.898	-560.896
T	-354.053	-708.104	-708.100
SJC	-339.544	-679.087	-679.083
Copula-GARCH			
Gumbel	**417.9847**	**835.9674**	**835.9611**
Rotated Gumbel	339.3431	678.6842	678.6780
Symmetrised Joe-Clayton	416.8612	833.7181	833.7057

***iii*) Copula-GARCH.**









3. Estimation of copula parameters bivariate EUR/PLN-EUR/HUF sub-portfolio.

i) Copula parameters.

Kendall's tau	Theoretical Rho of the sample	Gaussian	T-copula				Clayton			Frank		
		R	R	DoF	CI		θ	CI		θ	CI	
0.2238	0.3443	0.3313	0.3440	16.2281	5.5441	26.9121	0.3815	0.3304	0.4327	2.1826	1.9571	2.4081

Gumbel			Rotated Clayton			Rotated Gumbel			Plackett			SJC	
θ	CI		θ	CI		θ	CI		θ	CI		τ-Lower	τ-Upper
1.2589	1.2256	1.2923	0.4225	0.3713	0.4737	1.2481	1.2147	1.2815	2.9357	2.7102	3.1611	0.1181	0.1884

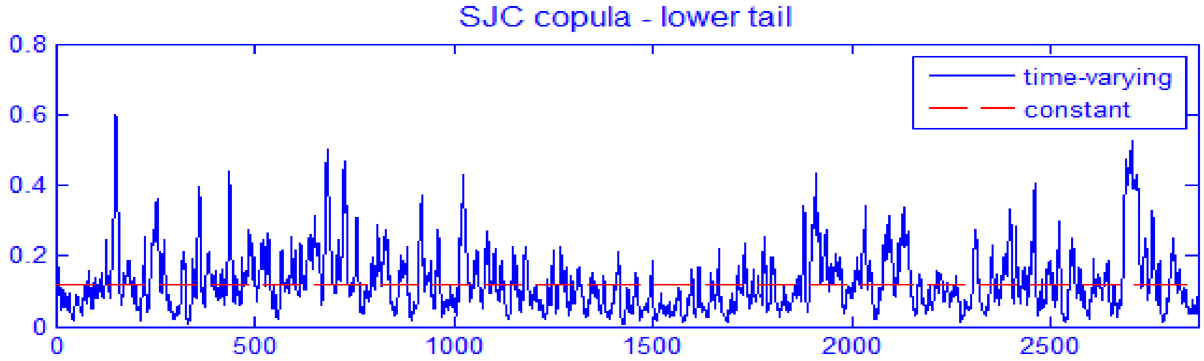
Tyme varying Rotated Gumbel			Tyme varying Rotated Gumbel			Tyme varying SJC					
Ω	β	α	Ω	β	α	Ω-Lower	β-Lower	α-Lower	Ω-Upper	β-Upper	α-Upper
0.9591	-0.0755	-1.4112	-0.1557	0.6135	-0.4331	1.3151	-8.4214	-3.5242	-0.0334	-9.0312	1.5326

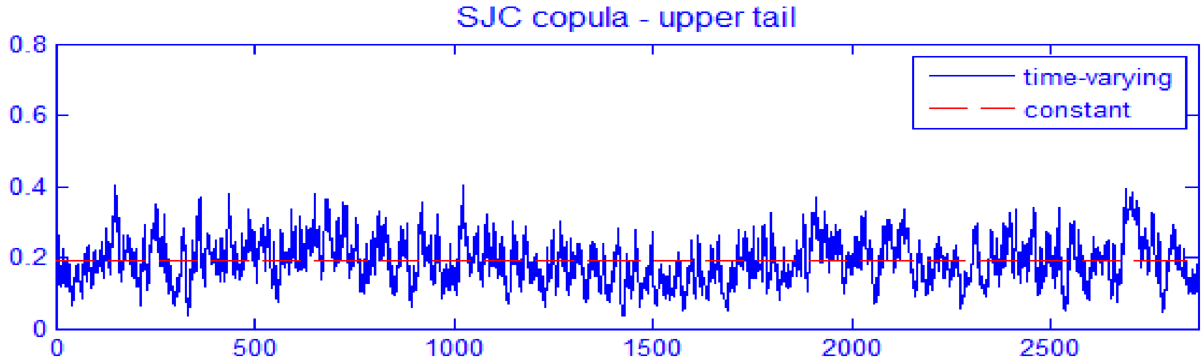
ii) Tail Dependence and Information Criteria.

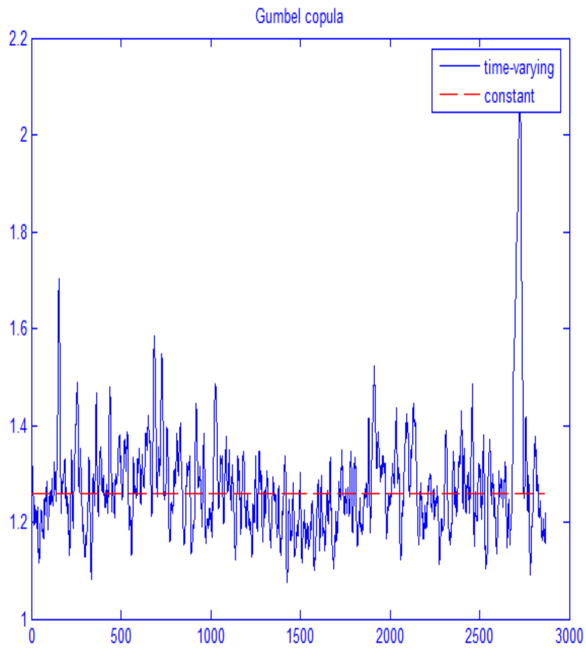
	Tail Dependence	
Copula	Lower	Upper
Gaussian	0	0
Clayton	0.1627	0
Rotated Clayton	0	0.1939
Plackett	0	0
Frank	0	0
Gumbel	0	0.2657
Rotated Gumbel	0.2574	0
T	0.0099	0.0099
SJC	0.1181	0.1884

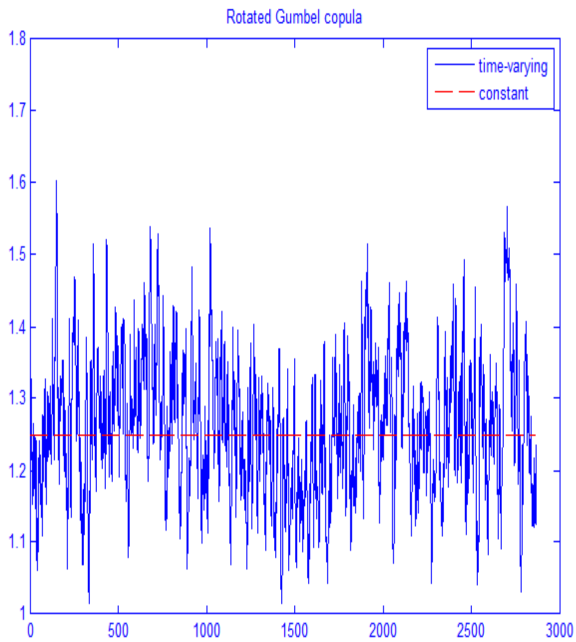
	Information Criteria		
Copula	NLL	AIC	BIC
Gaussian	-166.8175	-333.634	-333.632
Clayton	-112.2710	-224.541	-224.539
Rotated Clayton	-138.9871	-277.974	-277.971
Plackett	**-172.1960**	**-344.391**	**-344.389**
Frank	**-166.1478**	**-332.295**	**-332.293**
Gumbel	-159.1701	-318.339	-318.337
Rotated Gumbel	-142.3737	-284.747	-284.745
T	-171.8631	-343.725	-343.721
SJC	-163.9774	-327.953	-327.949
Copula-GARCH			
Gumbel	-172.0255	-344.049	-344.043
Rotated Gumbel	-156.7138	-313.426	-313.419
Symmetrised Joe-Clayton	**-176.9928**	**-353.981**	**-353.969**

***iii*) Copula-GARCH.**







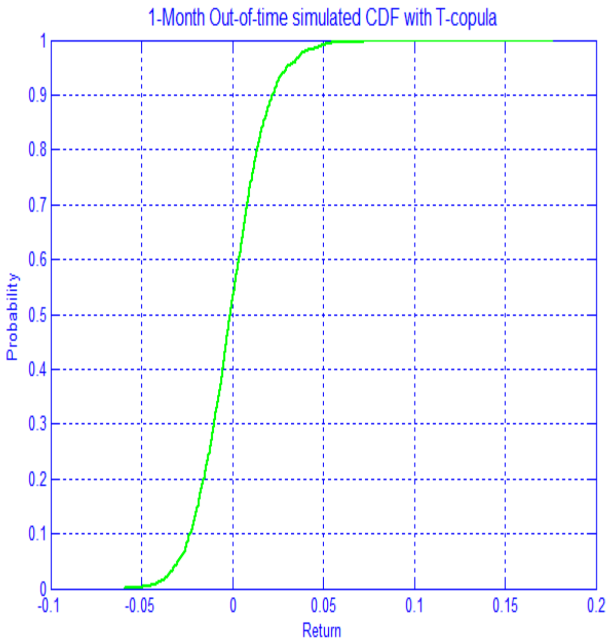


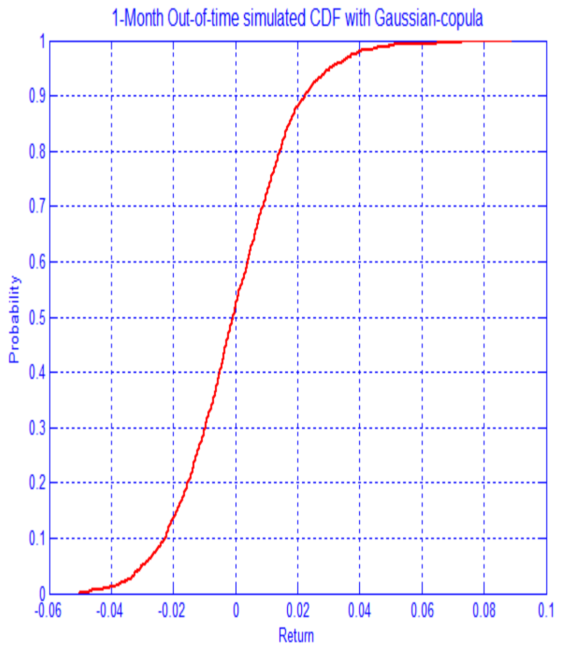
Appendix V

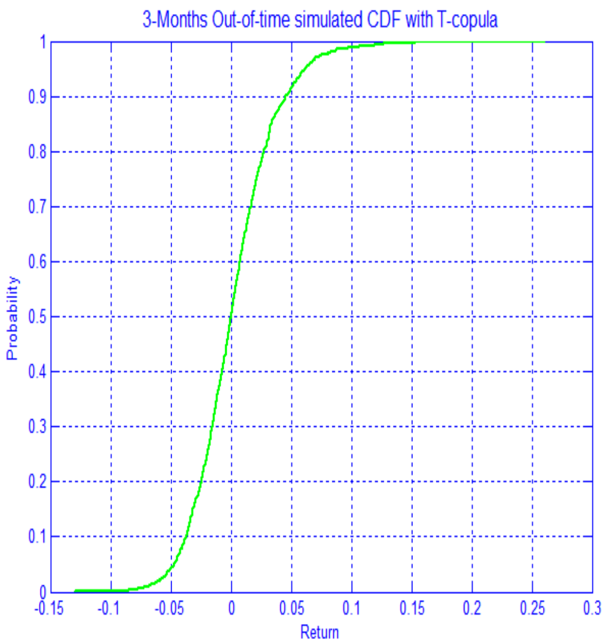
1. Portfolio risk measure; ***i*) In-sample estimation of VaR and CvaR**

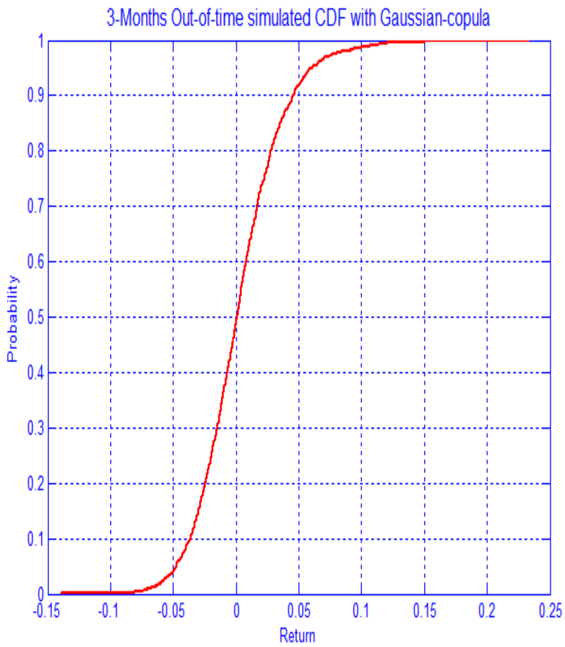
Horizon	Confidence level		T-copula VaR	Gaussian-Copula VaR		T-copula VaR	Gaussian-Copula VaR	Min. and max. empirical return	Out-of-Time realized return
1 day	95%	VaR	-0.7672	-0.7458	CVaR	-0.9694	-0.9368	-2.4476	**0.1997**
	99%		-1.0586	-1.0424		-1.2604	-1.2104		
	5%		0.6942	0.6876		1.0186	0.9724	2.9239	
	1%		1.1610	1.0788		1.6193	1.4892		
5 days	95%	VaR	-1.5504	-1.5285	CVaR	-1.9738	-1.9305	-5.3024	**-1.2347**
	99%		-2.1861	-2.1500		-2.7503	-2.6303		
	5%		1.6023	1.5935		2.1388	2.1383	6.4693	
	1%		2.4294	2.4947		2.8754	3.0795		
10 days	95%	VaR	-2.0100	-1.9870	CVaR	-2.5471	-2.5164	-5.6800	**-1.3248**
	99%		-2.9286	-2.8571		-3.3710	-3.3767		
	5%		2.2270	2.2400		3.1834	3.0859	7.2602	
	1%		3.6803	3.5362		4.9917	4.4554		
1 month	95%	VaR	-3.0695	-3.1148	CVaR	-3.8660	-3.8056	-5.1677	**-2.9442**
	99%		-4.2903	-4.3055		-5.0748	-4.8053		
	5%		3.3703	3.4187		4.8860	4.8636	9.1040	
	1%		5.6392	5.5865		7.5849	7.4142		
3 months	95%	VaR	-4.9829	-5.0366	CVaR	-6.4952	-6.3666	-9.2584	**1.2677**
	99%		-7.4981	-7.2513		-8.7605	-8.4968		
	5%		6.8738	6.8910		10.0183	10.0590	18.7701	
	1%		11.9919	11.9431		16.3543	15.5711		

***ii*) Monte-Carlo simulation of CDF**







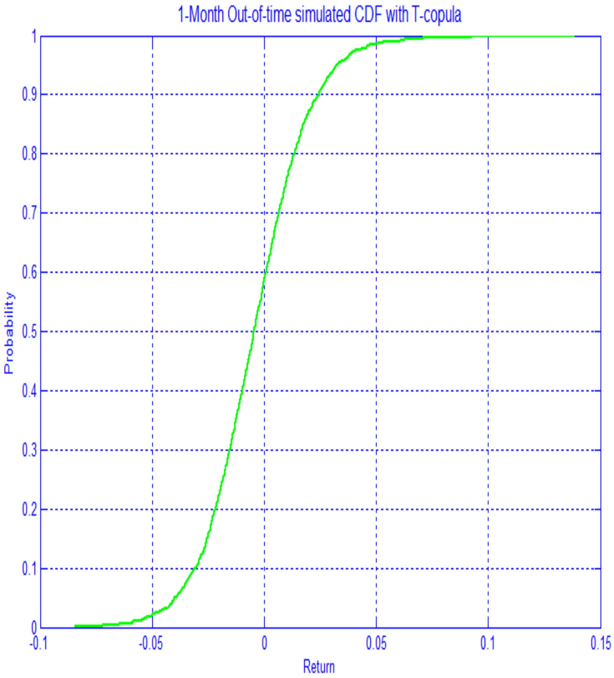


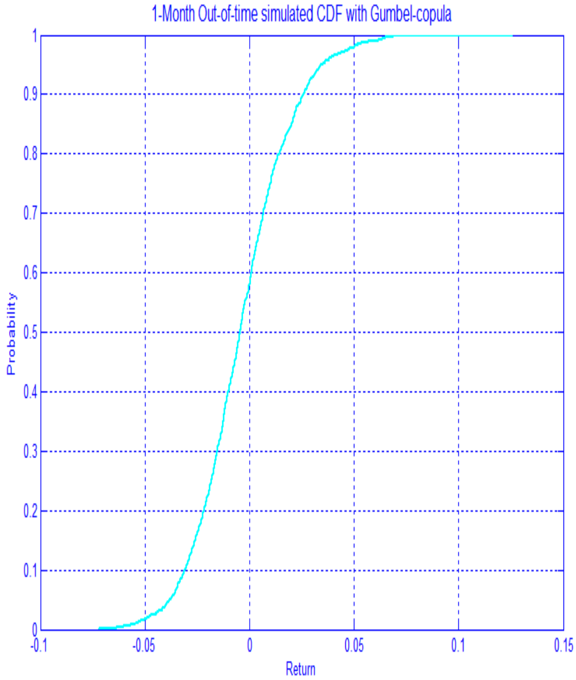
2. EUR/PLN-EUR/CZK sub-portfolio risk measure

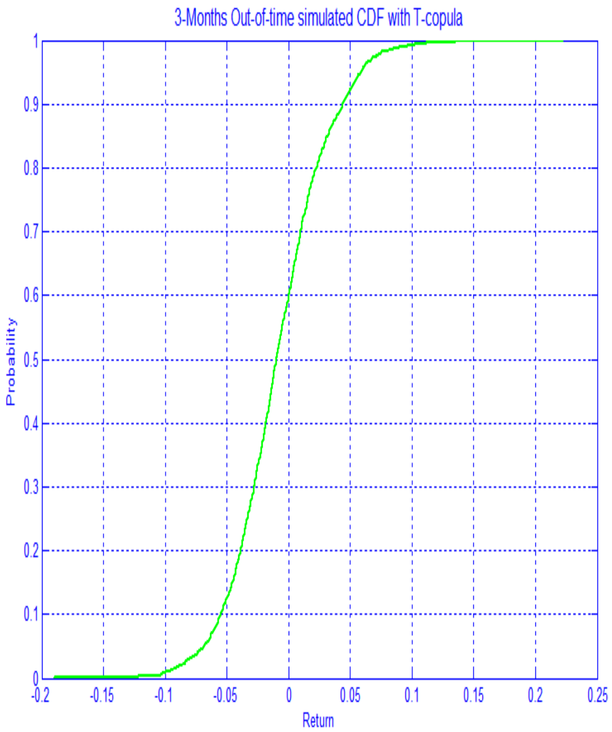
***i*) In-sample estimation of VaR and CvaR**

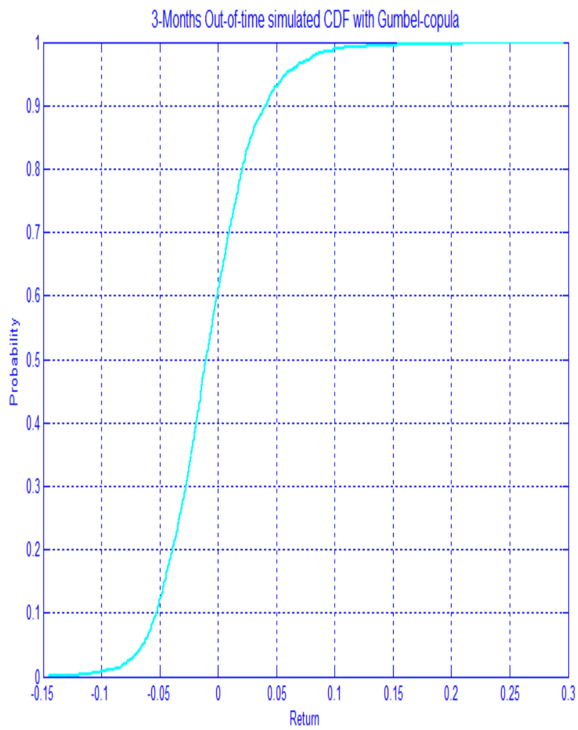
Horizon	Confidence level		T-copula VaR	Gumbel-Copula VaR		T-copula VaR	Gumbel-Copula VaR	Min. and max. empirical return	Out-of-Time realized return
1 day	95%	VaR	-0.9257	-0.9279	CVaR	-1.1922	-1.2045	-2.6805	**0.1748**
	99%		-1.3556	-1.3499		-1.6351	-1.6002		
	5%		0.8817	0.7710		1.2421	1.2279	3.1735	
	1%		1.4329	1.4865		1.8942	1.9810		
5 days	95%	VaR	-2.0562	-1.9067	CVaR	-2.5697	-2.3575	-7.2016	**-1.4244**
	99%		-2.8617	-2.6298		-3.5580	-2.9903		
	5%		1.8991	1.7797		2.6586	2.6064	8.2211	
	1%		3.1737	3.1102		3.9587	3.8942		
10 days	95%	VaR	-2.6055	-2.7462	CVaR	-3.3623	-3.4626	-6.6230	**-2.1492**
	99%		-3.7940	-3.7864		-4.6173	-4.5741		
	5%		2.3839	2.5540		3.4456	3.5967	8.0561	
	1%		3.9764	4.2246		5.1431	5.6460		
1 month	95%	VaR	-4.0449	-4.0077	CVaR	-5.3040	-5.0007	-7.6330	**-3.8425**
	99%		-5.8364	-5.6978		-7.2352	-6.3841		
	5%		4.0071	3.7864		5.9100	5.8640	10.9775	
	1%		6.5271	7.3033		10.1099	9.4476		
3 months	95%	VaR	-7.2029	-6.8356	CVaR	-8.9414	-8.7052	-11.0234	**0.4159**
	99%		-10.0390	-9.8307		-11.8241	-11.9075		
	5%		6.4088	6.6884		9.3623	11.3436	22.3665	
	1%		11.1877	13.3398		15.0395	20.9592		

***ii*) Monte-Carlo simulation of CDF**







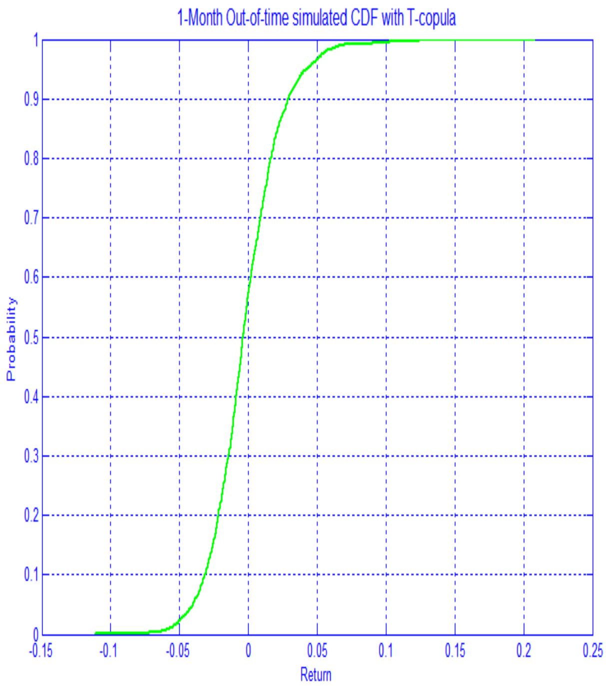


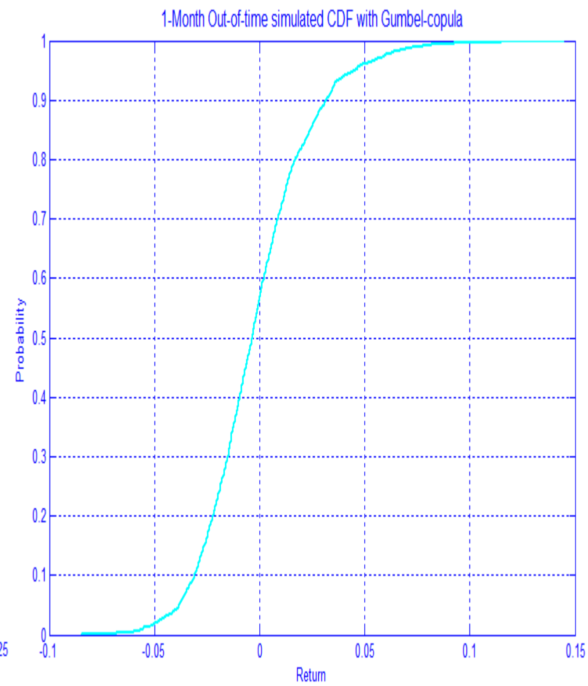
2. EUR/PLN-EUR/HUF sub-portfolio risk measure

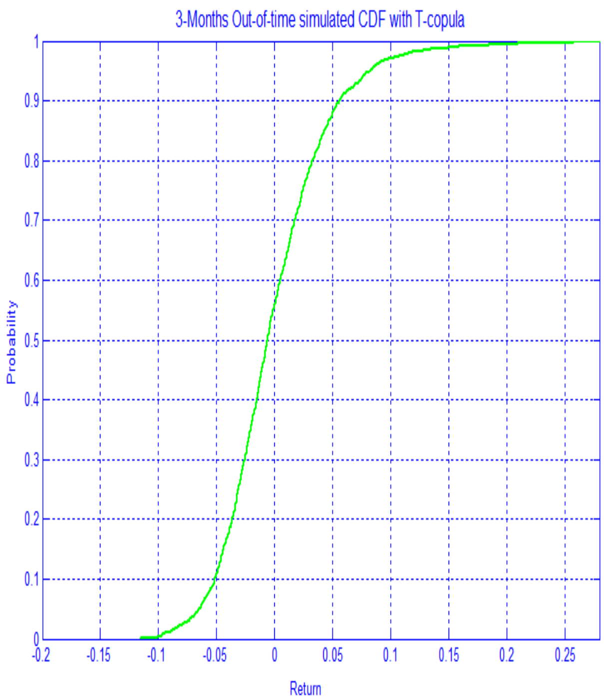
***i*) In-sample estimation of VaR and CvaR**

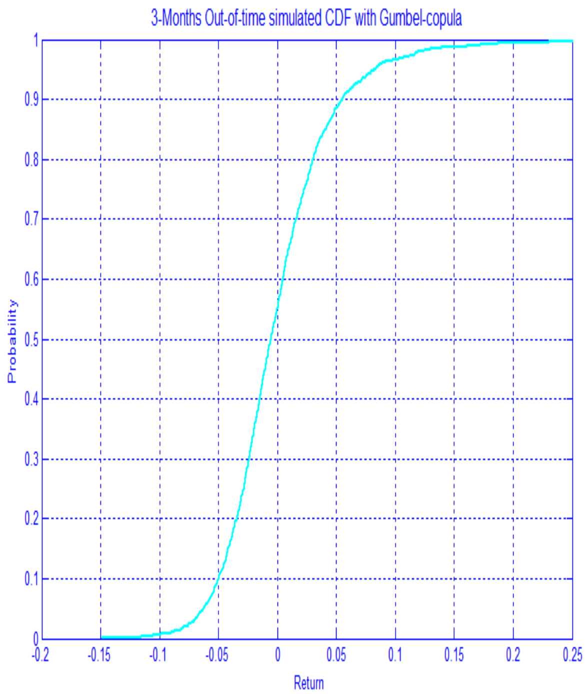
Horizon	Confidence level		T-copula VaR	Gumbel-Copula VaR		T-copula VaR	Gumbel-Copula VaR	Min. and max. empirical return	Out-of-Time realized return
1 day	95%	VaR	-1.0888	-1.0755	CVaR	-1.4094	-1.3781	-2.9986	**0.5517**
	99%		-1.6011	-1.5137		-1.9167	-1.8433		
	5%		0.9821	0.9200		1.3930	1.4393	4.1159	
	1%		1.6072	1.7173		2.0539	2.3175		
5 days	95%	VaR	-2.2232	-2.0826	CVaR	-2.9005	-2.6689	-9.2020	**-1.5119**
	99%		-3.2520	-3.0598		-3.8961	-3.4373		
	5%		2.1244	2.0942		3.2111	3.0368	9.2936	
	1%		3.6069	3.5849		4.8605	4.6940		
10 days	95%	VaR	-2.9652	-3.0645	CVaR	-3.7032	-3.8050	-7.9383	**-1.4125**
	99%		-4.2056	-4.3303		-4.8996	-5.0764		
	5%		2.9689	3.0843		4.2654	4.3156	9.0617	
	1%		5.2296	5.3492		6.4678	6.4423		
1 month	95%	VaR	-4.5654	-4.2590	CVaR	-5.9553	-5.4747	-7.4851	**-4.0792**
	99%		-6.8459	-6.2733		-8.0956	-7.2053		
	5%		4.5619	4.6203		6.9346	7.1122	15.4599	
	1%		8.0534	8.3552		11.7663	10.8031		
3 months	95%	VaR	-7.3782	-7.2217	CVaR	-9.5191	-9.3648	-11.5424	**2.5407**
	99%		-10.4988	-10.7371		-12.5633	-13.4850		
	5%		7.8492	8.6982		11.8615	14.2239	21.5802	
	1%		13.8947	17.1297		18.6205	25.7856		

***ii*) Monte-Carlo simulation of CDF**







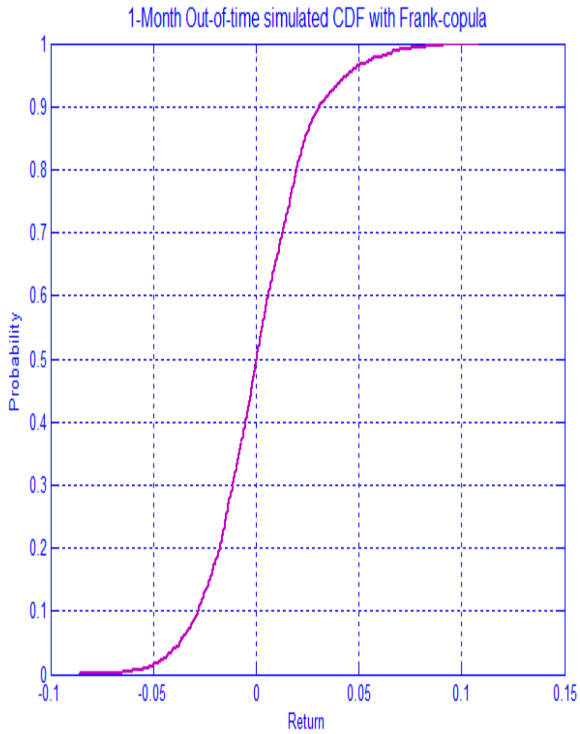


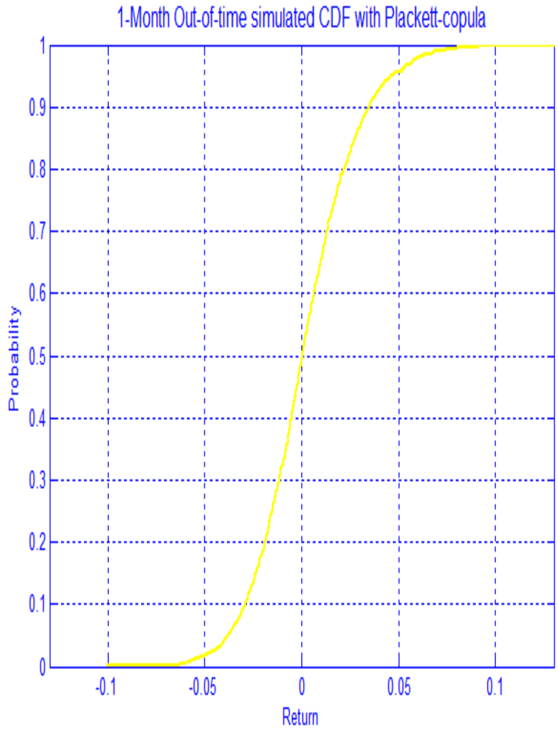
2. EUR/PLN-EUR/HUF sub-portfolio risk measure

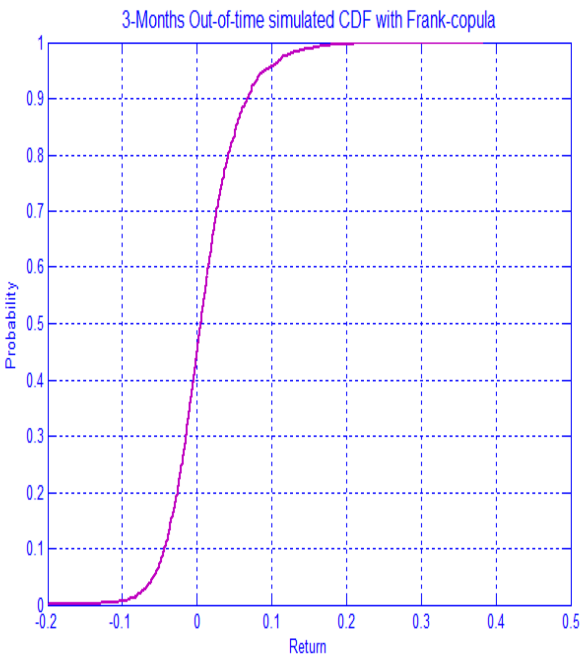
i) In-sample estimation of VaR and CvaR

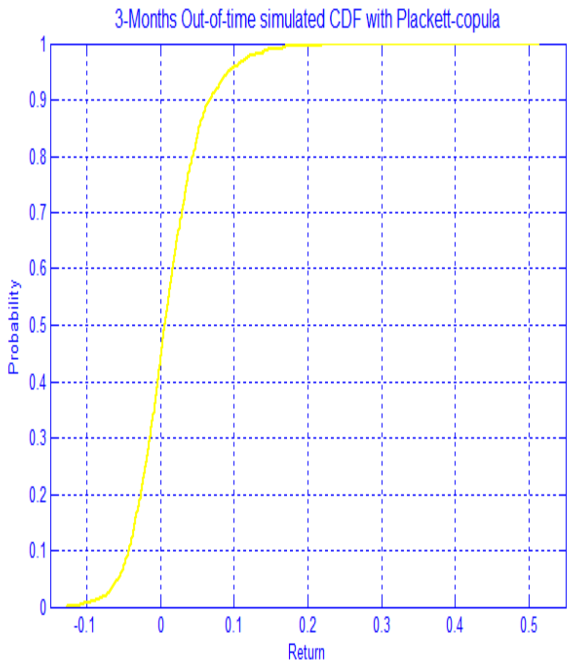
Horizon	Confidence level		Frank-copula VaR	Plackett-Copula VaR		Frank-copula VaR	Plackett-Copula VaR	Min. and max. empirical return	Out-of-Time realized return
1 day	95%	VaR	-0.9485	-0.9936	CVaR	-1.2759	-1.3022	-4.0511	**0.4116**
	99%		-1.4261	-1.4874		-1.7752	-1.8018		
	5%		0.9036	0.9167		1.2445	1.3030	5.8615	
	1%		1.4666	1.5054		1.7806	2.0016		
5 days	95%	VaR	-2.0450	-2.0300	CVaR	-2.7283	-2.6029	-4.8666	**-1.1141**
	99%		-3.2014	-2.9115		-3.7722	-3.4631		
	5%		1.8590	1.8269		2.5676	2.5472	6.5516	
	1%		2.9923	2.9373		3.7096	3.7520		
10 days	95%	VaR	-2.8445	-2.8308	CVaR	-3.7579	-3.5763	-5.5519	**-1.0764**
	99%		-4.1471	-4.2037		-5.2994	-4.6958		
	5%		2.7004	2.9144		3.7518	3.9300	7.5991	
	1%		4.4391	4.5338		5.3432	5.4369		
1 month	95%	VaR	-4.4102	-4.3280	CVaR	-5.6651	-5.5022	-5.5050	**-3.1584**
	99%		-6.2009	-6.1906		-7.6919	-7.2893		
	5%		4.1188	4.5157		5.7739	6.3479	9.6777	
	1%		6.7624	7.1696		7.9586	9.6292		
3 months	95%	VaR	-7.7137	-7.4042	CVaR	-10.2423	-9.8925	-10.2125	**1.9176**
	99%		-11.7625	-10.9088		-14.9252	-14.4789		
	5%		7.5080	8.6671		11.7263	13.1176	22.1169	
	1%		13.2778	15.5488		18.1708	21.1178		

***ii*) Monte-Carlo simulation of CDF**


1-Month Out-of-time simulated CDF with Frank-copula
Probability
Return
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-0.1
-0.05
0
0.05
0.1
0.15


1-Month Out-of-time simulated CDF with Plackett-copula
Probability
Return
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-0.1
-0.05
0
0.05
0.1

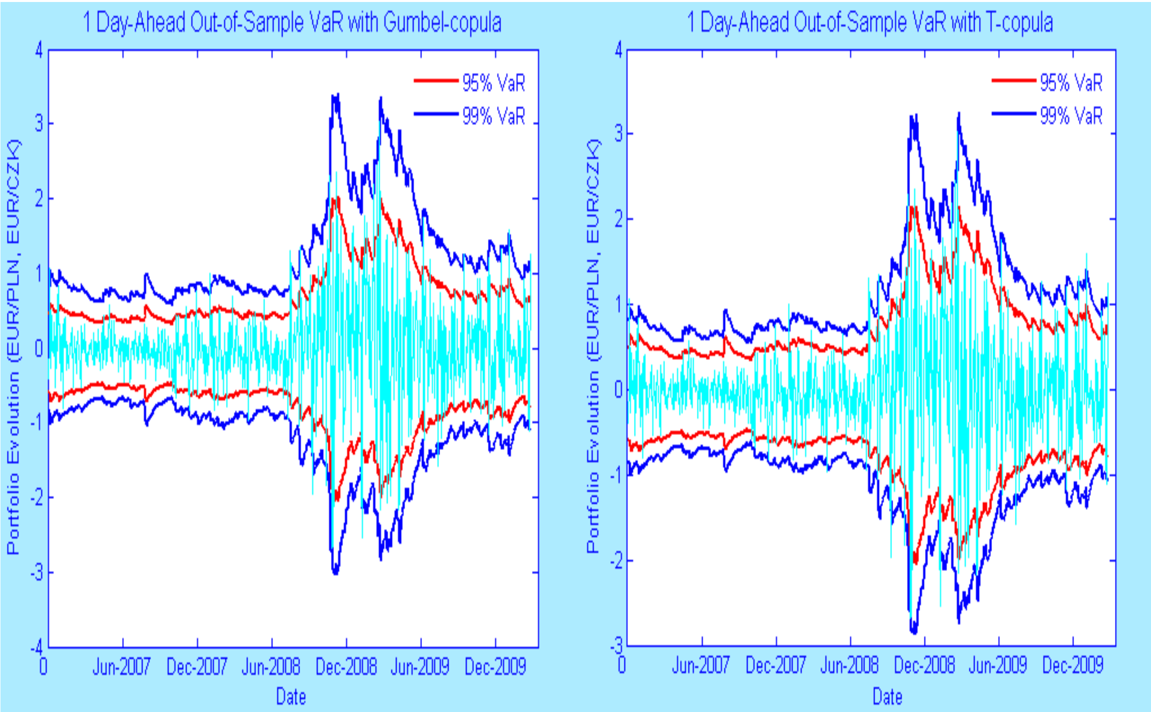

3-Months Out-of-time simulated CDF with Frank-copula
Probability
Return
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5


3-Months Out-of-time simulated CDF with Plackett-copula
Probability
Return
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-0.1
0
0.1
0.2
0.3
0.4
0.5

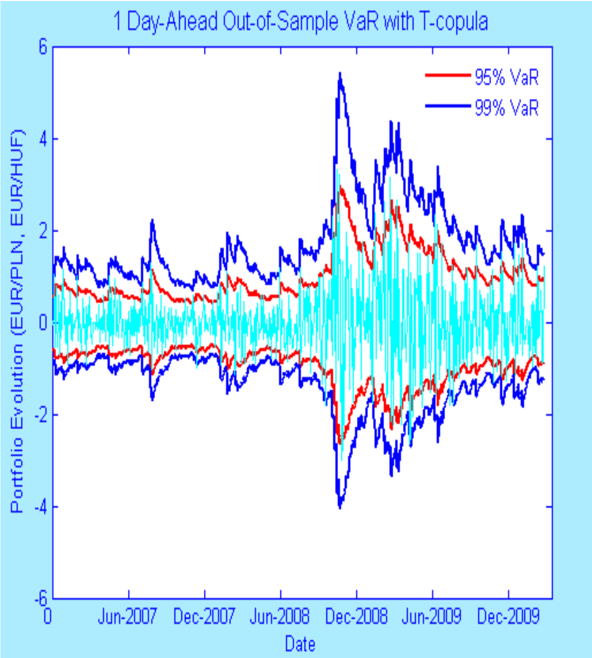
Appendix VI

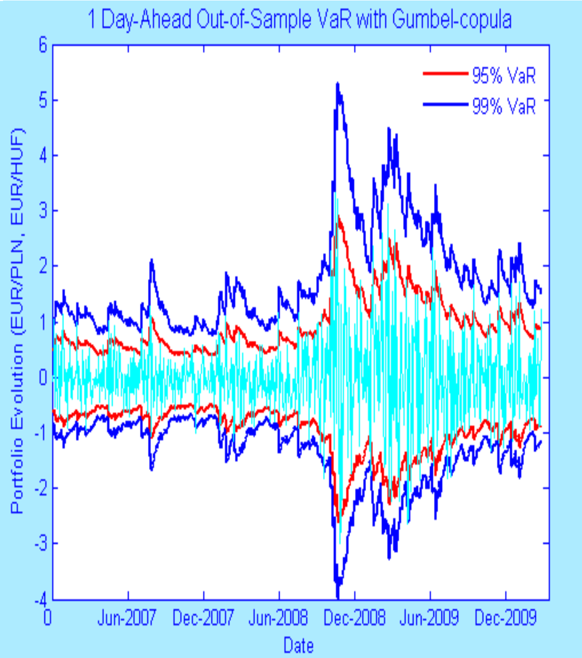
1. Out-of-Sample forecast of VaR against empirical returns

EUR/PLN-EUR/CZK

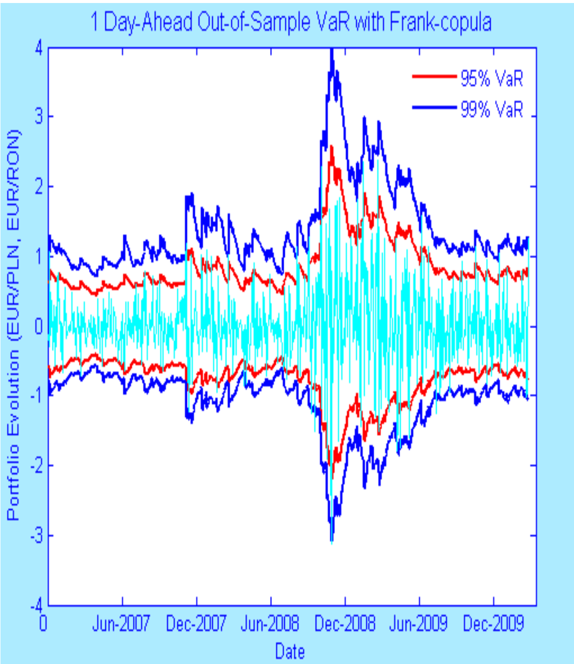


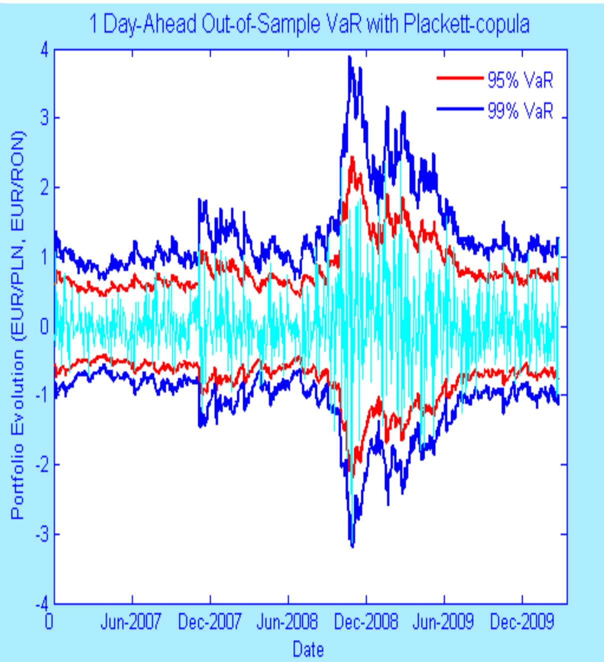
EUR/PLN-EUR/HUF



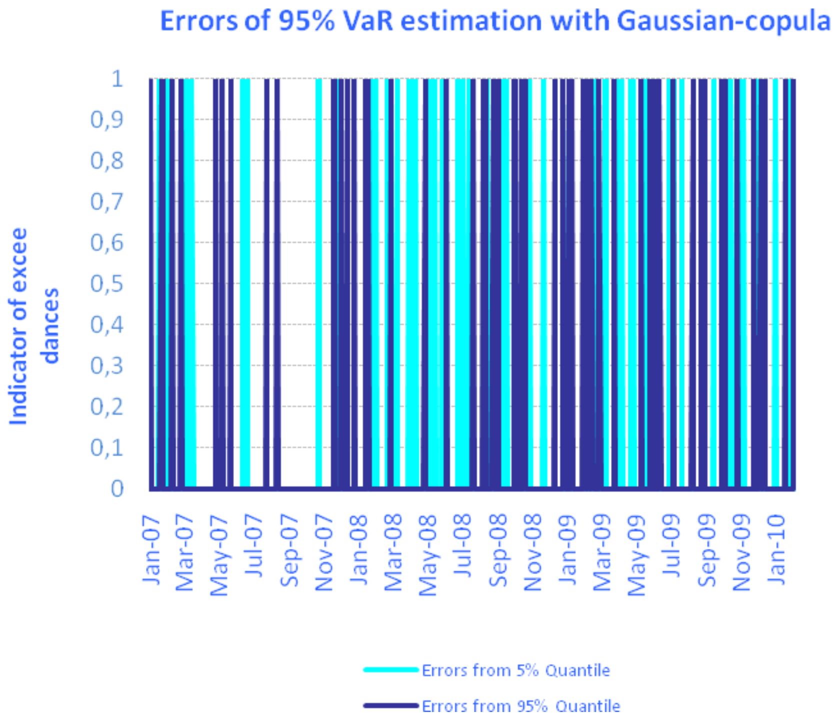


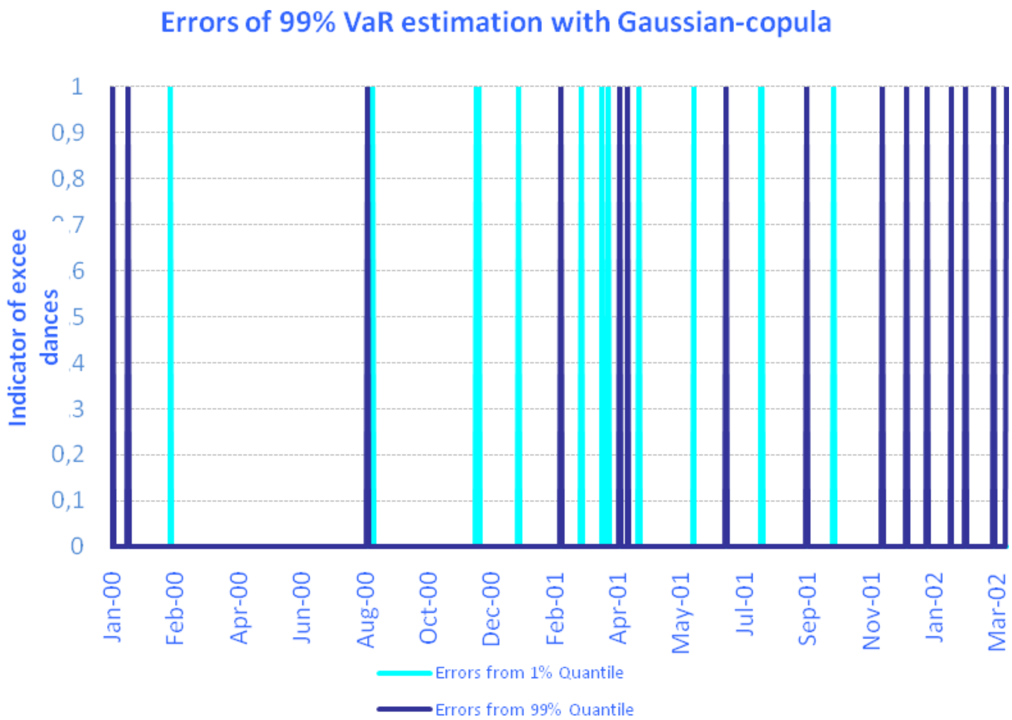
<u>EUR/PLN-EUR/RON</u>



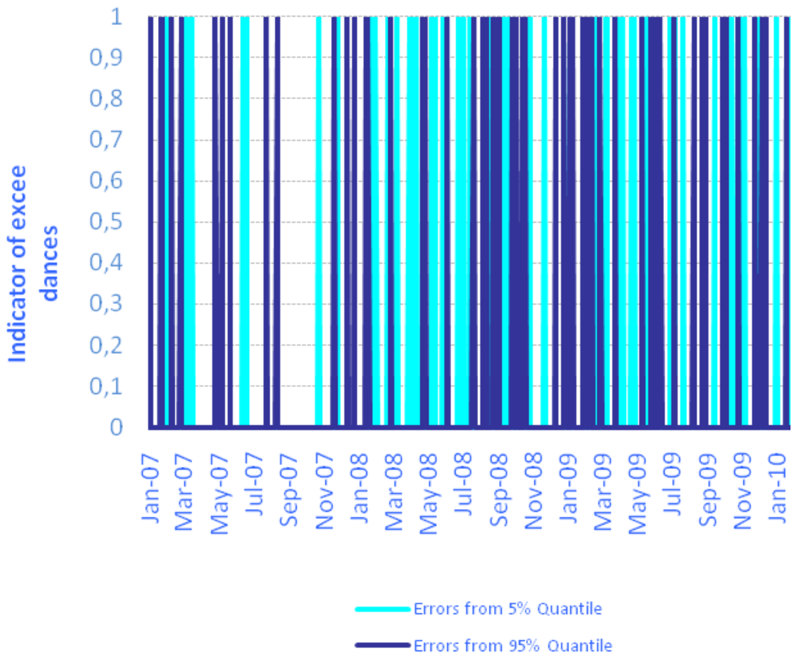


2. Evolution out-of-sample forecast's errors for large portfolio.





Errors of 95% VaR estimation with T-copula



Errors of 99% VaR estimation with T-copula