# MAXIMAL ELEMENTS OF NON NECESSARILY ACYCLIC BINARY RELATIONS*

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## MAXIMAL ELEMENTS OF NON NECESSARILY ACYCLIC BINARY RELATIONS

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### ABSTRACT

The existence of maximal elements for binary preference relations is analyzed without imposing transitivity or convexity conditions. From each preference relation a new acyclic relation is defined in such a way that some maximal elements of this new relation characterize maximal elements of the original one. The result covers the case whereby the relation is acyclic.

## 1. INTRODUCTION

In Bergstrom [1] and Walker [10], it is proved that if an acyclic binary relation defined on a topological space X is lower continuous, then every compact subset of X will contain a maximal element. In Campbell & Walker [5], the lower continuity is replaced by a weaker property [weak lower continuity] and, assuming stronger conditions than that of acyclicity [interval order], the result that maximal elements exist on any compact set is again obtained.

If the relation does not satisfy the property of acyclicity [and, therefore, is not a full transitive, extratransitive, pseudotransitive, nor transitive relation](1), some conditions of convexity and continuity are used to prove the existence of maximal elements on topological compact sets (see, for example, Sonnenschein [8], and Shafer & Sonnenschein [7]).

In the context of social choice, acyclicity is often considered as an assumption of rationality. However, in some important economic examples [committee selection, uncertainty choice, decision under risk, ...] the property of acyclicity is not a condition linked to rationality.

On the other hand, the assumption of convexity is meaningless when the set of alternatives is finite or countable. Even in the case of

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1 We follow the notation and terminology of Campbell & Walker [5].

non-countability, the convexity of the alternative set turns out a necessary condition in order to define convex preferences.

We analyze here the existence of maximal elements without imposing conditions of transitivity and/or convexity on the preference relation. We then obtain conditions which guarantee the existence of maximal elements on compact sets, by using the condition of lower continuity.

In the following Section, notation and definitions are presented. Then, new relations defined from the initial one are introduced, and an acyclic relation is constructed, in such a way that some maximal elements of this new relation characterize the maximal elements of the original one. In the third Section, the existence results of maximal elements are presented.

## 2. NOTATION, DEFINITIONS AND PRELIMINARY RESULTS

Let X denote a set of alternatives; let $\succ$ denote a preference relation [i.e., an asymmetric binary relation] on X; and let $\succsim$ denote the completion of $\succ$ [i.e., $x \succsim y$ means that $y \succ x$ does not hold]. A *cycle* is a finite list $(x_1, x_2, \ldots, x_n)$ that satisfies

$$x_1 \succ x_2 \succ \ldots \succ x_n \succ x_1.$$

We say that the relation $\succ$ is

i) *Acyclic* if it has no cycles

ii) *Interval order* if $x \succ y \succsim z \succ w \Rightarrow x \succ w$

iii) *Transitive* if $x \succ y \succ z \Rightarrow x \succ z$

iv) *Fully transitive* (*preorder*) if its completion $\succsim$ is transitive, i.e. $x \succsim y \succsim z \Rightarrow x \succsim z$

All implications in the following diagram hold, while none of their converses do:

Fully transitive $\Rightarrow$ Interval order $\Rightarrow$ Transitive $\Rightarrow$ Acyclic

The relation $\succ$, defined on a topological space X, is said to be *lower continuous* if for each $x \in X$ the set $\{y \in X \mid x \succ y\}$ is open.

From the initial preference relation $\succ$ we define the next two binary relations:

(1) $x \succ\succ y \iff \exists\, x_1, x_2, \ldots, x_m \mid x = x_1 \succ x_2 \succ \ldots \succ x_m = y$

(2) $x \approx y \iff x = y$ or $x \succ\succ y \succ\succ x$

The first one is known as the *transitive closure* of $\rangle$, since it is always a transitive relation; the second one turns out to be an equivalence relation, and we denote by $X/\approx$ the quotient set [the set of the equivalence classes]. The idea is to eliminate the cycles in the initial relation by taking each cycle as just one element [the class in which all the elements of the cycle are contained] on the set $X/\approx$.

Now it is possible to define a binary relation on the quotient set in the following way:

Let be $\mathfrak{x}, \mathfrak{y} \in X/\approx$, then

$$\mathfrak{x} \; P \; \mathfrak{y} \iff \mathfrak{x} \neq \mathfrak{y} \text{ and there exist } x \in \mathfrak{x},\ y \in \mathfrak{y} \text{ such that } x \rangle\rangle y$$

We shall call P the *quotient relation* of $\rangle$, and it is an asymmetric relation since its definition does not depend on the chosen elements in each class. To see this, let be $x, x' \in \mathfrak{x}$, $y, y' \in \mathfrak{y}$ and suppose that $x \rangle\rangle y$ , $y' \rangle\rangle x'$. Then, there are four possibilities:

(a) $x = x'$ and $y = y'$

(b) $x = x'$ and $y \rangle\rangle y' \rangle\rangle y$

(c) $x \rangle\rangle x' \rangle\rangle x$ and $y = y'$

(d) $x \rangle\rangle x' \rangle\rangle x$ and $y \rangle\rangle y' \rangle\rangle y$

The four cases give us that $\mathfrak{x} = \mathfrak{y}$ , as opposed to $\mathfrak{x} \; P \; \mathfrak{y}$. Thus this relation is asymmetric and therefore a preference relation. Notice that, if the initial relation is acyclic, then $X/\approx \; = X$ and the new relation P coincides with the transitive closure of the initial one.

As we have said, in some way we attempt to eliminate the cycles in the quotient set X/≈. It is possible that P = ∅ (that is the case in which all of the elements in X are indifferent, or when all of them are on a cycle, ...).

The next lemma proves that the relation P has no cycles since it is transitive.

**Lemma 1.** *Given an asymmetric relation* ⟩ *on a set* X*, the relation* P *defined on the quotient set* X/≈ *is transitive.*

*Proof*. Suppose that for some $\alpha_1, \alpha_2, \alpha_3 \in X/\approx$ we have

$$\alpha_1 \; P \; \alpha_2 \; P \; \alpha_3$$

then, $\alpha_1 \neq \alpha_2 \neq \alpha_3$ and since P is asymmetric $\alpha_3 \neq \alpha_1$ ; moreover there exist $x_1 \in \alpha_1$ , $x_2, x'_2 \in \alpha_2$ ,$x_3 \in \alpha_3$ such that

$$x_1 \rangle\rangle x_2 \; ; \; x'_2 \rangle\rangle x_3$$

being $x_2 = x'_2$ or $x_2 \rangle\rangle x'_2 \rangle\rangle x_2$

Thus, it is possible to construct the cycle

$$x_1 \rangle\rangle x_m \rangle\rangle x_1$$

which implies $\alpha_1 \; P \; \alpha_3$. Thus, P is a transitive relation.

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The next result shows us the relationship between the maximal elements of the initial preference relation and those of the quotient relation P.

**Theorem 1.** *Let* $\succ$ *be a preference relation defined on a set* X. *If* P *is the quotient relation of* $\succ$ *defined on* $X/\approx$ *then:*

$$x^* \text{ is a maximal element on } X \iff \mathfrak{x}^* = \{x^*\} \text{ is maximal on } X/\approx$$

*Proof.* First, note that if $x^*$ is maximal on X, from the definition of the quotient set $X/\approx$, necessarily $\mathfrak{x}^* = \{x^*\}$. Now, if we suppose that there exists $\mathfrak{y} \in X/\approx$ such that $\mathfrak{y}$ P $\mathfrak{x}^*$, then there is some $y \in \mathfrak{y}$ such that $y \succ\succ x^*$, which contradicts the fact that $x^*$ is maximal. Conversely, if $\mathfrak{x}^*$ is a maximal element on $X/\approx$ and the cardinal of this class is $\#\mathfrak{x}^* = 1$, then let be $\mathfrak{x}^* = \{x^*\}$. If we suppose that $x^*$ is not a maximal element on X, then there is some $y \in X$ such that $y \succ x^*$. This implies, from maximality of $\mathfrak{x}^*$, that necessarily $y \in \mathfrak{x}^*$, against $\#\mathfrak{x}^* = 1$. Then, $x^*$ is a maximal element on X.

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In view of the above results, the problem of the existence of maximal elements in the initial relation, is reduced to the existence of maximal elements [of cardinality 1] in the quotient relation, which is an acyclic relation. We can then apply to the quotient relation P those known results of existence of maximal elements for acyclic relations. This is the purpose of the next section.

## 3. EXISTENCE OF MAXIMAL ELEMENTS

To analyze the existence of maximal elements for a preference relation we use some known results. It is clear that an acyclic relation will always have a maximal element on a finite set. Thus we obtain the following result as an immediate consequence of Theorem 1:

**Corollary 1.** *Let* $\succ$ *be a preference relation defined on a set* X *such that the quotient set* $X/\approx$ *is finite. Then:*

*a) The quotient set* $X/\approx$ *has a maximal element for the relation* P

*b) There exists a maximal element in* X *for the relation* $\succ$ *if and only if there is some maximal* $\alpha^*$ *in* $X/\approx$ *for the relation* P *with* $\#\alpha^* = 1$.

When set X is non finite, some additional conditions of continuity must be added to obtain results of the existence of maximal elements.

**Theorem 2.** [Bergstrom, 1975; Walker, 1977] *Let* X *be a topological space, and let* $\succ$ *be a lower continuous and acyclic relation on* X. *Then in every non empty compact subset of* X *there is a maximal element.*

**Theorem 3.** *Let* X *be a compact topological space, and let* $\succ$ *be a preference relation such that for each* $x \in X$ *the set* $\{y \in X \mid x \succ\succ y, \text{ and not } y \succ\succ x\}$ *is open. Then:*

*a) The quotient set* $X/\approx$ *has a maximal element for the relation* P

*b) There exists a maximal element in* X *for the relation* $\rangle$ *if and only if there is some maximal* $\mathcal{x}^*$ *in* $X/\approx$ *for the relation* P *with* $\#\mathcal{x}^* = 1$.

*Proof.* From the topology in X we define the quotient topology in $X/\approx$ in the usual way: a set $\mathcal{D} \subseteq X/\approx$ is open if and only if the set $D = \bigcup\{x \in \mathcal{x},\ \mathcal{x} \in \mathcal{D}\}$ is open in X. With this topology since X is compact the quotient set $X/\approx$ is compact too. Furthermore for each $\mathcal{x} \in X/\approx$ the set $\mathcal{A} = \{\mathcal{y} \in X/\approx \mid \mathcal{x}\ P\ \mathcal{y}\}$ is open and then the relation P is lower continuous; to see this we first prove that for each $\mathcal{x}$, $A = \bigcup_{\mathcal{x} P \mathcal{y}} \{y \in \mathcal{y}\} = \bigcup_{x \in \mathcal{x}} \{y \in X \mid x \rangle\rangle y, \text{ and not } y \rangle\rangle x\}$.
In order to do this, let $y \in A$ be; then there exist $x_1 \in \mathcal{x}$, $y_1 \in \mathcal{y}$ such that $x_1 \rangle\rangle y_1$. As $y, y_1 \in \mathcal{y}$, thus $y = y_1$ or $y_1 \rangle\rangle y \rangle\rangle y_1$. In any case, $x_1 \rangle\rangle y$ and not $y \rangle\rangle x_1$ since $\mathcal{x} \neq \mathcal{y}$. Conversely, if for some $x \in \mathcal{x}$, $x \rangle\rangle y$ and not $y \rangle\rangle x$, then $\mathcal{x}\ P\ \mathcal{y}$ and $y \in A$. Then A is open and therefore in the quotient topology the set $\mathcal{A}$ is open. By applying Theorems 1 and 2 we obtain the desired result.

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Note that, if the relation is acyclic, the result of Bergstrom [1] and Walker [10] is a particular case of our result, since the condition that the sets

$$\{y \in X \mid x \rangle\rangle y, \text{ and not } y \rangle\rangle x\}$$

are open is a weaker condition than that of lower continuity as we can see in the following example:

Let $\succ$ be the preference relation defined on a subset X of $\mathbb{R}^n$, with the usual topology, as

$$x \succ y \iff \|x-y\| \geq 1,\ \|x\| > \|y\|,$$

this relation is not lower continuous, but if we consider its transitive closure, it holds that

$$\{y \in X \mid x \succ\succ y, \text{ and not } y \succ\succ x\} = \{y \in X \mid \|x\| > \|y\|\}$$

and therefore this set is open.

The condition introduced by Campbell & Walker [5], namely weak lower continuity, is still weaker than our condition, but in [5] some additional stronger conditions on transitivity are required.

EN BLANCO

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