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ON EQUILIBRIUM EXISTENCE IN INFINITE HORIZON ECONOMIES

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Abstract. In sequential economies with finite or infinite-lived real assets in positive net supply, we introduce constraints on the amount of borrowing in terms of the market value of physical endowments. We show that, when utility functions are either unbounded and separable in states of nature or separable in commodities, these borrowing constraints not only preclude Ponzi schemes but also induce endogenous Radner bounds on short-sales. Therefore, we obtain existence of equilibrium. Moreover, equilibrium also exists when both assets are numerarie and utility functions are quasilinear in the commodity used as numerarie.

Keywords: Equilibrium, Infinite horizon incomplete markets, Infinite-lived real assets.

1. Introduction

Ponzi schemes need to be avoided in order to obtain existence of equilibrium in infinite horizon incomplete markets. Indeed, debt constraints or transversality conditions have been required to assure that agents do not postpone, *ad infinitum*, the payments of their commitments. Within this context, many authors had shown that equilibrium exists when financial markets are composed by short-lived numeraire or nominal assets (see, for instance, Kehoe and Levine (1993), Magill and Quinzii (1994), Florenzano and Gourdel (1996), Hernández and Santos (1996), Levine and Zame (1996), and Araujo, Monteiro and Páscoa (1996)). Also, Hernández and Santos (1996) prove the existence of equilibrium when only one infinite-lived real asset, in positive net supply, is available for trade.

However, when financial markets include non-numerarie finite-lived real assets or more than one infinite-lived real asset, equilibrium existence has been guaranteed at most for dense subsets of economies (see, for instance, Hernández and Santos (1996) and Magill and Quinzii (1996)). In fact, in this scenario, Ponzi schemes are not the unique possible reason for non-existence of equilibrium. Precisely, since the rank of returns matrices become dependent on asset prices and conventional debt constraints bound the portfolio markets value but not the amount of borrowing, short-sales may fail to have endogenous upper bounds. Thus, agents can have more access to credit in any asset just by increasing their investment in the other securities. As a consequence, finite horizon economies, that

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are obtained by truncating the infinite horizon economy in order to prove equilibrium existence, may not have equilibrium.

The aim of this paper is to show the existence of equilibrium in a market where real assets in positive net supply can be traded. To prevent Ponzi schemes, the amount of borrowing that each agent is able to get becomes dependent on the market value of (individual or aggregated) physical endowments. We remark that, since assets may be infinite-lived, positive net supply is a necessary requirement for equilibrium existence in our model. Indeed, with zero net supply assets, finite asset prices might be incompatible with non-arbitrage conditions (as we remark after our main result). This difficulty was also pointed out by Hernández and Santos (1996, Example 3.9) in their model with debt constrained agents.

We prove that equilibrium exists when utility functions are either separable in the states of nature and unbounded or separable in commodities. Since we require utility functions to be unbounded only in those commodities in which real assets make promises, in the particular case in which assets are numerarie, to assure equilibrium existence it suffices to have utility functions which are quasi-linear in the commodity used as numerarie.

To prove our results, we follow the classical approach that finds an equilibrium as a limit of equilibria corresponding to a sequence of finite horizon economies. As a first step, we show a result of equilibrium existence for truncated economies by defining associated generalized games and showing that equilibrium asset prices are uniformly bounded. We remark that a positive lower bound for asset prices leads to short-sales constraints (Radner bounds) induced by borrowing restrictions. Since utility functions are unbounded in commodities in which assets pay, in equilibrium the market value of the positive net supply need to have a bounded purchase power, node by node. Thus, as positive net supply of assets neither depreciates nor disappear from the economy, there are endogenous upper bounds for asset prices. These upper bounds leads to a natural restriction on the set of prices that is selected in the generalized game. Thus, we can guarantee the non-emptiness of the interior of the budget constraint correspondences. In a second step, we check the asymptotic properties of individual debt, namely, transversality conditions, which are actually obtained as a consequence of the structure of restrictions on borrowing. Indeed, since we show that under Kuhn-Tucker multipliers the discounted value of individual wealth is finite, borrowing constraints prevent agents to be borrowers at infinity.

We remark that economies where physical endowments have no strictly positive lower bound are included within the framework stated in this paper. Furthermore, although utility functions are required to be separable, non-stationary intertemporal discounting is also compatible with our assumptions. In addition, when at each node of the economy there is only one asset to be traded, we can go further and assure that borrowing constraints become non-binding.

The remainder of the paper is organized as follows. In Section 2 we present the model. In Section 3 we state our main result of equilibrium existence whose proof is relegated to a final Appendix. In Section 4 we include some comments which connect our existence results and the required assumptions with the related literature. Moreover, we also present remarks on non-binding borrowing constraints, uniform impatience and rational asset pricing bubbles. We finish the paper with a concluding remarks section.

2. Model

We consider a discrete time economy with infinite horizon. Let S be the non-empty set of states of nature. At each date, individuals have common information about the realization of the uncertainty. Let $\mathcal{F}_t$ be the information available at date $t \in \{0, 1, \ldots\}$ which is given by a finite partition of S. For simplicity, we assume that there is no loss of information along the event-tree, i.e., $\mathcal{F}_{t+1}$ is finer than $\mathcal{F}_t$, for each $t \geq 0$. Moreover, no information is available at $t = 0$, i.e., $\mathcal{F}_0 = S$.

A pair $\xi = (t, \sigma)$, where $t \geq 0$ and $\sigma \in \mathcal{F}_t$, is called a *node* of the economy. The date associated to ξ is denoted by $t(\xi)$. The set of all nodes, called the *event-tree*, is denoted by D. Given $\xi = (t, \sigma)$ and $\mu = (t', \sigma')$, we say that μ is a *successor* of ξ, and we write $\mu \geq \xi$, if $t' \geq t$ and $\sigma' \subset \sigma$. Let ξ^+ be the set of immediate successors of ξ, that is, the set of nodes $\mu \geq \xi$, where $t(\mu) = t(\xi) + 1$. The (unique) predecessor of ξ is denoted by ξ^- and ξ_0 is the node at $t = 0$. Let $D(\xi) := \{\mu \in D : \mu \geq \xi\}$, $D^T(\xi) := \{\mu \in D(\xi) : t(\mu) \leq T + t(\xi)\}$ and $D_T(\xi) := \{\mu \in D(\xi) : t(\mu) = T + t(\xi)\}$.

At each $\xi \in D$ there is a finite ordered set, L, of perishable commodities that can be traded in spot markets. Let $p(\xi) = (p_l(\xi); l \in L) \in \mathbb{R}^L_+$ be the vector of commodity prices at ξ. Also, the process of commodity prices is denoted by $p = (p(\xi); \xi \in D)$.

There is an ordered set J of real assets that can be negotiated in the economy. Each asset $j \in J$ is characterized by the node at which it is issued, $\xi_j \in D$, by the maximum number of period in which it can be negotiated, $T_j \in \mathbb{N} \cup \{+\infty\}$, and by (unitary) real payments, $A(\mu, j) \in \mathbb{R}^L_+$, where $\mu \in D^{T_j}(\xi_j) \setminus \{\xi_j\}$. We assume that, for each $j \in J$, $(A(\mu, j); \mu \in D^{T_j}(\xi_j) \setminus \{\xi_j\}) \neq 0$. Thus, by construction, we avoid *fiat money* in our economy.

At each node the number of issued assets is finite. That is, the set $J(\xi) = \{j \in J : (\xi \in D^{T_j - 1}(\xi_j)) \wedge (\exists \mu > \xi, A(\mu, j) \neq 0)\}$, formed by the assets that can be negotiated at ξ, is either empty or finite. If for every $T > 0$ there exists $\xi \in D_T(\xi_j)$ such that $j \in J(\xi)$, then we say that asset j is infinite-lived.

Let $q(\xi) = (q_j(\xi); j \in J(\xi))$ be the vector of asset prices at ξ. Also, $q = (q(\xi); \xi \in D)$ denotes the process of asset prices in the economy. Define $D(J) = \{(\xi, j) \in D \times J : j \in J(\xi)\}$.

A finite number of agents, $h \in H$, trade securities and buy commodities at each node in the event-tree. Each $h \in H$ is characterized by her physical and financial endowments, $(w^h(\xi), e^h(\xi)) \in$

$\mathbb{R}^L_{++} \times \mathbb{R}^{J(\xi)}_+$, at each $\xi \in D$, and by her preferences on consumption, which are represented by an utility function $U^h : \mathbb{R}^{D\times L}_+ \to \mathbb{R}_+ \cup \{+\infty\}$.

For each $j \in J(\xi)$, $\overline{e}^h_j(\xi) = \sum_{\xi_j \le \mu \le \xi} e^h_j(\mu)$ denotes the vector of aggregated financial endowments received by agent h up to node ξ, where $e^h_j(\mu)$ is the quantity of asset j received by agent h at μ. Essentially, we assume that assets' net supply does not disappear or depreciate, before its terminal nodes. We denote by $W^h(\xi) = w^h(\xi) + \sum_{j\in J(\xi^-)} A(\xi,j)\overline{e}^h_j(\xi^-)$ the agent h's aggregated physical endowments up to node $\xi \in D$, where $A(\xi_0, j) = 0$, for each $j \in J(\xi_0)$. Also, we write $W(\xi) = \sum_{h\in H} W^h(\xi)$.

Let $x^h(\xi) = (x^h_l(\xi); l \in L)$ be the consumption bundle of agent h at ξ. Analogously, $\theta^h_j(\xi)$ and $\varphi^h_j(\xi)$ denote, respectively, the quantity of asset $j \in J(\xi)$ that agent h buys and sells at ξ. Thus, given commodity and asset prices (p, q), each agent $h \in H$ maximizes her preferences by choosing an allocation, $(x^h, \theta^h, \varphi^h) := \big((x^h(\xi), \theta^h(\xi), \varphi^h(\xi)); \xi \in D\big) \in \mathbb{E} := \mathbb{R}^{D\times L}_+ \times \mathbb{R}^{D(J)}_+ \times \mathbb{R}^{D(J)}_+$, which belongs to her budget set $B^h(p, q)$, which is given by the collection of allocations $(x, \theta, \varphi) \in \mathbb{E}$ such that, for every $\xi \in D$, the following two inequalities hold,

$$\begin{aligned}
p(\xi)\left(x(\xi) - w^h(\xi)\right) + q(\xi)\left(\theta(\xi) - \varphi(\xi) - e^h(\xi)\right) \quad &\le \quad \sum_{j\in J(\xi^-)} \left(p(\xi)A(\xi,j) + q_j(\xi)\right)\left(\theta_j(\xi^-) - \varphi_j(\xi^-)\right), \\
q(\xi)\varphi(\xi) \quad &\le \quad \kappa p(\xi) w^h(\xi),
\end{aligned}$$

where $\kappa > 0$ and $(\theta(\xi_0^-), \varphi(\xi_0^-)) = 0$. Note that, at each $\xi \in D$, agent h only choose short-positions $\varphi(\xi)$ that maintain an amount of borrowing which is less than or equal to a fixed proportion $\kappa > 0$ of her initial wealth (alternatively, we can assume that borrowing constraints depend on the market value of aggregated wealth (see the next section for details)). We introduce this *borrowing constraint* in order to prevent agents from entering into Ponzi schemes.

Definition. An *equilibrium* for our economy is given by a vector of prices (p, q) jointly with allocations $\big((x^h, \theta^h, \varphi^h); h \in H\big)$, such that,

(a) For each agent $h \in H$, $(x^h, \theta^h, \varphi^h) \in \operatorname{argmax}_{(x,\theta,\varphi)\in B^h(p,q)} U^h(x)$.

(b) At each $\xi \in D$, both physical and asset markets clear,

$$\sum_{h\in H} x^h(\xi) = W(\xi); \qquad \sum_{h\in H} \theta^h_j(\xi) = \sum_{h\in H} \overline{e}^h_j(\xi) + \sum_{h\in H} \varphi^h_j(\xi), \quad \forall j \in J(\xi).$$

3. Existence of Equilibrium

In this section we formalize our main result which assures that equilibrium exists in our economy.

Theorem. *Suppose that the following assumptions hold,*

(A1) *For each* $(\xi, h) \in D \times H$, $w^h(\xi) \gg 0$.

(A2) *For any asset* $j \in J$, $\sum_{h\in H} \overline{e}_j^h(\xi_j) > 0, \;\; \forall j \in J.$

(A3) *For each* $h \in H$, $U^h(x) = \sum_{\xi\in D} u^h(\xi, x(\xi))$, *where* $u^h(\xi,\cdot) : \mathbb{R}_+^L \to \mathbb{R}_+$ *is continuous, concave and strictly increasing. Moreover,* $U^h(W) < +\infty$.

(A4) *For each* $(\xi,h) \in D \times H$,

$$\lim_{x\in\mathbb{R}_{++}^L;\|x\|_{L(J)}\to+\infty} u^h(\xi,x) = +\infty,$$

where $L(J) := \{l \in L : \exists(\mu,j) \in D \times J, \;\; A_l(\mu,j) > 0\}$ *and* $\|x\|_{L(J)} = \max_{l\in L(J)} |x_l|$.

Then, our economy has an equilibrium.

The objective of Assumptions (A2) and (A4) is just to get bounds for equilibrium asset prices. Precisely, we prove that, if intertemporal utility functions go to infinity as consumption increases (on commodities in which assets pay), assets prices are bounded away from zero. Moreover, when assets have positive net supply, Assumption (A4) will allow us to assure that assets prices have an upper bound as well (see example below).

Our financial constraints allow us to establish a link between the asymptotic amount of borrowing and the asymptotic value of initial endowments. Thus, to prove optimality of individual allocations, that will be obtained as limit of optimal allocations in finite horizon economies, it is enough to assure that the discounted value of individual wealth is finite (using as deflators the cluster point of the Kuhn-Tucker multipliers corresponding to finite horizon economies). This will be the case, as it is proved in the Appendix (see the discussion after Lemma 2).

Note that, in the particular case in which assets are numerarie (that is, $L(J) = \{l\}$, for some $l \in L$), any utility function that is quasilinear in the commodity used as numerarie satisfy Assumption (A4).

Corollary 1. *Suppose that Assumptions (A1)-(A3) hold. If all assets pay in a commodity* $l \in L$ *and, for any* $(h,\xi) \in H \times D$,

$$u^h(\xi,x) = x_l + v^h(\xi, x_{-l}), \;\; \forall\, x = (x_l, x_{-l}) \in \mathbb{R}_+ \times \mathbb{R}_+^{L-1},$$

then our economy has an equilibrium.

It is also important to remark that our Theorem does not hold if we assume that there exist some asset in zero net supply. We illustrate this point with the following example, adapted from Hernández and Santos (1996, Example 3.9, page 118). Assume that there is no uncertainty in the economy (i.e., $D = \{0,1,2,\ldots\}$) and that there is only one commodity and only one consumer, which has a physical endowment $w_t = 1$ at period (node) $t \in D$. Also, the preferences of the consumer are represented by the utility function $U(x) = \sum_{t=0}^{+\infty} \beta_t u(x_t)$, where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, concave, strictly increasing and derivable function satisfying Assumption (A4). Moreover, for any $t \geq 0$, β_t is strictly positive and $\sum_{t=0}^{+\infty} \beta_t < +\infty$. Assume also that there is only one asset which is infinite-lived and is issued at $t = 0$. This asset promises a unitary real payment A_t at period $t > 0$. It follows that Assumptions (A1), (A3) and (A4) hold for this economy.

However, there is no equilibrium for the economy when $\sum_{t=0}^{+\infty} \beta_t A_t = +\infty$. Note that this possibility may happen for a variety of discounted factors and asset payments (for instance, when $(\beta_t, A_t) = ((3/4)^t, 2^t)$, for each $t > 0$).

Essentially, if there is an equilibrium for the economy above, then first order conditions of the consumer's problem implies that, for any $T \geq 1$, the unitary asset price at $t = 0$ satisfies,

$$q_0 = \frac{1}{\beta_0} \sum_{t=1}^{T} \beta_t A_t + \frac{1}{\beta_0} \beta_T q_T.$$

Therefore, $q_0 \geq \frac{1}{\beta_0} \sum_{t=1}^{T} \beta_t A_t$, for any $T \geq 1$. Thus, $\sum_{t=0}^{+\infty} \beta_t A_t < +\infty$.

4. Comments and remarks

In this Section, we present some comments and remarks which connect our existence results with related papers. We also analyze the assumptions that have been required to get existence of equilibrium in relation with other hypotheses stated in the literature.

◇ *Uniform impatience is not required to prove equilibrium existence.*

The uniform impatience properties used in the literature are joint requirements on preferences and endowments (see, for instance, Hernández and Santos (1996, Assumption C.3) or Magill and Quinzii (1996, Assumptions B2 and B4)). In particular, uniform impatience is satisfied when (i) individuals' endowments are uniformly bounded from above and away from zero, and (ii) intertemporal discount factors are constant. However, although in our model utility functions are separable in time and states of nature, intertemporal discount factors (when are well defined) are not necessarily constant and/or endowments are not necessarily bounded. For more details, see the characterization of uniform impatience in Páscoa, Petrassi and Torres-Martínez (2010).

◇ *Equilibria with bounded utilities.*

In our model, agents are not restricted to select *bounded* consumption plans. However, if we suppose that consumers can only choose plans $x^h = (x^h(\xi); \xi \in D)$ in $l^\infty_+(L \times D) := \{y \in \mathbb{R}^{L \times D}_+ :$

$\max_{(l,\xi)\in L\times D}\ y_l(\xi) < \infty\}$, then Assumption (A4) can be removed when both aggregated endowments are bounded and (A3) is strengthened by requiring also separability on commodities. Precisely, we can adapt the proof of our theorem to obtain the following result.

Corollary 2. *Suppose that consumption bundles are restricted to belong to $l^\infty_+(L \times D)$, that Assumptions (A1)-(A3) hold, and that the following hypotheses are satisfied,*

(A5) $W = (W(\xi); \xi \in D) \in l^\infty_+(L \times D)$.

(A6) *For any $(\xi, l, h) \in D \times L(J) \times H$, there are functions $f^h_l(\xi, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that,*

$$u^h(\xi, x) = v^h(\xi, x_{-l}) + \sum_{l \in L(J)} f^h_l(\xi, x_l), \quad \forall\, x = (x_l, x_{-l}) \in \mathbb{R}_+ \times \mathbb{R}^{L-1}_+.$$

Then, there exists an equilibrium for our economy.

$\diamond$ *Alternative borrowing constraints.*

Assume that for every $(\xi, h) \in D \times H$ we have $\rho W(\xi) \leq w^h(\xi)$, for some $\rho \in (0, 1)$. Then, we can bound the growth of borrowing by requiring that, at each node ξ, $q(\xi)\varphi(\xi) \leq \kappa p(\xi)W(\xi)$. Thus, borrowing constraints depend on the value of the *aggregated* wealth. Alternatively, the constraint $q(\xi)\varphi(\xi) \leq p(\xi)M$, where $M \in \mathbb{R}^L_+ \setminus \{0\}$, can be implemented provided that initial endowments, as in Magill and Quinzzi (1996), are uniformly bounded away from zero, i.e., $\exists \underline{w} \in \mathbb{R}^L_{++} : w^h(\xi) \geq \underline{w},\ \ \forall\, (\xi, h) \in D \times H$.

Actually, maintaining Assumptions (A1)-(A4) of our Theorem, in any of the cases above the same technique of proof will operate: truncated economies will also have equilibrium, given that asset prices will be bounded away from zero and from above, node by node. The main point is that transversality condition will also hold (see equations (5)-(7) in the Appendix).

$\diamond$ *Bounds on net financial debt.*

As a consequence of Assumption (A3) and (A4), for any $\xi \in D$, there exists an scalar $a(\xi) > 0$ such that, $\min_{h\in H}\ u^h(\xi, (a(\xi), \ldots, a(\xi))) > \max_{h\in H}\ U^h(W)$.

Thus, given an equilibrium $\big[(p, q), \big((x^h, \theta^h, \varphi^h); h \in H\big)\big]$, for any agent $h \in H$, the net investment at a node ξ, which is given by $\max\{q(\xi)(\theta^h(\xi) - \varphi^h(\xi)); 0\}$, is lower than $a(\xi)\|p(\xi)\|_\Sigma$.[1] In other case, instead of negotiating assets at ξ, the agent may use the resources to buy the bundle $(a(\xi), \ldots, a(\xi))$ at this node, which gives more utility to them than those that she may receive if she consumes at any node departing from ξ the aggregated endowment of the economy.

[1] Given $z = (z_1, \ldots, z_n) \in \mathbb{R}^n_+$, $\|z\|_\Sigma = \sum_{i=1}^n z_i$.

Therefore, financial market feasibility implies that, for each $h \in H$, we have that

$$-a(\xi)(\#H-1)\|p(\xi)\|_{\Sigma} \leq q(\xi)(\theta^h(\xi)-\varphi^h(\xi)) \leq a(\xi)\|p(\xi)\|_{\Sigma}.$$

In particular, since we may assume, without loss of generality, that commodity prices satisfy $\|p(\xi)\|_{\Sigma} = 1$, $\forall \xi \in D$, the net financial debt of any agent is bounded, node by node, independently of the equilibrium allocation.

Furthermore, there are some situations in which the sequence $(a(\xi); \xi \in D)$ is also uniformly bounded and, therefore, individuals' net debt is uniformly bounded along the event-tree. For instance, assume that $u^h(\xi,x) = \beta_h^{t(\xi)}\rho^h(\xi)u^h(x)$, where $\beta_h \in (0,1)$ represents an intertemporal discount factor, $\rho^h(\xi)$ is the probability of reach node ξ at period $t(\xi)$ and satisfies, $\rho^h(\xi) = \sum_{\mu\in\xi^+}\rho^h(\mu)$ with $\rho^h(\xi_0) = 1$. Moreover, suppose that Assumption (A5) holds. Then, it follows that, for any agent $h \in H$, $U^h(W) \leq \frac{u^h(\overline{W})}{1-\beta_h}$, where $\overline{W}$ is an upper bound for the agregated endowments of the economy. Taking a number $\overline{a}$ such that $\min_{h\in H}\ u^h(\overline{a},\ldots,\overline{a}) > \max_{h\in H}\ \frac{u^h(\overline{W})}{1-\beta_h}$, it follows that $a(\xi) \leq \overline{a}$, $\forall \xi \in D$.

$\diamond$ *On non-binding debt constraints.*

Note that, when there is at each node in the event-tree only one asset (finite or infinite-lived) available for trade, the uniform bound on net debt founded above induces an uniform bound on borrowing. Within this context, for values of κ large enough, our borrowing constraints are not binding at equilibrium. Previously, Hernández and Santos (1996) have shown equilibrium existence in an economy with debt constraints, when only one infinite-lived asset in positive net supply is traded. We assure more when agents are burden by borrowing constraints, namely, restrictions on the amount of borrowing became non-binding.

$\diamond$ *About the existence of rational bubbles.*

Suppose that Assumptions (A1)-(A5) hold and that initial endowments are uniformly bounded away from zero.[2] If $u^h(\xi,x) = \beta_h^{t(\xi)}\rho^h(\xi)u^h(x)$, where β_h and $\rho^h(\xi)$ satisfy the conditions previously stated, it follows from the previous comments that, for the equilibrium allocation we construct, (i) marginal rates of substitution will be summable (see equation (7) in the Appendix), and (ii) net debts will be uniformly bounded along the event-tree. In particular, as assets have positive net supply, their prices will be uniformly bounded along the event-tree. Therefore, the discounted value of asset prices, using the marginal rates of substitution as deflators, goes to zero as time goes to infinity. A necessary and sufficient condition for the absence of rational asset pricing bubbles. That

[2]That is, there exists $\underline{w} \in \mathbb{R}^L_{++}$ such that, $w^h(\xi) \geq \underline{w}$, $\forall (h,\xi) \in H \times D$.

is, analogous to Magill and Quinzii (1996) and Santos and Woodford (1997), the positive net supply assures that equilibrium asset prices are free of bubbles when uniform impatience holds.[3]

5. Conclusion

In this paper we give conditions which assure that, when finite or infinite-lived real assets in positive net supply are available for trade, equilibrium always exists in infinite horizon economies with incomplete financial markets. Borrowing constraints depending of the value of endowments (either individual or aggregated) avoid Ponzi schemes and assure equilibrium existence if utility functions are either unbounded and separable in states of nature or separable in commodities. With numerarie assets and utility functions that are quasilinear in the commodity used as numerarie, equilibrium also exists.

However, this results depend crucially on the positive net supply of assets. In fact, as we exemplify, in our model equilibrium does not necessarily exist when assets have zero net supply. This also happens in the models of Hernández and Santos (1996) and Magill and Quinzii (1996). As we can infer from the proof of equilibrium existence and from the example of non-existence of equilibria, the main difficulty is to find endogenous lower and upper bounds on assets prices, in order to obtain equilibria for truncated economies (which lead to get an equilibrium allocation as a limit equilibria in the sequence of truncated economies). It is in the second of these steps—the determination of upper bounds on asset prices—that the positive net supply and the unboundedness of utility functions become crucial. As a matter of future research, it is interesting to find conditions to prove equilibrium existence even with zero net supply long-lived assets, since within this type of financial contracts rational asset pricing bubbles with real effects may appear (see Magill and Quinzii (1996, Proposition 6.3)).

[3]Since utilities satisfies a strong version of Assumption (A3) and endowments are uniformly bounded form above and away from zero, uniform impatience holds, as was proved by Páscoa, Petrassi, and Torres-Martínez (2010, Proposition 1).

Appendix

To prove our main result we show, firstly, that there exists equilibrium in finite horizon truncated economies. Then, we find an equilibrium for the original economy as the limit of a sequence of equilibria corresponding to the truncated economies, when the time horizon increases.

Truncated economies. For each $T \in \mathbb{N}$, we define a truncated economy, $\mathcal{E}^T$, in which agents consume commodities and trade assets in the restricted event-tree $D^T(\xi_0)$.

Let $J^T(\xi) = \{j \in J(\xi) : \exists \mu \in D^{T-t(\xi)}(\xi),\, \mu \neq \xi,\, A(\mu, j) \neq 0\}$ be the set of available securities at $\xi \in D^{T-1}(\xi_0)$. At each $\xi \in D_T(\xi_0)$, we define $J^T(\xi) = \emptyset$. It follows that, given $\xi \in D$, $J^T(\xi) = J(\xi)$ for every T large enough. Let $D^T(J) = \{(\xi, j) \in D^T(\xi_0) \times J : j \in J^T(\xi)\}$.

Each individual $h \in H$ is characterized by her physical, $(w^h(\xi); \xi \in D^T(\xi_0))$, and financial, $(e^h(\xi); \xi \in D^{T-1}(\xi_0))$, endowments. Also, when agent h chooses a consumption plan $(x(\xi))_{\xi \in D^T(\xi_0)}$, her utility is given by $U^{h,T}(x) = \sum_{\xi \in D^T(\xi_0)} u^h(\xi, x(\xi))$.

For each truncated economy $\mathcal{E}^T$, we can consider, without loss of generality, prices (p,q) in

$$\mathbb{P}^T := \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta_+^L \times \mathbb{R}_+^{J^T(\xi)}\right) \times \prod_{\xi \in D_T(\xi_0)} \Delta_+^L,$$

where $\Delta_+^L := \{p \in \mathbb{R}_+^L : \|p\|_\Sigma = 1\}$. Then, given $(p,q) \in \mathbb{P}^T$, agent $h \in H$ solves the following optimization problem:

$$(P^{h,T}) \qquad \begin{array}{ll} \max & U^{h,T}(x) \\ \text{s.t.} & \left\{ \begin{array}{lll} y(\xi) = (x(\xi), \theta(\xi), \varphi(\xi)) \geq 0, & \forall \xi \in D^T(\xi_0), \\ g_\xi^{h,T}(y(\xi), y(\xi^-); p, q) & \leq 0, & \forall \xi \in D^T(\xi_0), \\ q(\xi)\varphi(\xi) - \kappa p(\xi) w^h(\xi) & \leq 0, & \forall \xi \in D^{T-1}(\xi_0), \\ (\theta(\xi), \varphi(\xi)) & = 0, & \forall \xi \in D_T(\xi_0), \end{array} \right. \end{array}$$

where $y(\xi_0^-) = 0$ and, for each $\xi \in D^T(\xi_0)$,

$$\begin{aligned} g_\xi^{h,T}(y(\xi), y(\xi^-); p, q) := p(\xi)\left(x(\xi) - w^h(\xi)\right) &+ \sum_{j \in J^T(\xi)} q_j(\xi)\left(\theta_j(\xi) - \varphi_j(\xi) - e_j^h(\xi)\right) \\ &- \sum_{j \in J^T(\xi^-)} \left(p(\xi)A(\xi, j) + q_j(\xi)\right)\left(\theta_j(\xi^-) - \varphi_j(\xi^-)\right). \end{aligned}$$

Let $B^{h,T}(p,q)$ be the truncated budget set of agent h, i.e., the set of plans $(y(\xi))_{\xi \in D^T(\xi_0)}$ that satisfy the restrictions of the problem $P^{h,T}$ above.

Definition 1. An *equilibrium* for the economy $\mathcal{E}^T$ is given by prices $(p^T, q^T) \in \mathbb{P}^T$ and individual allocations $(y^{h,T}(\xi))_{\xi \in D^T(\xi_0)} \in \mathbb{E}^T := \mathbb{R}_+^{D^T(\xi_0) \times L} \times \mathbb{R}_+^{D^T(J)} \times \mathbb{R}_+^{D^T(J)}$, such that:

(1) For each $h \in H$, $(y^{h,T}(\xi))_{\xi \in D^T(\xi_0)}$ is an optimal solution for $P^{h,T}$ at prices (p^T, q^T);

(2) Physical and financial markets clear at each $\xi \in D^T(\xi_0)$.

Equilibrium existence in the truncated economies. In order to show the existence of equilibria in $\mathcal{E}^T$ we follow a generalized game approach. For each $(\mathcal{X},\Theta,\Psi,M) \in \mathbb{F}^T := \mathbb{E}^T \times \mathbb{R}_{++}^{D^T(J)}$, consider the convex and compact set $\mathcal{K}(\mathcal{X},\Theta,\Psi) = [0,\mathcal{X}] \times [0,\Theta] \times [0,\Psi] \subset \mathbb{E}^T$ and define,

$$\mathbb{P}_M^T = \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta_+^L \times [0, M_\xi]\right) \times \prod_{\xi \in D_T(\xi_0)} \Delta_+^L.$$

Let $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ be a generalized game where each consumer is represented by a player $h \in H$ and, at each $\xi \in D^T(\xi_0)$, there is also a player who behaves as an auctioneer.

More precisely, in $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ each player $h \in H$ behaves as price-taker and, given $(p,q) \in \mathbb{P}_M^T$, she chooses strategies in the truncated budget set $B^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ in order to maximize the function $U^{h,T}$. Also, at each $\xi \in D^{T-1}(\xi_0)$ (resp. $\xi \in D_T(\xi_0)$) the corresponding auctioneer chooses commodity and asset prices $(p(\xi), q(\xi)) \in \Delta_+^L \times [0, M_\xi]$ (resp. just commodity prices $p(\xi) \in \Delta_+^L$) in order to maximize the function $\sum_{h\in H} g_\xi^{h,T}(y^h(\xi), y^h(\xi^-); p, q)$, where $y^h = (y^h(\xi))_{\xi \in D^T(\xi_0)}$ are the strategies selected by player $h \in H$.

Definition 2. A strategy profile $\left[(p^T(\xi), q^T(\xi)); (y^{h,T}(\xi))_{h\in H}\right]_{\xi \in D^T(\xi_0)} \in \mathbb{P}_M^T \times (\mathcal{K}(\mathcal{X},\Theta,\Psi))^H$ is a *Nash equilibrium* for $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ if each player maximizes her objective function, given the strategies chosen by the other players, i.e., no player has an incentive to deviate.

Lemma 1. Let $T \in \mathbb{N}$ and $(\mathcal{X},\Theta,\Psi,M) \in \mathbb{F}^T$. Under Assumptions (A1) and (A3) the set of Nash equilibria for the game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ is non-empty.

Proof. Note that each player's strategy set is non-empty, convex and compact. Further, it follows from Assumption (A3) that the objective function of each player is continuous and quasi-concave in her own strategy. Assumption (A1) assures that the correspondences of admissible strategies are continuous, with non-empty, convex and compact values. Therefore, we can find an equilibrium of the generalized game by applying Kakutani Fixed Point Theorem to the correspondence defined as the product of the optimal strategy correspondences. $\square$

Lemma 2. Let $T \in \mathbb{N}$. Under Assumptions (A1)-(A4) there exists (Θ^T, Ψ^T) such that, if $(\Theta,\Psi) \gg (\Theta^T, \Psi^T)$, then every Nash equilibrium of the game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ is an equilibrium of the economy $\mathcal{E}^T$ whenever $\mathcal{X}$ and M are large enough.

Proof. Let $\left[(p^T(\xi),q^T(\xi));(y^{h,T}(\xi))_{h\in H}\right]_{\xi\in D^T(\xi_0)}$ be a Nash equilibrium for $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$, with allocations given by $y^{h,T}(\xi)=(x^{h,T}(\xi),\theta^{h,T}(\xi),\varphi^{h,T}(\xi))$. Note that, for each $h\in H$,

$$(y^{h,T}(\xi))_{\xi\in D^T(\xi_0)}\in \operatorname{argmax}_{B^{h,T}(p^T,q^T)\cap\mathcal{K}(\mathcal{X},\Theta,\Psi)}\ U^{h,T}(x).$$

Then, as each auctioneer maximizes his objective function, we have that, at each $\xi\in D^T(\xi_0)$,

$$\sum_{h\in H}x^{h,T}(\xi)\le \Upsilon^T(\Theta,\xi):=\sum_{h\in H}\left(w^h(\xi)+\sum_{j\in J^T(\xi^-)}A(\xi,j)\Theta(\xi^-,j)\right).$$

It follows from Assumptions (A3) and (A4) that, for each $\xi\in D^T(\xi_0)$, there exists a real number $a^T_\Theta(\xi)>0$ such that,

$$\min_{h\in H}\ u^h\left(\xi,(a^T_\Theta(\xi),\dots,a^T_\Theta(\xi))\right)>\max_{h\in H}\ U^{h,T}(\Upsilon^T(\Theta)),$$

where $\Upsilon^T(\Theta):=(\Upsilon^T(\Theta,\xi);\xi\in D^T(\xi_0))$.

Suppose that $\mathcal{X}(\xi,l)>a^T_\Theta(\xi)$, for every $(\xi,l)\in D^T(\xi_0)\times L$. As $\|p^T(\xi)\|_\Sigma=1$, it follows from individual optimality that the value of accumulated individual financial endowments, at any $\xi\in D^T(\xi_0)$, is necessarily less than $p^T(\xi)(a^T_\Theta(\xi),\dots,a^T_\Theta(\xi))=a^T_\Theta(\xi)$. Therefore, for each $j\in J^T(\xi)$,

$$q^T_j(\xi)\le M^T_\Theta(\xi,j):=\frac{a^T_\Theta(\xi)\,\#H}{\sum_{h\in H}\overline{e}^h_j(\xi)}.$$

Let $M^T_\Theta=(M^T_\Theta(\xi,j);(\xi,j)\in D^T(J))$. We conclude that if $M\gg M^T_\Theta$, then in any Nash equilibrium of $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ the upper bounds of asset prices, which were previously imposed, are non-binding. Along the rest of this proof we assume that this property holds.

Step 1. Physical markets clear. For each $\xi\in D^T(\xi_0)$, let

$$\Gamma(\xi)=\sum_{h\in H}x^{h,T}(\xi)-W(\xi)\,,\qquad \Omega(\xi)=\sum_{h\in H}\theta^{h,T}(\xi)-\sum_{h\in H}\overline{e}^h(\xi)-\sum_{h\in H}\varphi^{h,T}(\xi).$$

Summing up the budget constraints at ξ_0 we have $p^T(\xi_0)\Gamma(\xi_0)+q^T(\xi_0)\Omega(\xi_0)\le 0$. Since the auctioneer at ξ_0 maximizes $p(\xi_0)\Gamma(\xi_0)+q(\xi_0)\Omega(\xi_0)$, we obtain that $\Gamma(\xi_0)\le 0$. Assume now that $\Omega(\xi_0,j)>0$, for some $j\in J^T(\xi_0)$. By the construction of the plan M, we know that $q^T_j(\xi_0)<M_{\xi_0,j}$, which leads us to obtain a contradiction with the optimal behaviour of the auctioneer at ξ_0. Thus $\Omega(\xi_0)\le 0$. Hence, if $\mathcal{X}(\xi_0,l)>\max\{W(\xi_0,l),a^T_\Theta(\xi_0)\}$ for each $l\in L$, then the upper bound on consumption is non-binding at ξ_0, allowing us to conclude, as a consequence of the monotonicity of preferences, that commodity markets clear at the initial node ξ_0, i.e., $\Gamma(\xi_0)=0$. Moreover, $q^T(\xi_0)\Omega(\xi_0)=0$.

Consider now a node ξ with $t(\xi)=1$, and recall that the corresponding auctioneer at ξ chooses prices in $\Delta^L_+\times[0,M_\xi]$ in order to maximize the function $\sum_{h\in H}g^{h,T}_\xi(y^{h,T}(\xi),y^{h,T}(\xi_0);p,q)$. Using the fact that $\Omega(\xi_0)\le 0$, we can deduce that $p^T(\xi)\Gamma(\xi)+q^T(\xi)\Omega(\xi)\le 0$, for every ξ with $t(\xi)=1$.

As before, $\Gamma(\xi) \leq 0$ and $\Omega(\xi) \leq 0$. Furthermore, if $\mathcal{X}(\xi) > \max\{W(\xi, l), a_\Theta^T(\xi)\}$ for every $l \in L$, then the upper bound on consumption is not binding at ξ, which implies that $\Gamma(\xi) = 0$.

By applying successively analogous arguments to the nodes with periods $t = 2, \ldots, T$, we conclude that $\Gamma(\xi) = 0$ for every $\xi \in D^T(\xi_0)$, provided that, for each $l \in L$, $\mathcal{X}(\xi, l) > \max\{W(\xi, l),\, a_\Theta^T(\xi)\}$. That is, physical markets clear in the economy $\mathcal{E}^T$. Furthermore, there is no excess of demand for financial markets, i.e., $\Omega(\xi) \leq 0$, for every $\xi \in D^{T-1}(\xi_0)$.

Step 2. Lower bounds for asset prices. Given $(\xi, j) \in D^T(J)$, fix a node $\mu(\xi, j)$ that belongs to the non-empty set *argmin*$\{t(\mu) \,:\, \mu \in D^{T-t(\xi)}(\xi),\, \mu \neq \xi,\, A(\mu, j) \neq 0\}$.

By Assumptions (A1), (A3) and (A4), there exists $b(\xi, j) \in (0, 1)$, independent of T, such that, for every $h \in H$, the following inequality holds,

$$u^h\left(\mu(\xi, j), w^h(\mu(\xi, j)) + \frac{A(\mu(\xi, j), j)\min_{l \in L} w_l^h(\xi)}{b(\xi, j)}\right) > U^h(W). \tag{1}$$

Suppose that,

$$\Theta(\xi, j) > \widehat{\Theta}(\xi, j) := \max_{h \in H} \frac{\min_{l \in L} w_l^h(\xi)}{b(\xi, j)},$$

and for every $\mu \in D^{T-t(\xi)}(\xi)$ with $j \in J^T(\mu)$,

$$\min_{l \in L} \mathcal{X}(\mu, l) > \mathcal{X}_{\Theta,\xi}^T(\mu, j) := \max_{(l,h) \in H \times L} \left\{W(\mu, l),\, a_\Theta^T(\mu),\, w_l^h(\mu) + \frac{A_l(\mu, j)\min_{l' \in L} w_{l'}^h(\xi)}{b(\xi, j)}\right\}.$$

We claim that $q_j^T(\xi) > b(\xi, j)$. In fact, if $q_j^T(\xi) \leq b(\xi, j)$ then, as by Step 1 $x^{h,T}(\mu) \leq W(\mu)$ for every $\mu \in D^T(\xi_0)$, it follows from Assumption (A3) and inequality (1) that any agent $h \in H$ has an incentive to deviate by choosing any budget feasible strategy $(x^h, \theta^h, \varphi^h)$ that satisfies,

$$\begin{aligned} \theta_j^h(\xi) &= \frac{\min_{l \in L} w_l^h(\xi)}{b(\xi, j)}, \\ x^h(\mu) &= w^h(\mu) + A(\mu, j)\theta_j^h(\xi), \quad \text{if } \mu = \mu(\xi, j). \end{aligned}$$

Therefore, if for each $\eta \in D^T(\xi_0)$,

$$\begin{aligned} \Theta(\eta, j) &> \widehat{\Theta}(\eta, j),\ \forall j \in J^T(\eta), \\ \mathcal{X}(\eta, l) &> \mathcal{X}_\Theta^T(\eta) := \max_{\substack{(\xi,j) \in D^T(J): \\ \eta > \xi,\, j \in J^T(\eta)}} \mathcal{X}_{\Theta,\xi}^T(\eta, j),\ \forall l \in L, \end{aligned}$$

then equilibrium asset prices have a positive lower bound away from zero. In fact, for each $(\eta, j) \in D^T(J)$, we have that $q_j^T(\eta) > b(\eta, j)$.

Step 3. Non-binding short-sales constraints. Define $\widehat{\Theta}^T = (\widehat{\Theta}(\eta, j); (\eta, j) \in D^T(J))$ and $\mathcal{X}_\Theta^T = (\mathcal{X}_\Theta^T(\eta); \eta \in D^T(\xi_0))$. If $\Theta \gg \widehat{\Theta}^T$ and $\mathcal{X} \gg \mathcal{X}_\Theta^T$, asset prices are bounded away from zero. Thus, using the borrowing constraints, we conclude that, for every player $h \in H$,

$$\varphi_j^{h,T}(\xi) < \widehat{\Psi}_j(\xi) := \kappa \, \frac{\max_{(h,l)\in H\times L} w_l^h(\xi)}{b(\xi,j)}, \qquad \forall (\xi,j) \in D^T(J).$$

Let $\Psi^T = (\widehat{\Psi}_j(\xi); (\xi,j) \in D^T(J))$. If $\Psi \gg \Psi^T$ then short-sales restrictions induced by $\mathcal{K}(\mathcal{X},\Theta,\Psi,M)$ are non-binding.

Step 4. Financial markets clear and upper bounds for long-positions are non-binding. Suppose that $(\Theta,\Psi) \gg (\widehat{\Theta}^T,\Psi^T)$ and $\mathcal{X} \gg \mathcal{X}_\Theta^T$. Now, by Step 1 we have that $q^T(\xi)\Omega(\xi) = 0$ and $\Omega(\xi) \leq 0$, for each $\xi \in D^{T-1}(\xi_0)$. Thus, if for some $(\xi,j) \in D^T(J)$, $\Omega_j(\xi) < 0$, then $q_j^T(\xi) = 0$, which is in contradiction with the lower bound on asset prices find in Step 2.

On the other hand, for each $\xi \in D^{T-1}(\xi_0)$, $(\varphi^{h,T}(\xi))_{h\in H}$ is bounded. Thus, as $\Omega(\xi) \leq 0$, $\sum_{h\in H} \theta^{h,T}(\xi)$ is also bounded. We conclude that there exists $\Theta^T \geq \widehat{\Theta}^T$ such that, if $\Theta \gg \Theta^T$ then upper bounds on long positions are non-binding.

Step 5. Individual optimality. As a consequence of all previous steps, if $(\Theta,\Psi) \gg (\Theta^T,\Psi^T)$ and $(\mathcal{X},M) \gg (\mathcal{X}_\Theta^T, M_\Theta^T)$ then, for each $h \in H$, the optimal allocation $y^{h,T}$ belongs to the interior of $\mathcal{K}(\mathcal{X},\Theta,\Psi,M)$ (relative to $\mathbb{E}^T$). As budget correspondences has finite-dimensional convex values, we conclude that,

$$(y^{h,T}(\xi))_{\xi\in D^T(\xi_0)} \in \operatorname{argmax}_{B^{h,T}(p^T,q^T)} \sum_{\xi\in D^T(\xi_0)} u^h(\xi,x(\xi)).$$

Therefore, since $(\Theta,\Psi) \gg (\Theta^T,\Psi^T)$ and $(\mathcal{X},M) \gg (\mathcal{X}_\Theta^T, M_\Theta^T)$, any Nash equilibrium of the game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ is an equilibrium of the truncated economy $\mathcal{E}^T$. $\square$

Recall that, given $\xi \in D$, $J^T(\xi) = J(\xi)$ for T large enough. Thus, by construction, the upper bounds $(\Theta^T(\xi),\Psi^T(\xi))$ are independent of $T > t(\xi)$, when T is large enough. Therefore, node by node, independently of the truncated horizon T, individual equilibrium allocations are uniformly bounded and commodity prices belong to the simplex.

Moreover, under Assumptions (A2)-(A4) asset prices are uniformly bounded by above, node by node. In fact, as consumption allocations are bounded by the aggregated resources, by analogous arguments to those made in the proof of Lemma 2, we can conclude that,

$$q_j^T(\xi) \leq \frac{a(\xi)\,\#H}{\sum_{h\in H} \bar{e}_j^h(\xi)}, \qquad \forall j \in J^T(\xi),$$

where $a(\xi) > 0$ is independent of $T > t(\xi)$ and is defined implicitly by

$$\min_{h\in H}\; u^h\left(\xi,(a(\xi),\ldots,a(\xi))\right) > \max_{h\in H} U^h(W).$$

Asymptotic equilibria. In order to find an equilibrium of our original economy, we look for an uniform bound (node by node) for the Kuhn-Tucker multipliers associated to the truncated individual problems.

To attempt this aim, for each $T \in \mathbb{N}$, consider an equilibrium $\left[p^T(\xi), q^T(\xi); (y^{h,T}(\xi))_{h\in H}\right]_{\xi\in D^T(\xi_0)}$ for the economy $\mathcal{E}^T$. Then. there exist *non-negative* multipliers $\left((\gamma_\xi^{h,T})_{\xi\in D^T(\xi_0)};\, (\rho_\xi^{h,T})_{\xi\in D^{T-1}(\xi_0)}\right)$ such that,

$$\gamma_\xi^{h,T} g_\xi^{h,T}(y^{h,T}(\xi),\, y^{h,T}(\xi^-); p^T, q^T) \;=\; 0, \quad \forall \xi \in D^T(\xi_0); \tag{2}$$

$$\rho_\xi^{h,T}\left(\kappa p^T(\xi) w^h(\xi) - q^T(\xi)\varphi^{h,T}(\xi)\right) \;=\; 0, \quad \forall \xi \in D^{T-1}(\xi_0). \tag{3}$$

Moreover, for each plan $(x(\xi), \theta(\xi), \varphi(\xi))_{\xi\in D^T(\xi_0)} \geq 0$, with $(\theta(\eta), \varphi(\eta))_{\eta\in D_T(\xi_0)} = 0$, the following saddle point property is satisfied (see Rockafellar (1997), Section 28, Theorem 28.3),

(4)

$$U^{h,T}(x) - \sum_{\xi\in D^T(\xi_0)} \gamma_\xi^{h,T} g_\xi^{h,T}(y(\xi), y(\xi^-);\, p^T, q^T) + \sum_{\xi\in D^{T-1}(\xi_0)} \rho_\xi^{h,T}(\kappa p^T(\xi) w^h(\xi) - q^T(\xi)\varphi(\xi)) \leq U^{h,T}(x^{h,T}).$$

Let us take $(x(\xi), \theta(\xi), \varphi(\xi))_{\xi\in D^T(\xi_0)} = (0,0,0)$ to obtain,

$$\sum_{\xi\in D^{T-1}(\xi_0)} p^T(\xi) w^h(\xi)\left[\gamma_\xi^{h,T} + \rho_\xi^{h,T}\kappa\right] \leq U^h(W) < +\infty. \tag{5}$$

Since commodity prices are in the simplex, node by node, for every $\xi \in D$ and for all $T > t(\xi)$, we conclude that,

$$0 \leq \gamma_\xi^{h,T} \leq \frac{U^h(W)}{\underline{w}_\xi^h}, \qquad 0 \leq \rho_\xi^{h,T} \leq \frac{U^h(W)}{\kappa\, \underline{w}_\xi^h},$$

where, by Assumption (A1), $\underline{w}_\xi^h := \min_{l\in L} w_l^h(\xi) > 0$.

In short, for each $\xi \in D$, the sequence formed by equilibrium prices, equilibrium allocations and Kuhn-Tucker multipliers, $((p^T(\xi), q^T(\xi)); (y^{h,T}(\xi), \gamma_\xi^{h,T}, \rho_\xi^{h,T})_{h\in H})_{T>t(\xi)}$, is bounded. Applying Tychonoff Theorem we can find a common subsequence $(T_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that, for each $\xi \in D$,

$$\lim_{k\to+\infty} ((p^{T_k}(\xi), q^{T_k}(\xi)); (y^{h,T_k}(\xi), \gamma_\xi^{h,T_k}, \rho_\xi^{h,T_k})_{h\in H}) = \left((\overline{p}(\xi), \overline{q}(\xi)); (\overline{y}^h(\xi), \overline{\gamma}_\xi^h, \overline{\rho}_\xi^h)_{h\in H}\right).$$

Hence, for each $h \in H$, $\left(\overline{y}^h(\xi)\right)_{\xi\in D} \in B^h(\overline{p}, \overline{q})$. Moreover, limit allocations are cluster points, node by node, of equilibria in truncated economies and then market clearing follows. Therefore, in order to conclude that $\left[(\overline{p}(\xi), \overline{q}(\xi)); (\overline{y}^h(\xi))_{h\in H}\right]_{\xi\in D}$ is an equilibrium it remains to show that, for each agent $h \in H$, $(\overline{y}^h(\xi))_{\xi\in D}$ is an optimal choice when prices are $(\overline{p}, \overline{q})$.

Lemma 3. Under Assumptions (A1)-(A4), $U^h(\tilde{x}) \leq U^h(\overline{x})$, for every $\tilde{y} := (\tilde{x}, \tilde{\theta}, \tilde{\varphi}) \in B^h(\overline{p}, \overline{q})$.

PROOF. Fix a node $\xi \in D$. Let us take $T > t(\xi)$ large enough to assure that $J^T(\mu) = J(\mu)$ for each $\mu \leq \xi$ and consider the allocation,

$$(x(\mu), \theta(\mu), \varphi(\mu)) = \begin{cases} (x^{h,T}(\mu), \theta^{h,T}(\mu), \varphi^{h,T}(\mu)), & \text{if } \mu \neq \xi, \\ (\tilde{x}(\xi), \tilde{\theta}(\xi), \tilde{\varphi}(\xi)), & \text{if } \mu = \xi. \end{cases}$$

Then, it follows from inequality (4) that, under Assumption (A3),

$$u^h(\xi, \tilde{x}(\xi)) - u^h(\xi, x^{h,T}(\xi)) \leq -\rho_\xi^{h,T} \left(\kappa p^T(\xi) w^h(\xi) - q^T(\xi)\tilde{\varphi}(\xi) \right)$$

$$+\gamma_\xi^{h,T} g_\xi^h(\tilde{y}(\xi), y^{h,T}(\xi^-);\, p^T, q^T) + \sum_{\mu \in \xi^+} \gamma_\mu^{h,T} g_\mu^h(y^{h,T}(\mu), \tilde{y}(\xi);\, p^T, q^T),$$

where $g_\xi^h \leq 0$ denotes the budget constraint at $\xi \in D$. As $\tilde{y}$ is budget feasible at prices $(\overline{p}, \overline{q})$, taking the limit as $T = T_k$ goes to infinity, we obtain that,

$$u^h(\xi, \tilde{x}(\xi)) - u^h(\xi, \overline{x}(\xi)) \leq \overline{\gamma}_\xi^h g_\xi^h(\tilde{y}(\xi), \overline{y}^h(\xi^-);\, \overline{p}, \overline{q}) + \sum_{\mu \in \xi^+} \overline{\gamma}_\mu^h g_\mu^h(\overline{y}^h(\mu), \tilde{y}(\xi);\, \overline{p}, \overline{q}).$$

As $\tilde{y}$ and $(\overline{y}^h(\xi))_{\xi \in D}$ belongs to $B^h(\overline{p}, \overline{q})$, adding previous inequality over the nodes in $D^N(\xi_0)$, with $N \in \mathbb{N}$, it follows that,

$$U^{h,N}(\tilde{x}) - U^{h,N}(\overline{x}) \leq \sum_{\mu \in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h g_\mu^h(\overline{y}^h(\mu), \tilde{y}(\mu^-);\, \overline{p}, \overline{q}).$$

Thus, as $\tilde{y}$ is budget feasible, borrowing constraints imply that,

$$(6) \qquad U^{h,N}(\tilde{x}) - U^{h,N}(\overline{x}) \leq \sum_{\mu \in D_{N+1}(\xi_0)} \overline{\gamma}_\mu^h \left(\overline{p}(\mu)\overline{x}^h(\mu) + \overline{q}(\mu)(\overline{\theta}^h(\mu) - \overline{\varphi}^h(\xi)) + \kappa \overline{p}(\mu) w^h(\mu) \right).$$

Define $L_\xi^{h,T} = p^T(\xi)x^{h,T}(\xi) + q^T(\xi)(\theta^{h,T}(\xi) - \varphi^{h,T}(\xi))$ and consider the allocation,

$$(x(\mu), \theta(\mu), \varphi(\mu)) = \begin{cases} (x^{h,T}(\mu), \theta^{h,T}(\mu), \varphi^{h,T}(\mu)), & \text{if } \mu \neq \xi\,, \\ (0, 0, 0), & \text{if } \mu = \xi. \end{cases}$$

Using inequality (4), Assumption (A3) assures that,

$$\begin{aligned} \gamma_\xi^{h,T} L_\xi^{h,T} &\leq u^h(\xi, x^{h,T}(\xi)) + \sum_{\mu \in \xi^+} \gamma_\mu^{h,T} L_\mu^{h,T}, \quad \forall \xi \in D^{T-1}(\xi_0); \\ \gamma_\xi^{h,T} L_\xi^{h,T} &\leq u^h(\xi, x^{h,T}(\xi)), \quad \forall \xi \in D_T(\xi_0). \end{aligned}$$

Thus, by monotonicity of preferences,

$$\sum_{\xi \in D_{N+1}(\xi_0)} \gamma_\xi^{h,T} L_\xi^{h,T} \leq \sum_{\mu \in D \setminus D^N(\xi_0)} u^h(\mu, W(\mu)), \quad \forall T > N+1.$$

Taking the limit as T goes to infinity we obtain,

$$\sum_{\xi \in D_{N+1}(\xi_0)} \overline{\gamma}_\xi^h \left(\overline{p}(\xi)\overline{x}^h(\xi) + \overline{q}(\xi)(\overline{\theta}^h(\xi) - \overline{\varphi}^h(\xi)) \right) \leq \sum_{\mu \in D \setminus D^N(\xi_0)} u^h(\mu, W(\mu)).$$

Thus, it follows from inequality (6) that,

$$U^{h,N}(\tilde{x}) - U^{h,N}(\overline{x}) \leq \sum_{\mu \in D \setminus D^N(\xi_0)} u^h(\mu, W(\mu)) + \kappa \sum_{\mu \in D_{N+1}(\xi_0)} \overline{\gamma}^h_\mu \overline{p}(\mu) w^h(\mu).$$

Now, inequality (5) assures that,

$$\sum_{\xi \in D} \overline{\gamma}^h_\xi \overline{p}(\xi) w^h(\xi) < +\infty. \tag{7}$$

Therefore, it follows from Assumption (A3) that: For each $\varepsilon > 0$ there exists $\overline{N}_\varepsilon > 0$ such that,

$$\sum_{\xi \in D^N(\xi_0)} u^h(\xi, \tilde{x}(\xi)) < \varepsilon + U^h(\overline{x}), \quad \forall N > \overline{N}_\varepsilon$$

Finally, we conclude that, for each $\varepsilon > 0$, $U^h(\tilde{x}) \leq \varepsilon + U^h(\overline{x})$, which ends the proof. □

Proof of the Corollary 2. Given $(\xi, h) \in D \times H$, define

$$\tilde{u}^h(\xi, x) = v^h\left(\xi, (x_l)_{l \in L \setminus L(J)}\right) + \sum_{l \in L(J)} \left(f^h_l\left(\xi, \min\{x_l, 2W_l(\xi)\}\right) + \rho(\xi, l) \max\{x_l - 2W_l(\xi), 0\}\right),$$

where $x = (x_l; l \in L) \in \mathbb{R}^L_+$ and $\rho(\xi, l) \in \partial f^h_l(\xi, 2W_l(\xi))$.[4] It follows from the separability of the inter-temporal utilities on commodities in $L(J)$ that the functions,

$$\tilde{U}^h(x) := \sum_{\xi \in D} \tilde{u}^h(\xi, x(\xi)),$$

satisfy Assumptions (A3) and (A4). Therefore, there exists an equilibrium $\left[(\overline{p}(\xi), \overline{q}(\xi)); (\overline{y}^h(\xi))_{h \in H}\right]_{\xi \in D}$, being $\overline{y}^h(\xi) = (\overline{x}^h(\xi), \overline{\theta}^h(\xi), \overline{\varphi}^h(\xi))$, for the economy in which each $h \in H$ has preferences represented by the function $\tilde{U}^h$ instead of U^h. Moreover, this equilibrium is actually an equilibrium for the original economy. In fact, since agents are restricted to choose bounded consumption plans, if there exists a budget feasible allocation $(x^h, \theta^h, \varphi^h)$ such that $U^h(x^h) > U^h(\overline{x}^h)$ then there is $\lambda \in (0,1)$ such that, the consumption plan $x(\lambda) := \lambda x^h + (1-\lambda)\overline{x}^h$, with $x(\lambda) = (x_l(\lambda, \xi); \xi \in D)$, satisfies $x_l(\lambda, \xi) < 2W_l(\xi)$, $\forall l \in L(J)$. Thus,

$$\tilde{U}^h(x(\lambda)) = U^h(x(\lambda)) > \lambda U^h(x^h) + (1-\lambda) U^h(\overline{x}^h) > U^h(\overline{x}^h) = \tilde{U}^h(\overline{x}^h),$$

which is a contradiction. □

[4] We denote by $\partial f^h_l(\xi, x)$ the *super-gradient* of a concave function $f^h_l(\xi, \cdot)$ at point x. That is, $z \in \partial f^h_l(\xi, x)$ iff $f^h_l(\xi, y) - f^h_l(\xi, x) \leq z(y - x)$ for every $y \in \mathbb{R}_+$. Recall that, given $l \in L(J)$, $\partial f^h_l(\xi, x) \neq \emptyset$ at any point $x > 0$.

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