Estimation of Dynamic Discrete Games Using the Nested Pseudo Likelihood Algorithm: Code and Application

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Abstract

This document describes program code for the solution and estimation of dynamic discrete games of incomplete information using the Nested Pseudo Likelihood (NPL) method in Aguirregabiria and Mira (2007). The code is illustrated using a dynamic game of store location by retail chains, and actual data from McDonalds and Burger King.

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1 Introduction

This document describes program code for the solution and estimation of dynamic discrete games of incomplete information using the Nested Pseudo Likelihood (NPL) method in Aguirregabiria and Mira (2007). The code is written in *GAUSS* programming language and it is included in an Appendix and available online at `http://individual.utoronto.ca/vaguirre/`. Given that the code uses low-level commands in *GAUSS,* it should be straightforward to translate it to other matrix languages such as Matlab, Fortran 90, R, or C+. I illustrate the use of this software using a dynamic game of store location by retail chains and actual data for McDonalds and Burger King. The example is intentionally simple and it tries to provide a helpful starting point for the user of this code. The list of programs (*.prg*) and procedures (*.src*) is the following:

Program / Procedure	**Description**
`npl_dyngame.prg`	Main program for the NPL estimation of a dynamic game of store location
`npl_bkmd.src`	Given an initial vector of choice probabilities (CCPs), it computes an NPL fixed point estimator.
`miprobit.src`	Given a vector of choice probabilities (CCPs), it returns the pseudo ML estimator of a probit model.
`freqprob.src`	Calculates a frequency estimator of Conditional Choice Probabilities (CCPs)

The main program is `npl_dyngame.prg`. It includes all the procedures that it calls, such that the user does not have to create any GAUSS library with the procedures called by the main program.[1]

The rest of this document is organized as follows. Section 2 presents the model, data, and the estimation method in the empirical application that we use to illustrate the algorithm and code. Section 3 describes the different parts of the main program. Section 4 goes through the procedures or subroutines called by the main program. Section 5 describes the estimation output and comments the estimates in the empirical application. The code is included in an Appendix.

[1]Alternatively, the user might prefer to remove from the main program all the procedures and place them in a GAUSS library.

2 Empirical Application

2.1 Model

Time is discrete an indexed by $t \in \{1, 2, ..., T\}$, where T is the time horizon. There are two players in the game, and we use the indexes $i \in \{1, 2\}$ and $j \in \{1, 2\}$ to represent a player and his opponent, respectively. Every period, each player makes a binary choice. Within a given period players' actions are taken simultaneously. Let $Y_{it} \in \{0, 1\}$ represent the choice of player i at period t. Each player makes this decision to maximize its expected and discounted payoff $E_t(\sum_{s=0}^{T-t} \beta_i^s \Pi_{i,t+s})$, where $\beta_i \in (0, 1)$ is player i's discount factor and Π_{it} is his payoff at period t. Here we concentrate in Markovian decision models with infinite horizon, $T = \infty$. The payoff function has the following structure:

$$\Pi_{it} = \mathbf{z}_{it}(Y_{it}, Y_{jt})\ \boldsymbol{\theta}_i - Y_{it}\ \varepsilon_{it} \tag{1}$$

$\mathbf{z}_{it}(0, 0)$, $\mathbf{z}_{it}(0, 1)$, $\mathbf{z}_{it}(1, 0)$, and $\mathbf{z}_{it}(1, 1)$ are row vectors of known functions of state variables. $\boldsymbol{\theta}_i$ is a column vector of structural parameters, and $\boldsymbol{\theta} \equiv (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ is the vector with both players' parameters. Structural parameters and the vectors $\mathbf{z}_{it}(Y_{it}, Y_{jt})$ are common knowledge to the two players, up to the action of the other player. The variable ε_{it} is private information of firm i at period t. A player has uncertainty on the current value of his opponent's ε, and on future values of both his own and his opponent's $\varepsilon's$. The vectors $\mathbf{z}_{it}(Y_{it}, Y_{jt})$ have the following structure:

$$\mathbf{z}_{it}(Y_{it}, Y_{jt}) = z(\mathbf{W}_i, X_{it}, X_{jt}, Y_{it}, Y_{jt}) \tag{2}$$

$z()$ is a known vector-valued function. $\mathbf{W}_i$ is a vector of time-invariant exogenous characteristics of player i. And X_{it} is an endogenous 'stock' variable for player i that evolves over time according to the transition rule $X_{it+1} = X_{it} + Y_{it}$. The set of possible values for these stock variables is {0, 1, 2, ..., K} where $K > 1$ is a natural number that represents the maximum level of the stock. The variables ε_{1t} and ε_{2t} are independent of $(\mathbf{W}_1, \mathbf{W}_2)$, independent of each other, and independently and identically distributed over time. Their distribution functions, G_1 and G_2, are absolutely continuous and strictly increasing with respect to the Lebesgue measure on $\mathbb{R}$.

The model can easily accommodate depreciation (e.g., $X_{it+1} = (1-\delta)X_{it} + Y_{it}$, with $0 < \delta < 1$) and endogenous disinvestment (e.g., $Y_{it} \in \{-1, -, +1\}$). However, in the data that we use to illustrate the algorithm and code, the two retail chains never closed a store during the sample period. Therefore, we have preferred to present here the simple case without depreciation or disinvestment because that is the case in the empirical application.

EXAMPLE (Capacity Investment in an Oligopoly Industry). Consider a dynamic game of capacity investment between two firms competing in an oligopoly industry of an homogeneous product.[2] The demand function is $Q_t = S_t(b_0 - b_1 P_t)$, where b_0 and b_1 are parameters, Q_t represents aggregate output, S_t is the exogenous market size, and P_t is the product price. There are 2 firms operating in the industry. Every period t, these firms compete in quantities a la Cournot (static game), and choose whether to invest to increase their capacity (dynamic game). Production costs are linear in the quantity produced, i.e., $C_{it} = MC_{it}\ q_{it}$, where MC_{it} is the marginal cost, and q_{it} represents output. Marginal cost declines with installed capacity, i.e., $MC_{it} = c_i - d(X_{it} + Y_{it})$, where $c_i > 0$ and $d > 0$ are parameters, X_{it} is the installed capacity at the beginning of period t, and $Y_{it} \in \{0, 1\}$ represents capacity investment, that is a binary choice. It is simple to show that the Cournot equilibrium variable profit of firm i is:

$$\begin{aligned} VP_{it} &= \frac{S_t}{b_1}\left(\frac{b_0 + MC_{jt} - MC_{it}}{3}\right)^2 \\ &= \theta_{0i}^{VP}\ S_t\ 1\{X_{it} + Y_{it} > 0\} + \theta_{1i}^{VP}\ S_t(X_{it} + Y_{it} - X_{jt} - Y_{jt}) + \theta_{2i}^{VP}\ S_t(X_{it} + Y_{it} - X_{jt} - Y_{jt})^2 \end{aligned} \tag{3}$$

where $1\{.\}$ is the indicator function, and $\theta_{0i}^{VP} > 0$, θ_{1i}^{VP}, and θ_{2i}^{VP} are structural parameters that are known functions of the 'deep' parameters b_0, b_1, c_i, c_j, and d. More specifically, it is simple to verify that $\theta_{0i}^{VP} \equiv (b_0 + c_j - c_i)^2$, $\theta_{1i}^{VP} \equiv 2d(b_0 + c_j - c_i)$, and $\theta_{2i}^{VP} \equiv d^2$. Given a value of the vector of parameters $(\theta_{0i}^{VP}, \theta_{1i}^{VP}, \theta_{2i}^{VP} : i = 1, 2)$, we can (over-)identify the 'deep' structural parameters d, b_0, and $(c_j - c_i)$. Here we concentrate on the identification and estimation of the parameters $(\theta_{0i}^{VP}$, θ_{1i}^{VP}, $\theta_{2i}^{VP} : i = 1, 2)$ together with the parameters in fixed costs.[3] The set of possible capacity levels

[2] See Besanko and Doraszelski (2004), or Ryan (2009) for related dynamic games of firm capacity.

[3] We may consider a more flexible model of competition between McDonalds and Burger King. Suppose that these firms have differentiated products. The demand function form firm i is $q_i = A_i - b(P_i - P_j)$, where A_i and b are structural demand parameters. The specification of the marginal cost function is the same as above. Firms compete in prices a la Nash-Bertrand. It is straightforward to show that variable profit of firm i in the Nash-Bertrand equilibrium is:

$$VP_i = \frac{S}{b}\ \frac{2A_i + A_j + b(MC_i - MC_j)}{3}^2$$

This expression is useful to interpret the empirical results. In particular, some differences between the parameters is $\{0, 1, 2, ..., K\}$ where $K - 1 > 1$ is a natural number that represents the maximum feasible level of capacity. A firm's capacity evolves over time according to the transition rule $X_{it+1} = X_{it} + Y_{it}$. The firm's total profit function is:

$$\Pi_{it} = VP_{it} - \theta_{0i}^{FC}\ 1\{Y_{it} + X_{it} > 0\} - \theta_{1i}^{FC}(Y_{it} + X_{it}) - \theta_{2i}^{FC}(Y_{it} + X_{it})^2 - Y_{it}\ \varepsilon_{it} \tag{4}$$

where θ_{0i}^{FC}, θ_{1i}^{FC} and θ_{2i}^{FC} are parameters in the fixed cost function of firm i. The variable ε_{it} is a private information shock in the firm's investment cost, and it is normally distributed. In this example, the vector of structural parameters for firm i is:

$$\boldsymbol{\theta}_i \equiv (\theta_{0i}^{VP},\ \theta_{1i}^{VP},\ \theta_{2i}^{VP},\ \theta_{0i}^{FC},\ \theta_{1i}^{FC},\ \theta_{2i}^{FC})' \tag{5}$$

and the vector $\mathbf{Z}_{it}(Y_{it}, Y_{jt})$ is:

$$\begin{aligned}\mathbf{Z}_{it}(Y_{it}, Y_{jt}) \quad \equiv \quad & \{\ S_t 1\{X_{it} + Y_{it} > 0\},\ S_t(X_{it} + Y_{it} - X_{jt} - Y_{jt}),\ S_t(X_{it} + Y_{it} - X_{jt} - Y_{jt})^2 \\ & (-1\{X_{it} + Y_{it} > 0\},\ -(X_{it} + Y_{it}),\ -(X_{it} + Y_{it})^2\}\end{aligned} \tag{6}$$

In our empirical application, we consider the industry of fast-food burger restaurants in UK. The two companies are McDonalds and Burger King who compete in the number of stores. A local market is a district. X_{it} represents the number of installed stores, and Y_{it} is the decision to open a new store. During the sample period (1991-1996), these firms did not close any existing store. That is the reason why there is not an exit decision in the model. The model assumes that the decision to open a new store is completely irreversible.

Players' strategies are the result of a Markov Perfect Equilibrium (MPE). In a MPE, players' strategies depend only on payoff relevant state variables. In this model, the payoff-relevant information of firm i at period t is $(S_t,\ X_{1t},\ X_{2t},\ \varepsilon_{it})$. We use $\mathbf{X}_t$ to represent the vector of common knowledge state variables: $\mathbf{X}_t \equiv (S_t,\ X_{1t},\ X_{2t})$. Let $\mathcal{X}$ be set with all the possible values of $\mathbf{X}_t$. Let $\boldsymbol{\sigma} \equiv \{\sigma_i(\mathbf{X}_t, \varepsilon_{it}) : i = 1, 2\}$ be a set of strategy functions, one for each player. $\boldsymbol{\sigma}$ is a MPE if, for every player i, the strategy σ_i maximizes the expected value of firm i at every state $(\mathbf{X}_t, \varepsilon_{it})$ and taking as given the opponent's strategy. It is convenient to represent players' strategies and MPE in terms of players' *Conditional Choice Probabilities* (CCPs). Let $P_i(\mathbf{X}_t)$ represents firm $i's$

in variable profits of McDonalds and Burger King can better interpreted in a model where firms are heterogeneous both in marginal costs and in quality or consumers' willingness to pay (i.e., $A_i \neq A_j$).

probability of increasing its capacity (i.e., of $Y_{it} = 1$) given that the state is $\mathbf{X}_t$. This probability is defined as the integral of the strategy function $\sigma_i(\mathbf{X}_t, \varepsilon_{it})$ over the distribution of ε_{it}.

$$P_i(\mathbf{X}_t) \equiv \int 1\left\{\sigma_i(\mathbf{X}_t, \varepsilon_{it}) = 1\right\} \, dG_1(\varepsilon_{it}) \tag{7}$$

where $1\{.\}$ is the indicator function. We can represent a MPE as a pair of probability functions $\mathbf{P} \equiv \{P_i(\mathbf{X}_t) : i = 1, 2;\ \mathbf{X}_t \in \mathcal{X}\}$ such that the strategy P_i maximizes the expected value of firm i at every state $\mathbf{X}_t$ taking as given the opponent's strategy P_j.

The equilibrium mapping in the space of CCPs is the key component of this class of dynamic games. It summarizes all the relevant structure in the model. The form of this equilibrium mapping depends on the payoff function, the transition rule of the state variables, and the distribution of the private information shocks ε_{it}. As shown above, in our model the one-period profit of firm i can be written as $\Pi_{it} = \mathbf{Z}_{it}(Y_{it}, Y_{jt})\boldsymbol{\theta}_i - Y_{it}\ \varepsilon_{it}$. Therefore, the one-period *expected* profit of firm i is:

$$\begin{aligned} \Pi_{it}^{\mathbf{P}}(Y_{it}) &= (1 - P_j(\mathbf{X}_t))\ \mathbf{Z}_{it}(Y_{it}, 0)\boldsymbol{\theta}_i + P_j(\mathbf{X}_t)\ \mathbf{Z}_{it}(Y_{it}, 1)\boldsymbol{\theta}_i - Y_{it}\varepsilon_{it} \\ &= \mathbf{z}_{it}^{\mathbf{P}}(Y_{it})\ \boldsymbol{\theta}_i - Y_{it}\varepsilon_{it} \end{aligned} \tag{8}$$

where $\mathbf{z}_{it}^{\mathbf{P}}(Y_{it}) \equiv (S_t,\ S_t(X_{it} + Y_{it}),\ (1 - P_j(\mathbf{X}_t))S_t(X_{jt} + 0) + P_j(\mathbf{X}_t))S_t(X_{jt} + 1),\ -Y_{it},\ -Y_{it}X_{it})$. For the sake of illustration, let us consider first the equilibrium mapping for the case when firms are myopic, i.e., $\beta_1 = \beta_2 = 0$). The best response function in the space of a player's action is:

$$\{Y_{it} = 1\} \Leftrightarrow \left\{\mathbf{z}_{it}^{\mathbf{P}}(1)\ \boldsymbol{\theta}_i - \varepsilon_{it} \geq \mathbf{z}_{it}^{\mathbf{P}}(0)\ \boldsymbol{\theta}_i\right\} \tag{9}$$

And in the space of CCPs, firm i's best response is:

$$\Pr\left(Y_{it} = 1 \mid \mathbf{X}_t\right) = G_i\left(\left[\mathbf{z}_{it}^{\mathbf{P}}(1) - \mathbf{z}_{it}^{\mathbf{P}}(0)\right]\boldsymbol{\theta}_i\right) \tag{10}$$

A MPE in this static/myopic game (i.e., a Bayesian Nash Equilibrium) is a pair of probability functions that solves the system of equations:

$$\begin{aligned} P_1(\mathbf{X}_t) &= G_1\left(\left[\mathbf{z}_{1t}^{\mathbf{P}}(1) - \mathbf{z}_{1t}^{\mathbf{P}}(0)\right]\boldsymbol{\theta}_1\right) \\ P_2(\mathbf{X}_t) &= G_2\left(\left[\mathbf{z}_{2t}^{\mathbf{P}}(1) - \mathbf{z}_{2t}^{\mathbf{P}}(0)\right]\boldsymbol{\theta}_2\right) \end{aligned} \tag{11}$$

for every value of $\mathbf{X}_t$. Given our assumptions on the distributions G_i, Brower's Theorem implies that an equilibrium exits. The model may have multiple equilibria. Note that, for this myopic or static game, there is a separate system of equations for every value of $\mathbf{X}_t$. We could say that for each value of $\mathbf{X}_t$ we have a separate equilibrium. As shown below, this is not the case for a dynamic game. In a MPE of a dynamic game, the whole best response probability function of player i depends on the whole probability function of player j at every possible value of $\mathbf{X}_t$.

Now, we describe a MPE in a dynamic game where players are forward-looking, i.e., $\beta_i > 0$. Following Aguirregabiria and Mira (2007), a MPE can be described as a vector of CCPs, $\mathbf{P} \equiv \{P_i(\mathbf{X}_t) : i = 1, 2;\ \mathbf{X}_t \in \mathcal{X}\}$, such that for every firm i and every state $\mathbf{X}_t \in \mathcal{X}$ we have that:

$$P_i(\mathbf{X}_t) \quad = \quad G_i\left(\left[\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(1) - \widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(0)\right]\boldsymbol{\theta}_i - \left[\widetilde{e}_{it}^{\mathbf{P}}(1) - \widetilde{e}_{it}^{\mathbf{P}}(0)\right]\right) \tag{12}$$

where $\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(Y_{it})$ is the expected and discounted sum of current and future $\mathbf{z}$ vectors $\{\mathbf{z}_{it+s}(Y_{it+s}, Y_{jt+s}) : s = 0, 1, 2, ...\}$ which may occur along all possible histories originating from the choice of Y_{it} in state $\mathbf{X}_t$, if every player behaves according to their CCPs in $\mathbf{P}$. More formally,

$$\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(Y_{it}) \equiv \quad \mathbf{z}_{it}^{\mathbf{P}}(Y_{it}) \quad + \quad E\left(\sum_{s=1}^{\infty}\beta^s\ \mathbf{z}_{it+s}^{\mathbf{P}}(Y_{it+s}) \mid \mathbf{X}_t,\ Y_{it}\right) \tag{13}$$

Similarly, $\widetilde{e}_{it}^{\mathbf{P}}(Y_{it})$ is the expected and discounted sum of realizations of $\{\varepsilon_{it+s}Y_{it+s} : s = 0, 1, 2, ...\}$ originating from the choice of Y_{it} in state $\mathbf{X}_t$, when players behave according to their CCPs in $\mathbf{P}$:

$$\widetilde{e}_{it}^{\mathbf{P}}(Y_{it}) \quad \equiv \quad E\left(\sum_{s=1}^{\infty}\beta^s\ \varepsilon_{it+s}Y_{it+s} \mid \mathbf{X}_t,\ Y_{it}\right) \tag{14}$$

Now, we describe in detail the exact computation of the values $\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(0)$, $\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(1)$, $\widetilde{e}_{it}^{\mathbf{P}}(0)$, and $\widetilde{e}_{it}^{\mathbf{P}}(1)$, for every possible value of $\mathbf{X}_t$ in the space of $\mathcal{X}$. Let $f_i^{\mathbf{P}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t)$ be the transition probability of $\{\mathbf{X}_t\}$ from the point of view of player i who knows his own current action Y_{it} but ignores the current action of his competitor and only knows that it is a random draw from the probability distribution $P_j(\mathbf{X}_t)$. By definition,

$$f_i^{\mathbf{P}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) \equiv 1\{X_{it+1} = X_{it} + Y_{it}\}\ P_j(\mathbf{X}_t)^{1\{X_{jt+1}=X_{jt}+1\}}\,(1 - P_j(\mathbf{X}_t))^{1\{X_{jt+1}=X_{jt}\}} \tag{15}$$

Define also the value vector $\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P}}(\mathbf{X}_t) \equiv (1 - P_i(\mathbf{X}_t))\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(0) + P_i(\mathbf{X}_t)\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(1)$, and the scalar value $W_{ei}^{\mathbf{P}}(\mathbf{X}_t) \equiv (1 - P_i(\mathbf{X}_t))\widetilde{e}_{it}^{\mathbf{P}}(0) + P_i(\mathbf{X}_t)\widetilde{e}_{it}^{\mathbf{P}}(1)$. It is straightforward to see that, by definition:

$$\widetilde{\mathbf{z}}_{it}^{\mathbf{P}}(Y_{it}) \equiv \quad \mathbf{z}_{it}^{\mathbf{P}}(Y_{it}) \quad + \quad \beta \sum_{\mathbf{X}_{t+1}\in\mathcal{X}} f_i^{\mathbf{P}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t)\ \mathbf{W}_{\mathbf{Z}i}^{\mathbf{P}}(\mathbf{X}_{t+1}) \tag{16}$$

and

$$\widetilde{e}_{it}^{\mathbf{P}}(Y_{it}) \quad \equiv \quad \beta \sum_{\mathbf{X}_{t+1} \in \mathcal{X}} f_i^{\mathbf{P}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) \; W_{ei}^{\mathbf{P}}(\mathbf{X}_{t+1}) \tag{17}$$

The matrix of values $\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P}} \equiv \{\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P}}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}\}$ and the vector of values $\mathbf{W}_{\mathbf{e}i}^{\mathbf{P}} \equiv \{W_{ei}^{\mathbf{P}}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}\}$ are obtained by solving systems of linear equations with dimension $|\mathcal{X}|$. The solution to these systems of equations has the following closed-form analytical expression:

$$\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P}} = \left(\mathbf{I} - \beta \; \mathbf{F}_{\mathbf{X}}^{\mathbf{P}}\right)^{-1} \left[(1 - \mathbf{P}_i) * \mathbf{Z}_i^{\mathbf{P}}(0) + \mathbf{P}_i * \mathbf{Z}_i^{\mathbf{P}}(1)\right] \tag{18}$$

and

$$\mathbf{W}_{ei}^{\mathbf{P}} = \left(\mathbf{I} - \beta \; \mathbf{F}_{\mathbf{X}}^{\mathbf{P}}\right)^{-1} \mathbf{e}_i^{\mathbf{P}} \tag{19}$$

$\mathbf{P}_i$ is a $|\mathcal{X}| \times 1$ vector with the stacked CCPs of player i for every possible value of $\mathbf{X}_t$. $\mathbf{Z}_i^{\mathbf{P}}(0)$ and $\mathbf{Z}_i^{\mathbf{P}}(1)$ are matrices with $|\mathcal{X}|$ rows and the same number of columns as $\mathbf{z}_{it}^{\mathbf{P}}(Y)$ such that a row of $\mathbf{Z}_i^{\mathbf{P}}(Y)$ is equal to the vector $\mathbf{z}_{it}^{\mathbf{P}}(Y)$ associated with a given value of $\mathbf{X}_t$. $*$ represents the Hadamard or element-by-element product. $\mathbf{I}$ represents the identity matrix with dimension $|\mathcal{X}|$. $\mathbf{F}_{\mathbf{X}}^{\mathbf{P}}$ is the transition matrix of $\{\mathbf{X}_t\}$ induced by the vector of CCPs $\mathbf{P}$ such that the elements of this matrix are $(1 - P_i(\mathbf{X}_t))f_i^{\mathbf{P}}(\mathbf{X}_{t+1}|0, \mathbf{X}_t) + P_i(\mathbf{X}_t))f_i^{\mathbf{P}}(\mathbf{X}_{t+1}|1, \mathbf{X}_t)$, or what is equivalent, $\prod_{i=1}^{2} P_i(\mathbf{X}_t)^{1\{X_{it+1}=X_{it}+1\}} \left(1 - P_i\left(\mathbf{X}_t\right)\right)^{1\{X_{it+1}=X_{it}\}}$. Finally, $\mathbf{e}_i^{\mathbf{P}}$ is a vector that contains the expected values $E(\varepsilon_{it}Y_{it}|\mathbf{X}_t, Y_{it} \text{ is optimal})$ for every value of $\mathbf{X}_t$. These conditional expectations only depend on the probability distribution of ε_{it} and on the choice probability $P_i(\mathbf{X}_t)$. For the logit and probit models we have the following closed expressions. When ε_{it} is extreme value distributed (logit):

$$E\left(\varepsilon_{it}Y_{it}|\mathbf{X}_t, Y_{it} \text{ optimal}\right) = \mathit{Euler} - (1 - P_i(\mathbf{X}_t)) \ln\left(1 - P_i(\mathbf{X}_t)\right) - P_i(\mathbf{X}_t) \ln\left(P_i(\mathbf{X}_t)\right) \tag{20}$$

where *Euler* represents Euler's constant. And when ε_{it} has a standard normal distribution (probit):

$$E\left(\varepsilon_{it}Y_{it}|\mathbf{X}_t, Y_{it} \text{ optimal}\right) = \phi\left(\Phi^{-1}\left(P_i(\mathbf{X}_t)\right)\right) \tag{21}$$

where $\phi\left(.\right)$ and $\Phi^{-1}\left(.\right)$ are the PDF and the inverse-CDF of the standard normal.

Equation (12) represents a MPE as a fixed point of a mapping in the space of CCPs. Given our assumptions, Brower's Theorem guarantees the existence of a MPE. In general, there may be multiple equilibria. This equilibrium mapping is the cornerstone of the NPL estimation method.

2.2 Data

To illustrate the algorithm and code, we estimate a dynamic game of store location by McDonalds (MD) and Burger King (BK) using data for United Kingdom during the period 1991-1995. The dataset comes from the paper Toivanen and Waterson (2005).[4] It is a panel of 422 local markets (districts) and five years with information on the stock of stores and the flow of new stores of MD and BK in each local market, as well as local market characteristics such as population, density, age distribution, average rent, income per capita, local retail taxes, and distance to the headquarters of the firm in UK.

We index firms by $i \in \{BK,\ MD\}$, local markets by m, and years by t. The specification of the model is the one in the Example on *Capacity Investment in an Oligopoly Industry* in section 2.1 above. X_{imt} represents the number of installed stores of firm i in market m at the beginning of the year. The maximum value of X_{imt} in the sample is 13, and we consider that the set of possible values of X_{imt} is $\{0, 1, ..., 15\}$. Therefore, the state space $\mathcal{X}$ is $\{0, 1, ..., 15\} \times \{0, 1, ..., 15\}$ that has 256 grid points. Y_{imt} is the binary indicator of the event "firm i opens a new store in market m at year t". For the code that we provide here, we consider that market characteristics are constant over time, and use market-specific mean values of these variables. However, it is straightforward to extend the code to accommodate exogenous state variables that evolve over time according to first order Markov processes. The measure of market size S_m is total population in the district. For some specifications, we allow the cost of investment to depend on market characteristics such as average rent, retail taxes, population density, or average income.

2.3 NPL Estimation

For an arbitrary vector of players' CCPs, $\mathbf{P} \equiv \{P_i(\mathbf{X}) : i = 1, 2;\ \mathbf{X} \in \mathcal{X}\}$, define the pseudo log-likelihood function:

$$\begin{aligned} Q(\boldsymbol{\theta}, \mathbf{P}) \;=\; & \sum_{m=1}^{M} \sum_{i=1}^{2} \sum_{t=1}^{T} Y_{imt}\ \ln G_i\left(\left[\widetilde{\mathbf{z}}_{imt}^{\mathbf{P}}(1) - \widetilde{\mathbf{z}}_{imt}^{\mathbf{P}}(0)\right] \boldsymbol{\theta}_i - \left[\widetilde{e}_{imt}^{\mathbf{P}}(1) - \widetilde{e}_{imt}^{\mathbf{P}}(0)\right]\right) \\ & + \; (1 - Y_{imt}) \ln\left(1 - G_i\left(\left[\widetilde{\mathbf{z}}_{imt}^{\mathbf{P}}(1) - \widetilde{\mathbf{z}}_{imt}^{\mathbf{P}}(0)\right] \boldsymbol{\theta}_i - \left[\widetilde{e}_{imt}^{\mathbf{P}}(1) - \widetilde{e}_{imt}^{\mathbf{P}}(0)\right]\right)\right) \end{aligned} \tag{22}$$

[4] We would like to thank Otto Toivanen and Michael Waterson for generously sharing their data with us.

In this likelihood function, choice probabilities are best responses to an arbitrary $\mathbf{P}$. The arbitrary probabilities in $\mathbf{P}$ may be interpreted as players' beliefs about other players' expected behavior. These beliefs $\mathbf{P}$ are parameters to estimate together with $\boldsymbol{\theta}$. When G_i is the logistic function (or the CDF of the standard normal), the function $Q(\boldsymbol{\theta}, \mathbf{P})$ is the likelihood of a Logit (Probit) model where the parameter associated with the explanatory variable $\widetilde{e}^{\mathbf{P}}_{imt}(1) - \widetilde{e}^{\mathbf{P}}_{imt}(0)$ is restricted to be -1. For every possible value of $\mathbf{P}$, the likelihood $Q(\boldsymbol{\theta}, \mathbf{P})$ is globally concave in $\boldsymbol{\theta}$. This property simplifies significantly the implementation of the NPL algorithm.

Let $\mathbf{P}^0$ be the true vector of CCPs in the population under study, and let $\hat{\mathbf{P}}^0$ be a nonparametric consistent estimator of $\mathbf{P}^0$. For instance, a frequency estimator of $\mathbf{P}^0$ is:

$$\hat{P}^0_i(\mathbf{X}) = \frac{\sum_{m=1}^{M} \sum_{t=1}^{T} Y_{imt} \ 1\{\mathbf{X}_{mt} = \mathbf{X}\}}{\sum_{m=1}^{M} \sum_{t=1}^{T} 1\{\mathbf{X}_{mt} = \mathbf{X}\}} \tag{23}$$

The two-step estimator is defined as the value of $\boldsymbol{\theta}$ that maximizes the pseudo likelihood $Q(\boldsymbol{\theta}, \hat{\mathbf{P}}^0)$. The estimator is consistent and asymptotically normal. Its main computational cost is in the calculation of the present values $\widetilde{\mathbf{z}}^{\hat{\mathbf{P}}^0}_{imt}$ and $\widetilde{e}^{\hat{\mathbf{P}}^0}_{imt}$ following the procedure described in section 2.1. However, for the example we consider here the dimension of the state space $\mathcal{X}$ is small and the computation of these present values is quite simple. The main limitations of the two-step estimator are its asymptotic inefficiency, its large finite sample bias, and its problems to accommodate unobserved variables for the econometrician which are common knowledge to players, such as unobserved market characteristics.

If the equilibrium that generates the data is Lyapunov stable,[5] then a recursive version of the two-step estimator, i.e., a K-step estimator, has better asymptotic and finite sample properties than the two-step estimator (see Aguirregabiria and Mira, 2007 and 2009, and Kasahara and Shimotsu, 2008). Given an initial nonparametric estimator $\hat{\mathbf{P}}^0$, the sequence of K-step estimators $\{\hat{\boldsymbol{\theta}}^K, \hat{\mathbf{P}}^K : K \geq 1\}$ is defined as:

$$\hat{\boldsymbol{\theta}}^K \quad = \quad \arg\max \ Q(\boldsymbol{\theta}, \hat{\mathbf{P}}^{K-1}) \tag{24}$$

[5] Let $\mathbf{P} = \Psi(\theta, \mathbf{P})$ be the fixed point problem that defines an equilibrium for a given vector of structural parameters θ. In our model, the equilibrium mapping $\Psi(\theta, \mathbf{P})$ is $G_i([\mathbf{z}^{\mathbf{P}}_{it}(1) - \mathbf{z}^{\mathbf{P}}_{it}(0)]\theta_i - [e^{\mathbf{P}}_{it}(1) - e^{\mathbf{P}}_{it}(0)])$ for every player i and state $\mathbf{X}$. An equilibrium $\mathbf{P}^*$ is Lyapunov stable if the Jacobian matrix $\partial\Psi(\theta, \mathbf{P}^*)/\partial\mathbf{P}$ has a spectral radius smaller than 1. The spectral radius is the maximum absolute eigenvalue.

where the probabilities in $\hat{\mathbf{P}}^K$ are updated using the recursive formula

$$\hat{P}_i^K(\mathbf{X}_t) \quad = \quad G_i\left(\left[\widetilde{\mathbf{z}}_{it}^{\hat{\mathbf{P}}^{K-1}}(1) - \widetilde{\mathbf{z}}_{it}^{\hat{\mathbf{P}}^{K-1}}(0)\right]\hat{\boldsymbol{\theta}}^K - \left[\widetilde{e}_{it}^{\hat{\mathbf{P}}^{K-1}}(1) - \widetilde{e}_{it}^{\hat{\mathbf{P}}^{K-1}}(0)\right]\right) \tag{25}$$

This recursive procedure is called the *Nested Pseudo Likelihood (NPL) algorithm.* The limit K-step estimator, as K goes to infinity, is an *NPL fixed point* associated with the initial estimator $\hat{\mathbf{P}}^0$. In general, an NPL fixed point $(\hat{\boldsymbol{\theta}}_{NPL-FP}, \hat{\mathbf{P}}_{NPL-FP})$ is defined by two conditions: (1) $\hat{\boldsymbol{\theta}}_{NPL-FP} = \arg\max \quad Q(\boldsymbol{\theta}, \hat{\mathbf{P}}_{NPL-FP})$; and (2) $\hat{\mathbf{P}}_{NPL-FP}$ is an equilibrium of the model given $\hat{\boldsymbol{\theta}}_{NPL-FP}$. The model may have multiple NPL fixed points. If the equilibrium that generates the data is Lyapunov stable, then a *NPL fixed point* that is obtained by initializing the NPL algorithm with a consistent initial estimator $\hat{\mathbf{P}}^0$ is consistent, asymptotically normal, and it has smaller asymptotic variance and finite sample bias than the two-step estimator. If a *NPL fixed point* is obtained by initializing the NPL algorithm with an inconsistent initial $\hat{\mathbf{P}}^0$, then the *NPL fixed point* is not necessarily consistent. In that context, the *NPL estimator* is defined as the NPL fixed point with the largest value of the pseudo likelihood function. The *NPL estimator* is consistent, asymptotically normal, and it has smaller asymptotic variance and finite sample bias than the two-step estimator (Aguirregabiria and Mira, 2007).

The NPL method has been used in different applications of empirical games or single-agent dynamic decision models, such as models of entry in oligopoly markets (Aguirregabiria, Mira, and Roman, 2007, and Suzuki, 2008), plant turnover and productivity (Collard-Wexler, 2008, and Tomlin, 2009), supermarket pricing strategies (Ellickson and Misra, 2008, and Kano, 2006), dynamic games of competition between airline networks (Aguirregabiria and Ho, 2009), competition between real estate agents (Han and Hong, 2008), adoption of new technologies (Lenzo, 2007), land use and deforestation (De Pinto and Nelson, 2007 and 2009), quality competition between nursing homes (Lin, 2008), firm investment (Sanchez-Mangas, 2002), entry and competition in the religion industry (Walrath, 2008), demand of durable goods (Lorincz, 2005), or dynamic labor demand (Aguirregabiria and Alonso-Borrego, 2009), among others.

3 Main Program (`npl_dyngame.prg`)

The file `npl_dyngame.prg` contains the main program where all the primitives are specified and the different procedures are called. This program is divided into five parts.

PART 1: Specification of Some Constants.

The user should specify the values of the following constants and parameters.

Program Constant	Description
`filedat`	Name and address of the datafile.
`nvar`	Number of variables in the datafile.
`nobs`	Number of observations in the datafile.
`nmarket`	Number of markets in the dataset.
`nyear`	Number of years in the dataset.
`maxstore`	Maximum number of stores (i.e., max, value of X_{it})
`namesb1`	Vector with names of parameters that vary across firms
`namesb2`	Vector with names of parameters that do not vary across firms
`nplayer`	Number of players
`maxiter`	Maximum number of iterations for the NPL algorithm
`dfact`	Discount factor parameter, β

Given this information, the program generates the matrix `vstate` with all the possible values of the vector of state variables $\mathbf{X}_t$. Each row of `vstate` represents one value of (X_{1mt}, X_{2mt}).

PART 2: Reading Data and Construction of Vectors with observations of state and decision variables

The vectors `x_bk` and `x_md` contain the observations of the state variables X_{1mt} and X_{2mt}. The vectors `a_bk` and `a_md` contain the observations of the decision variables Y_{1mt} and Y_{2mt}.

PART 3: Procedures

This part of the program contains the different procedures called by the main program: `freqprob`, procedure for the initial estimates of CCPs; `miprobit`, procedure for the maximum likelihood estimation of a Probit model with constrains on parameters; and `npl_bkmd`, procedure for the NPL algorithm.

PART 4: Estimation of Initial Probabilities.

It calls the procedure `freqprob` for the frequency estimation of CCPs. In the version of the program that we provide here by default, initial CCPs are estimated separately market by market. Alternatively, the user could include time invariant market characteristics (i.e., population, average income, density, etc) as explanatory variables and call the procedure `freqprob` only once but including all the markets. The user could also prefer to use a Kernel estimator instead of the frequency estimator.

By running this part of the program, we get a vector of estimated initial CCPs called `prob_freq`. Alternatively, if we want to search for multiple NPL fixed points, we can replace this estimated vector by an arbitrary value of `prob_freq`. For instance, initialized the NPL algorithm with a vector of constant probabilities, e.g., `prob_freq = (1/2)*ones(nmarket*nstate,nplayer);`, or with random draws from a uniform distribution, e.g., `prob_freq = rndu(nmarket*nstate,nplayer);`.

PART 5: NPL Estimation.

This part of the program calls the procedure for NPL estimation, `npl_bkmd`, that generates `maxiter` iterations of the NPL algorithm given an initial vector of CCPs.

4 Procedures

4.1 `freqprob`

This procedure obtains frequency or 'cell' estimates of the probability distribution of a vector of discrete random variables Y conditional on a vector of discrete random variables X.

	Procedure `freqprob`
	Format: `{ prob } = freqprob(yobs, xobs, xval)`
	INPUT VARIABLES
Name	**Description**
`yobs`	$(nobs \times q)$ matrix with sample observations of $Y = Y1\tilde{}Y2\tilde{}...\tilde{}Yq$
`xobs`	$(nobs \times k)$ matrix with sample observations of $X = X1\tilde{}X2\tilde{}...\tilde{}Xq$
`xval`	$(numx \times k)$ matrix with values of X at which we want to estimate the conditional probability function $Pr(Y\|X)$.
	OUTPUT VARIABLES
Name	**Description**
`prob`	$(numx \times q)$ matrix with estimates of $Pr(Y\|X)$ for every value of $\not{X}$ in `xval` $Pr(Y1 = 1\|X)\tilde{}Pr(Y2 = 1\|X)\tilde{}...\tilde{}Pr(Yq = 1\|X)$

Frequency estimators for empty cells (i.e., values in `xval` for which there are zero observations in `xobs`) are defined to be zero. This frequency or 'cell' estimator of the conditional probability of discrete random variables is consistent under very weak conditions.

4.2 `miprobit`

This procedure obtains the Maximum Likelihood estimates (MLE) of a binary Probit model. The parameters of some explanatory variables can be restricted to take specific values. The algorithm to obtain the MLE is Newton's method with analytical expressions for gradient and Hessian. The log-likelihood function of this Probit model is globally concave in the parameters. Therefore, in the absence of multicollinearity problems or other numerical issues (e.g., choice probabilities too close to zero or one), Newton's algorithm always converges to the MLE regardless the value of the parameters that we use to initialize the algorithm.

Procedure `miprobit`	
Format: `{best,varest,llike} = miprobit(ydum,x,rest,b0,nombres,out)`	
INPUT VARIABLES	
Name	**Description**
`ydum`	($nobs \times 1$) vector with sample observations of the dependent variable
`x`	($nobs \times K$) matrix with sample observations of explanatory variables associated with the unrestricted parameters
`rest`	($nobs \times 1$) vector with observations of the sum of the explanatory variables whose parameters are restricted to be 1
`b0`	($K \times 1$) vector with values of parameters to initialized Newton's method
`nombres`	($K \times 1$) vector with names of parameters to estimate
`out`	Binary scalar that specifies screen output: 0=no table of results; 1=table with estimation results
OUTPUT VARIABLES	
Name	**Description**
`best`	($K \times 1$) vector with maximum likelihood estimates
`varest`	($K \times K$) matrix with estimated variances-covariances of estimates
`llike`	Scalar with value of log-likelihood function at the MLE

4.3 `npl_bkmd`

This procedure iterates in the Nested Pseudo Likelihood algorithm given an initial vector of CCPs. At each NPL iteration, the algorithm performs three main tasks.

Task 1: Computing the matrices `ztilda_bk` and `ztilda_md` and the vectors `etilda_bk` and `etilda_md` for every market and every sample observation. From a computational point of view, this is the most demanding part of an NPL iteration. The update of these matrices and vectors is done market by market because each market has its own CCPs. This task is divided into three sub-tasks: (a) construction of one-period expected profits; (b) construction of transition probabilities; and (c) computation of `ztilda_bk`, `ztilda_md`, `etilda_bk`, and `etilda_md`.

Task 2: Call to the procedure `miprobit` for the pseudo maximum likelihood estimation of structural parameters.

Task 3: Update of the CCPs.

Procedure `npl_bkmd`

Format: `{thetaest,varest,pchoice,pest_obs} = npl_bkmd(yobs,xobs,msize,zmarket,pchoice,mstate,beta,kiter,namesb)`

INPUT VARIABLES

Name	**Description**
`yobs`	($nobs \times 2$) matrix with sample observations of players' choice variables
`xobs`	($nobs \times 2$) matrix with sample observations of players' endogenous state variables
`msize`	($nmarket \times 1$) vector with sample observations of market size (population)
`zmarket`	($nmarket \times Kz$) matrix with sample observations of time-invariant market characteristics
`pchoice`	($nstate * nmarket \times 2$) matrix with initial vector of CCPs for each market, each state, and each player
`mstate`	($nstate \times$ 2) matrix with all the possible values of the endogenous state variables
`beta`	Scalar with value of the discount factor
`kiter`	Scalar natural number with number of NPL iterations
`namesb`	($K \times 1$) vector with names of the structural parameters to estimate

OUTPUT VARIABLES

Name	**Description**
`thetaest`	($K \times 1$) vector with parameter estimates at the last NPL iteration
`varest`	($K \times K$) matrix of variances and covariances
`pest`	($nstate * nmarket \times 2$) matrix with estimates of CCPs for every market, state and player
`pest_obs`	($nobs \times 2$) matrix with estimates of CCPs for every observations and player

5 Output and Empirical Results

For each NPL iteration, the code provides the following screen output:

- **At every Pseudo ML (PML) iteration (within an NPL iteration)**. Index of the PML iteration; value of pseudo log-likelihood; value of the criterion for convergence, $\max_j |\hat{\theta}_j^q - \hat{\theta}_j^{q-1}|$, where q represents the index for the PML iteration. For instance:

```
Pseudo MLE Iteration = 2.000
Log-Likelihood function = -2883.
Criterion = 1.595
```

- **At the end of every NPL iteration**.

Total number of PML iterations; Final value of the pseudo log-likelihood; Likelihood Ratio Index (measure of goodness-of-fit); Pseudo R-square; Parameter estimates and asymptotic standard errors; Number of NPL iterations and value of the NPL convergence criterion, i.e., $\max_j |\hat{\theta}_j^K - \hat{\theta}_j^{K-1}|$ where K represents the index for the NPL iteration. For instance:

```
Number of Iterations = 10.00
Log-Likelihood function = -655.7
Likelihood Ratio Index = 0.3699
Pseudo-R2 = 0.3228
-----------------------------------------------------------------
 Parameter         Estimate     Standard Error     t-ratio
-----------------------------------------------------------------
 VP0_BK         0.5849         0.1077            5.430
 VP1_BK         -0.2096         0.0552             -3.792
 VP2_BK         -0.0110         0.0029             -3.761
 FC0_BK         0.0784         0.0213            3.674
 FC1_BK         0.0790         0.0445            1.775
 FC2_BK         -0.0078         0.0059             -1.333
 VP0_MD         0.8303         0.2968            2.798
 VP1_MD         -0.0024         0.0392             -0.0615
 VP2_MD         0.0008         0.0027            0.3184
 FC0_MD         0.0822         0.0332            2.473
 FC1_MD         0.1076         0.0400            2.689
 FC2_MD         -0.0034         0.0023             -1.435
 DENSITY         10.74         2.817            3.811
 GDP             0.0003         0.0002            1.374
 RENT              -0.0016         0.0006              -2.606
 TAX             -1.746e-005     5.489e-005         -0.3181
-----------------------------------------------------------------
NPL ITERATION = 2.000 Criterion = 10.74
```

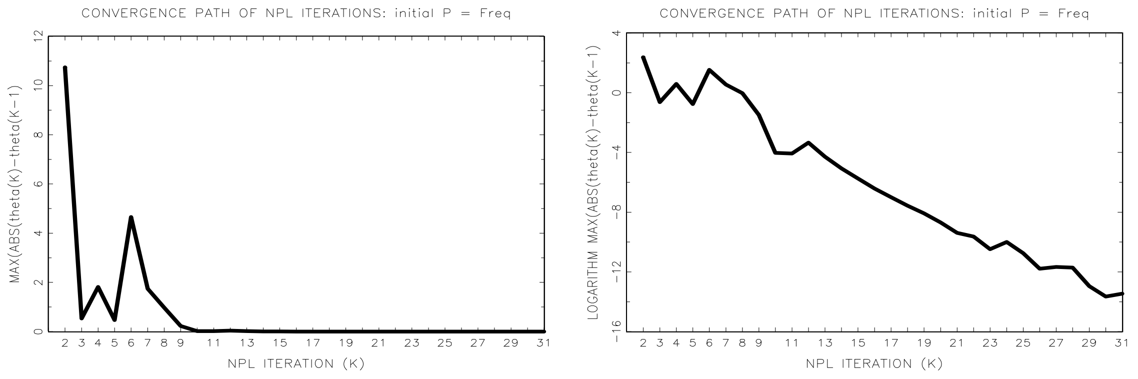
The following table contains estimates using the McDonalds-Burger King dataset and the model described in section 2.

Dynamic Game of Entry for McDonalds and Burger King
Under the Assumption that Players' Beliefs are in Equilibrium
Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)			
	Two Step Estimates		**NPL Estimates**	
	Burger King	**McDonalds**	**Burger King**	**McDonalds**
Variable Profits:				
θ_0^{VP}	0.5849 (0.1077)*	0.8303 (0.2968)*	1.098 (0.2169)*	0.9737 (0.3091)*
θ_1^{VP} cannibalization	-0.2096 (0.0552)*	-0.0024 (0.0392)	-0.0765 (0.0725)	0.2874 (0.0986)*
θ_2^{VP} competition	-0.0110 (0.0029)*	0.0008 (0.0027)	-0.0129 (0.0065)*	-0.0074 (0.0073)
Fixed Costs:				
θ_0^{FC} fixed	0.0784 (0.0213)*	0.0822 (0.0332)*	0.0788 (0.0307)*	0.0773 (0.0261)*
θ_1^{FC} linear	0.0790 (0.0420)*	0.1076 (0.0400)*	0.1509 (0.0282)*	0.1302 (0.0185)*
θ_2^{FC} quadratic	-0.0078 (0.0059)	-0.0034 (0.0023)	-0.0054 (0.0026)*	0.0001 (0.016)
Log-Likelihood	-655.7		-893.4	
Distance $\|\|P^K - P^{K-1}\|\|$	4831.26		0.00	
# NPL iterations	1		31	

(a) Convergence: The NPL fixed point reported in this table is the one that we converge to when the NPL algorithm is initialized with the nonparametric frequency estimator. It takes 31 NPL iterations to converge to this fixed point. Figure 1 presents the NPL convergence criterion, $\max_j |\hat{\theta}_j^K - \hat{\theta}_j^{K-1}|$, at every NPL iteration. We present the convergence criterion both in levels and in logarithms because the representation in logarithms provides a better picture of the convergence rate. Notice that convergence is not monotonic. This is because the consistent NPL fixed point is just a local contraction, and not a global contraction. In fact, as mentioned above, convergence of the NPL algorithm is not guaranteed. In particular, it is possible that the algorithm converges to a cycle of two or more than two values of $\hat{\theta}$. In that case, NPL iterations can be combined with techniques to deal with cycles in fixed point iteration algorithms. For instance, if we find a cycle, a possible solution is to re-start the NPL algorithm using as initial probabilities the mean values of the probabilities in the different point of the cycle. This is a very simple but it tends to be very effective.

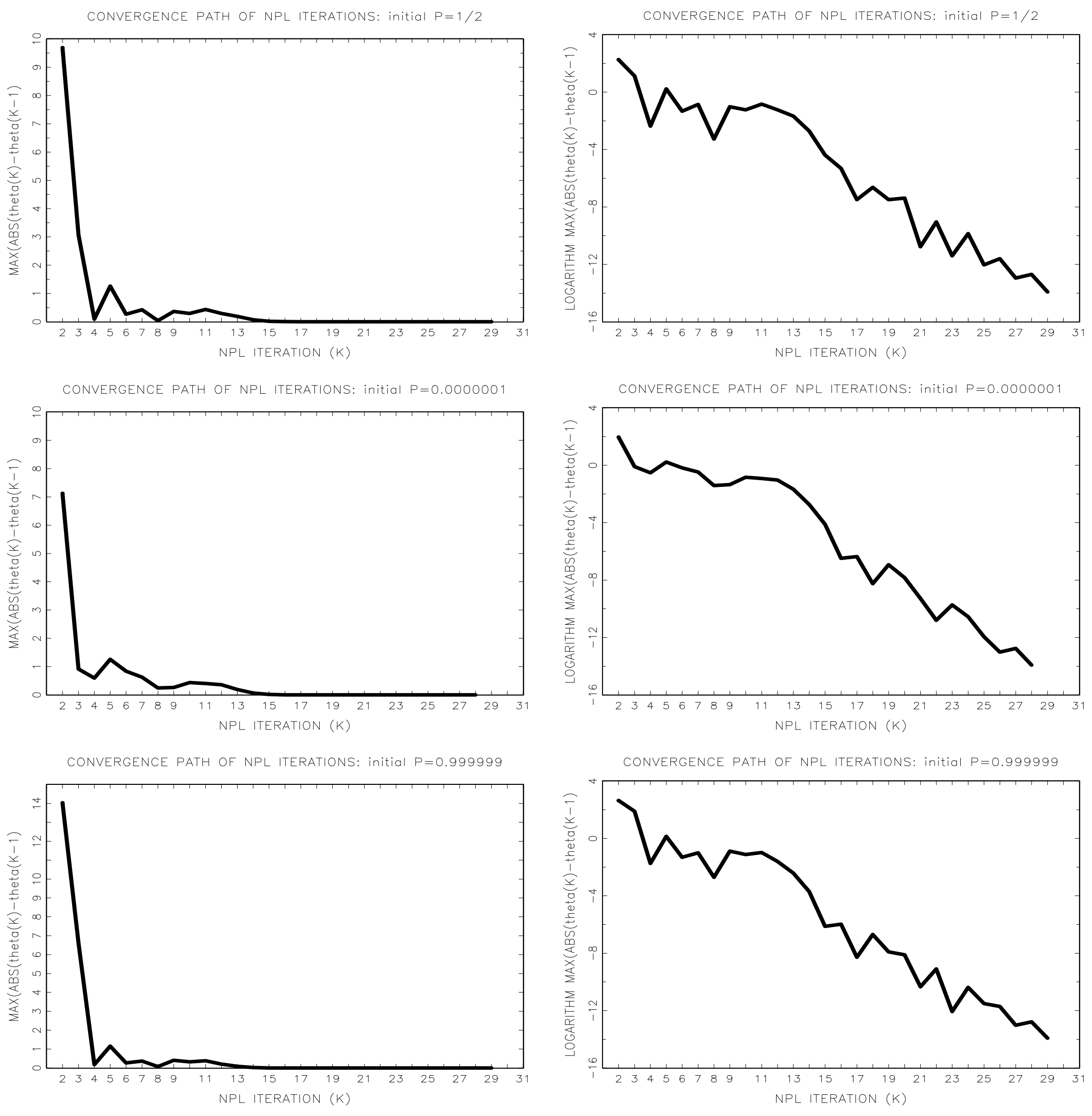
Figure 1



(b) NPL Estimator (Global search): The NPL fixed point associated with the initial frequency estimator is consistent. However, it is not necessarily the NPL estimator because there may be other NPL fixed points with higher value of the pseudo likelihood function. We have implemented the NPL algorithm for different values of the initial $\mathbf{P}_0$: e.g., $\mathbf{P}_0 = 0.5$ for every market, player and state; $\mathbf{P}_0 = 0.0000001$ for every market, player and state; $\mathbf{P}_0 = 0.999999$ for every market, player and state; and $\mathbf{P}_0 =$ vector of random draws from $Uniform(0,1)$. In this application, and for all the initial values we have tried, we have converge always to the same NPL fixed point. The

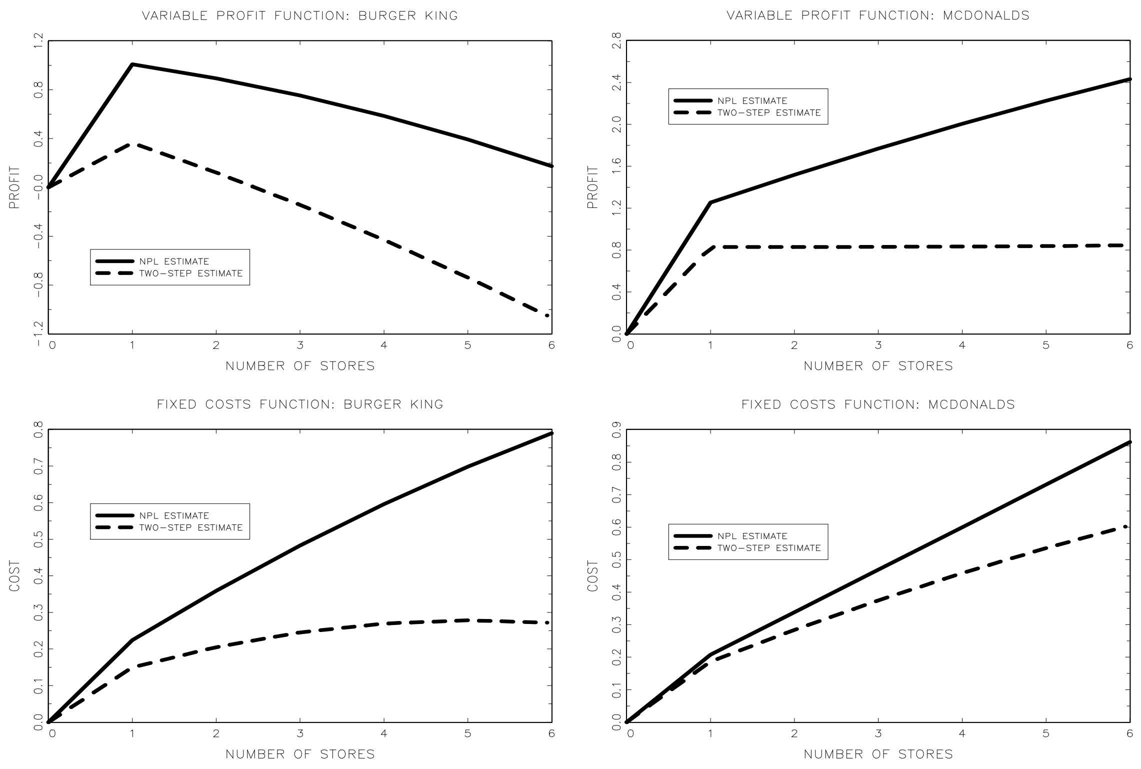
convergence path is quite similar in all the cases we have tried. Figure 2 presents the convergence paths for three cases: $\mathbf{P}_0 = 0.5$, $\mathbf{P}_0 = 0.0000001$, and $\mathbf{P}_0 = 0.999999$.

Figure 2



Note that convergence to the NPL fixed point is quite slow. In fact, as shown by the figures in logarithms, the rate of convergence declines when we approach to the NPL fixed point. Regardless the vector of CCPs that we use to initialize the NPL algorithm, 15 NPL iterations or less take us very close to an NPL fixed point. However, it takes other 15 iterations to really converge to that fixed point. The convergence criterion that we use is $\max_j |\hat{\theta}_j^K - \hat{\theta}_j^{K-1}| < 10^{-6}$. We might want to relax that convergence criterion. Alternatively, it is possible to use "accelerate NPL iterations" as proposed by Kasahara and Shimotsu (2008). For instance, we can apply more than one iteration in the best response mapping at each NPL iteration. Iterations in the best response mapping are computationally costly, but this additional cost might be compensated by a smaller number of NPL iterations.

(c) Comparing Two-Step and NPL estimators. In this application, we find important differences between the parameter estimates using two-step and NPL methods. Figures 3 presents the estimated variables profit functions and fixed cost functions for BK and MD under the two estimation methods.



Appendix: Gauss code

```
// *********************************************************************
// NPL_DYNAGAME_150909.prg
//
// THIS PROGRAM ESTIMATES A DYNAMIC GAME OF ENTRY-EXIT USING
// THE NESTED PSEUDO LIKELIHOOD (NPL) METHOD, AND ACTUAL DATA
// ON MCDONALDS AND BURGER KING LOCATION OF OUTLETS IN UK
//
// by VICTOR AGUIRREGABIRIA
//
// SEPTEMBER 2009
//
// *********************************************************************
//
// SPECIFICATION OF ONE-PERIOD PROFIT FUNCTION
// The profit function for firm i is:
//
// Ui = zi(ai,aj) * thetai - ai * epsi
//
// where ai is the new entry decision of firm i, aj is the
// new entry decision of firm j, zi(ai,aj) are vectors
// of variables, and thetai is a vector of parameters.  More specifically,
//
// thetai = (VP0i, VP1i, VP2i, FC1i, FC2i)
//
// where VP0i, VP1i, and VP2i are parameters in the variable profit function,
// FC1i and FC2i are parameters in fixed costs.  And
//
// zi(ai,aj) = { S * 1(xi + ai > 0) }
// ~{ S * (xi + ai - xj - aj) }
// ~{ S * (xi + ai - xj - aj)^2 }
// ~{ -1(xi + ai > 0) }
// ~{ -(xi + ai) }
// ~{ -(xi + ai)^2 }
//
new ;
closeall ;
library pgraph gauss ;
format /mb1 /ros 16,4 ;
// ****************************************
// PART 1:  SPECIFICATION OF SOME CONSTANTS
// ****************************************
// Constants of the datafile
// Name and address of data file
filedat =
"c:\\mypapers\\arvind_rationalizability\\data\\toivanen_waterson_nolondon_120809.dat";

nobs = 2110 ; // Number of observations in data file
nmarket = 422 ; // Number of local markets
nyear = 5 ; // Number of years
nvar = 27 ; // Number of variables in dataset
// Constants of the model
maxstore = 15 ; // maximum number of stores
nplayer = 2 ;
```

```
dfact = 0.95 ; // Discount factor
maxiter = 50 ;
namesb1 = "VP0_BK" | "VP1_BK" | "VP2_BK" | "FC0_BK" | "FC1_BK" | "FC2_BK"
 | "VP0_MD" | "VP1_MD" | "VP2_MD" | "FC0_MD" | "FC1_MD" | "FC2_MD" ;
namesb2 = "DENSITY" | "GDP" | "RENT" | "TAX" ;
namesb = namesb1 | namesb2 ;
// Calculating some constants
vstate = seqa(0,1,maxstore) ;
vstate = (vstate.*.ones(maxstore,1)) ~(ones(maxstore,1).*.vstate) ;
 // Matrix with all possible values of the state variables
nstate = rows(vstate) ;
kp1 = rows(namesb1)/2 ;
kp2 = rows(namesb2) ;
kparam = rows(namesb) ;
// ****************************************************
// PART 2.  READING DATA AND CONSTRUCTION OF VARIABLES
// ****************************************************
open dtin = ^filedat for read varindxi ;
data = readr(dtin,nobs);
dtin = close(dtin) ;
county_name = data[.,1] ;
district_name = data[.,2] ;
county_code = data[.,3] ;
district_code = data[.,4] ;
year = data[.,5] ;
mcd_stock = data[.,6] ;
mcd_entry = data[.,7] ;
mcd_entdum = data[.,8] ;
bk_stock = data[.,9] ;
bk_entry = data[.,10] ;
bk_entdum = data[.,11] ;
district_area = data[.,12] ;
population = data[.,13] ;
pop_0514 = data[.,14] ;
pop_1529 = data[.,15] ;
pop_4559 = data[.,16] ;
pop_6064 = data[.,17] ;
pop_6574 = data[.,18] ;
avg_rent = data[.,19] ;
ctax = data[.,20] ;
ecac = data[.,21] ;
ue = data[.,22] ;
gdp_pc = data[.,23] ;
dist_bkhq_miles = data[.,24] ;
dist_bkhq_minu = data[.,25] ;
dist_mdhq_miles = data[.,26] ;
dist_mdhq_minu = data[.,27] ;
// Construction of variables
x_bk = bk_stock ; // Stock of stores for BK
x_md = mcd_stock ; // Stock of stores for MD
a_bk = (bk_entry.>0) ; // Dummy of new entry for BK
a_md = (mcd_entry.>0) ; // Dummy of new entry for MD
population = population/1000 ; // Population in millions
density = population./district_area ;
```

```
// Market specific mean values of some exogenous explanatory variables
marketsize = meanc(reshape(population,nmarket,nyear)') ;
zmarket = meanc(reshape(density,nmarket,nyear)')
 ~meanc(reshape(gdp_pc,nmarket,nyear)')
 ~meanc(reshape(avg_rent,nmarket,nyear)')
 ~meanc(reshape(ctax,nmarket,nyear)') ;
// ******************
// PART 3.  PROCEDURES
// ******************
// --------------------------------------
// A. PROCEDURE for FREQUENCY ESTIMATOR
// --------------------------------------
proc (1) = freqprob(yobs,xobs,xval) ;
// ---------------------------------------------------------------------------
// FREQPROB.SRC Procedure that obtains a frequency estimation
// of Prob(Y|X) where Y is a vector of binary
// variables and X is a vector of discrete variables
// FORMAT:
// freqp = freqprob(yobs,xobs,xval)
// INPUTS:
// yobs - (nobs x q) vector with sample observations
// of Y = Y1 ~Y2 ~...  ~Yq
//
// xobs - (nobs x k) matrix with sample observations of X
//
// xval - (numx x k) matrix with the values of X for which
// we want to estimate Prob(Y|X).
// OUTPUTS:
// freqp - (numx x q) vector with frequency estimates of
// Pr(Y|X) for each value in xval.
// Pr(Y1=1|X) ~Pr(Y2=1|X) ~...  ~Pr(Yq=1|X)
// ---------------------------------------------------------------------------
 local numx, numq, prob1, t, selx, denom, numer ;
 numx = rows(xval) ;
 numq = cols(yobs) ;
 prob1 = zeros(numx,numq) ;
 t=1 ;
 do while t<=numx ;
 selx = prodc((xobs.==xval[t,.])')  ;
 denom = sumc(selx) ;
 if (denom==0) ;
 prob1[t,.]  = zeros(1,numq) ;
 else ;
 numer = sumc(selx.*yobs) ;
 prob1[t,.]  = (numer')./denom ;
 endif ;
 t=t+1 ;
 endo ;
 retp(prob1) ;
endp ;
// ----------------------------------------------
// B. PRECEDURE for CONSTRAINED PROBIT ESTIMATOR
// ----------------------------------------------
proc (3) = miprobit(ydum,x,rest,b0,nombres,out) ;
```

```
// -----------------------------------------------------------------------
// MIPROBIT - Estimation of a Probit Model by Maximum Likelihood
// The optimization algorithm is a Newton's method
// with analytical gradient and hessian
//
// FORMAT {best,varest,llike} = miprobit(ydum,x,rest,b0,nombres,out)
//
// INPUTS
// ydum - (nobs x 1) vector with observations of the dependent variable
// x - (nobs x k) matrix with observations of explanatory variables
// associated with the unrestricted parameters
// rest - vector with observations of the sum of the explanatory
// variables whose parameters are restricted to be 1
// (Note that the value 1 is without loss of generality
// if the variable rest is constructed appropriately)
// b0 - (k x 1) vector with values of parameters to initialized
// Newton's methos
// nombres - (k x 1) vector with names of parameters to estimate
// out - 0=no table of results; 1=table with estimation results
//
// OUTPUTS
// best - ML estimates
// varest - estimate of the covariance matrix
// llike - value of log-likelihood function at the MLE
// -----------------------------------------------------------------------
 local myzero, nobs, nparam, eps, iter, llike,
 criter, Fxb0, phixb0, lamdab0, dlogLb0,
 d2logLb0, b1, lamda0, lamda1, Avarb, sdb, tstat,
 numy1, numy0, logL0, LRI, pseudoR2, k ;
 myzero = 1e-36 ;
 nobs = rows(ydum) ;
 nparam = cols(x) ;
 eps = 1E-6 ;
 iter=1 ;
 llike = 1000 ;
 criter = 1000 ;
 do while (criter>eps) ;
 if (out==1) ;
 "" ;
 "Pseudo MLE Iteration = " iter ;
 "Log-Likelihood function = " llike ;
 "Criterion = " criter ;
 "" ;
 endif ;
 Fxb0 = cdfn(x*b0+rest) ;
 Fxb0 = Fxb0 + (myzero - Fxb0).*(Fxb0.<myzero)
 + (1-myzero - Fxb0).*(Fxb0.>(1-myzero));
 llike = ydum'*ln(Fxb0) + (1-ydum)'*ln(1-Fxb0) ;
 phixb0 = pdfn(x*b0+rest) ;
 lamdab0 = ydum.*(phixb0./Fxb0) + (1-ydum).*(-phixb0./(1-Fxb0)) ;
 dlogLb0 = x'*lamdab0 ;
 d2logLb0 = -((lamdab0.*(lamdab0 + x*b0 + rest)).*x)'*x ;
 b1 = b0 - inv(d2logLb0)*dlogLb0 ;
 criter = maxc(abs(b1-b0)) ;
 b0 = b1 ;
 iter = iter + 1 ;
 endo ;
 Fxb0 = cdfn(x*b0 + rest) ;
 Fxb0 = Fxb0 + (myzero - Fxb0).*(Fxb0.<myzero)
 + (1-myzero - Fxb0).*(Fxb0.>(1-myzero));
 llike = ydum'*ln(Fxb0) + (1-ydum)'*ln(1-Fxb0) ;
 phixb0 = pdfn(x*b0 + rest) ;
 lamda0 = -phixb0./(1-Fxb0) ;
 lamda1 = phixb0./Fxb0 ;
 Avarb = ((lamda0.*lamda1).*x)'*x ;
 Avarb = inv(-Avarb) ;
 sdb = sqrt(diag(Avarb)) ;
 tstat = b0./sdb ;
 numy1 = sumc(ydum) ;
 numy0 = nobs - numy1 ;
 logL0 = numy1*ln(numy1) + numy0*ln(numy0) - nobs*ln(nobs) ;
 LRI = 1 - llike/logL0 ;
 pseudoR2 = 1 - ( (ydum - Fxb0)'*(ydum - Fxb0) )/numy1 ;
 if (out==1) ;
 "Number of Iterations = " iter ;
 "Log-Likelihood function = " llike ;
 "Likelihood Ratio Index = " LRI ;
 "Pseudo-R2 = " pseudoR2 ;
 "" ;
 "----------------------------------------------------------------------";
 " Parameter Estimate Standard t-ratios";
 " Errors" ;
 "----------------------------------------------------------------------";
 k=1;
 do while k<=nparam;
 print $nombres[k];;b0[k];;sdb[k];;tstat[k];
 k=k+1 ;
 endo;
 "----------------------------------------------------------------------";
 endif ;
 retp(b0,Avarb,llike) ;
endp ;
// ------------------------------
// C. PROCEDURE for NPL ESTIMATOR
// ------------------------------
proc (4) = npl_bkmd(yobs, xobs, msize, zmarket, pchoice, mstate, beta, kiter, namesb);
 // ---------------------------------------------------------------------
 // NPL_BKMD
 // This procedure iterates in the NPL algorithm given an initial vector
 // of CCPs.  The model is the dynamic game of local market entry for
 // McDonalds and Burger King.  The procedure returns the vector of parameter
 // estimates, the variance matrix, and the matrix with players choice
 // probabilities at every state, and at every sample point.
 //
 // FORMAT (thetaest,varest,pest,pest_obs) =
 // npl_bkmd(yobs, xobs, msize, zmarket, pchoice, mstate, beta, kiter, namesb)
 //
 // INPUTS
```

```
    // yobs - (nobs x 2) matrix with observations of players' choices
    // xobs - (nobs x 2) matrix with observations of players' endogenous
    // state variables
    // msize - (nmarket x 1) vector with observations of market size (population)
    // zmarket - (nmarket x kz) matrix with observations of time-invariant
    // market chracteristics
    // pchoice - (nstate*nmarket x 2) matrix with initial vector of CCPs for
    // every market, state and player
    // mstate - (nstate x 2) matrix with all the possible values of the
    // endogenous state variables
    // beta - Scalar with value of the discount factor
    // kiter - Scalar natural number with number of NPL iterations
    // namesb - (K x 1) vector with names of the structural parameters
    //
    // OUTPUTS
    // thetaest - (K x1) vector with parameter estimates at the last NPL iteration
    // varest - (K xK) matrix of variances and covariances
    // pest - (nstate*nmarket x 2) matrix with estimates of CCPs for
    // every market, state and player
    // pest_obs - (nobs x 2) matrix with estimates of CCPs for every
    // observation and state
    // -----------------------------------------------------------------------

    local myzero, nobs, nmarket, nyear, nplayer, ns, numx, ktot, kvpfc, kz, indxobs,
j,
    xbk, xmd, npliter, criterion, conv_const, theta0,
    p_bk, p_md, ztilda_bk, ztilda_md, etilda_bk, etilda_md,
    ztilda_obs_bk, ztilda_obs_md, etilda_obs_bk, etilda_obs_md,
    m, valmsize, valzmarket,
    zbk_00, zbk_01, zbk_10, zbk_11, zmd_00, zmd_01, zmd_10, zmd_11,
    eprofbk_0, eprofbk_1, eprofmd_0, eprofmd_1,
    tranxbk_bk0, tranxbk_bk1, tranxmd_md0, tranxmd_md1, tranxmd_bk, tranxbk_md,
    tottran_bk0, tottran_bk1, tottran_md0, tottran_md1, uncontran,
    value_z_bk, value_z_md, value_e_bk, value_e_md,
    zt_bk, zt_md, et_bk, et_md, count1, count2,
    zobs, eobs, thetaest, varest, likelihood, theta_bk, theta_md, pest_obs ;
    // ---------------
    // Some constants
    // ---------------
    myzero = 1e-12 ; // Constant for truncation of CCPs to avoid numerical errors
    nobs = rows(yobs) ; // Total number of market*year observations
    nmarket = rows(msize) ; // Total number of markets in the sample
    nyear = nobs/nmarket ; // Number of years in the sample (balanced panel)
    if nyear/=int(nyear) ; "ERROR: Number of years is not an integer"; end; endif;

    nplayer = cols(yobs) ;
    ns = rows(pchoice)/nmarket ; // number of states in a single market
    if ns/=int(ns) ; "ERROR: Number of states in a single market is not an integer";
end; endif;
    numx = sqrt(ns) ; // number of values of xbk or xmd
    if numx/=int(numx) ; "ERROR: Number of values of xbk or xmd is not an integer";
end; endif;
    ktot = rows(namesb) ; // Total number of parameters
    kz = cols(zmarket) ; // Number of parameters associated with the control variables
in zmarket
    kvpfc = (ktot-kz)/2 ; // Number of parameters in var profits and fixed costs for
a single firm
    if kvpfc/=int(kvpfc) ; "ERROR: Number of parameters in var profits and fixed costs
is not an integer"; end; endif;
    xbk = mstate[.,1] ; // vector stock of stores for BK
    xmd = mstate[.,2] ; // vector stock of stores for MD
    // ------------------------------------------
    // Vector with indexes for the observed state
    // ------------------------------------------
    indxobs = zeros(nobs,1) ;
    j=1 ;
    do while j<=ns ;
    indxobs = indxobs + j.*prodc((xobs.==mstate[j,.])')  ;
    j=j+1 ;
    endo ;
    // -------------
    // NPL algorithm
    // -------------
    criterion = 1000 ;
    conv_const = 1e-6 ;
    theta0 = zeros(ktot,1) ;
    npliter=1 ;
    do while (npliter<=kiter).and(criterion>conv_const) ;
    "NPL ITERATION =";; npliter ;; "Criterion =";; criterion ;
    "" ;
    // -------------------------------------------------------
    // TASK 1:  Computing the matrices ztilda_bk and ztilda_md
    // and the vectors etilda_bk and etilda_md
    // for every market and every sample observation
    // -------------------------------------------------------
    ztilda_bk = zeros(nmarket*ns,kvpfc+kz) ;
    ztilda_md = zeros(nmarket*ns,kvpfc+kz) ;
    etilda_bk = zeros(nmarket*ns,1) ;
    etilda_md = zeros(nmarket*ns,1) ;
    ztilda_obs_bk = zeros(nobs,kvpfc+kz) ;
    ztilda_obs_md = zeros(nobs,kvpfc+kz) ;
    etilda_obs_bk = zeros(nobs,1) ;
    etilda_obs_md = zeros(nobs,1) ;
    m=1;
    do while m<=nmarket ;
    valmsize = msize[m] ;
    valzmarket = zmarket[m,.]  ;
    // ------------------------------------------------------------------
    // Selection of probabilities for the market and
    // truncation of probabilities to avoid inverse Mill's ratio = +INF
    // ------------------------------------------------------------------
    p_bk = pchoice[(m-1)*ns+1:m*ns,1] ;
    p_md = pchoice[(m-1)*ns+1:m*ns,2] ;
    p_bk = (p_bk.<=myzero).*myzero
    + (p_bk.>=(1-myzero)).*(1-myzero)
    + (p_bk.>=myzero).*(p_bk.<=(1-myzero)).*p_bk ;
    p_md = (p_md.<=myzero).*myzero
    + (p_md.>=(1-myzero)).*(1-myzero)
    + (p_md.>=myzero).*(p_md.<=(1-myzero)).*p_md ;
    // ------------------------------
    // Vectors of expected profits
    // ------------------------------
    zbk_00 = (valmsize.*(xbk.>0)) ~(valmsize.*(xbk-xmd))
   ~(valmsize.*(xbk-xmd).*(xbk-xmd))
    ~(-(xbk.>0)) ~(-xbk) ~(-xbk.*xbk)
    ~(valzmarket.*xbk) ;
    zbk_01 = (valmsize.*(xbk.>0)) ~(valmsize.*(xbk-xmd-1))
   ~(valmsize.*(xbk-xmd-1).*(xbk-xmd-1))
    ~(-(xbk.>0)) ~(-xbk) ~(-xbk.*xbk)
    ~(valzmarket.*xbk) ;
    zbk_10 = (valmsize.*((xbk+1).>0))~(valmsize.*(xbk+1-xmd))
   ~(valmsize.*(xbk+1-xmd).*(xbk+1-xmd))
    ~(-((xbk+1).>0)) ~(-(xbk+1)) ~(-(xbk+1).*(xbk+1))
    ~(valzmarket.*(xbk+1)) ;
    zbk_11 = (valmsize.*((xbk+1).>0))~(valmsize.*(xbk+1-xmd-1))
   ~(valmsize.*(xbk+1-xmd-1).*(xbk+1-xmd-1))
    ~(-((xbk+1).>0)) ~(-(xbk+1)) ~(-(xbk+1).*(xbk+1))
    ~(valzmarket.*(xbk+1)) ;
    zmd_00 = (valmsize.*(xmd.>0)) ~(valmsize.*(xmd-xbk))
   ~(valmsize.*(xmd-xbk).*(xmd-xbk))
    ~(-(xmd.>0)) ~(-xmd) ~(-xmd.*xmd)
    ~(valzmarket.*xmd) ;
    zmd_01 = (valmsize.*(xmd.>0)) ~(valmsize.*(xmd-xbk-1))
   ~(valmsize.*(xmd-xbk-1).*(xmd-xbk-1))
    ~(-(xmd.>0)) ~(-xmd) ~(-xmd.*xmd)
    ~(valzmarket.*xmd) ;
    zmd_10 = (valmsize.*((xmd+1).>0))~(valmsize.*(xmd+1-xbk))
   ~(valmsize.*(xmd+1-xbk).*(xmd+1-xbk))
    ~(-((xmd+1).>0)) ~(-(xmd+1)) ~(-(xmd+1).*(xmd+1))
    ~(valzmarket.*(xmd+1)) ;
    zmd_11 = (valmsize.*((xmd+1).>0))~(valmsize.*(xmd+1-xbk-1))
   ~(valmsize.*(xmd+1-xbk-1).*(xmd+1-xbk-1))
    ~(-((xmd+1).>0)) ~(-(xmd+1)) ~(-(xmd+1).*(xmd+1))
    ~(valzmarket.*(xmd+1)) ;
    eprofbk_0 = (1-p_md).*zbk_00 + p_md.*zbk_01 ; // Expected Profit BK if a=0
    eprofbk_1 = (1-p_md).*zbk_10 + p_md.*zbk_11 ; // Expected Profit BK if a=1
    eprofmd_0 = (1-p_bk).*zmd_00 + p_bk.*zmd_01 ; // Expected Profit MD if a=0
    eprofmd_1 = (1-p_bk).*zmd_10 + p_bk.*zmd_11 ; // Expected Profit MD if a=1
    // --------------------------
    // Transition probabilities
    // --------------------------

    // Remember:  vstate = xbk ~xmd
    // where:  xbk = (1|2| ...  |14).*.(1|1|....|1)
    // xmd = (1|1| ...  |1) .*.(1|2|....|14)

    // Transition xbk for BK: abk = 0
    // Pr( xbk' | xbk, xmd, abk =0) = 1{xbk' = xbk)
    tranxbk_bk0 = eye(numx) ; // Transition of xbk in the space of xbk
    tranxbk_bk0 = tranxbk_bk0.*.ones(numx,numx) ; // Transition of xbk in the space
of xbk, xmd
```

```
    // Transition xbk for BK: abk = 1
    // Pr( xbk' | xbk, xmd, abk =0) = 1{xbk' = xbk+1)
    tranxbk_bk1 = (zeros(numx-1,1) ~eye(numx-1))
    | (zeros(1,numx-1) ~1); // Transition of xbk in the space of xbk
    tranxbk_bk1 = tranxbk_bk1.*.ones(numx,numx) ; // Transition of xbk in the space
of xbk, xmd
    // Transition xmd for MD: amd = 0
    // Pr( xmd' | xbk, xmd, amd =0) = 1{xmd' = xmd)
    tranxmd_md0 = eye(numx) ; // Transition of xmd in the space of xmd
    tranxmd_md0 = ones(numx,numx).*.tranxmd_md0 ; // Transition of xmd in the space
of xbk, xmd
    // Transition xmd for MD: amd = 1
    // Pr( xmd' | xbk, xmd, amd =1) = 1{xmd' = xmd+1)
    tranxmd_md1 = (zeros(numx-1,1) ~eye(numx-1))
    | (zeros(1,numx-1) ~1); // Transition of xmd in the space of xmd
    tranxmd_md1 = ones(numx,numx).*.tranxmd_md1 ; // Transition of xmd in the space
of xbk, xmd
    // Transition xmd from the point of view of BK who doesn't know amd
    // Pr( xmd' | xbk, xmd) = (1-pmd) * 1{xmd' = xmd) + pmd * 1{xmd' = xmd+1)
    tranxmd_bk = (1-p_md).* tranxmd_md0 + p_md.* tranxmd_md1 ;
    // Transition xbk from the point of view of MD who doesn't know abk
    // Pr( xbk' | xbk, xmd) = (1-pbk) * 1{xbk' = xbk) + pbk * 1{xbk' = xbk+1)
    tranxbk_md = (1-p_bk).* tranxbk_bk0 + p_bk.* tranxbk_bk1 ;
    // Total transition matrix of (xbk,xmd) for BK if abk = 0
    // Pr( xbk',xmd' | xbk, xmd, abk=0) = Pr( xbk' | xbk, abk =0) * Pr( xmd' | xbk, xmd)
    tottran_bk0 = tranxbk_bk0 .* tranxmd_bk ;
    if sumc(sumc(tottran_bk0').>(1.00001)) or sumc(sumc(tottran_bk0').<(0.99999))
;
    "ERROR: Transition matrix does not sum 1" ; end ;
    endif ;
    // Total transition matrix of (xbk,xmd) for BK if abk = 1
    // Pr( xbk',xmd' | xbk, xmd, abk=1) = Pr( xbk' | xbk, abk =1) * Pr( xmd' | xbk, xmd)
    tottran_bk1 = tranxbk_bk1 .* tranxmd_bk ;
    if sumc(sumc(tottran_bk1').>(1.00001)) or sumc(sumc(tottran_bk1').<(0.99999))
;
    "ERROR: Transition matrix does not sum 1" ; end ;
    endif ;
    // Total transition matrix of (xbk,xmd) for MD if amd = 0
    // Pr( xbk',xmd' | xbk, xmd, amd=0) = Pr( xmd' | xmd, amd =0) * Pr( xbk' | xbk, xmd)
    tottran_md0 = tranxmd_md0 .* tranxbk_md ;
    if sumc(sumc(tottran_md0').>(1.00001)) or sumc(sumc(tottran_md0').<(0.99999))
;
    "ERROR: Transition matrix does not sum 1" ; end ;
    endif ;
    // Total transition matrix of (xbk,xmd) for MD if amd = 1
    // Pr( xbk',xmd' | xbk, xmd, amd=1) = Pr( xmd' | xmd, amd =1) * Pr( xbk' | xbk, xmd)
    tottran_md1 = tranxmd_md1 .* tranxbk_md ;
    if sumc(sumc(tottran_md1').>(1.00001)) or sumc(sumc(tottran_md1').<(0.99999))
;
    "ERROR: Transition matrix does not sum 1" ; end ;
    endif ;
    // Unconditional transition matrix
    uncontran = (1-p_bk) .* tottran_bk0 + p_bk .* tottran_bk1 ;
    if sumc(sumc(uncontran').>(1.00001)) or sumc(sumc(uncontran').<(0.99999)) ;
    "ERROR: Transition matrix does not sum 1" ; end ;
    endif ;
    // ----------------------------------------------------------------------
    // ztilda_bk, ztilda_md, etilda_bk, etilda_md for every possible state
    // ----------------------------------------------------------------------
    uncontran = inv(eye(ns) - beta*uncontran) ; // Matrix (I - beta*F)^-1
    value_z_bk = (1-p_bk).*eprofbk_0 + p_bk.*eprofbk_1 ;
    value_z_bk = uncontran * value_z_bk ; // Value Z function BK
    value_e_bk = uncontran * pdfn(cdfni(p_bk)) ; // Value e function BK
    value_z_md = (1-p_md).*eprofmd_0 + p_md.*eprofmd_1 ;
    value_z_md = uncontran * value_z_md ; // Value Z function MD
    value_e_md = uncontran * pdfn(cdfni(p_md)) ; // Value e function MD
    zt_bk = (eprofbk_1 - eprofbk_0) + beta*(tottran_bk1-tottran_bk0)*value_z_bk ;
    zt_md = (eprofmd_1 - eprofmd_0) + beta*(tottran_md1-tottran_md0)*value_z_md ;
    et_bk = beta*(tottran_bk1-tottran_bk0)*value_e_bk ;
    et_md = beta*(tottran_md1-tottran_md0)*value_e_md ;
    // ------------
    // Filling
    // ------------
    count1 = (m-1)*ns + 1 ;
    count2 = m*ns ;
    ztilda_bk[count1:count2,.]  = zt_bk ;
    ztilda_md[count1:count2,.]  = zt_md ;
    etilda_bk[count1:count2,.]  = et_bk ;
    etilda_md[count1:count2,.]  = et_md ;
    count1 = (m-1)*nyear + 1 ;
    count2 = m*nyear ;
    ztilda_obs_bk[count1:count2,.]  = zt_bk[indxobs[count1:count2],.]  ;
    ztilda_obs_md[count1:count2,.]  = zt_md[indxobs[count1:count2],.]  ;
    etilda_obs_bk[count1:count2,.]  = et_bk[indxobs[count1:count2],.]  ;
    etilda_obs_md[count1:count2,.]  = et_md[indxobs[count1:count2],.]  ;

    m=m+1;
    endo ;

    // --------------------------------------------
    // TASK 2:  Pseudo Maximum Likelihood Estimation
    // --------------------------------------------
    zobs = (ztilda_obs_bk[.,1:kvpfc] | zeros(nobs,kvpfc))
    ~(zeros(nobs,kvpfc) | ztilda_obs_md[.,1:kvpfc])
    ~(ztilda_obs_bk[.,kvpfc+1:kvpfc+kz] | ztilda_obs_md[.,kvpfc+1:kvpfc+kz]) ;
    eobs = etilda_obs_bk | etilda_obs_md ;
    {thetaest,varest,likelihood}
    = miprobit((yobs[.,1]|yobs[.,2]),zobs,eobs,zeros(ktot,1),namesb,1) ;
    // ------------------------------------------------
    // TASK 3:  Updating Conditional Choice Probabilities
    // ------------------------------------------------
    theta_bk = thetaest[1:kvpfc] | thetaest[2*kvpfc+1:ktot] ;
    theta_md = thetaest[kvpfc+1:2*kvpfc] | thetaest[2*kvpfc+1:ktot] ;
    pchoice = cdfn(ztilda_bk*theta_bk + etilda_bk) ~cdfn(ztilda_md*theta_md + etilda_md)
;
    // -------------------------
    // Checking for Convergence
```

```
     // -------------------------
     criterion = maxc(abs(thetaest-theta0)) ;

     theta0 = thetaest ;
     npliter = npliter+1 ;
     endo ;

     // ---------------------------------------------
     // Observed Conditional Choice Probabilities:
     // ---------------------------------------------
     pest_obs = cdfn(ztilda_obs_bk * theta_bk + etilda_obs_bk)
     ~cdfn(ztilda_obs_md * theta_md + etilda_obs_md) ;

     retp(thetaest,varest,pchoice,pest_obs) ;
    endp ;
    // ***********************************
    // PART 4:  ESTIMATION OF INITIAL CCPs
    // ***********************************
    prob_freq = zeros(nmarket*nstate,nplayer) ;
    market = 1 ;
    do while market<=nmarket ;
     count1 = (market-1)*nstate + 1 ;
     count2 = market*nstate ;
     yyy = a_bk[(market-1)*nyear+1:market*nyear] ~a_md[(market-1)*nyear+1:market*nyear]
;
     xxx = x_bk[(market-1)*nyear+1:market*nyear] ~x_md[(market-1)*nyear+1:market*nyear]
;
     buff = freqprob(yyy,xxx,vstate) ;
     prob_freq[count1:count2,.]  = freqprob(yyy,xxx,vstate) ;
     market = market+1 ;
    endo ;
    // Alternatively, the user could initialize the NPL algorithm using
    // a vector of constant probabiliies, e.g.,
    // prob_freq = (1/2)*ones(nmarket*nstate,nplayer);
    // or using random draws from a uniform distribution, e.g.,
    // prob_freq = rndu(nmarket*nstate,nplayer);

    // ***********************
    // PART 5:  NPL ESTIMATION
    // ***********************
    {best,varb,pstate,pobs} =
    npl_bkmd((a_bk~a_md),(x_bk~x_md),marketsize,zmarket,prob_freq,vstate,dfact,maxiter,namesb);
    end ;
```

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