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Price-taking Strategy Versus Dynamic Programming in Oligopoly

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Working Paper No: 2009/04

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Price-taking Strategy versus Dynamic Programming in Oligopoly

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Version: May 21, 2009

In a quantity-competed duopoly, one firm is a naive price-taker (who responses only to the last period's price) while the other has all the market information so as be able to optimize its profit stream (either discounted or un-discounted) dynamically over a finite or infinite horizon. With a traditional linear economy, we are able to derive algebraically the optimal policies of all periods for the dynamic optimizer. A counter-intuitive phenomenon is then observed: regardless of the planning horizon and the discounted factor, there exists a relative profitability range of initial prices, starting with which the price-taker make higher profit than the dynamic optimizer. Furthermore, with the increase in the planning horizon, the price-taker's relative profitability range increases accordingly and finally covers the entire economically meaningful range.

Contents

Keywords: Economics; dynamic programming; Bellman's optimality principle; applied OR; duopoly

1. INTRODUCTION

Since the classic work by Cournot in 1838, research interests in oligopoly were almost entirely concentrated on analyzing competitions between firms which were profit-maximizers. Given limited information about the market as well as its rivals' behavior, which varied from case to case, each firm is invariably assumed to maximize myopically its instantaneous absolute payoff with a best response (as a reaction function of its rival's expected output for the period). It is economically irrational if a firm either ignores or is ignorant about its market power but behaves as a price-taker who determines its output by equating the marginal cost to the price of last period. However, such beliefs were questioned in Huang (2002) where an oligopoly that consists of a price-taker and many sophisticated firms, with identical technology, was studied. A counter-intuitive phenomena is revealed – no matter what strategies the sophisticated firms may adopt, the price-taker always triumphs over them in terms of relative profitability at any dynamic equilibrium. It is further demonstrated in Huang (2008) that, either in dynamical transitionary periods or when the economy turns cyclic or chaotic, a combination of the price-taking strategy with a simple cautious adjustment strategy could also lead to relatively higher average profits for a firm than its rival, should the latter adopt a myopic Cournot best-response.

The above results naturally motivate us to further investigate whether such peculiar phenomenon will be again observed if the sophisticated firm, instead of being myopic, optimizes its discounted payoff stream over the entire planning horizon T (either finite or infinite). To accomplish this goal, a multi-period dynamic optimization problem in the context of a heterogeneous duopoly model which comprises of a sophisticated firm and a price-taker is studied. The dynamic programming approach and reasoning framework adopted in the early studies of duopoly game such as Friedman (1968), Cyert and DeGroot (1970) and Diricky (1973) are revisited. With a traditional linear economy (linear demand and marginal cost), we are able to derive algebraically the optimal policies for all periods. It is found that, regardless of the planning horizon and the discounted factor, there exists a relative profitability range of initial prices, starting with which the price-taking firm makes higher profit than the sophisticated firm. This relative profitability range expands with increasing planning horizon. When the planning horizon is sufficiently long, the relative profitability range covers the entire economically meaningful price regime, that is, the price-taker always enjoys a higher average profit relative to the sophisticated firm for any economically meaningful initial price level.

The remaining discussion is organized as follows. In Section 2, T-periods dynamic optimization model involving the heterogeneous duopoly is formulated and the complete set of analytical recursive formulas for the optimal plan and optimal payoffs are then derived. Section 3 explores the long-run stationary property of the optimal plan, the turnpike property of optimal plan and address its link to the conventional static optimization problem. Section 4 then analyzes the relative profitability of the price-taking strategy. Finally, Section 5 comprises of the conclusion of the research as well as remarks on the future research directions.

2. DYNAMIC OPTIMIZATION

Consider a duopoly industry in which two firms X and Y produce a homogeneous product at period t with quantity X_t and Y_t, respectively. The inverse market demand for the product is given by $P_t = D(Q_t^d)$, with $D' \leq 0$. The conventional assumption that $Q_t^d = X_t + Y_t$ applies, i.e., *the actual market price adjusts to the demand so as to clear the market at every period*.

Both firms are assumed to have an identical technology and hence an identical cost function $C(q)$.

Firm X is assumed to be a price-taker, whose current production X_t is determined by equating the marginal cost incurred with the naive price expectation at period t as $P_t^e = P_{t-1}$, that is,

$$X_t = MC^{-1}\left(P_{t-1}\right) = MC^{-1}\left(D\left(X_{t-1} + Y_{t-1}\right)\right), \tag{1}$$

where MC^{-1} denotes the inverse function of the marginal cost C'.

Firm Y, in contrast, is a dynamic optimizer whose objective is to maximize its discounted profit over an horizon of T periods with a given an initial price P_0, or, equivalently, the price-taker's output $X_1 = MC^{-1}\left(P_0\right)$.

Let $\Pi^y\left(X_t, Y_t\right) = D\left(X_t + Y_t\right) Y_t - C\left(Y_t\right)$ be the instantaneous profit of the dynamic optimizer at period t. The the objective function for the dynamic optimizer is

$$S^y = \max_{Y_1, Y_2, \ldots, Y_T} \sum_{t=1}^{T} \rho^{t-1} \Pi^y\left(X_t, Y_t\right), \tag{2}$$

where $0 < \rho \leq 1$ is the discount factor applied by the dynamic optimizer[1].

[1] $r = (1-\rho)/\rho$ is the discount rate (Friedman 1968).

However, for the convenience of mathematical manipulation, we shall insert the subscript k as an indication of *the periods remained before the end of planning horizon*[2].

For $1 \leq k \leq T$, let the outputs of the two firms at the stage $(T-k+1)$ be $x_k \doteq X_{T-k+1}$ and $y_k = Y_{T-k+1}$, respectively. Since under an optimal plan, y_k is a function of x_k, we are able to represent the profits of the two firms as functions of x_k only, that is, $\pi^x(x_k) \doteq \Pi^x(X_{T-k+1}, Y_{T-k+1})$ and $\pi^y(x_k) \doteq \Pi^y(X_{T-k+1}, Y_{T-k+1})$.

Define $s_k^y(x_k)$ as the weighted sum of the maximized profit for the dynamic optimizer that could possibly be accumulated in the remaining k periods of planning horizon, should the optimal plan $\{y_j\}_{j=1}^k$ be implemented. Then we have

$$s_k^y(x_k) = \max_{Y_{T-k+1}, Y_{T-k+2}, \ldots, Y_T} \sum_{j=1}^{k} \rho^{k-j} \Pi^y(X_{T-j+1}, Y_{T-j+1}) = \max_{y_1, y_2, \ldots, y_k} \sum_{j=1}^{k} \rho^{k-j} \pi^y(x_j).$$

From Bellman's principle of optimality, when the optimal plan is implemented, we have the following recursion:

$$s_k^y(x_k) = \max_{y_k} \{\pi^y(x_k) + \rho s_{k-1}^y(x_{k-1})\} \tag{3}$$

with the boundary condition $s_0^y(\cdot) \equiv 0$.

The state transition equation $X_t = MC^{-1}(D(P_{t-1}))$ is then recast as

$$x_{k-1} = MC^{-1}(D(x_k + y_k)) \tag{4}$$

while the maximized objective function (2) is thus given by $S^y = s_T^y(x_T) = s_T^y(X_1)$.

Working with (x_k, y_k) instead of (X_t, Y_t) provides us with an unique advantage of deriving a full set of optimal policies for various planning horizon k for $k \geq 1$.

The general formulations (3) and (4) also provide us with a framework to discuss the qualitative properties of optimal solutions as well as the optimized objective function. However, the best way to explore the relative profitability quantitatively, that is, to compare the (average of) the accumulated profits earned by both firms, is to work with a model that leads to the solutions with analytically closed forms. For such consideration, we shall proceed our discussion for the widely studied **Linear Model**, by which we mean: i) the market demand is linear, that is, $P_t = D(X_t + Y_t) = 1 - X_t - Y_t$ and ii) the marginal cost is linear so that the cost function adopts the form of $C(q) = cq^2/2$, where $c > 1$ is the cost parameter[3]. However, although adopting Linear Model brings about the possibility of deriving analytically closed solutions, it also generates extra difficulty, that is, the possibility of "market crash" resulting from the nonpositive price and/or over production (i.e., the industrial outputs exceeds unity). To have a general picture of the dynamic interaction between the price-taker and the dynamic optimizer while keeping the generality of the Linear Model, we shall focus mainly on the situations in which *the optimal plans that compose with the interior solution at each and every period* so that $x_i \in (0,1)$, $y_i \in (0, 1 - x_i)$ and

[2] To avoid confusion, a usage convention will be adopted in this paper so that all capital symbols together with subscript t indicate the forward sequences while the corresponding little cases together with subscript $k \doteq T - t + 1$ indicate all the backward sequences.

[3] This is because, if $c < 1$, the price-taker's response to the market price $x_{k-1} = MC^{-1}(D(x_k + y_k)) = (1 - x_k - y_k)/c$ may be invalid (that is, may not stay in the interior of $[0,1]$).

$p_i \in (0,1)$, for all $i = 1, 2..., k$, can be guaranteed. For interior optimal solution, Eqs. (3) and (4) simplify to[4].

$$s_k^y(x_k) = \max_{0<y_k<1-x_k} \{(1-(x_k+y_k))\, y_k - c y_k^2/2 + \rho s_{k-1}^y(x_{k-1})\}, \tag{5}$$

with

$$x_{k-1} = (1-(x_k+y_k))/c. \tag{6}$$

Fortunately, for a k-periods planning, the interior optimal solutions exist for all $c > 1$ regardless of ρ, so long as the initial state x_k is restrained by an *initial upper bound* $X_k^u \in [0,1]$.

Theorem 1 (**Optimal policy and payoffs**). *For the Linear model with $c > 1$, we have*

i) For any $0 < x_k < X_k^u$, an optimal policy is a linear function of x_k given by

$$y_k = u_k - v_k x_k, \tag{7}$$

from which the optimal payoff to the dynamic optimizer is a quadratic function of x_k that takes the form[5]:

$$s_k^y(x_k) = \left(\alpha_k (1-x_k)^2 + 2\beta_k x_k + \gamma_k\right)/2, \tag{8}$$

where α_k, β_k and γ_k are constant ***payoff coefficients*** *that can be determined recursively through*

$$\alpha_k = \frac{c(c+\rho\alpha_{k-1})}{c^2(2+c) - \rho\alpha_{k-1}}, \tag{9}$$

$$\beta_k = \frac{\rho c(c+1)(\alpha_{k-1}-\beta_{k-1})}{c^2(2+c)-\rho\alpha_{k-1}}, \tag{10}$$

$$\gamma_k = \rho(\gamma_{k-1} + \frac{\rho\beta_{k-1}(\beta_{k-1}-2\alpha_{k-1}) + c\left(c^2-2\right)\alpha_{k-1} + 2c\beta_{k-1}(c+1)}{c^2(c+2)-\rho\alpha_{k-1}}), \tag{11}$$

with the boundary conditions $\alpha_0 = \beta_0 = \gamma_0 = 0$.

ii) ***Policy coefficients*** *u_k and v_k are determined by*

$$\left.\begin{array}{l} u_k = (1-\alpha_k+\beta_k)/(c+1) \\ v_k = (1-\alpha_k)/(c+1) \end{array}\right\}. \tag{12}$$

iii) ***Initial upper-bound*** *X_k^u is determined by*

$$X_k^u = 1 - \beta_k/(c+\alpha_k). \tag{13}$$

Proof. See Appendix A. ∎

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[4]To take into the possibility of corner solution, the recursion (3) needs to be reformulated as

$$\begin{aligned} s_k^y(x_k) &= \max_{0\le y_k\le 1-x_k} \{\pi^y(x_k) + \rho s_{k-1}^y(x_{k-1})\} \\ &= \begin{cases} (1-x_k-y_k)\, y_k - c y_k^2/2 + \rho s_{k-1}^y((1-x_k-y_k)/c), & \text{if } 0<y_k<1-x_k, \\ \rho s_{k-1}^y(\min\{1,(1-x_k)/c\}), & \text{if } y_k = 0, \\ \rho s_{k-1}^y(0) - c(1-x_k)^2/2, & \text{if } y_k = 1-x_k, \end{cases} \end{aligned}$$

with $0 \le x_i \le 1$ for $i = 1, 2, ..., k$. The analysis of this type of constrained dynamic optimization problem can only be carried out with the recursive technique (Stokey and Lucas (1995)) and the meaningful conclusions are generally obtained through numerical simulations.

[5]The particular expression is selected for $s_k^y(x_k)$ by trial and error so as to keep the recursive formula for the coefficients to their simplest forms and at same retain the economic meaning for each coefficient.

First, we analysis the long-run convergency in recursive relationships for payoff coefficients, economically meaningful range and the optimal policy parameters and then discuss the nice characteristics of the Turnpike property.

The following observations can be verified straightforwardly for $c > 1$ and any $0 < \rho \leq 1$.

PROPOSITION 1. *i) While* $\{\alpha_k\}$ *is a monotonically increasing sequence with*

$$0 < \alpha_1 < \alpha_2 < ... < \alpha_\infty \leq 1,$$

$\{v_k\}$ *is a monotonically decreasing sequence with*

$$1 > v_1 > v_2 > ...v_\infty = \bar{v}(\rho) > 0.$$

ii) $\{\beta_k\}$, $\{u_k\}$ *and* $\{X_k^u\}$ *are positive sequences that converges* cyclically *to their stationary value* β_∞, u_∞ *and* X_∞^u, *respectively.*

iii) When $\rho < 1$, $\{\gamma_k\}$ *converges to a constant* γ_∞, *otherwise,* $\{\gamma_k\}$ *approaches infinity.*

Proof. See Appendix A. ∎

Stationary values of relevant coefficients are listed in Table 1.

k	$k=1$	$k=2$	$\cdots$	$k=\infty$
α_k	$\frac{1}{c+2}$	$\frac{c(c(c+2)+\rho)}{c^2(c+2)^2-\rho}$	$\cdots$	$\frac{c}{2\rho}((c+2)c-\rho-\eta)$
β_k	0	$\frac{\rho c(c+1)}{c^2(c+2)^2-\rho}$	$\cdots$	$c\frac{c(c+2)(c+1)-(c+1)(\rho+\eta)}{c(c+2)+\rho(3+2c)+\eta}$
γ_k	0	$\frac{\rho(c^2-2)c}{c^2(c+2)^2-\rho}$	$\cdots$	$\frac{\rho\left(\beta_\infty(\rho\beta_\infty-2\rho\alpha_\infty+2c(c+1))+c(c^2-2)\alpha_\infty\right)}{(1-\rho)(c^2(c+2)-\rho\alpha_\infty)}$
u_k	$\frac{1}{c+2}$	$\frac{c^2(c+2)-\rho+c\rho}{c^2(c+2)^2-\rho}$	$\cdots$	$\frac{c^2+3\rho+\eta}{c^2+3\rho+2c(1+\rho)+\eta}$
v_k	$\frac{1}{c+2}$	$\frac{c^2(c+2)-\rho}{c^2(c+2)^2-\rho}$	$\cdots$	$\frac{(c+2)(\rho-c^2)+c\eta}{2(c+1)\rho}$
X_k^u	1	$1-\frac{\rho}{c(c+1)(c+2)}$	$\cdots$	$\frac{2(c+1+\rho)}{c^2+4c+2+\rho-\eta}$
Remarks	$\eta \doteq \sqrt{((c+2)^2-\rho)(c^2-\rho)}$ and $\gamma_\infty(1)=\infty$			

Table 1: Optimal policy

Based on Proposition 1, all recursive coefficients converge to their stationary values in the long-run. Although convergence demands that k approaches infinity in theory, in reality a "Turnpike property" does exhibit so that the convergence is accomplished in limited periods (less than 10 in our example). Typical trajectories of (u_k, v_k) and X_k^u are provided in Fig. 1, from which the speed of convergency to the stationary values, that is, the speed to reach the "Turnpike", can be appreciated[6]. Therefore, when the planning horizon is sufficiently long, facing any initial state $X_1 \in X_\infty^u$, the dynamic optimizer will choose its output according to

$$Y_t = R_\rho(X_t) \doteq \bar{u}(\rho) - \bar{v}(\rho) X_t, t \geq 1, \tag{14}$$

for almost the entire process, or more precisely, for all but the final 10 periods of the process. For the final 10 periods, the optimal plan will then change to (7).

[6]Stated loosely, the turnpike property describes a situation where an economy, which pursues optimality over a sufficiently long period, spends most of the periods performing nearly a steady state extremal path. Eventually, over an infinite horizon, any optimal trajectory should converge towards such an extremal steady state. See Haurie (1976) for the details.

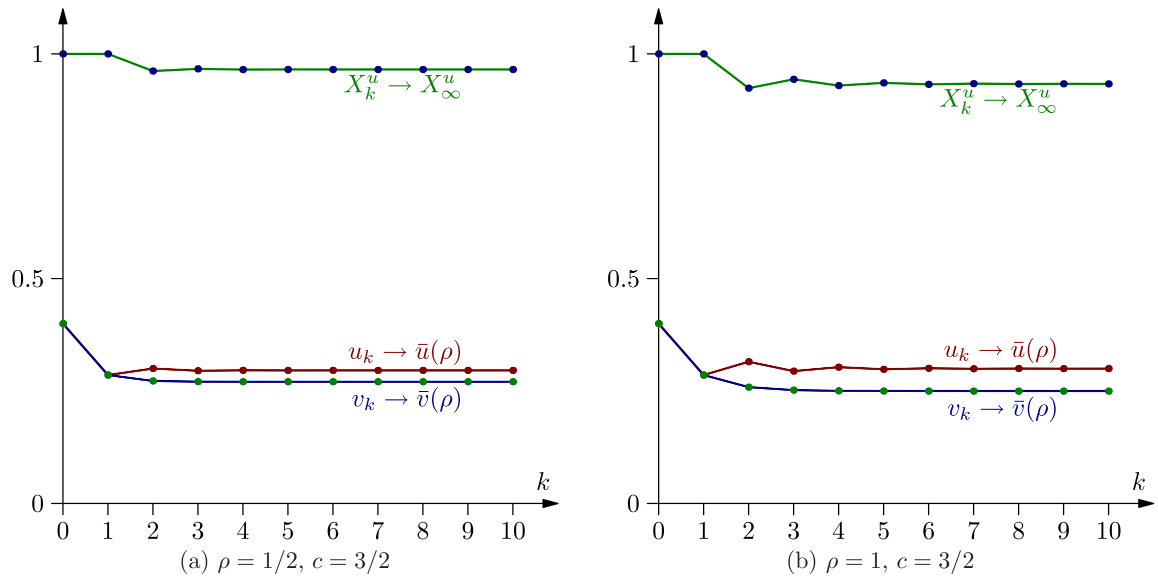


FIG. 1 Illustration of the Turnpike property

While the price-taker's response is fixed to Eq. (1), (14) will be applied repeatedly for long sequence of periods at the beginning of the process. (X_t, Y_t) will then converge to a *steady state* $(\bar{x}(\rho), \bar{y}(\rho))$ independent of the initial state X_1 (or, initial price P_0), where

$$\bar{x}(\rho) \doteq \frac{c+1+\rho}{c^2+c\rho+3c+1+2\rho} \text{ and } \bar{y}(\rho) \doteq \frac{c+\rho}{c^2+c\rho+3c+1+2\rho}. \tag{15}$$

So long as convergence to the stationary outputs $(\bar{x}(\rho), \bar{y}(\rho))$ is achieved, it will be produced most of periods until approximately the final 10 periods. In other words, except for the very early periods and the very late periods of the dynamical interaction, the output bundle will remain as $(\bar{x}(\rho), \bar{y}(\rho))$ for most of planning periods. *At each period,* an equilibrium profit pair $(\bar{\pi}^x(\rho), \bar{\pi}^y(\rho))$ is earned by the respectively firms, where

$$\bar{\pi}^x(\rho) \doteq \frac{1}{2}\frac{c(c+1+\rho)^2}{((c+2)(c+\rho)+c+1)^2} \text{ and } \bar{\pi}^y(\rho) \doteq \frac{1}{2}\frac{c(c+\rho)(c+2+\rho)}{((c+2)(c+\rho)+c+1)^2}. \tag{16}$$

Needless to say, this profit pair $(\bar{\pi}^x(\rho), \bar{\pi}^y(\rho))$ is equal to the long-run average profits for respective firm.

What we shall do now is to provide the economic interpretations for these stationary values by linking them to the relevant equivalences in an one-period myopic optimization problem.

Let the price-taker's response be fixed to (1), but assume that the dynamic optimizer's best-response is instead derived from the following first-order static optimization condition:

$$D + y_t D' + \mu \frac{dX_t}{dY_t} D' = C'(Y_t), \tag{17}$$

where $\mu \in [0, 1]$ is a variational parameter that reflects the dynamic optimizer's information accuracy and/or confidence about the counter-response from the price-taker, i.e., dX_t/dY_t.

The dynamic optimizer realizes that the price-taker's long run reaction to the market price P_{t-1} boils down to a direct "reaction" to its output Y_t in the long-run so that X_t must lie on a "stationary reaction curve" $X_t = R^x_w(Y_t)$ implicitly defined by

$$X_t = MC^{-1}(P_t) = MC^{-1}(D(X_t + Y_t)), \tag{18}$$

should an intertemporal equilibrium is arrived.

For the Linear Model, it turns out $X_t = R_w^x(Y_t) = (1 - Y_t)/(1+c)$ and $dX_t/dY_t = -1/(1+c)$. Consequently, the *variational best-response reaction* for the dynamic optimizer can be derived from (17) as

$$Y_t = r_{\mu(\rho)}(X_t) \doteq \frac{(c+1)(1-X_t)}{(c+1)(c+2) - \mu(\rho)}. \tag{19}$$

Then we immediately verify that Eq. (19), together with (1), will yield an intertemporal equilibrium that is identical to $(\bar{x}(\rho), \bar{y}(\rho))$ with

$$\mu(\rho) \doteq (c+1)\rho/(c+\rho). \tag{20}$$

Therefore, for any given c, there exists an one-to-one correspondence between the variational parameter μ (which characterizes the information availability or accuracy) for the static optimization (one-shot game) and the discounted factor ρ for the dynamic optimization. Moreover, from the comparative statics:

$$\frac{\partial \bar{x}(\rho)}{\partial \rho} < 0, \frac{\partial \bar{\pi}^x(\rho)}{\partial \rho} < 0,$$
$$\frac{\partial \bar{y}(\rho)}{\partial \rho} > 0, \frac{\partial \bar{\pi}^y(\rho)}{\partial \rho} > 0,$$

it is concurred that the minimum and the maximum of $\bar{\pi}^y(\rho)$ occur at $\rho = 0$ and $\rho = 1$ respectively.

Case I: $\rho = 0$, that is when the future payoffs are heavily discounted

It follows from the recursive formula of (9)-(11) that $\beta_k = 0$ and $\gamma_k = 0$ for all $k \geq 0$ while $\alpha_k = \alpha_1$ for all $k \geq 0$ so that the stationary optimal plan (14) for $\rho = 0$ simplifies to

$$Y_t = R_{\rho=0}(X_t) = (1 - X_t)/(c+2), t > 1. \tag{21}$$

We see immediately that $R_{\rho=0}$ is nothing but the instantaneous Cournot best response $r_{\mu(0)}$ specified in (19). In other words, when the future profit is discounted heavily, multi-periods dynamic optimization degenerates into infinitely repeated "static optimization"[7].

Case II. $\rho = 1$, that is when the future payoffs are not discounted, which leads to the maximum $\bar{\pi}^y(1)$.

The optimal response is given by

$$Y_t = R_{\rho=1}(X_t) = \bar{u}(1) - \bar{v}(1) X_t. \tag{22}$$

The following proposition confirms that $\bar{\pi}^y(1)$ is exactly the un-discounted average profit given by $\lim_{k\to\infty} s_k^Y(x_k)/k$.

PROPOSITION 2. $\bar{\pi}^y(1) = \lim_{k\to\infty} s_k^Y(x_k)/k = c/(2(c+3)(c+1))$.

Proof. See Appendix A. ∎

What is the economic interpretation of long-run average profit $\bar{\pi}^y(1)$? We note from (20) that $\mu(1) = 1$ so that (19) takes the form of

$$Y_t = r_{\mu(1)}(X_t) = \frac{(c+1)(1-X_t)}{c^2+3c+1}, \tag{23}$$

which is the standard reaction function of Walrasian-Stackelberg game in which the price-taker plays the role of a follower while the dynamic optimizer plays the role of Stackelberg leader.

The above discussions are summarized in the following theorem.

[7] The impact of short-run commitments in dynamic oligopolies have been explored by Maskin and Tirole (1987) and Dana and Montrucchio (1986, 1987), in which Markov strategies (or dynamic reaction functions) in deterministic infinite-horizon duopoly games with alternating moves have been derived for quadratic payoffs in particular. One of their main results is that the set of Markov-perfect equilibria converges to the one-shot best reply functions as the players get more and more impatient (i.e. the discount factor tends to zero), which is consistent to our analysis.

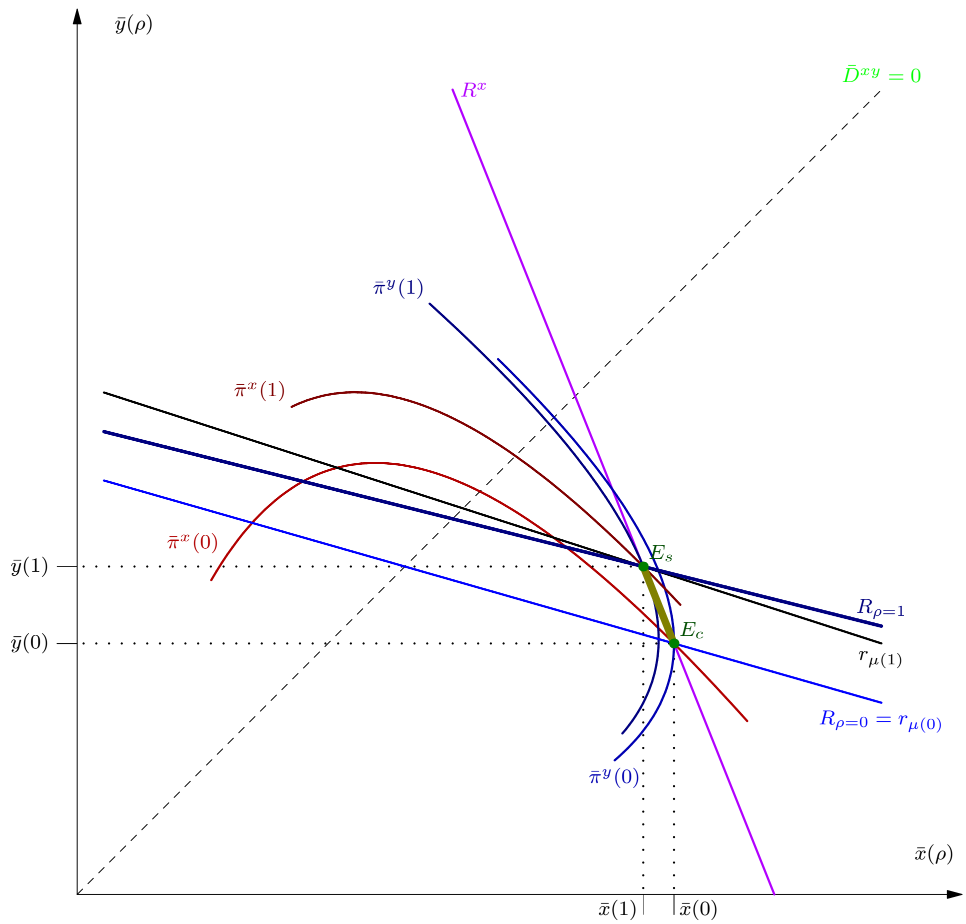


FIG. 2 Static equilibria vs. Stationary equilibria

THEOREM 2 (**Equivalency**). *When the planing horizon T is sufficiently long, the optimal plan for the multiperiod dynamic optimization becomes stationary. This results in a stationary equilibrium that coincides exactly with the intertemporal equilibrium resulted from the variational best-response (19) in an one-period static optimization problem. The stationary profits are then the long run average profits for the respective firm.*

In particular, when the dynamic optimizer is shortsighted (i.e., $\rho = 0$), the stationary optimal plan is to execute the Cournot reaction for all periods. The dynamic optimizer achieves a minimum possible long-run average profit that coincides with the one achieved with the Walrasian-Cournot equilibrium.

When the dynamic optimizer is provident (i.e., $\rho = 1$), the dynamic optimizer achieves a maximum possible long-run average profit that coincides with the one achieved with the Walrasian-Stackelberg equilibrium.

Remark 1. It is worthwhile to emphasize that, although an one-to-one correspondence can be established between the intertemporal equilibrium of the static optimization problem and the long run stationary outcome of the dynamic optimization, there exists fundamental difference in general in interpreting these outcomes. In terms of best-response reaction, unless $\rho = 0$, we have

$$R_\rho \neq r_{\mu(\rho)}$$

in general. In terms of average profit, if $\rho < 1$, for any initial state of X_1 (orP_0), we have

$$\lim_{T\to\infty} s_T^y\left(X_1\right)/T \neq \lim_{T\to\infty} s_T^y\left(\bar{x}\left(\rho\right)\right)/T.$$

These distinctions are illustrated in Fig. 2, in which the thick portion drawn along the price-taker's long-run reaction curve R_w^x represents the stationary equilibrium $\left(\bar{x}\left(\rho\right), \bar{y}\left(\rho\right)\right)$, while $\left(\bar{x}\left(0\right), \bar{y}\left(0\right)\right)$ and $\left(\bar{x}\left(1\right), \bar{y}\left(1\right)\right)$ coincide the Walrasian-Stackelberg equilibrium E_s and Walrasian-Cournot equilibrium E_c, respectively. The improvement of stationary profit gained by the dynamic optimizer from $\rho = 0$ to $\rho = 1$ is evidenced by inward-shifting of iso-profit curve $\pi^y\left(\rho\right)$. $\pi^y\left(1\right)$ is the maximum profit that the dynamic optimizer can obtain since it is tangent to the price-taker's implicit reaction R_w^x. Also can be seen is the difference between the static reaction $R_{\rho=1}$ and the stationary reaction $r_{\mu(1)}$ (but $R_{\rho=0}$ coincides with $r_{\mu(0)}$).

4. RELATIVE PROFITABILITY OF PRICE-TAKING STRATEGY

As having been explored in the last section, for all the initial state $X_1 < X_\infty^u$, a stationary equilibrium profit bundle $\left(\bar{\pi}^x\left(\rho\right), \bar{\pi}^y\left(\rho\right)\right)$ always results if the planning horizon T is sufficiently long, at which the long-run average of the profit difference between the dynamic optimizer and the price-taker is given by

$$\Delta^{yx}\left(c,\rho\right) \doteq \bar{\pi}^y\left(\rho\right) - \bar{\pi}^x\left(\rho\right) = \frac{-c}{2\left(\left(c+2\right)\left(c+\rho\right)+c+1\right)^2} < 0. \tag{24}$$

That is to say, for all possible initial price $P_0 \in \left(0, \bar{p}_\infty\right)$, the long-run average profits made by the dynamic optimizer is always less than the one made by the price-taker. This fact is consistent with the conclusion for the intertemporal equilibrium discussed in Huang (2002).

How about the average relative profitability when the planning horizon T is relatively short? To answer this question, we can analyze the accumulated profit difference $s_k^{yx}\left(x_k\right) = s_k^y\left(x_k\right) - s_k^x\left(x_k\right)$ directly, where $s_k^i\left(x_k\right) = \sum_{j=1}^k \rho^{k-j}\Pi_{T-j+1}^i\left(x_j\right)$, for $i \in \{x, y\}$. Apparently, s_k^{yx} must be a quadratic function of x_k as well and it satisfies the recursive relation:

$$s_k^{yx}\left(x_k\right) = \Delta_k^{yx}\left(x_k\right) + \rho s_{k-1}^{yx}\left(x_{k-1}\right) \tag{25}$$

with $s_0^{yx}\left(\cdot\right) = 0$, where Δ_k^{yx} is the relative profit of the two firms at the stage k given by

$$\Delta_k^{xy}\left(x_k\right) \doteq \pi_k^y\left(x_k\right) - \pi_k^x\left(x_k\right) = \left(y_k - x_k\right)\left(1 - \left(1 + \frac{c}{2}\right)\left(y_k + x_k\right)\right).$$

We are able to arrive at the following recursive relationships for the parameters of s_k^{yx} as follows:

PROPOSITION 3. *The accumalted profit difference s_k^{yx} can be expressed as*

$$s_k^{yx}\left(x_k\right) = -a_k\left(q_k^u - x_k\right)\left(x_k - q_k^l\right) \tag{26}$$

where $q_k^{u,l} = \left(b_k \pm \sqrt{b_k^2 + a_k d_k}\right)/a_k$ and the following recursive relations hold for $k \geq 1$:

$$a_k = \left(2+c\right)\left(1-v_k^2\right)/2 + \rho\left(1-v_k\right)^2 a_{k-1}/c^2, \tag{27}$$

$$b_k = \rho\left(1-v_k\right)\left(1-u_k\right)a_{k-1}/c^2 + \left(v_k+1\right)/2 - \rho\left(1-v_k\right)b_{k-1}/c - u_k v_k\left(2+c\right)/2, \tag{28}$$

$$d_k = 2\rho\left(1-u_k\right)/cb_{k-1} + \rho d_{k-1} - \rho\left(1-u_k\right)^2 a_{k-1}/c^2 - u_k\left(1-\left(2+c\right)u_k/2\right), \tag{29}$$

with the boundary conditions

$$a_0 = b_0 = d_0 = 0.$$

Proof. Omitted since it can be verified straightforwardly. ∎

Remark 2. Although it is tedious, it can verified straightforwardly that $\lim_{k\to\infty} d_k/k = \Delta^{yx}(\rho)$.

Define $x_k^l \doteq \max\{0, q_l\}$, $x_k^u \doteq \min\{X_k^u, q_k^u\}$. Then $\Omega_k^x \doteq (x_k^l, q_k^l)$ is such a compact set in $[0, X_\infty^u]$ that $s_k^{yx}(x_k) < 0$ if and only if $x_k \in \Omega_k^x$. We shall call Ω_k^x the *relative profitability range for the price-taker*. The compact property of Ω_k^x suggests that if $x_a, x_b \in \Omega_k^x$, then for all $x_\epsilon = \epsilon x_b + (1-\epsilon)x_a$,with $\epsilon \in (0,1)$, we have $x_\epsilon \in \Omega_k^x$.

Due to the cyclically converging characteristics of policy parameter u_k, the recursive relations (27) to (29) suggest that both q_k^u and q_k^l must be cyclically converging sequences as well. Therefore, when k is small, there does not exist monotonically inclusive relationships among Ω_k^x. On the other hand, the Turnpike property of the optimal policy ensures that Ω_k^x do exhibit "expansion property" for $k > 10$, as depicted in Fig. 3. Formally, we have

THEOREM 3. *For Linear Model with $c > 1$ and arbitrary $0 < \rho \leq 1$, we have*

i) there always exists a compact set $\Omega_k^x \subset [0, X_k^u]$ such that the price-taker can make higher average profit than the dynamic optimizer if $x_k \in \Omega_k^x$;

ii) there exists a $k^* > 1$ such that $\Omega_k^x \subset \Omega_{k+1}^x$ for all $k > k^*$;

iii) $\lim_{k\to\infty} \Omega_k^x = [0, X_\infty^u]$.

Proof. See Appendix A. ∎

Substituting $y_k = u_k - v_k x_k$ into (4), we are able to get the inverse recursive relation for the price-taker's output:

$$x_{k-1} = \theta(x_k) = (1-u_k)/c - \sigma_k x_k \tag{30}$$

where $\sigma_k \doteq (1-v_k)/c$.

For $k >> 10$, we have

$$x_{k-1} = (1-\bar{u}(\rho))/c - \sigma(\rho)\, x_k, \tag{31}$$

with $\sigma(c,\rho) = (1-\bar{v}(\rho))/c$. The convergency speed of X_t to the intertemporal equilibrium $\bar{x}(\rho)$ given in (15) is determined by the multiplier of $\sigma(c,\rho)$, which in turns accounts for the rate of expansion of Ω_k^x for large k. The larger the value of $\sigma(c,\rho)$ is, the faster the convergency speed of X_t to $\bar{x}(\rho)$ and the faster that Ω_k^x expands with increasing k.

Simple algebra manipulation reveals that

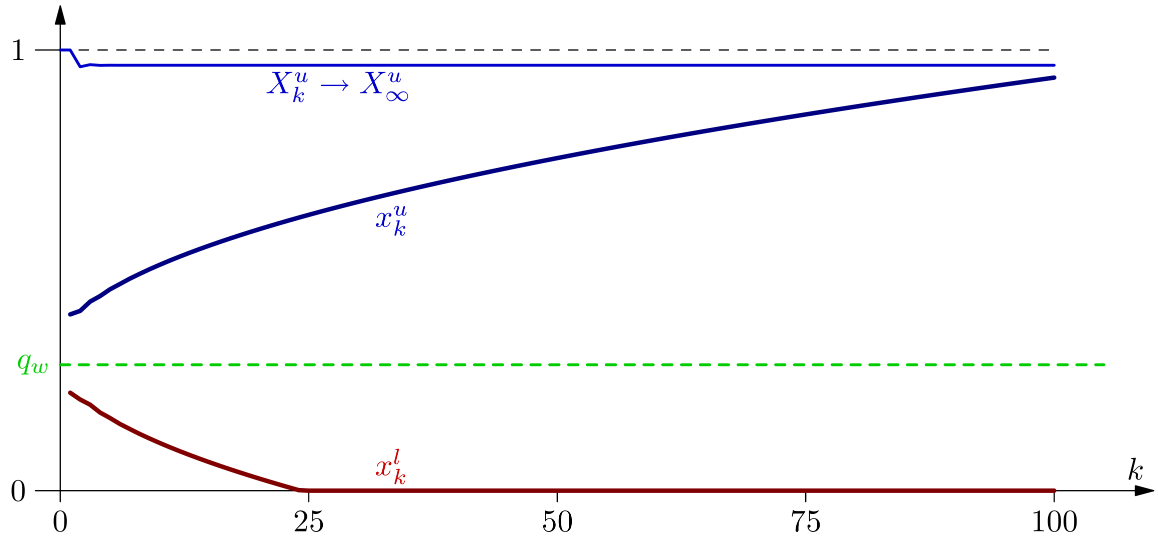
a) $\partial\sigma(c,\rho)/\partial\rho > 0$, i.e., for fixing c, increasing the discount factor ρ decreases the stability of (31) and hence decreases the expansion rate of Ω_k^x along increasing k; and

b) $\partial\sigma(c,\rho)/\partial c < 0$, i.e., for fixing ρ, increasing the cost c parameter increases the stability of (31) and hence increases the expansion rate of Ω_k^x along increasing k.

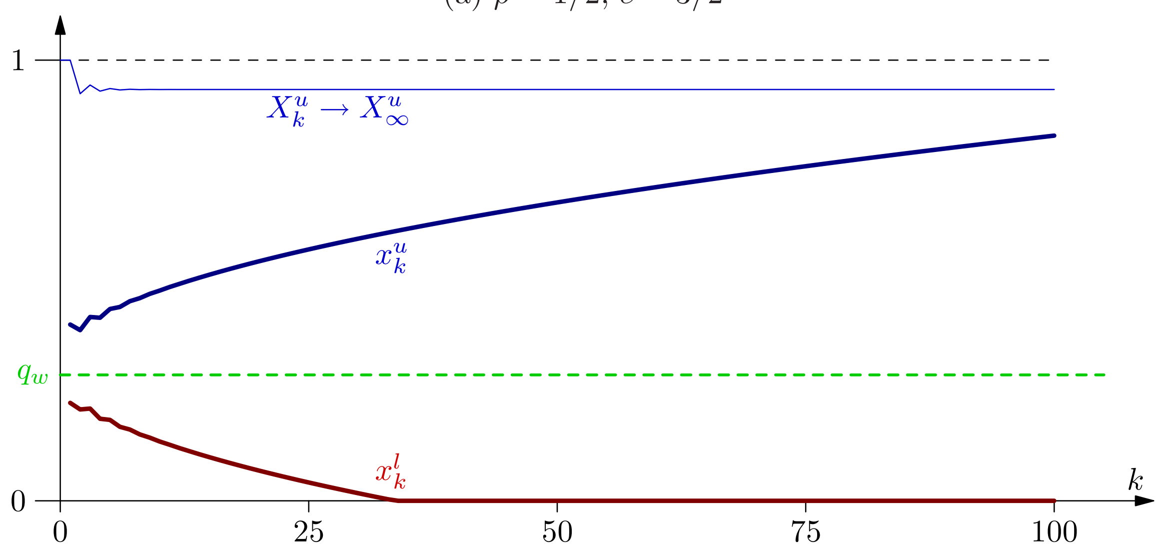
Let λ denote the Lebesgue measure, then $\lambda(\Omega_k^x)$ indicates the width of Ω_k^x. We have the following observations:

i) $\Omega_1^x = (1/(c+3), 1/(c+1))$ is independent of ρ while $\Omega_\infty^x = (0, X_\infty^u)$ depends on both c and ρ. However, we have $\partial\lambda(\Omega_1^x)/\partial c < 0$ but $\partial\lambda(\Omega_\infty^x)/\partial c > 0$. *While higher production cost c reduces the one-shot relative profitable range for the price-taker, it does benefit the price-taker in the long-run.* On the other hand, higher production cost c slows down the expansion speed of Ω_k. This is consistent with the fact that higher production cost c stabilizes the system (from $\partial\sigma(c,\rho)/\partial c < 0$). These facts can be confirmed by comparing Fig. 3(a) with Fig. 3(c).

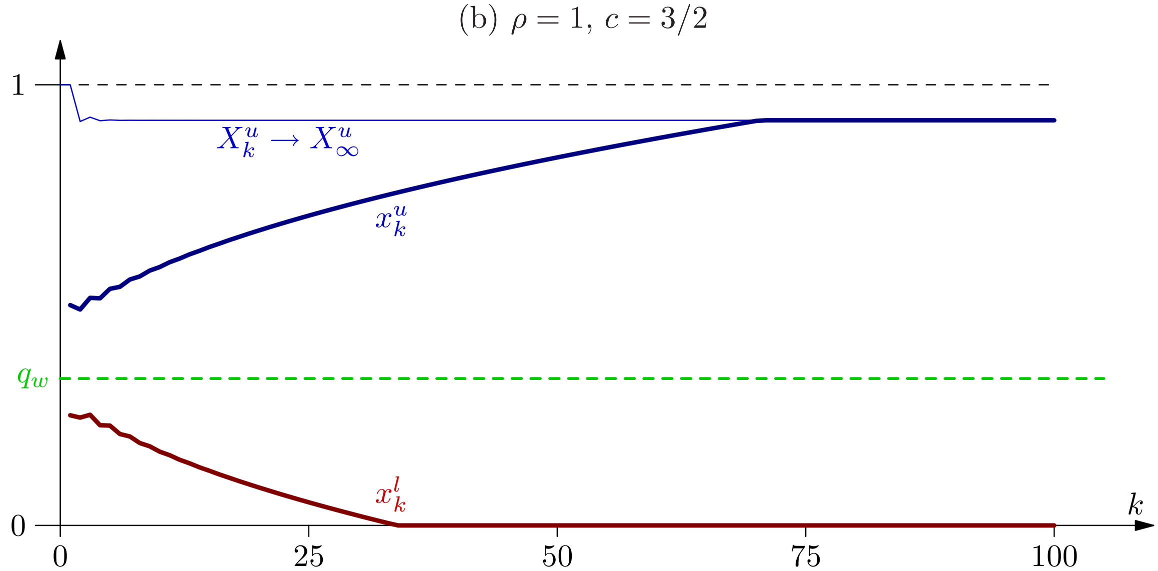
ii) $\partial\lambda(\Omega_\infty^x)/\partial\rho < 0$, that is, lower discount factor increases the long-run relative profitable range for the price-taker. Since lower discount factor increases the stability of the dynamic process as well, it speeds up the expansion rate of Ω_k. These facts can be confirmed by comparing Fig. 3(a) with Fig. 3(b).



(a) $\rho = 1/2,\ c = 3/2$



(b) $\rho = 1,\ c = 3/2$



(c) $\rho = 1/2,\ c = 1$

FIG. 3 Relative profitability range for the price-taker

iii) The average profit difference in the long-run is given by $\lim_{k\to\infty} s_k^{yx} = \Delta^{yx}(c,\rho)$, where $\Delta^{yx}(c,\rho)$ is defined in (24). It can be verified that $\partial\Delta^{yx}(c,\rho)/\partial\rho > 0$ and $\partial\Delta^{yx}(c,\rho)/\partial c > 0$. Therefore, the more advanced the technology (the smaller c) is, and/or the more the future profit is discounted, the larger the long-run profit difference is.

5. FINAL REMARKS

We have proved theoretically and demonstrated numerically with a Linear Model that relative profitability of price-taking strategy can still be preserved even when a rival firm strives to maximize its discounted profit over finite or infinite planning horizon. In particular, when the marginal cost is not too small ($c > 1$), an interior optimal plan can always be implemented so that for any finite planning horizon, when the initial price falls within the a compact relative profitability range that centered around the Walrasian equilibrium price, the price-taker always ends up with a higher relative profit as compared to the dynamic optimizer in terms of average profit. This relative profitability range expands with increasing planning horizon and finally covers almost all of the price-domain when the planning horizon approaches infinity.

It is, however, worthwhile to mention that

a) although our analysis is carried out to a duopoly, the analysis can be carried out to the general oligopoly model consisting of n price-takers and m dynamic optimizers who form *a monopolistic cartel and produce at an identical output level*.[8] Analogous conclusions can be arrived for such generalization except that the stability condition, the economically meaningful range and the relative profitability range vary with the distribution parameters n and m. However, the implication is straightforward: even the higher relative profitability enjoyed by the price-taker may induce additional price-takers to enter the market so that long run profit difference with respect to the dynamic optimizer will be reduced, but the relative profitability advantage always prevails, no matter how insignificant it is, unless free entry is allowed[9].

b) the adoption of Linear Model enables us to present all results in analytical closed forms so that the exact optimal plan as well as related concepts can be derived recursively and evaluated rigorously. Our numerical simulations do confirm that the same conclusions can be arrived when general nonlinear demand and/or marginal cost are adopted.

It should be warned that the higher relative profit is not intentionally but unconsciously achieved by the price-taker. It is a free-rider and the outcome arises simply because its rival is concerned with its own absolute profit, i.e., the dynamic optimize's objective is to maximize discounted ABSOLUTE profit instead of RELATIVE profit as compared to the price-taker over the planning horizon. Even so, it is by no mean suggested that the dynamic optimizing is an inferior strategy. The truth is, with the cost-saving technology ($c < 1$) and limited planning horizon, the dynamic optimizer can effectively fulfill dual goals of achieving the absolute profitability (due to optimizing behavior) and at the same time maintaining the relative profitability under many circumstances (due to $x_k \notin \Omega_x^y$).

Needless to say, this research can be extended and generalized in many different ways.

First, as we have seen in the this part of research, the adoption of Linear Model inevitably brings unnecessary difficulty of market crash and force us to concentrate our attention to the interior optimal solution. Economically, corner solution (occurring for $c < 1$ and/or $x_k \in [0,1] \setminus \mathbb{X}_k$) may of interest in its own sake because it may be rational for the dynamic optimizer to stop production for one or two period(s) to allow the price level raise to a desired high level. It is similarly rational for the dynamic optimizer to intentionally over-supply to push the

[8] It is impossible to study the dynamic optimization problem when each dynamic optimizer acts individually unless a proper sequence of choice is assumed, as assumed in Cyert and DeGroot (1970).

[9] If so, the strategic advantage of price-takers should vanish in limit because the long run profits of price-takers become indistinguishable from dynamic profit maximizers as the oligopolistic competition converges to the perfect competition.

market price extremely low so that the price-taker are enticed to produce an extremely small quantity at the next period or even be forced to exit the market. Such consideration motivates us to explore the optimal plan that is not economically meaningful.

Secondly, instead of devising an optimal plan for long periods, the optimizer may also consider to break the planning horizon, say T, into a number of short periods, say L, and repeat the optimal plan cyclically for T/L times. Such sub-optimal planning may be taken as a measure to prevent the optimal plan from becoming stationary and thus avoid the “free-rider” benefit enjoyed by the price-taker. Apparently, a prerequisite for such implementation is an additional boundary constraint: $X_1 = X_{L+1}$. How the sub-optimal plan changes the relative profitability of the price-taker will be explored.

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7. APPENDIX: PROOFS

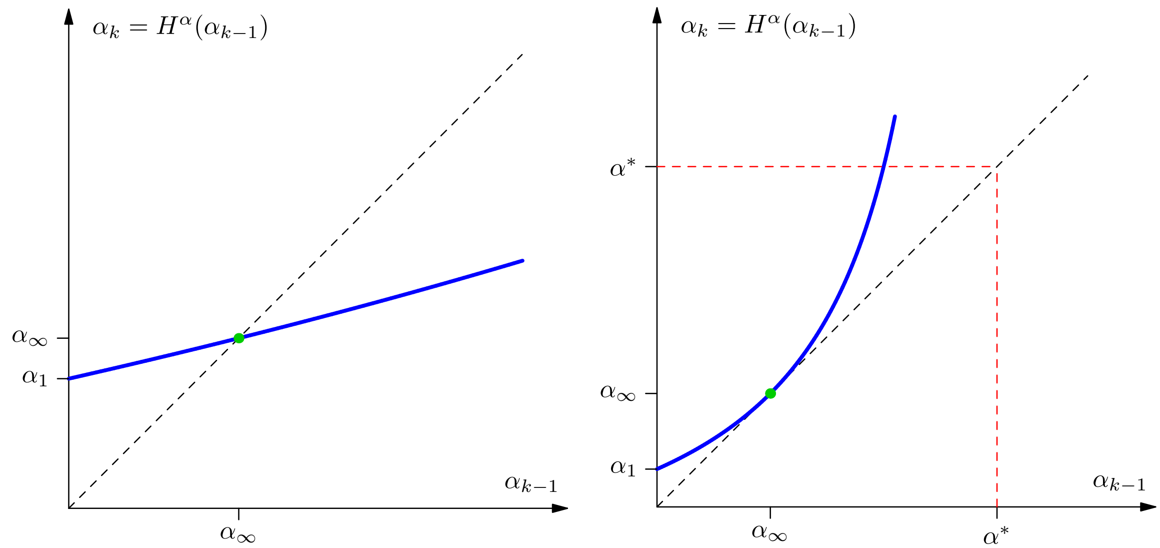
Proof of Theorem 1:

i) and ii): Substituting (8) and (6) into (5) and rearranging yields

$$s_k^y(x_k) = \max_{y_k < 1 - x_k} H(y_k)$$

where

$$H(y_k) = h_1 y_k^2 + h_2 y_k + h_3 \tag{32}$$



(a)$c = 3/2,\ \rho = 1$ (b) $c = 1,\ \rho = 1/2$

FIG. 4 Illustration of recursive map $\alpha_k = H^\alpha(\alpha_{k-1})$

with

$$h_1 \doteq -\left(c^2(2+c) - \rho\alpha_{k-1}\right) / \left(2c^2\right) \tag{33}$$

$$h_2 = \left(\rho\alpha_{k-1}(c + x_k - 1) - \rho c\beta_{k-1} + c^2(1 - x_k)\right) / c^2$$

and $$h_3 = \frac{\rho}{2c^2}\left(\alpha_{k-1}(c - 1 + x_k)^2 + c^2\gamma_{k-1} + 2c\beta_{k-1}(1 - x_k)\right)$$

If the second-order condition $d^2 s_k^y(x_k) / (dy_k)^2 = h_1 < 0$ is satisfied, that is,

$$\alpha_{k-1} < \alpha^* \doteq c^2(2+c)/\rho, \tag{34}$$

which will be shown indeed true for $c > 1$ in the proof of Proposition **1**, an interior solution is given by

$$y_k = -\frac{1}{2}\frac{h_2}{h_1} = \frac{\rho\alpha_{k-1} - c^2}{c^2(c+2) - \rho\alpha_{k-1}} x_k + \frac{\rho\alpha_{k-1}(c-1) - \rho c\beta_{k-1} + c^2}{c^2(c+2) - \rho\alpha_{k-1}} \tag{35}$$

so that

$$s_k^y(z_k) = \frac{1}{2}\frac{\rho\alpha_{k-1}c + c^2}{c^2(2+c) - 2\rho\alpha_{k-1}} x_k^2 + \frac{\rho c\left(\alpha_{k-1}c - (c+1)\beta_{k-1}\right) - c^2}{c^2(2+c) - \rho\alpha_{k-1}} x_k$$
$$+ \frac{1}{2}\frac{c^2 + \rho\beta_{k-1}\left(\rho\beta_{k-1} + 2c + 2c^2\right) - \rho\alpha_{k-1}\left(-c^3 + c + 2\rho\beta_{k-1}\right)}{2c^2 + c^3 - \rho\alpha_{k-1}} + \frac{1}{2}\rho\gamma_{k-1}.$$

Comparing the above expression with (8) leads to (9)-(11). Recasting (35) leads to (12).
The initial values follow from the boundary condition of $s_0^y(x_0) = 0$.
iii) With $y_k = u_k - v_k x_k$, we have

$$p_k = 1 - u_k - (1 - v_k)x_k.$$

$p_k > 0$ thus requires that $x_k < (1 - u_k)/(1 - v_k) = 1 - \beta_k/(c + \alpha_k)$. ***Q.E.D.***

Proof of Proposition 1:

i) The recursive relation (9), rewritten as

$$\alpha_k = H^\alpha(\alpha_{k-1}) \doteq \frac{c(c + \rho\alpha_{k-1})}{c^2(2+c) - \rho\alpha_{k-1}}$$

is essentially an one-dimensional discrete dynamic process with the derivative properties: $\partial H^{\alpha}/\partial\alpha_{k-1} > 0$ and $\partial^2 H^{\alpha}/\partial\alpha_{k-1}^2 > 0$. Hence, H^{α} is a monotonically increasing convex map. Starting with $\alpha_0 = 0$, $\alpha_1 = H^{\alpha}(0) > 0$, we obtain a monotonically increasing sequence. As illustrated in Fig.4, when $c > 1$, the map H^{α} has two fixed point(s) given by

$$\alpha_{\infty} \doteq \frac{1}{2}\alpha^* - \frac{c}{2} - \frac{c}{2\rho}\sqrt{(c^2-\rho)\left((c+2)^2-\rho\right)}, \tag{36}$$

$$\alpha'_{\infty} \doteq \frac{1}{2}\alpha^* - \frac{c}{2} + \frac{c}{2\rho}\sqrt{(c^2-\rho)\left((c+2)^2-\rho\right)}, \tag{37}$$

where α^* is defined in (34) and $0 < \alpha_{\infty} \leq \alpha'_{\infty} < \alpha^*$. More importantly, it can be verified that $\alpha_1 = H^{\alpha}(0) < \alpha'_{\infty}$.

By the analytical nature of H^{α}, we must have $0 < \frac{\partial H^{\alpha}}{\partial\alpha_{k-1}}|_{\alpha_{\infty}} < 1$ and $\frac{\partial H^{\alpha}}{\partial\alpha_{k-1}}|_{\alpha'_{\infty}} > 1$. Since $\alpha_1 < \alpha$ and H^{α} intersects the 45 degree line in the α_k-α_{k-1} plane from above, *the sequence* $\{\alpha_k\}$ *will converge monotonically to the lower fixed point* α_{∞}. Moreover, from the expression of (36) itself, we can see that $\alpha_{\infty} < \alpha^*$, which implies that $\alpha_1 < \alpha_2 < ... < \alpha_{\infty} < \alpha^*$.

While α_k converges monotonically to α_{∞}, Eq. (10) degenerates into a constant coefficient linear dynamic process:

$$\beta_k = H^{\beta}(\beta_{k-1}) = \frac{\rho c(c+1)(\alpha_{\infty} - \beta_{k-1})}{c^2(2+c) - \rho\alpha_{\infty}}$$

which will converges *cyclically* to its stationary value β_{∞} if the absolute value of the slope

$$\left|\frac{\partial H^{\beta}}{\partial\beta_{k-1}}\right|_{\alpha_{\infty}} = \frac{\rho c(c+1)}{c^2(c+2) - \rho\alpha_{\infty}}$$

is less than unity, which can be shown to be true when $c \geq 1$.

iii) The rest of conclusions follow directly from their relationships with α_k and β_k. ***Q.E.D.***

Proof of Proposition 2:

For T is sufficiently large, and (22) is implemented for sufficiently long time, $s_T^y(X_1) = s_T^Y(x_T)$ approaches infinity, which is confirmed by the facts that $\lim_{k\to\infty}\alpha_k = \alpha_{\infty}$ and $\lim_{k\to\infty}\beta_k = \beta_{\infty}$ for all $\rho \leq 1$ but $\lim_{k\to\infty}\gamma_k = \gamma_{\infty}$ exists only for $\rho < 1$. When $\rho = 1$, γ_k does not converge because the recursive relation (11) simplifies to

$$\gamma_k = \gamma_{k-1} + \frac{\beta_{k-1}(\beta_{k-1} - 2\alpha_{k-1}) + c\left(c^2-2\right)\alpha_{k-1} + 2c(c+1)\beta_{k-1}}{c^2(c+2) - \alpha_{k-1}}. \tag{38}$$

If we evaluate the average payoff $s_k^Y(x_k)/k$, then we have[10]

$$\lim_{k\to\infty} s_k^Y(x_k)/k = \begin{cases} 0, & \rho < 1, \\ \lim_{k\to\infty}\frac{\gamma_k}{2k}, & \rho = 1. \end{cases}$$

When $k \to \infty$, it follows from (38) that

$$\lim_{k\to\infty}\gamma_k/(2k) \simeq \langle\pi^y\rangle + \frac{\gamma_{k-1}}{2k} \text{ when } k \to \infty,$$

where

$$\langle\pi^y\rangle = \frac{\beta_{\infty}(\beta_{\infty} - 2\alpha_{\infty}) + c\left(c^2-2\right)\alpha_{\infty} + 2c(c+1)\beta_{\infty}}{2\left(c^2(c+2) - \alpha_{\infty}\right)},$$

[10] "Average" here is defined in the conventional sense so that the profit at each period carries an equal weight for the purpose of possible evaluation of long-run characteristics.

which then suggests that $\lim_{k\to\infty} s_k^Y(x_k)/k = \lim_{k\to\infty}\gamma_k/(2k) = \langle\pi^y\rangle$. Substituting α_∞ and β_∞ with their respective values given by (9) and (10) into above expression leads to $\langle\pi^y\rangle = c/(2(c+3)(c+1))$, which is exactly $\bar{\pi}^y(1)$. ***Q.E.D.***

Proof of Proposition 3:

ii).and iii)

Based on the Turnpike property, when $k >> 10$, the optimal response converges to

$$y_k = \bar{u}(\rho) - \bar{v}(\rho)x_k \tag{39}$$

and $\Delta_k^{yx}(x_k) < 0$ if and only if $x_k \in \Lambda_\infty \doteq (x_l, x_u)$, where

$$x_l = \frac{\bar{u}(\rho)}{\bar{v}(\rho)+1} \text{ and } x_u = \frac{2-(c+2)\bar{u}(\rho)}{(c+2)(1-\bar{v}(\rho))}.$$

Applying the inverse recursive relation (31), we get

$$\theta(x_l) = \frac{1-2\bar{u}(\rho)+\bar{v}(\rho)}{c(\bar{v}(\rho)+1)} \text{ and } \theta(x_u) = \frac{1}{c+2} = q_w$$

It can be verified that $\theta(x_u) > x_l$ and $\theta(x_l) \leq x_u$, or equivalently, $\theta(\Lambda_\infty) \subset \Lambda_\infty$. Therefore, if there exists a large enough k^* such that $\Lambda_\infty \subseteq \Omega_{k^*}^x$, then for all $k > k^*$, starting with $x_k \in \Lambda_\infty$, the implementation of the optimal plan given by (39) does not only lead to $\Delta_k^{yx}(x_k) < 0$ but also enforces x_{k-1} to fall into Λ_∞.[11]

Following from the continuity of the accumulated profit difference function s_k^{yx} and the inversely converging property of $\{x_k\}$, there exists such a $\epsilon_k < x_l$ that so long as $x_k \in (x_l-\epsilon, x_l)\cup(x_u, x_u+\epsilon)$, we have $\Delta_k^{yx}(x_k) > 0$, $x_{k-1} \in \Lambda_\infty \subset \Omega_{k-1}^x$ and

$$s_k^{yx}(x_k) = \underbrace{\Delta_k^{yx}(x_k)}_{(+)} + \rho \underbrace{s_{k-1}^{yx}(x_{k-1})}_{(-)\text{ due to } x_{k-1}\in\Lambda_\infty\subset\Omega_{k-1}^x} < 0, \tag{40}$$

that is, the possibility in which the negative accumulated profit difference in later periods outweighs the positive instantaneous profit difference, which justifies the expectations that there exists a k^* around 10 and Ω_k^x expands with increasing k for all $k > k^*$. In other words, we have $\Omega_{k^*}^x \subset \Omega_{k^*+1}^x \subset \cdots \subset \Omega_\infty^x = [0, X_\infty^u]$. The last equality $\Omega_\infty^x = [0, X_\infty^u]$ follows from the fact that $\Delta^{yx}(c,\rho) < 0$ for all $X_1 \in [0, X_\infty^u]$. ***Q.E.D.***

[11] In this sense, Λ_∞ is essentially a "super relative profitability range" for the price-taker in the sense that so long as the price-taker's current output happens to fall within it, the price-taker does not only make higher profit than the dynamic optimizer instantaneously at the current period but also makes a higher average profit than the dynamic optimizer for all the future periods.