Firm Behaviour under the Threat of Liquidation: Implications for Output, Investment, and Business Cycle Transmission

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Abstract

We study the optimal behaviour of a firm whose cash holdings are determined by a diffusion process and which is faced with the threat of liquidation if internal cash balances fall below some threshold. There is a conflict between the desire to pay out cash as dividends to satisfy relatively impatient shareholders and the need to retain cash as a barrier against possible liquidation (and consequent loss of all future revenues). Stochastic optimal control characterises the value function and yields optimal decision rules for the firm. The firm has a precautionary motive for retaining earnings, the internal cost of funds and local risk aversion are all decreasing functions of net worth, and (in natural extensions to our model) output and investment are both increasing functions of net worth. We show that for a population of such firms under threat of liquidation, aggregate shocks to net worth lead to substantial and prolonged deviations of economy wide aggregates from their steady state values, supporting the view that credit market constraints play a significant role in business cycle transmission.

1. Introduction

This paper considers the optimal decisions of a firm operating in a stochastic environment under an exogenous threat of liquidation (which we impose as a simple way of capturing credit market imperfections). The interaction of these two conditions causes a marked shift in firm behaviour compared with the deterministic perfect market case, inducing both local risk-aversion and a rise in the cost of capital. This shift in firm behaviour is a plausible mechanism of business cycle transmission.

Credit market constraints have often been suggested as an explanation of cyclical movements in corporate and household expenditures (such arguments can be traced back at least as far as Fisher (1932)), and have frequently been used to justify cash flow effects in econometric studies of investment (beginning with Meyer and Kuh (1957)). But it is only since the mid-1970s that a formal theoretical justification for such effects, based on information asymmetries, has begun to emerge.

The implications for firm behaviour of credit market imperfections in a static context, resulting from information asymmetries between borrower and lender, are now well understood (see *inter alia* Stiglitz and Weiss (1981), Gale and Hellwig (1985), Williamson (1986), De Meza and Webb (1987)). In that literature it has been possible both to characterise the optimal form of financial contract (which depends on the informational assumptions and monitoring technology) and to discuss the efficiency of the firm's decision making. However the extension of such models to a dynamic context resists explicit solution and the form of optimal contract in a multi-period context has not been fully analysed.[1]

[1] Gertler (1992) has solved one dynamic model of debt finance in which the state of nature is private information to an entrepreneur seeking debt finance. This implies a value function for the firm with a shape similar to that obtained in this paper, and distortion of firm decisions whenever net worth falls below a threshold level.

Hence it has not yet been possible to provide complete micro-economic foundations for a business cycle model based on credit market imperfections.

There is nonetheless a general case for the role of credit market imperfections in business cycle transmission. As Stiglitz (1991) points out, the conventional competitive paradigm of firm behaviour faces difficulties in explaining certain aspects of observed macroeconomic behaviour. These include the size of business cycles (in that the shocks to the production function in a real business cycle model required to generate cycles matching observed fluctuations seem implausibly large) and the seeming lack of smoothing behaviour by firms over the cycle, particularly with regard to inventory behaviour and investment.

Recent empirical studies have appealed to static models of credit market imperfections as an explanation of various dynamic aspects of firm behaviour. This has produced some notable empirical findings with both micro- and macro-data. These include the role of cash flow and firm size as a determinant of investment behaviour (Fazzari *et al.* (1988), Gertler and Gilchrist (1992)), the cyclical behaviour of bank and non-bank credit (Gertler and Gilchrist (1993) find that large firms increase bank-credit in recession, and supply trade-credit to small firms, who, in contrast, reduce their dependence on bank-credit), cash-flow effects on inventory investment which are restricted to companies in financial distress (Milne (1991)) and the effect of loan supply on portfolio composition and investment (Kashyap, Stein and Wilcox (1993)). A weakness of most of these studies is that because the assumed theory is static, the dynamic implications of the presence of credit market imperfections is not exploited in model specification.

The aim of this paper is to provide an intuitively plausible account of firm behaviour in the presence of credit market constraints, which offers a more precise characterisation of the dynamics of firm behaviour. Like Greenwald and Stiglitz (1993), in the absence of any tractable analysis of the optimal financial contract, we appeal to general arguments about informational asymmetries as a justification for imposing exogenous constraints on the availability of finance. The distinguishing feature of our analysis is that we apply the theory of control of continuous time diffusion processes, rather than discrete time stochastic dynamic programming. This delivers clear-cut analytical results about the implications of financial constraints for firm behaviour. In particular we demonstrate that the presence of a financial constraint influences firm behaviour even when the level of firm assets means that it is well away from the liquidation threshold, that the presence of constraints both increases the cost of capital and induces local risk-aversion; and that the degree of departure from unconstrained behaviour is greater the closer the firm is to the liquidation threshold. A further attraction of our framework is that it offers the prospect of straightforward extensions to several different aspects of firm decision making, some of which we explore in the paper.[2]

We start with an extremely simplified model of a firm, in which the only decision is about the payment of dividends. This allows clear identification of where our description of the firm departs from the conventional model, without the confusion of output or factor input decisions. This model is used to characterise the form of optimal behaviour and to show that the internal cost of finance depends on the net worth of the firm.

There are two other technical differences between our model and that of Greenwald and Stiglitz (1993). We assume that the only constraint on dividend payment is that dividends must be non-negative, and that firms seek to maximise the expected discounted value of dividend payments. The reason for doing this is that dividend policy then takes a particularly simple form, which in turn facilitates solution of our model. The second

[2] We also point out parallels with the literature on consumption and precautionary saving, especially the discrete time analysis made by Deaton (1991) of optimal consumption smoothing subject to liquidity constraints.

difference is less material. In our setup the deadweight costs of bankruptcy take the extreme form of complete loss of shareholder wealth when the firm is liquidated. This highlights the resulting change in firm behaviour, but we could derive qualitatively identical results with a model in which the firm continues in operation after bankruptcy, but shareholders are required to pay a fixed cost of bankruptcy.

Subsequent sections of the paper extend our setup to output and investment decisions and discuss aspects of behaviour towards risk. In section 3 we endogenise the output decision, showing that when output affects the rate of diffusion of cash balances, output itself depends on the degree of local risk aversion. We show also that local risk aversion is a decreasing function of internal net worth, that is, the firm is most risk averse when it is closest to liquidation. In section 4 we discuss the consistency of this latter finding with other standard models (including Greenwald and Stiglitz (1993)) which suggest the possibility of risk-loving behaviour when a firm is close to liquidation. Section 5 extends our model to include an investment decision. When there are convex costs of adjustment of the firm's capital stock, investment depends on an amended Tobin's q relationship, in which q is a function of net worth and (over an important range of values for the capital stock) investment declines as net worth is reduced.

Finally in section 6 we discuss the macro-economic implications of our model, using an aggregation technique similar to that applied to a model of consumer durables by Bertola and Caballero (1990) and Caballero (1993). We consider the way in which economy-wide shocks affect the aggregate of individual firm decisions and show that our model generates asymmetric and serially correlated aggregate responses to serially uncorrelated aggregate shocks, suggesting that financial constraints are indeed a possible transmission mechanism for the business cycle.

2. The Problem of Cash Management

In this section we consider the optimal policy for a firm receiving stochastic income, facing the threat of liquidation if cash-in-hand drops below some prescribed level. The firm seeks to maximise shareholder value defined as the value of a stream of dividend payments. We show that with constraints on the availability of external finance, the optimal policy is for the firm to make dividend payments when it has sufficient internal resources, but that otherwise no dividend payment is made in order to build up internal cash.

Assumptions

Let the cash-in-hand held by the firm be denoted by x. Over a small time period dt the increment to x is given by the brownian motion:

(1) $$dx=(rx+\mu-d)dt+\sigma dz$$

where r is the return on funds per unit time available to the firm, d is the (flow) dividend rate and firm's income over the time interval consists of two components, a deterministic part μdt (where $\mu>0$ so the expected income flow is positive) and a stochastic element σdz where z is a Wiener process. Neither μ nor σ is here a function of the state x, but in subsequent sections we consider cases where μ and σ are functions of further state and control variables.

The value of the firm (V) at a time t_0 is the expected value of the discounted flow of future dividend payments. If the payments at time t are d(t) and they are discounted at rate ρ, then V is given by:

(2) $$V= E \int_0^\infty e^{-\rho\tau}d(t_0+\tau)d\tau$$

The firm chooses a dividend rule to maximise V. In general V will be a function of the current cash-in-hand x and current time t_0, $V=V(x,t_0)$. For a stationary problem such as we have here the time dependence is suppressed: $V=V(x)$.

We now make three assumptions that generate a demand to hold cash inside the firm as a hedge against possible liquidation. The first is an equity finance constraint. We assume that dividend payments must be positive. This implies that the firm cannot raise new equity finance. Such a constraint can be justified as the consequence of shareholders and banks not being fully informed about the current state of the firm, so that dividend payments and capital issues take on a signalling role.[3] The particular form of the equity finance constraint is not important. What is essential is that there is some restriction on new equity finance so as to motivate the key distinction between internal and external funds explored in this paper. Only the former are effective as a hedge against possible liquidation. This implies that the firm must balance two conflicting objectives: keeping sufficient resources inside the firm to reduce the probability of future liquidation, and paying out enough resources to satisfy impatient shareholders. We propose this particular specification as a simple way of capturing this trade-off, not as an entirely realistic model of dividend policy.

Our second capital market constraint is an exogenously imposed liquidation trigger. If cash-in-hand x_t falls below a specified level (which we assume to be zero) the firm is liquidated and its value becomes zero from then on. This is effectively a cost of bankruptcy, in which the costs are the loss of future income.[4] Again this constraint on the availability

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[3] See Myers and Majluf (1984). Equity market imperfections are magnified if the management of the firm has objectives other than the maximisation of shareholder wealth, leading to the problems of agency cost discussed by Jensen and Meckling (1976).

[4] We could instead assume (as do Greenwald and Stiglitz) that the crossing of the liquidation trigger invokes a costly monitoring process, which reveals the state of the firm to shareholders and allows it to make a one-off issue of new equity capital.

of debt finance is justified as a consequence of information asymmetries between the providers of finance and the management of the firm.[5]

Finally we assume that the interest rate r paid on all cash holdings is less than the shareholders' discount rate, $r<\rho$.[6] This provides the firm with an incentive to pay out dividends. Were this not true, the firm would maximise value by accumulating cash balances internally such that the threat of liquidation was effectively inoperative. This possibility is removed by assumption.

There are thus three special aspects to our firm. It is unable to obtain external finance freely and faces an uncertain income with the threat of liquidation if cash balances drop too low, yet any cash it receives it would like to pay out as dividends. It is the interaction of these conflicting forces that we study.

The firm's problem is:

choose d(.) to maximise V(x) subject to

$$dx=(rx+\mu-d)dt+\sigma dz$$

$$d\geq 0$$

$$V(x)=0 \text{ if } x\leq 0 \text{ and the problem stops}$$

It is immediately obvious that the optimal dividend rule will be a function of cash-in-hand alone $d=d(x)$.

The Optimal Decision Rule

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[5] Instead of invoking liquidation it would be possible to work with a model without liquidation, but in which debt finance ($x<0$) was charged an interest rate $r'>\rho$. This penalty interest rate would induce a similar aversion to indebtedness as appears in our model. We choose to work with a liquidation threshold both because this is analytically more convenient, and because it seems to be a better reflection of actual debt contracts.

[6] This assumption corresponds to the assumption of Deaton (1989) that consumers are impatient, and rules out the indefinite accumulation of assets.

The Hamilton-Jacobi-Bellman equation for this problem implies that the value function V(x) and the optimal policy rule d(x) satisfy:[7]

(3) $$\rho V = \max_{d} \{ d + (rx+\mu-d)V_x + \tfrac{1}{2}\sigma^2 V_{xx} \}$$

In general the solution technique for such problems is to perform the maximisation in (3) to give the optimal policy rule. This can then be substituted back to give a differential equation in V. This can either be solved explicitly, or if, as is often the case for these types of problems, explicit solutions cannot be obtained, numerical methods can be used to characterise the behaviour of V (see Merton (1971), Brennan and Schwartz (1985), Pindyck (1980,1991), Dixit (1991) for examples and discussion). Our particular problem allows a slightly different approach. Since (3) is linear in the control variable d, the optimal control is bang-bang. That is, the dividend policy will be to pay zero dividends (the minimum possible) below some trigger value of x, say x^*; and to pay the maximum possible above this trigger.

$r<\rho$ implies that a policy of $x^*=\infty$ (never paying a dividend) would result in V=0.[8] Thus the firm can always do better by making a dividend payout for some finite value of x and x^* is finite. In principle x^* could be zero, in which case the optimal policy is to pay out

[7] The problem is a discounted continuous time dynamic stochastic programme, with the return function bounded below by the constraint that $d \geq 0$, and eventual termination within a finite period. The assumption of a Wiener stochastic process rules out any problems of measurability. There is a unique functional solution which satisfies the Hamilton-Jacobi-Bellman equation (see Malliaris (1982), Whittle (1983), Harrison (1983) for discussion of the technical aspects of such problems).

[8] A formal proof of this statement can be obtained by considering the finite horizon problem, where the firm operates for T periods and operates a policy of never paying a dividend (x^* infinite). In this case the value of the firm satisfies the inequality:

$$V \leq \int_0^T \mu e^{r(T-\tau)} d\tau$$

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where the right hand side of this inequality represents the present discounted value of the terminal payout to shareholders, assuming that there is no liquidation trigger. In the limit, as $\zeta \rightarrow \infty$, this becomes $V \leq 0$.

all cash and go into liquidation. This would clearly be an optimal response to a very poor outlook for the future. This case does not, however, generate any further behaviour of interest. We assume that the parameters are such that x^* is positive.

We sum up the above in the following:

Proposition 1.

For the problem described above in equations (1) and (2), the optimal dividend rule takes the following form:

$$\exists\ x^* \text{ s.t.}$$

$$d(x)=0 \text{ for } x \leq x^*$$

and $$\text{barrier control at } x=x^*$$

$$\text{where } 0<x^*<\infty$$

To the right of the dividend payment trigger $V(x)=V(x^*)+x-x^*$ (since control is then applied to bring the firm back to the dividend barrier by paying an instantaneous dividend of $x-x^*$).

Note that it is the interaction of the liquidation threat and the non-negativity assumption on dividends that drives this solution. If dividends could be negative then the firm would borrow if it received a bad shock at any time and pay out whenever it had cash so do to. The optimal level of cash held internally would be zero, and dividends would always be set to keep dx=0. The presence of the liquidation threat causes the dividend policy to switch to d=0 below the trigger value x^* and cash to be held internally.

The Characteristic Shape of the Value Function

Given the optimal dividend rule we may derive the value function V(x). For the region $0<x<x^*$ the dividend is zero so the maximand in the Hamilton-Jacobi-Bellman equation (3) yields the following differential equation for V:

$$\rho V = V_x(rx+\mu)+\tfrac{1}{2}\sigma^2 V_{xx}$$

or equivalently

$$\tfrac{1}{2}\sigma^2 V_{xx} + V_x(rx+\mu) - \rho V = 0 \qquad 0<x<x^* \tag{4}$$

The differential equation (4) has no general analytical solution, but we can solve analytically in particular cases, and can characterise the form of the solution in the general case.[9] For the specific case where μ and σ are exogenous and r=0 (an appropriate assumption if internal funds are held in zero interest accounts or as cash itself) we obtain:[10]

$$\tfrac{1}{2}\sigma^2 V_{xx} + \mu V_x - \rho V = 0 \qquad 0<x<x^* \tag{5}$$

which has the solution

$$V(x) = A \exp(m_1 x) + B \exp(m_2 x) \qquad 0<x<x^* \tag{6}$$

where m_1, m_2 are the roots of

$$\tfrac{1}{2}\sigma^2 m^2 + \mu m - \rho = 0 \tag{7}$$

i.e. $$m_{1,2} = \{-\mu \pm \sqrt{(\mu^2+2\rho\sigma^2)}\}/\sigma^2$$

There are three unknowns in the solution (6), the constants A and B and the dividend switch point x^*, all of which must be determined. Thus we need three boundary conditions. It is relatively straightforward to derive the solution for $x\leq 0$ and $x\geq x^*$. For $x\leq 0$ liquidation occurs and the value function is zero. For $x\geq x^*$, all cash above the amount x^* will be paid

9 Numerical techniques can also be used to investigate the solution, see appendix for an example.

10 It is also possible to obtain an explicit solution for the case where ρ=0 and r<0; in this case although the problem is undiscounted liquidation eventually takes place. The problem is thus one of transient dynamic programming and the value function exists and has a finite value (see Whittle (1983) for fuller discussion).

out as dividends, so that the value function increases linearly with x in this region. The required boundary conditions are:

(i) Continuity of V at the liquidation threshold x=0. This boundary condition obtains because the sample paths for x across the liquidation boundary are continuous (this condition is standard, see for example Whittle (1983) ch 37, Theorem 4.1).

(ii) $V_x=1$ *at* x^**, i.e. continuity of* V_x *at the upper boundary* x^**, using that the value function for* $x \geq x^*$ *is given by* $V(x)=V(x^*)+x-x^*$. This condition emerges because control is instantaneous at the x^* boundary (Dixit (1991) contrasts the boundary conditions which emerge under the alternatives of impulse and instantaneous control). This boundary condition will apply whether or not x^* is chosen optimally since diffusion paths across the boundary are continuous.

(iii) $V_{xx}=0$ *at* x^* *i.e. continuity of* V_{xx} *at the upper boundary* x^*. This is a consequence of x^* being chosen optimally. Otherwise the value function could be increased at x^*, by a small shift of x^* in the direction that $V_x<1$.[11]

These boundary conditions determine the full solution of the differential equations (4) and (5) when r=0. A solution is illustrated in Figure 1 where we graph the theoretical value function for a particular choice of the parameters. The general shape of the function is an upward sloping curve from the liquidation point (zero) with gradient declining, this curve meets the 45° line at x^* and continues along this line.

[11] Condition (iii) is conventionally referred to as smooth pasting (though the origin of this term relates to problems of impulse, not instantaneous control, see Dixit (1991)). The combination of conditions (ii) and (iii) is what Dumas (1991) describes as super-contact.

Where $r \neq 0$ we are unable to obtain an explicit solution of the differential equation. We can still obtain key qualitative features of the solution over the range $0<x<x^*$, from consideration of the stochastic co-state equation[12]:

$$\frac{1}{2}\sigma^2 V_{xxx} + V_{xx}(rx+\mu) - \rho V_x = 0 \qquad 0<x<x^* \tag{8}$$

This implies that $\lim_{x\uparrow x^*} V_{xxx}=\rho/\frac{1}{2}\sigma^2>0$, where the limit is understood as x approaches x^* from below (we express this as a limiting condition because V_{xxx} is discontinous at x^*). From this we can derive the following:

Proposition 2.

For all values of x, $0<x<x^*$, the value function satisfies (4) and (8); the boundary conditions (i), (ii) and (iii); and $V_{xxx}>0$, $V_{xx}\leq 0$, $V_x \geq 1$.

Proof: See Appendix, Corollary 1.

Proposition 2 shows that even when an explicit solution of the differential equation (4) cannot be found, it must still have the same shape as obtained when r=0. Diagrams of this kind are characteristic of models in which capital market imperfections lead to constraints on the availability of external capital; similar diagrams are presented by both Gertler (1992) and Greenwald and Stiglitz (1993). The diagram indicates that the internal cost of capital to the firm (V_x) is greater than the external cost of capital whenever $x<x^*$, and with the premium

[12] First-order conditions for the optimality of the control variables imply that small changes to control variables have only second-order effects on the value function (an envelope property). Hence the co-state equation is the same even when μ and σ are functions of controls which are themselves a function of x.

on internal funds increasing, and increasing more rapidly, as the firm approaches the liquidation threshold.[13] This distortion to the cost of capital results in modification of all dimensions of firm decision making that involve inter-temporal comparison of costs and benefits.

3. The Model with an Endogenous Output Decision

The cash management model described above allows investigation of the problems associated with calculating solutions for stochastic optimal control models, and has already shown certain aspects of firm behaviour in these situations. To understand more fully the behaviour of optimising firms in stochastic environments facing the threat of liquidation we extend the model within the same overall framework. In this section we describe a simple extension to the model to include the optimal choice of output. This allows us to capture real effects from the financial imperfections. The firm now chooses both the level of output and a dividend policy. Income is a function of the level of output chosen, but retains a stochastic element. We replace the cash evolution equation (1) by

$$dx=[rx+(y-ay^2/2)-d]dt+\sigma y dz \qquad (9)$$

where a is some positive constant. Here there are diminishing returns to producing further output calibrated by the constant a. There is also uncertainty about income measured by the term $\sigma y dz$. Note that this uncertainty is increasing in the level of output y, so that higher levels of production are associated with higher levels of income uncertainty. The firm's problem is now to choose a level of output y and a dividend policy d to maximise the

[13] Like Deaton (1989) we find that the presence of a liquidity constraint affects behaviour well away from the point at which the constraint binds.

objective function V(x) as in (2). The Hamilton-Jacobi-Bellman equation for this problem implies that the value function satisfies the following differential equation at the optimal choices of d and y

$$\rho V = \max_{d,y} \{ d + V_x[rx+(y-ay^2/2)-d] + \tfrac{1}{2}\sigma^2 y^2 V_{xx} \} \tag{10}$$

Applying the results of section 2 we note that (10) is linear in the control d so the optimal dividend rule will be still be bang-bang. We investigate the choice of output y in the region where no dividends are paid, d=0. Equation (10) then implies that y is chosen optimally as the solution to

$$\rho V = \max_{y} \{ V_x[rx+(y-ay^2/2)] + \tfrac{1}{2}\sigma^2 y^2 V_{xx} \} \tag{11}$$

in which the first order condition for the maximisation over y (the second order condition for a maximum holds) implies

$$(1-ay)V_x + \sigma^2 y V_{xx} = 0$$

$$\Rightarrow \qquad y = V_x/(aV_x-\sigma^2 V_{xx}) = 1/(a-\sigma^2 V_{xx}/V_x)$$

If the firm's income were non-stochastic and equal to $(y-ay^2/2)$ then the optimal choice of output would be given by $y^*=1/a$. Using the results on the boundary conditions derived the simple cash management case we see that at the upper boundary (where dividend payments start) $V_{xx}=0$ so that the optimal choice of output implied by equations (11) is 1/a, the same as in the deterministic case. In the non-dividend paying region we have $V_{xx}<0$ and $V_x>0$.

Let $W=-V_{xx}/V_x$ which we interpret as a measure of (local) risk aversion. As a consequence of proposition 2 we know that W>0. We can then write the optimal choice of output as

(12) $$y = 1/(a+\sigma^2 W) \; < \; 1/a = y^*$$

that is, the output level is reduced below the optimal level of the deterministic case, except at the boundary x^*. Note that W depends on the level of cash-in-hand x, so that the reduction in output is a function of x. Moreover we have the following:

Proposition 3

Over the entire region $0 \leq x \leq x^*$, $W_x < 0$ so that W is declining as x increases

Proof: See Appendix, Theorem 2

Thus the financial constraints on the firm generate a value function where the degree of local risk aversion increases as the firm moves closer to liquidation. The firm's value function exhibits prudence in the sense of Kimball (1990).[14]

4. The threat of liquidation and attitude towards risk.

In the last section we found that the threat of liquidation induces prudence (local risk-aversion increases as internal cash holdings fall) on the part of the firm. This is in apparent conflict with other studies (including Greenwald and Stiglitz (1993)) who find that firms close to bankruptcy may behave in a risk-loving rather than a risk-averse manner.

These different conclusions about behaviour towards risk rest on different assumptions about the stochastic disturbance to the net worth of the firm. Other studies have generally

[14] Deaton makes a similar point about the effect of liquidity constraints on the attitude of households to risk: "The precautionary motive for saving...is strengthened by the existence of liquidity constraints" (Deaton (1992) p197).

applied discrete time analysis, in which the shock (which corresponds to our σdz) may alter firm net worth by a discrete (non-infinitesimal) amount in each time period. In such cases an increase in the variance of the shock may increase the value of the firm, due to a convexity in the value function caused by the liquidation possibility.

As a specific example, consider in the discrete time framework some action on the part of the firm that would increase the variance of a (symmetric) stochastic disturbance to cash holdings that occurs each period. Suppose also that the mean of the disturbance is on the threshold of liquidation (this ensures that since the distribution is symmetric that the increase in variance has no effect on the probability of liquidation taking place). The widening of the left hand tail of the distribution (which lies entirely beyond the liquidation threshold) causes no cost to the firm. However the increase in the right-hand tail is of benefit to the firm. Thus close to liquidation the firm prefers a higher variance of the disturbance and, in situations where it has control over the variance, behaves in a risk-loving manner.

In contrast, we have modelled the stochastic disturbance as a brownian motion, so that all sample paths are continuous. Here firms diffuse gradually into liquidation rather than being pushed suddenly into liquidation by a discrete shock. Any increase in the variance of the disturbance to the cash balances of the firm is therefore costly, because it is always evaluated in terms of the local curvature of the value function (which exhibits local risk-aversion induced by the threat of liquidation); and the cost to the firm of an increase in the variance of the stochastic disturbance always rises when internal cash falls (because of the local prudence of the value function).

Risk-loving behaviour could still arise in our set-up in response to a non-continuous stochastic disturbance.[15] Consider for example an amendment to our model to include the threat of a natural hazard, for example such as a fire, that would result in a discrete (negative) shock to the net worth of the firm.[16] The financially unconstrained firm operating in a world of perfect capital markets will be completely risk-neutral with regard to this threat and be uninterested in insurance.[17] The financially constrained firm, which we have studied operating under the threat of liquidation, will normally be averse to this risk (because the curvature of the value function induces an aversion to variation in cash balances). It will thus be interested in insurance against the occurrence of such natural hazards. However when close to the liquidation threshold, the convexity of the value function at that threshold means that the firm no longer worries about such risks and is thus not interested in insurance. It therefore acts in a risk-loving way towards this discrete stochastic disturbance, even though the value function is locally risk-averse.

From this it can be seen that there is no real conflict between our finding that the value function is locally risk-averse and the threat of liquidation induces prudence on the part of the firm, and the examples of risk-loving behaviour provided by other studies. The attitude of the firm towards discrete shocks is affected both by local risk aversion (over those regions of the value function where the shock does not result in liquidation) *and* by the convexity at the liquidation threshold. For discrete shocks where a substantial part of the probability mass lies beyond the liquidation threshold, the firm prefers higher to lower variance even though

[15] The standard technique for analysing such a non-continuous disturbance in a continuous time framework is to treat them as a "jump process" in which the probability of a change taking place over a small period dt is proportional to dt (the period until a jump takes place having a poisson distribution).

[16] Assuming that the risk is sufficiently small that proposition 2 still applies.

[17] This will be the case even if shareholders are risk averse, because the risk of fire is diversifiable.

value function is locally risk-averse throughout the range of cash balances for which the firm remains in operation.

General statements about risk aversion and prudence, independent of the distribution of the shocks, can be made only if the value function exhibits global risk aversion (or prudence) i.e. local risk-aversion (or prudence) for all values of cash $-\infty<x<\infty$. This is not the case for our model: we have demonstrated local risk-aversion and prudence only for the range of values $0 \le x \le x^*$ and so statements about the attitude of the firm towards risk are dependent on the distribution of the relevant stochastic disturbance.

Other examples of risk-loving behaviour can be found which, while they do not correspond precisely to our our set-up, can be interpreted in terms of curvature of the value function. One which is often mentioned is the behaviour of US savings and loans which took advantage of Federal deposit insurance to expand rapidly in the early 1980s. The deposit insurance meant that depositors had little incentive to monitor the financial situations of each savings and loans. Crucially however prudential supervision was also weak and this allowed many savings and loans to continue trading even when they had made substantial loan losses and were insolvent. Such savings and loans had strong incentives to make risky investments. If these paid off they could escape insolvency while any downside risk would be borne by the Federal deposit insurance fund.

Viewed in terms of our framework these Savings and Loans were, effectively, operating to the left of our liquidation trigger ($x=0$). The consequence of this is a convexification of the value function, with curving in the opposite direction to that shown in figure 1 and $V_{xx}>0$. W would thus be negative corresponding to locally risk loving behaviour.

5. The Model with an Investment Decision

In this section we describe an alternative extension of the basic model to include an additional state variable other than cash holdings. Output is produced by a stock of quasi-fixed factors of production which is costly to adjust. This we label k and think of as capital stock, though they could represent any factor of production where there are convex costs of adjustment. In standard manner we assume costs to adjusting this productive input which are quadratic in the flow of gross investment i. There are thus two state variables at time t_0, the cash-in-hand of the firm x and the inherited capital stock k. The controls are the dividend paid d, and the level of investment i.

$$y=y(k) \tag{13}$$

$$dk=(i-\delta k)dt \tag{14}$$

$$dx=(rx+py(k)-p_i i-\tfrac{1}{2}\theta i^2/k-d)dt+\sigma dz \tag{15}$$

where p is the price of output, and p_i the price of investment. The evolution of the capital stock is governed by equation (14), increased by investment i and depreciating at a rate δ. This equation is deterministic, but could be made stochastic. Cash flow is given by the diffusion process (15). Again the firm seeks to maximise the present value of a stream of dividend payments as in (2). In a non-stochastic world the firm would choose the capital level such that the cost of incrementing the capital stock by one more unit, p_i, is just matched by the increase in the value of the firm arising from that one unit, that is, if the value of the firm is V(x,k), then optimality requires $V_k=p_i$. For the stochastic case we have the Hamilton-Jacobi-Bellman equation

$$\rho V= \max_{d,i,l}\{d + V_x[rx+py-p_i i-\tfrac{1}{2}\theta i^2/k-d]+\tfrac{1}{2}\sigma^2 y^2 p^2 V_{xx} + V_k(i-\delta k)\} \tag{16}$$

Again the partial differential equation implied by the maximisation of (16) cannot be solved analytically. We use the intuition gained in the simple cash management problem to investigate the nature of the solution. Consider the first order condition for i in the no dividend paying region

(17) $$V_k - \{p_i+\theta i/k\}V_x = 0$$

or $$i/k = \theta^{-1}p_i\{q - 1\}, \text{ where } q=V_k/(V_x p_i)$$

which expresses the investment rate per unit of capital as a function of the ratio of the marginal increment to firm value from undertaking the investment to the price of the investment as in the non-stochastic case, but here deflated by the term V_x, that is the marginal valuation of cash balances to the firm. The boundary condition for this term V_x discussed earlier is that $V_x=1$ at the boundary of the region in which the threat of liquidation starts to affect behaviour. Thus here the formula reduces to the standard q-theory, with $q = V_k/p_i$.

It is shown in the Appendix (theorem 1 corollary 2) that proposition 2 continues to apply to the q-model of investment, as long as the firm has a level of capital stock which exceeds (or is only slightly less than) the level of capital stock k^* which results in zero net investment on the dividend paying frontier (k^* is the unique level of capital which satisfies $q(k^*,x^*)=\theta^{-1}\delta$.). Proposition 2 in turn implies a larger reduction in investment the closer the firm is to the liquidation barrier.

Once again internal cash resources have real effects on the firm’s decisions, but in this case these arise because the internal cost of capital (V_x) exceeds the external cost of capital ($V_x>1$) and hence induces a greater discounting of the future benefits of investment.

6. Aggregation

The purpose of this final section is to describe the behaviour of an economy which comprises a large number of firms facing the liquidation threshold and behaving in the manner analysed in the preceding sections of this paper. We shall confine our attention to the simple cash management problem of section 2 with a zero interest rate on cash balances held by the firm, although the results would be similar for the more general models which we have analysed in sections 3 and 5. We consider the steady state distribution achieved by such an economy, characterised as the cross-sectional distribution of cash holdings amongst firms, and then calculate the behaviour of such an economy in response to an aggregate shock using a simple numerical approximation.

We assume that the economy consists of a large number of firms (measure one) indexed i facing the stochastic cash management problem with identical parameters but independent shocks. The cash holdings of firm i, x_i, evolve according to:

$$dx_i = (\mu - d_i)dt + \sigma\, dz_i \tag{17}$$

where the dz_i are independent Weiner processes. All firms face liquidation if cash balances fall below zero, so they choose identical dividend rules d as in section 2, and all choose the same x^*. Firms will then hold cash balances in the range $(0,x^*)$. We assume that firms are being created at the same rate as they diffuse across the liquidation threshold so as to keep the measure of firms constant. Our results assume further that new firms have cash holdings x^*, i.e. they are created at the dividend paying boundary, though we have investigated other rebirth assumptions which generate qualitatively similar results. Although individual firms' cash holdings vary between zero and x^* in a random manner, the distribution of firms settles down to what we shall refer to as the stable distribution on $(0,x^*)$. This can be thought of either as the cross-sectional distribution function for a large number of firms, or as describing probabilistically the evolution of a particular firm's position on $(0,x^*)$ over time.[18] The distribution can be solved for analytically via the Kolmogorov equations, at least in the simple zero interest rate cash management problem we consider here. The intuitive argument is that for each point $x\varepsilon(0,x^*)$ the stable distribution, g(x), is characterised by movements away from that point being exactly balanced by movements to the point. This allows one to write down a differential equation satisfied by the stable distribution, with boundary conditions g(0)=0 (the probability of a firm existing at the liquidation threshold is zero), and that the probabilities over $(0,x^*)$ sum to one. With rebirth at the boundary x^* this equation takes the form:

$$\mu(x)\frac{\partial g(x)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 g(x)}{\partial x^2} = 0 \tag{18}$$

which has exponential solutions.[19]

This steady state can be solved for using appropriate boundary conditions.[20] However we find it more convenient to calculate both the steady state distribution and out of steady state behaviour, using a numerical solution of the following discrete approximation to this continuous-time problem. We divide the d=0 region $(0,x^*)$ into discrete steps of size η, and consider cash holdings as evolving according to the following rule. Let the cash holdings at time t be x(t) (suppressing the i subscript)

$$\text{then} \quad x(t+dt) = \begin{cases} x(t) - \eta & \text{with probability } p = \tfrac{1}{2}(1-\mu dt/\eta) \\ x(t) + \eta & \text{with probability } 1-p = \tfrac{1}{2}(1+\mu dt/\eta) \end{cases}$$

---

[18] This assertion is an implication of the Glivenko-Cantelli lemma, see Billingsley (1986). For a similar application of the technique, see Caballero and Engle (1990).

[19] See Karlin and Taylor (1981) section 15.5 for a formal derivation of this equation.

[20] These are g(0)=0 and $\int_0^{x^*} g(x)dx = 1$.

then letting $\eta = \sigma\sqrt{dt}$ as $dt \to 0$ this discrete process converges to (17). At the boundaries the behaviour is governed by modified probabilities, firms at x^* remain at x^* with probability 1-p, and go to $x^*-\eta$ with probability p. Firms at 0 are liquidated with probability p (and return at x^*) or go to η with probability 1-p. We can thus divide the state space as $s=[0,\eta,2\eta,.....,x^*-\eta,x^*]$. Let the probability density for each firm's initial position in the state space be $g_0=[g_0(0),g_0(\eta),.....,g_0(x^*-\eta),g_0(x^*)]'$. Then the distribution of firms at time t f_t is given by the adjustment rule

$$g_t = Pg_{t-dt} \tag{19}$$

where P is the transition matrix (the same for all firms) given by:

$$P = \begin{bmatrix} 0 & p & 0 & 0 & . & . & . & . & 0 \\ 1-p & 0 & p & 0 & 0 & & & & \\ 0 & 1-p & 0 & p & 0 & & & & \\ . & . & . & . & . & . & 1-p & 0 & p \\ p & 0 & 0 & 0 & . & . & 0 & 1-p & 1-p \end{bmatrix}$$

The discrete approximation to the continuous stable distribution can then be derived by iterating forward. That is, starting from an arbitrary initial distribution we iterate to convergence (measured as when the change in components of g_t falls below some level). This approximates the steady state solution to (18).

We have chosen parameters values which we believe are of a realistic order of magnitude to undertake these calculations (and the same parameter values used to calculate figure 1). We have mean increment to cash per period of $\mu=0.14$, a variance of the brownian motion $\sigma^2=0.005$, and the discount rate $\rho=.05$. If cash holdings as regarded as being measured in units of \$100,000 and the time period as annual, then equation (1) describes a firm expecting to receive an income of \$14,000 per annum (think of this as after expenses profit flow), with a standard deviation of \$7070, so that the firm expects to lose money approximately one year in twenty. The firm's shareholders are discounting the future at 5% per annum. With these parameters the firm will hold cash balances of about 18000 before paying a dividend, so that in our discrete approximation η corresponds to about 600. Recall that the discrete approximation sets $\eta=\sigma\sqrt{dt}$ where dt will now measure in years, so that each "period" of the discrete approximation corresponds to $(\eta/\sigma)^2$ years or about .007 years, and 140 periods corresponds to about one year.

Figure 2 graphs the resulting steady state distribution for the model parameters listed in Figure 1. Although the distribution is heavily weighted towards to dividend paying boundary, the average value of V_x turns out to be 1.0386, compared with the perfect capital market value of 1. This implies an average premium on internal cash holdings of nearly 4 percent per annum. The average value of W (our measure of local risk aversion) is 1.226, again suggesting a considerable departure from the perfect capital markets value of 0.

We now turn to the behaviour of firms in the face of a single aggregate deterministic shock. We consider this as a shock coordinated across firms that moves all their cash holdings by the same amount, and then trace behaviour as the distribution of firms gradually returns to the stable distribution discussed above. In particular we look at the evolution of the average values of V_x, V_{xx} and the ratio V_{xx}/V_x (=-W). We implement the aggregate shock by starting from the stable distribution of firms and then increase all cash holdings by 2η, giving a new initial distribution of firms. The economy is then allowed to evolve according to the transition matrix P. The exercise was repeated for a reduction in cash holdings by 2η (note that firms pushed into liquidation by this act are replaced by new firms with cash holdings x^*). These shocks correspond either to all firms receiving an identical good or bad shock at a particular time (a shock to net worth) or to banks raising or lowering the liquidation threshold by the same amount for all firms (a credit shock).

Figures 3,4,5 and 6 graph the evolution of the average value of cash holdings, V_x, V_{xx} and W after the shock. Three points are worth noting from these charts. First the absolute size of the response of this economy is greater after a negative shock to net worth than after a positive shock to net worth. In part this is because a positive shock increases the number of firms at the dividend paying threshold, while a negative shock reduces the number of firms at the threshold. Thus, following a positive shock, there is a relatively effect on average dividend payout and a correspondingly smaller increase in average cash holdings, compared to the case of a negative shock. The extent of the assymmetry (the ratio of the absolute response to positive and negative shocks) is however greater for the other measures (V_x, V_{xx} and W) than for average cash holdings; this is because the second derivatives of each of these measures with respect to x are non-zero.

Second V_x, V_{xx} and W all exhibit a much more substantial response to the shock than does average cash holdings. We have chosen an aggregate shock which alters average cash holdings by around 5% of its steady state value. After the negative shock of this size the average premium on internal cash more than doubles to over 8 percent (V_x rises from 1.039 to 1.081), V_{xx} also more than doubles and local risk aversion (W) rises from 1.2 to 2.3.

Finally the response to the aggregate shock persists for a considerable period. An appropriate measure of persistence is the time taken for half the initial effect of the shock to be eradicated (the half-life of the shock).[21] For the shocks we have investigated the average

[21] We could have presented a formal analysis of the return to steady state values, by applying the theory of finite-dimensional markov chains (this theory is set out in standard texts on stochastic processes such as Karlin and Taylor (1975)). This establishes that the discrepancy between our econonomy wide averages and their steady state values are a sum of terms each of which decays exponentially, with rates of decay which are the eigen-values of the matrix P (other than the largest eigenvalue of 1). This means that the half-life of each econonomy wide average can depend on both the size and magnitude of the initial shock, and will alter over time (as the

value of cash balances takes around 20 periods (corresponding to some 1½ months in the interpretation we have placed on our chosen parameter values), while local risk aversion W takes takes around 40 periods (3½ months) and V_x and V_{xx} each take around 60 periods (5 months), to return to within half of their steady state values. 140 periods (one year) after the negative shock V_x has declined from it's initial value of 1.081 to 1.044 but is still some 0.05 above its steady state value of 1.038. Thus there is a premium on internal funds which is still some ½% greater than the steady state value of 3.8 %, a full year after the initial shock has taken place.

In this section we have traced the response to aggregate shocks in an economy consisting of firms facing stochastic income flows and the threat of liquidation, who manage cash optimally so as to maximise the returns to impatient shareholders. We find that relatively modest shocks to aggregate net worth lead to substantial responses in the economy wide averages both of the internal cost of capital (V_x) and of local risk aversion (W). The absolute size of response is greater for negative shocks to net worth. And the return to steady state is protracted.

The analysis of firm output and investment decisions offered in sections 3 and 5, indicate that both the internal cost of capital (V_x) and local risk aversion (W) have a direct impact on firm decision making; effectively any firm decision which involves an inter-temporal calculation of costs and benefits will exhibit a substantial and protracted response to an initial shock to net worth. We have not, in this paper, begun to analyse the general equilibrium response to such a shock, which would take account of the feedback from firm decision making on aggregate demand, but we believe that we have already done enough to

relative importance of different components of the decay evolve over time).

demonstrate that capital market constraints are a potentially powerful source of business cycle transmission.[22]

7. Conclusions

We analyse a model of firm behaviour, in which internal funds are determined by a diffusion process, and where the firm is subject to liquidation (and consequent loss of all future revenue) when internal funds cross an exogenously imposed threshold. We show that when the firm's objective is the maximisation of the expected present value of future dividends, and when the firm's shareholders are impatient, the firm must balance the conflicting objectives of maintaining cash inside the firm (to serve as a barrier against possible liquidation) and paying cash out. Linearity of the objective function means that the optimal policy is bang-bang control, with dividend payment for all levels of internal funds above some threshold x^*.

The value function satisfies a second order non-linear differential or partial differential equation, with boundary conditions determined by the liquidation and dividend payment thresholds. We provide an explicit solution for one simple example and prove that the value function takes a characteristic shape, with a first derivative V_x (the cost of internal funds) which is greater than unity for $x<x^*$ and monotonically decreases from $x=0$ until it reaches the value $V_x=1$ at $x=x^*$; and a second derivative V_{xx} which is negative and monotonically increases over the range $0<x<x^*$, until it reaches the boundary value $V_{xx}=0$ at x^*.

[22] Nor have we been able to take account of the effect of a decline in either internal net worth or of asset values on the credit worthiness of firms, which can provide a further amplification of business cycle disturbances (on which see Kiyotaki and Moore (1993)).

These qualitative results on the value function (which we show generalise to situations where the firm sets control variables other than dividends, and where there are state variables in addition to cash-in-hand that affect the value of the firm), imply that firm behaviour, subject to the threat of liquidation, becomes both locally more risk-averse and more impatient, as the level of cash-in-hand falls. We provide two specific examples of how this can affect firm behaviour. If the level of output affects the variance of returns, so exposing the firm to additional risk as output is increased, then shortage of internal funds increases risk aversion and leads to a reduction of output. If there are costs of adjusting the capital stock then an amended q-relationship arises, in which a shortage of internal funds effectively increases the firms rate of discount, lowers the value of q for any given level of capital stock and reduces the level of investment.

Finally we consider the implications our model has for aggregate behaviour. A relatively modest shock to the net worth of firms leads to a substantial and protracted deviation of economy wide averages, both of the internal cost of funds and of local risk aversion, from steady state values. We also find that the response to shocks is asymmetric with respect to the sign of the shock, with a greater response in absolute terms following a negative shocks to net worth. The size of these responses indicates that the effect of credit market imperfections on the dynamics of firm behaviour in a stochastic environment, provided they are realistically captured by our model of liquidation threat, provides a powerful mechanism of business cycle transmission.

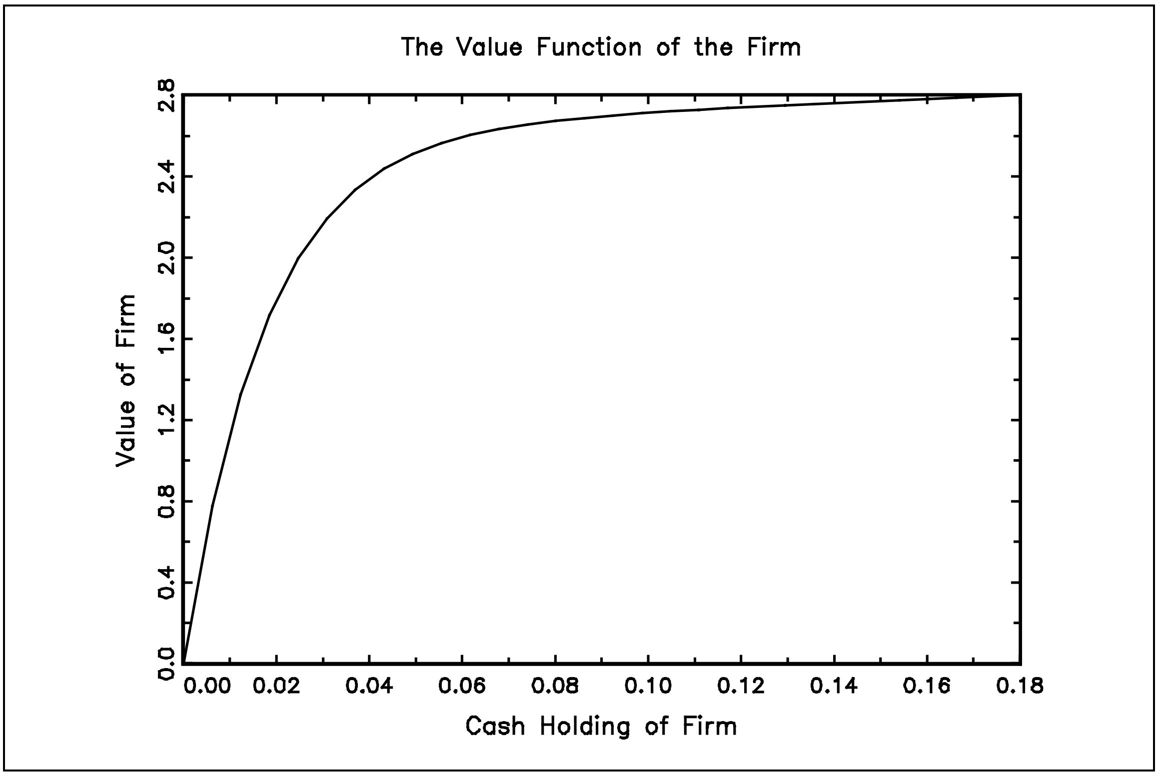


Figure 1

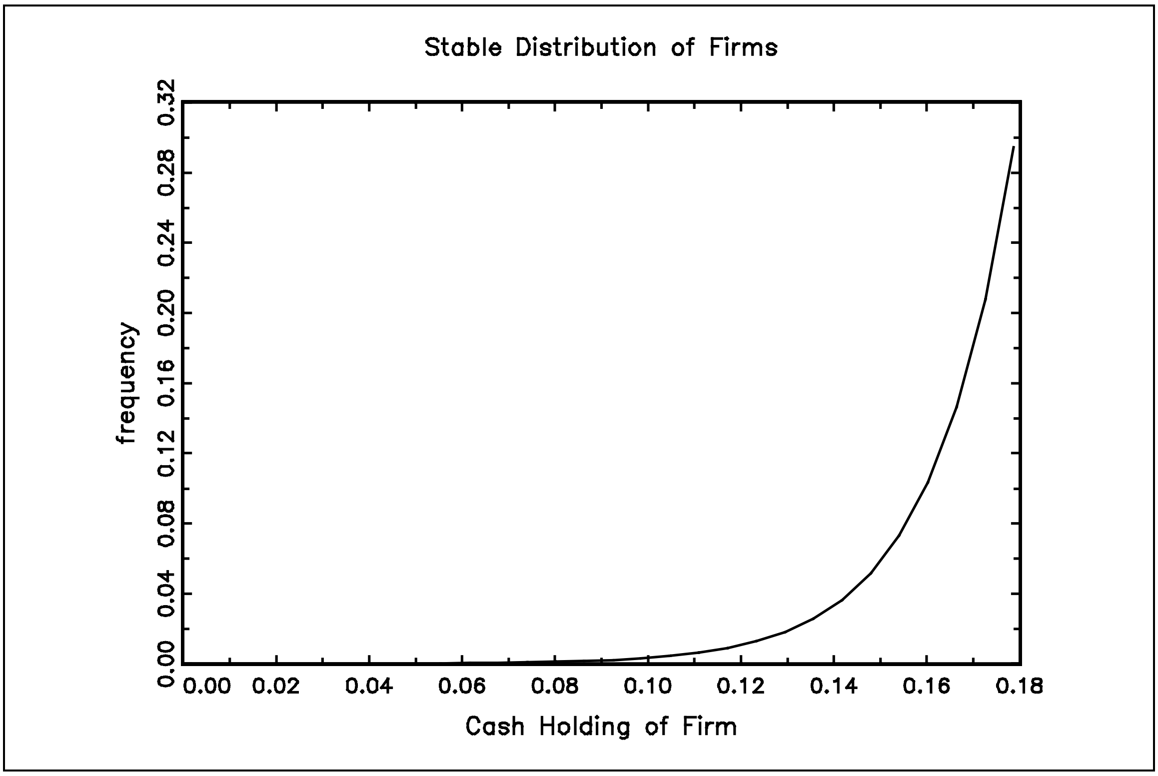


Figure 2

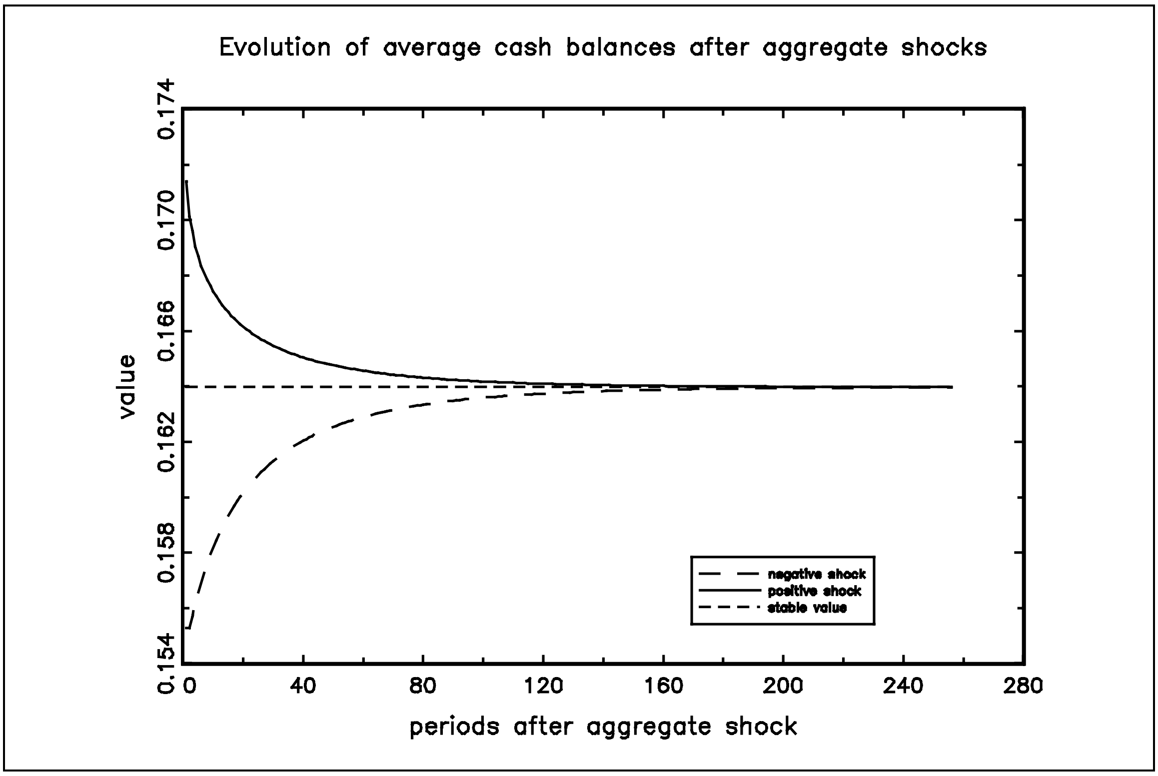


Figure 3

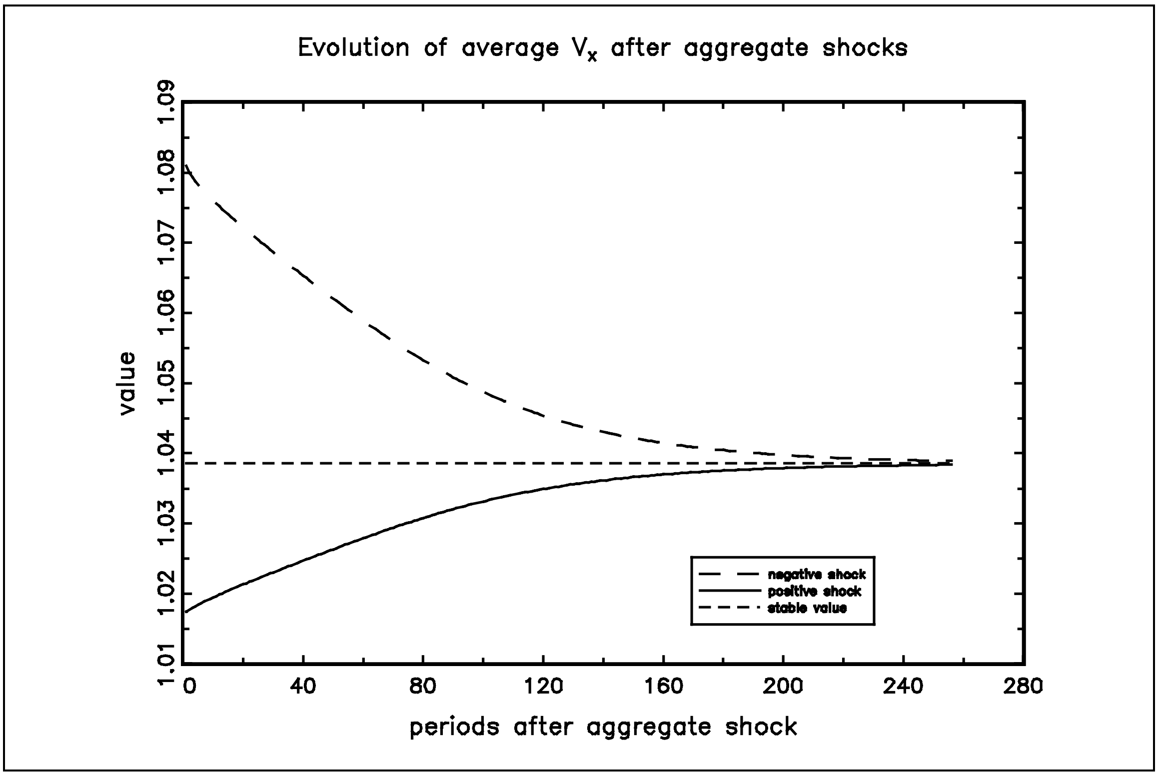


Figure 4

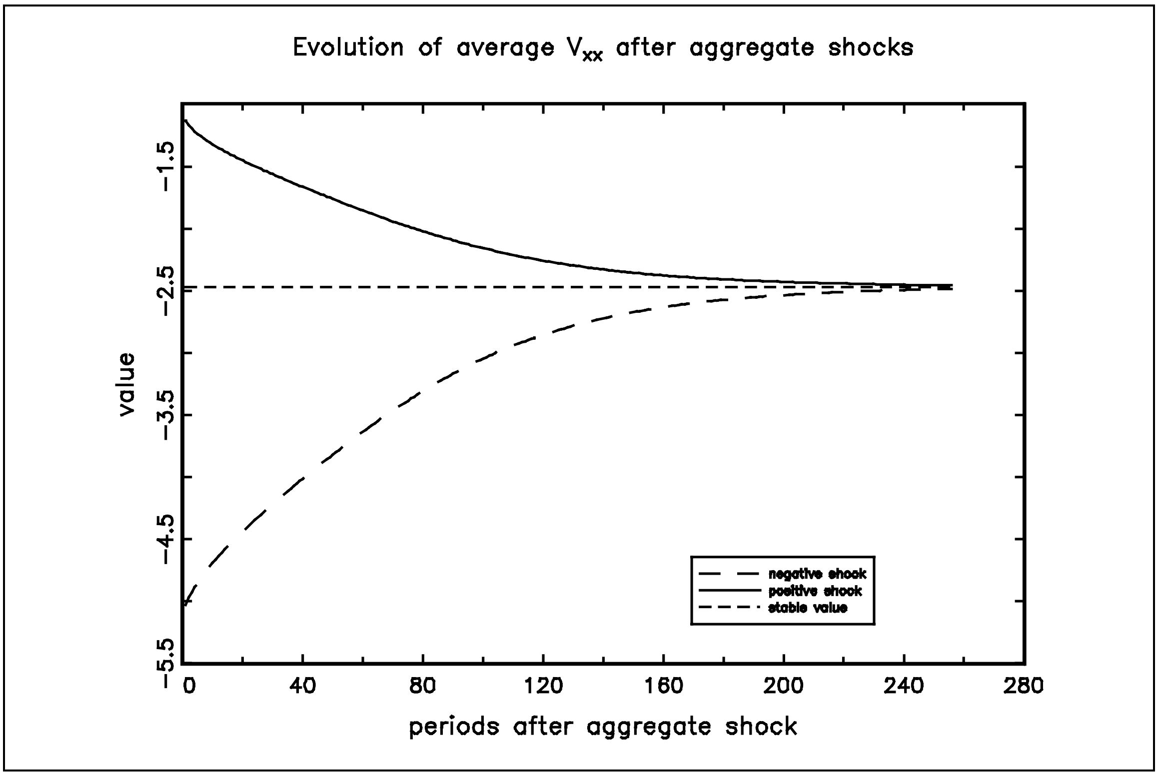


Figure 5

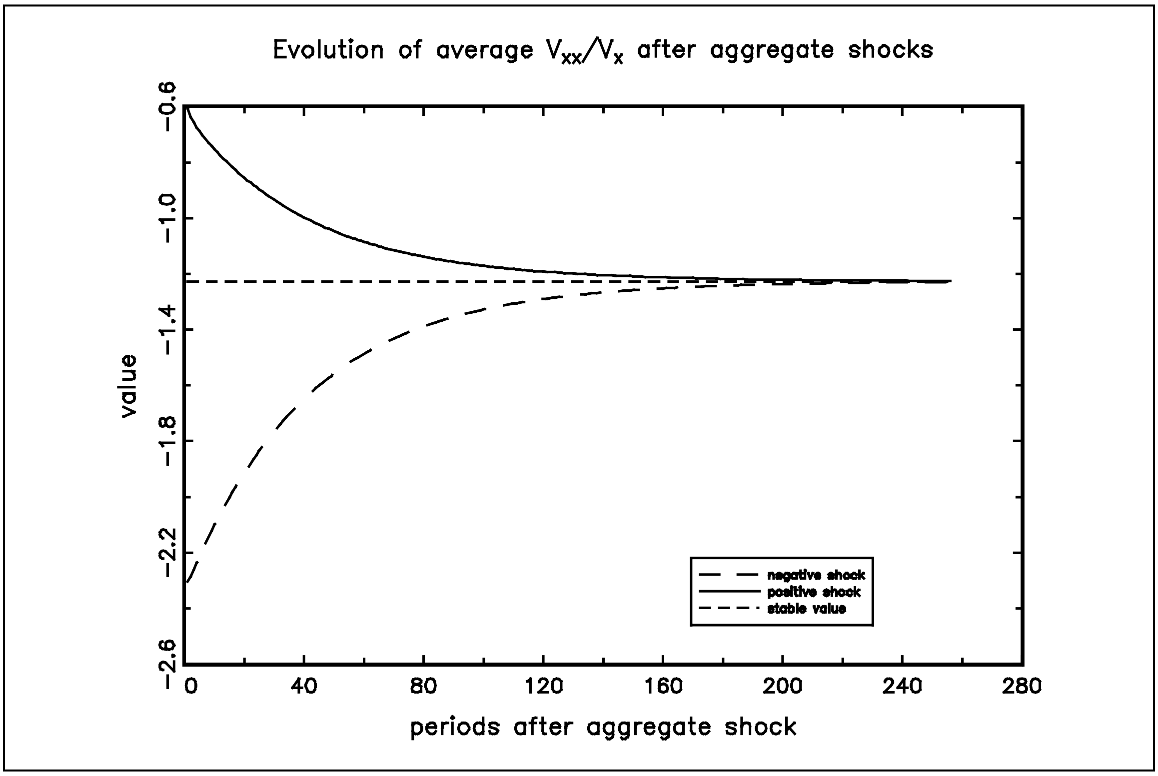


Figure 6

Notes to Figures 1-6

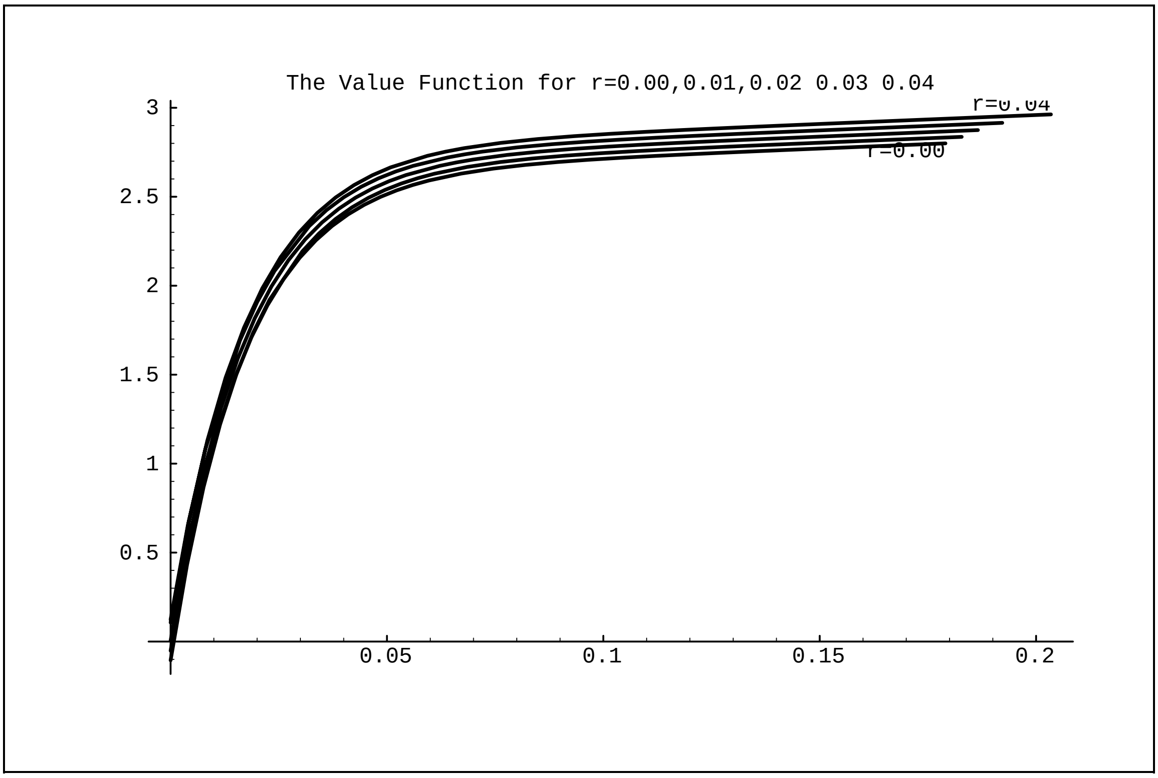
(i) All figure are based on the following underlying parameters for the diffusion process and the optimisation problem: μ=0.14, σ^2=0.005, ρ=.05, r=0. The liquidation threshold is at zero. These parameters imply a value of x^* of 0.17872.

(ii) The discrete markov process on $(0,x^*)$ has 30 steps.

Appendix

The numerical solution for non-zero r

We are unable to provide explicit solutions to the Hamilton-Jacobi-Bellman equations in the general case of non-zero r. In this appendix we present numerical solutions to the differential equation for such cases. The solutions for V(x) below show that as r varies between zero and ρ the general shape of V(x) does not change that much. In particular this means that the dynamic behaviour derived for the zero interest case after an aggregate shock provides a reasonable guide to the behaviour for non-zero r. The numerical solutions were computed by the program *Mathematica*. The nature of the boundary conditions in the HJB equation (4) is such that we know the values of V′(x) and V″(x) (and hence V(x)) at x^* but don't know x^*, it is identified by the further condition V(0)=0. We therefore use a repeated bisection search to discover x^* for each r. That is, nominate x^*, calculate the numerical solution V(x) and check the sign of V(0). If it is positive raise x^* and vice versa. Using this method we can identify the numerical solution to any desired degree of accuracy. The figure below graphs for each r the solution from zero to the implied x^* for that r.



Appendix

Throughout this appendix we shall assume such smoothness properties of the value function V(.) as required. In particular derivatives of sufficient order exist and are continuous except possibly at points where control regimes shift. Note that in the interior of control regions any order differentiability of the value function will follow by repeated differentiation of equation (8), so long as the forcing term has derivatives of the required order, and the controls (other than dividend payment) are unconstrained.

We first demonstrate sufficient conditions for Proposition 2. These are derived in a general situation in which there are further state variables, in addition to cash, that may or may not be controlled. We assume these other states are non-stochastic, although the condition under which Proposition (2) holds with stochastic states is similar to that derived below.

Suppose the state variables are cash x and a vector of other states s. The value function is then $V=V(x,s)$. Write the state equations as

$$ds = \mu_s dt$$
$$dx = (\mu_x - d)dt + \sigma dz$$

where μ_s may be controlled, but we assume it does not depend on x. The firm's problem is the maximisation of the expected dividend stream d (equation (2) of the text).

Theorem 1

Proposition (2) holds if

$$\mu_x V_{xx} + \mu_s V_{sx} \leq 0 \qquad (*)$$

NB This condition is sufficient, but not necessary. Necessary conditions will be problem specific, and in particular will depend on $\mathrm{sgn}(V_{sx})$ which cannot generally be allocated *a priori*.

Proof

The optimal control of the dividend process is bang-bang. In the no dividend paying region we have the state and co-state equations

$$\tfrac{1}{2}\sigma^2 V_{xx} + \mu_x V_x + \mu_s V_s - \rho V = 0 \qquad 0<x<x^*$$

and:

$$\tfrac{1}{2}\sigma^2 V_{xxx} + \mu_x V_{xx} + V_x\, d\mu_x/dx + \mu_s V_{sx} - \rho V_x = 0 \qquad 0<x<x^*$$

with boundary conditions as in the text.

We begin with the following:

Lemma 1 $\lim_{x\to x^*} V_{xxx}>0$.

Proof This follows from second-order conditions for the optimal choice of the dividend payment threshold x^*.

We now show by contradiction that, when (*) holds, the same inequality applies throughout the range $0<x<x^*$:

We have at x^*: $V_{xx}=0$ and $V_x=1$ (Note that these conditions *define* x^*)

Continuity of V_x and V_{xx} and the lemma imply

$$\exists \text{ region } \Re=(x^*-\delta,x^*),\ \delta>0 \text{ s.t. } V_{xxx}>0 \text{ in } \Re$$

$$\Rightarrow\quad V_{xx} \text{ increasing with x in } \Re, \text{ so } V_{xx}(x^*-\delta,s)<V_{xx}(x^*,s)=0 \text{ in } \Re$$

$$\Rightarrow\quad V_x \text{ decreasing with x, so } V_x(x^*-\delta,s)>V_x(x^*,s)=1 \text{ in } \Re.$$

Note that x^* will in general depend upon the state vector s, so that the region $\Re$ will be s-dependent. We now show that $\Re$ contains the interval $(0,x^*)$.

Suppose not, let $\tilde{x} = \sup\{ x\varepsilon(0,x^*) \text{ s.t. } V_{xxx}\leq 0 \}$ (the largest value of x^* not in $\Re$).

At $\tilde{x}$ we know $V_x>1$ and $V_{xx}<0$ because $V_{xxx}>0$ in $(\tilde{x},x^*)$ and continuity of the derivatives. as above. Substitute in the co-state equation

$$\tfrac{1}{2}\sigma^2 V_{xxx} = (\rho-d\mu_x/dx)V_x - \{\mu_x V_{xx}+\mu_s V_{sx}\}$$

Now for x^* to exist we have $(\rho-d\mu_x/dx)>0$. Otherwise cash is valued more highly in the firm than paid as dividends, and the optimal control policy is non-payment of dividends, and cash accumulates in the firm.

Given $\mu_x V_{xx}+\mu_s V_{sx}\leq 0$ then

$$\tfrac{1}{2}\sigma^2 V_{xxx}(\tilde{x},s) \geq (\rho-d\mu_x/dx)V_x(\tilde{x},s) > (\rho-d\mu_x/dx) > 0$$

which contradicts the definition of $\tilde{x}$. Thus $(0,x^*)\subset\Re$

The rest of Proposition (2) then follows immediately.

Corollary 1

Proposition (2) holds for the cash management problem of Section 2.

Proof

We have μ_x=rx+μ>0, $d\mu_x/dx<\rho$, $V_{sx}=0$ as there are no other states. Hence condition (*) is satisfied for the optimal cash management problem of Section 2.

Corollary 2

Proposition (2) holds for the optimal investment problem of Section 4, at least for a range of values of the capital stock described below.

Proof The states are cash x and capital k. We denote the expected rate of increase per unit time of x and k by μ_x and μ_k. These are given by:

$$\mu_x = rx - p_i i - \tfrac{1}{2}\theta i^2/k$$

and $$\mu_k = (i - \delta k)$$

x^* is defined by the relation $V_x(x^*,k)=1$ and $V_{xx}(x^*,k)=0$. These implicitily define a dividend payout boundary which is a function of the state k, $x^*=x^*(k)$.

At x^* $V_{xk}(x^*,k)=0$ (since a small perturbation of x has no effect on future dividend payments). Thus condition (*) of Theorem 1 holds on x^*.

Lemma 2 There $\exists$ a value of k, $\hat{k}$ at which $\mu_k=0$ on x^* (investment is at replacement level on the dividend paying threshold) and where $\mathcal{L}_k<0$ for all $k>\hat{k}$ and $x=x^*(k)$.

Proof From section 5 investment is given by $i/k=\theta^{-1}p_i\{V_k/(V_x p_i)-1\}$. On x^* $V_x=1$ and this becomes $i/k=\theta^{-1}p_i\{V_k/p_i-1\}$. For large k V_k is very small (a consequence of the declining marginal product of capital) and $i/k<\delta$ (implying $\mu_k<0$). At k=0 $\mu_k \geq 0$. V_k is a continuous function of x,k.

Moreover, V_k on x^* falls as k increases, since the effect of any change in k along x^* on V_k can be written:

$$dV_k = \{V_{kk} + V_{kx} \tfrac{\partial x}{\partial k}|_{x^*}\} \; dk = V_{kk} \, dk$$

and the term in V_{kx} equals 0.

Hence there is a unique value of k ($\hat{k}$) for which $\mu_k=0$ on x^*, and for which $\mu_k<0$ for greater values of k.

We interpret $\hat{k}$ as the stable value of k on the x^* boundary (it is not a steady state since, for the reasons advanced in section 6, the steady state of this model corresponds to a probability distribution rather than a single value of k).

<u>Lemma 3</u> For $x<x^*$ $V_{xk}>0$. This can be shown by considering a small +ve perturbation to the stock of cash. For a sub-set of the space of sample paths this perturbation is sufficient to avoid liquidation, increasing the (marginal) value of capital. For all other sample paths the marginal value of capital is unaffected. Hence $V_{xk}>0$.

We now show that proposition 2 applies for all values of $k \geq \hat{k}$.

<u>Proof</u> The proof proceeds by establishing (*) for all values of $x<x^*(k)$, $k \geq \hat{k}$. As already shown (*) applies on x^*. Morevover on x^* for $k \geq \hat{k}$ $\mathbb{I}k <0$ and $\mu x>0$. The proof now applies a similar contradiction argument to the proof of proposition 2 itself.

Continuity of V_x, V_{xx}, V_k and V_{kx} imply that for each $k \geq \hat{k}$

$$\exists \text{ region } \mathfrak{R}=(x^*-\delta,x^*),\ \delta>0 \ \text{ s.t. } \mu_k<0,\ \mu_x>0 \text{ and hence (*) applies in } \mathfrak{R}$$

$$\Rightarrow \quad V_{xx} \text{ decreasing with x in } \mathfrak{R}, \ \text{ so (by lemma 3) } q_x > 0 \text{ in } \mathfrak{R}$$

$$\Rightarrow \quad \text{i increasing with x in } \mathfrak{R}.$$

We now show that $\mathfrak{R}$ contains the interval $(0,x^*)$.

Suppose not, let $\tilde{x} = \sup\{ x \varepsilon (0,x^*) \text{ s.t. } \mu_k=0 \}$ (the largest value of x^* not in $\mathfrak{R}$).

At $\tilde{x}$ we know $\mu_x>0$ (since the reduction of μ_k is reflected in an increase in μ_x for $x<x^*$).

Hence (*) still applies at $\tilde{x}$, $V_{xx}<0$ and $q_x>0$ and μ_{kx} declines with x in $(\tilde{x},x^*)$. But this contradicts our assumption that $\mu_k=0$ at .

We have therefore shown that for all values of $k \geq \tilde{x}$, q and hence investment increases with the net cash holdings of the firm.[23]

[23] If it is assumed that the liquidation threshold is independent of the stock of capital k, then an alternative proof may be used to establish that $q_x>0$ for all values of x.

Theorem 2

For the model of section 3, $W=-V_{xx}/V_x$ is monotone decreasing.

Proof

We allow for arbitrary control variables affecting μ and σ but no state variables other than cash x.

Let $W=-V_{xx}/V_x$. This implies

$$W_x = \{-V_{xxx}V_x + (V_{xx})^2\}/(V_x)^2 = W\, V_{xxx}/V_{xx} + W^2 \qquad (A)$$

From rearrangement of the state equation and co-state equation we obtain, respectively:

$$W = (\tfrac{1}{2}\sigma^2)^{-1}\{-\rho V/V_x+\mu\} \qquad (B)$$

$$V_{xxx}/V_{xx} = (\tfrac{1}{2}\sigma^2)^{-1}\{(\rho-r)V_x/V_{xx}-\mu\} \qquad (C)$$

Summing (B) and (C) and substituting in (A) we get:

$$W_x/W = (\tfrac{1}{2}\sigma^2)^{-1}\{(\rho-r)V_x/V_{xx}-\rho V/V_x\} < 0 \qquad (D)$$

The last inequality is a consequence of Theorem 1, which implies that $V_x/V_{xx}<0$ and $V/V_x<0$. The result follows on noting that $W>0$.

Theorem 2 can be extended to the case where there is a vector of state variables. In this case (B), (C) and (D) become

$$W = (\tfrac{1}{2}\sigma^2)^{-1}\{-\rho V/V_x+\mu+\mu_s V_s/V_x\} \qquad (B)$$

$$V_{xxx}/V_{xx} = (\tfrac{1}{2}\sigma^2)^{-1}\{(\rho-r)V_x/V_{xx}-\mu-\mu_s V_{sx}/V_{xx}\} \qquad (C)$$

$$W_x/W = (\tfrac{1}{2}\sigma^2)^{-1}\{(\rho-r)V_x/V_{xx}-\rho V/V_x+\mu_s[V_s/V_x-V_{sx}/V_{xx}]\} \qquad (D)$$

where the other state variables obey

$$ds = \mu_s\, dt$$

with μ_s (possibly) a controlled vector, where if controlled it may be set optimally (this ensures envelope properties apply). Then W declines monotonically if $\mu_s[V_s/V_x-V_{sx}/V_{xx}]<0$. This holds under the conditions of Theorem 1 if V_s and V_{sx} are both positive in regions where the state variables s are being decumulated. Note $V_s>0$ if the value of the firm is increased by additions to the states, i.e. they are valuable. Note $V_{sx}>0$ if at higher values of the other state variables the insurance value of cash held internally rises.

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