A WEIGHTED POSITION VALUE

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Abstract

We provide a generalization of the position value (Meessen 1988) that allows players to benefit from transfers of worth by investing in their communication links. The player who invests the most in a communication link obtains a compensation from the second one. We characterize this new allocation rule on the class of communication situations with cycle-free graphs by means of six axioms. The first two axioms, component efficiency and superfluous link property, are used to characterize the position value (Borm, Owen, and Tijs (1992)). Quasi-additivity is a weak version of the standard additivity axiom. Link decomposability captures the fact that the insurance system only allows compensations between players who share a link. Weak positivity states that if the communicative strength of a link is non null, its adjacent players cannot obtain a null payoff. Finally, weak power inversion reflects the compensation mechanism.

Keywords: Weighted position value; Monotonicity

1 Introduction

Many economic or social projects are carried out by groups of agents, called players in the sequel, who cooperate to achieve a common goal. These situations can be appropriately formalized via cooperative games with transferable utility, or TU games. A TU game summarizes the worth produced by each coalition when its players agree to cooperate. It is assumed that every coalition of players can form.

Oftentimes, the coordination of activities between these players takes place through communication networks, which restrict the possibility of coalitions to form. Myerson (1977) suggests to use undirected graphs to model

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such networks. He introduces communication situations which combine TU games and undirected graphs. Vertices of an undirected graph represent the players and edges represent the bilateral communication links between players. In order to measure the impact of restrictions on communication on the worth produced by coalitions, Myerson (1977) suggests to associate to each communication situation a graph-restricted TU game. This game provides an assessment of the worths from cooperation that are obtainable by coalitions in the face of restricted communication possibilities. Then the author defines a set of attractive properties on the class of communication situations that suffices to determine a unique allocation rule, the so-called Myerson value. Myerson (1980), Borm, Owen, and Tijs (1992) and Slikker and van den Nouweland (2001) provide various characterizations of the Myerson value that are valid on different classes of communication situations.

Meessen (1988) introduces an alternative associated TU game that highlights the role of links in the production of worth. In this TU game, called link game, the set of players is the set of links. The worth associated with a set of links is the worth obtainable by the grand coalition when only this set of links is available. A link game measures the communicative strength of each subgraph. To compute the position value of a communication situation, one first determines the Shapley value of each link in the link game. These values can be seen as the average communicative strength of the links. Then, the Shapley value of each link is equally divided between its two incident players. The total amount that a player obtains in that way is his position value. Borm, Owen, and Tijs (1992) and Slikker (2005) provide characterizations of this allocation rule.

In this article, we generalize the position value in order to allow players to benefit from transfers of worth by investing in communication links. The idea behind the position value is that the worth produced by the cooperation between players is due to the presence of communication links. If there is no link, players cannot coordinate their actions and then cannot cooperate nor produce worth. Thus one can argue that the player who invests the most in a communication link should benefit from an insurance system that allows him to obtain transfers of worth from the other player.

We suggest to model the level of investment of players in communication links through a weight scheme that is in the same spirit as the one used by Haeringer (2006) to generalize the Shapley value. This weight scheme permits to define a compensation mechanism in which players can obtain different shares of the Shapley value of a link, depending on the sign of this Shapley value: if the Shapley value of a link is positive, the higher the level of investment of a player is, the higher his share of this Shapley value. On the contrary, if the Shapley value of a link is negative, the higher the level of investment of a player is, the lower his share of this Shapley value. We characterize this allocation rule using six axioms. Following in Kalai and Samet (1987), Chun (1991) and Haeringer's (2006) footsteps, the weights do not appear explicitly in the axioms. Superfluous link property states that if the presence or absence of a link in a communication situation does not change the worth of the grand coalition, then the removal of this link does not change the payoffs of the players. Component efficiency states that the payoffs received by the players of a component add up to the worth of this component. These two axioms are satisfied by the Myerson value and the position value. Quasi-additivity is a weak version of additivity. Link decomposability captures the fact that the insurance system only allows compensations between players who share a link. Weak positivity states that if the average communicative strength of a link is non null, its adjacent players cannot obtain a null payoff. Finally, weak power inversion reflects the compensation mechanism: it states that the relative share a player obtains of the Shapley value of a link can depend on the sign of this Shapley value. We show that the combination of these six axioms determines the weighted position value uniquely on the class of communication situations such that the game is zero normalized and the graph is cycle-free, a class considered by Borm, Owen, and Tijs (1992).

This article pursues the literature on weighted values initiated by Shapley (1953b), who generalizes the Shapley value in order to take into account information that is external to the game, like bargaining abilities or levels of effort. This external information is modelled through weights. Kalai and Samet (1987) extend this weighted Shapley value enabling weights to be equal to zero for some players. Chun (1991) provides alternative characterizations of this allocation rule. Owen (1968) shows that the weight systems used by Shapley (1953b) and Kalai and Samet (1987) measures the slowness of players to reach the grand coalition rather than their bargaining abilities or their levels of effort. Then Haeringer (2006) suggests an alternative way to define weights so that they can be interpreted as a measure of power. He obtains a weighted Shapley value that is increasing with the weights of players.

Haeringer (1999) and Slikker and van den Nouweland (2000) generalize the weighted Shapley value defined by Kalai and Samet (1987) to communication situations and to hierarchical structures respectively. Kamijo and Kongo (2009) extend the position value in order to take into account two different sources of asymmetry: asymmetries among links and among players. Asymmetries among links are obtained by applying the weighted Shapley value of Shapley (1953b) to the link game. Asymmetries among players are obtained by dividing unequally the Shapley value of a link between its two incident players. Unlike our weighted position value, all these asymmetric extensions of allocation rules to TU games with a communication structure cannot be seen as an insurance system: the payoffs of the players are not increasing with respect to their weights.

This article is organized as follows. In section 2, we introduce the definitions and notations. In section 3, we define and characterize our weighted position value.

2 Preliminaries

2.1 TU games

Let $N = \{1, \ldots, n\}$ be a finite set of players. Denote by 2^N the set of all subsets of N. A coalition S is an element of 2^N of which players cooperate to achieve a common goal. For a coalition $S \in 2^N$, $|S|$ denotes its cardinal. A TU game is a pair (N, v) consisting of player set N and a characteristic function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, that associates to every coalition $S \subseteq N$ the worth its players create by agreeing to cooperate. A game (N, v) is zero normalized if $v(\{i\}) = 0$ for each $i \in N$. A carrier of a game (N, v) is a coalition $R \in 2^N \setminus \{\emptyset\}$ such that for each $S \in 2^N \setminus \{\emptyset\}$, $v(S) = v(S \cap R)$. Consider (N, v) and $S \in 2^N$. The subgame $(S, v_{|S})$ of (N, v) on any nonempty $S \subseteq N$ is given by $v_{|S}(T) = v(T)$ for each $T \subseteq S$. An allocation rule on a class of TU games is a function Y that assigns a payoff vector $Y(N, v) \in \mathbb{R}^N$ to every TU game in that class.

For each nonempty $S \subseteq N$, the unanimity game (N, u_S) is defined as $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. Every characteristic function v can be written as a unique linear combination of unanimity games in the following way:

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S^v u_S,$$

where for each $S \in 2^N \setminus \{\emptyset\}$, the unanimity coefficients α_S^v are given by:

$$\alpha_S^v = \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} (-1)^{|S|-|T|} v(T).$$

The Shapley value (Shapley 1953a) Sh is the allocation rule on the class of all TU games given by:

$$Sh_i(N, v) = \sum_{\substack{S \in 2^N \\ S \ni i}} \frac{\alpha_S^v}{|S|}$$

for each (N, v) and each $i \in N$.

We now state five properties, satisfied by the Shapley value, that will be useful to prove the main result of this article. The four first properties are provided by Shubik (1962) to characterize the Shapley value.

Efficiency requires that the payoffs of the players add up to the worth of the grand coalition.

Efficiency: an allocation rule Y on a class of TU games is efficient if $\sum_{i \in N} Y_i(N, v) = v(N)$ for each TU game (N, v) in that class.

Symmetry requires that symmetric players obtain the same payoff. Formally, two players i and j of N are symmetric in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for each $S \subseteq N \setminus \{i, j\}$.

Symmetry: an allocation rule Y on a class of TU games is symmetric if for each TU game (N, v) in that class and for any two symmetric players $i, j \in N$, $Y_i(N, v) = Y_j(N, v)$.

Null player property requires that null players, i.e. players whose presence or absence does not change the worth of any coalition, obtain a payoff equal to zero. Formally, a player $i \in N$ is null in (N, v) if $v(S \cup \{i\}) = v(S)$ for each $S \subseteq N \setminus \{i\}$.

Null player property: an allocation rule Y on a class of TU games satisfies the null player property if for each TU game (N, v) in that class, $Y_i(N, v) = 0$ if $i \in N$ is a null player.

Additivity requires that the allocation rule is an additive operator on the class of games on which it is defined.

Additivity: an allocation rule Y on a class of TU games is additive if for any two TU games (N, v) and (N, w) in that class, it holds that $(N, v + w)$, where $(v + w)(S) = v(S) + w(S)$ for each $S \subseteq N$, is in that class and $Y(N, v + w) = Y(N, v) + Y(N, w)$.

The null player out property is provided by Derks and Haller (1999). It requires that if a game admits a null player, the payoffs of the game resulting from the deletion of the null player are the same as the payoffs of the original game for the remaining players.

Null player out property: an allocation rule Y on a class of TU games satisfies the null player out property if for each TU game in that class and each null player $j \in N$, $(N \setminus \{j\}, v_{|N\setminus\{j\}})$ is in that class and $Y_i(N, v) = Y_i(N \setminus \{j\}, v_{|N\setminus\{j\}})$ for each $i \in N$, $i \neq j$.

Theorem 1 *(Shubik, 1962)*
The Shapley value is the unique allocation rule on the class of all TU games satisfying efficiency, symmetry, additivity and null player property.

Proposition 1 *(Derks and Haller, 1999)*
The Shapley value satisfies the null player out property on the class of all TU games.

2.2 Communication situations

A comunication graph is a pair (N, L) where the set of vertices N is the set of players and edges of $L \subseteq L^N = \{\{i, j\} \,|\, i, j \in N, i \neq j\}$ represent bilateral communication links. A sequence of k different vertices $(i_1, \ldots, i_k)$ is a path in (N, L) if $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k-1$. A cycle is a sequence of vertices $(i_1, \ldots, i_{k+1})$, $k \geq 3$, such that $(i_1, \ldots, i_k)$ is a path, $\{i_k, i_{k+1}\} \in L$ and $i_{k+1} = i_1$.

Two vertices $i, j \in N$ are connected in (N, L) if $i = j$ or there exists a path $(i_1, \ldots, i_k)$ with $i_1 = i$ and $i_k = j$. For any $S \subseteq N$, $(S, L(S))$ denotes the subgraph of (N, L) induced by S, where $L(S) = \{\{i, j\} \in L \,|\, i, j \in S\}$. For each $S \subseteq N$, $(S, L(S))$ is connected if any two vertices $i, j \in S$ are connected. A coalition $S \subseteq N$ is a connected component in (N, L) if $(S, L(S))$ is connected and for each $i \in N \setminus S$, $(S \cup \{i\}, L(S \cup \{i\}))$ is not connected. Note that for each graph (N, L), the set of connected components, denoted by N/L, partitions the set of players N in a unique way. A tree is a cycle-free graph such that $|N/L| = 1$. For each $L \subseteq L^N$ and $i \in N$, let $L_i = \{\{i, j\} \,|\, j \in N \text{ and } \{i, j\} \in L\}$ be the set of player i's links in (N, L). For each $A \subseteq L$, $N(A) = \{i \in N \,|\, \exists\, j \in N : \{i, j\} \in A\}$ is the set of players of N who have a link in A.

A communication situation is a triple (N, v, L) where (N, v) is a TU game and (N, L) is a communication graph. For the remainder of this article, we restrict ourselves to communication situations with a fixed player set N and a zero normalized TU game. The class of communication situations such that the player set is N, the game is zero normalized and the graph is cycle-free is denoted by $\mathcal{CS}^N$.

In order to assess the impact of restrictions on communication on the worth created by coalitions, Meessen (1988) suggests to associate to each communication situation $(N,\, v,\, L)$ a link game $(L,\, r^v)$ defined as:

$$r^v(A) = \sum_{C \in N/A} v(C)$$

for each $A \subseteq L$. The link game associated with $(N,\, v,\, L)$ is a TU game in which the set of players is the set of links in $(N,\, L)$. The worth of a set of links $A \subseteq L$ is the worth obtainable by the grand coalition if only links in A are available. As the grand coalition partitions in connected components, the worth obtainable by N is the sum of the worths obtainable by the connected components of N/A. Note that as $(N,\, v)$ is zero normalized, $r^v(\emptyset) = 0$.

Let $(N,\, L)$ be a cycle-free graph. The connected hull of a coalition $S \subseteq N$, provided by Borm, Owen, and Tijs (1992), is defined as $H(S) = \cap\{T \subseteq N \,|\, S \subseteq T \text{ and } T \text{ is connected}\}$. As the graph is cycle-free, the connected hull of a coalition $S \subseteq N$ consists of the players whose cooperation is both necessary and sufficient to enable the players in S to communicate. If $S \subseteq C \in N/L$, as $(C,\, L(C))$ is a tree, then $H(S)$ is connected. Moreover, if S is connected, $H(S) = S$. If $S \not\subseteq C$ for each $C \in N/L$, then $H(S) = \emptyset$. For each $A \subseteq L$, let $\Delta(A) = \{S \subseteq N \,|\, S \subseteq C \in N/L,\, A = L(H(S))\}$ be the set of coalitions of which the connected hull is $N(A)$. Note that $A = L(H(S))$ if and only if $N(A) = H(S)$.

The following lemma, provided by Borm, Owen, and Tijs (1992), states the relation between the unanimity coefficients of the link game and the unanimity coefficients of the underlying coalitional game.

Lemma 1 *(Borm, Owen, and Tijs 1992)*
For each $(N,\, v,\, L) \in \mathcal{CS}^N$ and $A \subseteq L$,

$$\alpha_A^{r^v} = \begin{cases} \displaystyle\sum_{\substack{S \in 2^N \setminus \{\emptyset\} \\ S \in \Delta(A)}} \alpha_S^v & \text{if } N(A) \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

An allocation rule on a class of communication situations is a function Y that assigns a payoff vector $Y(N,\, v,\, L) \in \mathbb{R}^N$ to each communication situation in that class. The position value P is the allocation rule for the class of zero-normalized communication situations defined as:

$$P_i(N,\, v,\, L) = \sum_{l \in L_i} \frac{1}{2} Sh_l(L,\, r^v)$$

for each zero normalized $(N,\, v,\, L)$ and each $i \in N$.

Now consider the following example.

Example 1
Let $(N,\, v,\, L)$ be a communication situation such that $N = \{1,\, 2,\, 3\}$,

$$v(S) = \begin{cases} -20 & \text{if } S = \{1,\, 2\}, \\ 40 & \text{if } S = \{2,\, 3\}, \\ 40 & \text{if } S = \{1,\, 2,\, 3\}, \\ 0 & \text{otherwise,} \end{cases}$$

and $L = \{\{1,\, 2\},\, \{2,\, 3\}\}$. The Shapley value of $(L,\, r^v)$ is given by $Sh_{\{1,\, 2\}}(L,\, r^v) = -10$, $Sh_{\{2,\, 3\}}(L,\, r^v) = 50$, and the position value of $(N,\, v,\, L)$ equals $P(N,\, v,\, L) = (-5,\, 20,\, 25)$. □

In this example, players 1 and 2 suffer equally from the low-achieving of coalition $\{1,\, 2\}$. Now, suppose that one player, for instance player 2, invests more in the creation or the maintaining of link $\{1,\, 2\}$ than player 1. One can argue that player 2 should be protected against loss, to a certain extent, and should benefit from a transfer of worth from player 1. An allocation rule encompassing this mechanism is described in the following section.

3 The value

We provide a generalization of the position value that is in the same spirit as the weighted Shapley value defined by Haeringer (2006). The levels of investment of players in their links are formalized through the set of weights $\lambda^{L,\,+} = \{\lambda^{L,\,+}_{i,\,\{i,\,j\}} \in \mathbb{R}_{++} \,|\, i \in N(L) \text{ and } \{i,\, j\} \in L_i\}$. The element $\lambda^{L,\,+}_{i,\,\{i,\,j\}}$ can reflect the level of investment realised by player i in link $\{i,\, j\}$. From $\lambda^{L,\,+}$, we define $\lambda^{L,\,-} = \{\lambda^{L,\,-}_{i,\,\{i,\,j\}} \in \mathbb{R}_{++} \,|\, \lambda^{L,\,-}_{i,\,\{i,\,j\}} = 1/\lambda^{L,\,+}_{i,\,\{i,\,j\}}\}$. We will use $\lambda^{L,\,+}$ to share the Shapley values of links that are positive and $\lambda^{L,\,-}$ to share the Shapley values of links that are negative between their incident players. To lighten notation, we omit L and write, for example, λ^{+} and $\lambda^{+}_{i,\,\{i,\,j\}}$.

The position value of a communication situation $(N,\, v,\, L)$ weighted by λ^+, denoted by P^{λ^+}, is defined as:

$$P_i^{\lambda^+}(N,\, v,\, L) = \sum_{\{i,\,j\} \in L_i} \frac{\bar{\lambda}_{i,\,\{i,\,j\}}}{\bar{\lambda}_{i,\,\{i,\,j\}} + \bar{\lambda}_{j,\,\{i,\,j\}}} Sh_{\{i,\,j\}}(L, r^v) \tag{2}$$

for each $i \in N$, where $\bar{\lambda}_{i,\,\{i,\,j\}} = \lambda^{+}_{i,\,\{i,\,j\}}$ if $Sh_{\{i,\,j\}}(L, r^v) \geq 0$ and $\bar{\lambda}_{i,\,\{i,\,j\}} = \lambda^{-}_{i,\,\{i,\,j\}}$ if $Sh_{\{i,\,j\}}(L, r^v) < 0$. In this allocation rule, the share of each player is determined according to the sign of the Shapley value and the relative weights of involved players. If the Shapley value of a link is positive, the player who invests the most to maintain a link obtains a higher part of its Shapley value than the other player. On the contrary, if the Shapley value of a link is negative, the player who invests the most to maintain the link obtains a lower part of its Shapley value than the other player. Thus the relative level of investment of a player determines his level of protection against loss. This weighted position value is a generalisation of the position value. To see this, note that if $\lambda^+_{i,\{i,j\}} = a$, $a \in \mathbb{R}_{++}$, for each $i \in N$ and each $\{i, j\} \in L_i$, then $P^{\lambda^+}(N, v, L) = P(N, v, L)$.

The following example highlights the transfer of worth induced by the weighted position value.

Example 2
Consider the communication situation (N, v, L) defined in Example 1, and suppose that $\lambda^+ = \{1, 9, 1, 1\}$, i.e. players 1 and 2 invest up to 1 and 9 in link $\{1, 2\}$ respectively, and players 2 and 3 both invest up to 1. As the Shapley value of link $\{1, 2\}$ is negative, we use $\lambda^-_{1,\{1,2\}} = 1$ and $\lambda^-_{2,\{1,2\}} = 1/9$ to share it between players 1 and 2. As the level of investment of players 2 and 3 in the link $\{2, 3\}$ is the same, the Shapley value of link $\{2, 3\}$ is shared equally between its two incident players. We obtain $P^{\lambda^+}(N, v, L) = (-9, 24, 25)$. Then player 2 obtains a transfer of an amount of 4 from player 1. □

Now, we introduce a set of axioms used to characterize the weighted position value on $\mathcal{CS}^N$. Component efficiency, defined by Myerson (1977), is a standard axiom. It is satisfied by the Myerson value and the position value. It requires that the payoffs of the players of a component add up to the worth of this component.

Component efficiency: an allocation rule Y on $\mathcal{CS}^N$ is component efficient if for each $(N, v, L) \in \mathcal{CS}^N$ and each connected component $C \in N/L$,

$$\sum_{i \in C} Y_i(N, v, L) = v(C).$$

The superfluous link property is defined by Borm, Owen, and Tijs (1992) to characterize the position value. A link $\{i, j\} \in L$ is superfluous in a communication situation (N, v, L) if its presence or absence does not change the worth obtainable by the grand coalition: $r^v(A) = r^v(A \setminus \{i, j\})$ for all $A \subseteq L$. In other words, $\{i, j\}$ is null in r^v. The superfluous link property requires that the removal of a superfluous link does not change the payoffs of the players.

Superfluous link property: an allocation rule Y on $\mathcal{CS}^N$ satisfies the superfluous link property if for each communication situation $(N, v, L) \in \mathcal{CS}^N$ and each superfluous link $\{i, j\} \in L$, it holds that:

$$Y(N, v, L) = Y(N, v, L \setminus \{i, j\}).$$

The third axiom is based on the link unanimity property provided by van den Brink, van der Laan, and Pruzhansky (2007). A communication situation is link unanimous if $r^v = \left[\sum_{C \in N/L} v(C)\right] u_L$. This means that the grand coalition produces a value of zero if some links of L are not available, i.e. all the links are veto players in (L, r^v). Link decomposability captures the fact that when the links of a communication situation are veto players in (L, r^v), compensations are only allowed between players who share a link.

Link decomposability: an allocation rule Y on $\mathcal{CS}^N$ is link decomposable if for each link unanimous communication situation $(N, v, L) \in \mathcal{CS}^N$ there exists $c \in \mathbb{R}$ such that for each $i \in N$:

$$Y_i(N, v, L) = \begin{cases} c \displaystyle\sum_{\{i,j\} \in L_i} Y_i(N, u_{\{i,j\}}, \{\{i, j\}\}) & \text{if } \sum_{C \in N/L} v(C) \geq 0, \\ c \displaystyle\sum_{\{i,j\} \in L_i} Y_i(N, -u_{\{i,j\}}, \{\{i, j\}\}) & \text{if } \sum_{C \in N/L} v(C) < 0. \end{cases}$$

Note that $(N, u_{\{i,j\}}, \{\{i, j\}\})$ and $(N, -u_{\{i,j\}}, \{\{i, j\}\})$ are in $\mathcal{CS}^N$, and $(N, u_{\{i,j\}})$ and $(N, u_{\{i,j\}})$ are zero-normalized for each pair $i, j \in N$.

The fourth axiom is a weak version of the standard additivity property that relies on the following definition. Two communication situations $(N, v, L) \in \mathcal{CS}^N$ and $(N, w, L) \in \mathcal{CS}^N$ are comparable if $\alpha_A^{r^v} \alpha_A^{r^w} \geq 0$ for each $A \subseteq L$, i.e. if for each set of links, the unanimity coefficients of the link games have the same sign.

Quasi-additivity: an allocation rule Y on $\mathcal{CS}^N$ is quasi-additive if for each comparable pair $(N, v, L), (N, w, L) \in \mathcal{CS}^N$,

$$Y(N, v + w, L) = Y(N, v, L) + Y(N, w, L).$$

Weak positivity states that if the communicative strength of a link is different from zero, it's incident players cannot obtain a null payoff.

Weak positivity: an allocation rule Y on $\mathcal{CS}^N$ satisfies weak positivity if for each $(N, u_{\{i,j\}}, \{\{i, j\}\}) \in \mathcal{CS}^N$ such that $i, j \in N$,

$$Y_k(N, u_{\{i,j\}}, \{\{i, j\}\}) > 0$$

for $k \in \{i, j\}$, and for each $(N, -u_{\{i,j\}}, \{\{i, j\}\}) \in \mathcal{CS}^N$ such that $i, j \in N$,

$$Y_k(N, -u_{\{i,j\}}, \{\{i, j\}\}) < 0$$

for $k \in \{i, j\}$.

Finally, weak power inversion can be interpreted as follows. Suppose that a pair of players has to share a worth equal to 1. The relative share a player obtains of this worth can be different from the share he obtains if the pair of players has to share a worth equal to -1.

Weak power inversion: an allocation rule Y on $\mathcal{CS}^N$ satisfies weak power inversion if for each $(N, u_{\{i,j\}}, \{\{i, j\}\}) \in \mathcal{CS}^N$ such that $i, j \in N$,

$$Y_i(N, u_{\{i,j\}}, \{\{i, j\}\})Y_i(N, -u_{\{i,j\}}, \{\{i, j\}\}) = \\ Y_j(N, u_{\{i,j\}}, \{\{i, j\}\})Y_j(N, -u_{\{i,j\}}, \{\{i, j\}\}).$$

Note that weak power inversion and weak positivity gives:

$$\frac{Y_i(N, u_{\{i,j\}}, \{\{i, j\}\})}{Y_j(N, u_{\{i,j\}}, \{\{i, j\}\})} = \frac{Y_j(N, -u_{\{i,j\}}, \{\{i, j\}\})}{Y_i(N, -u_{\{i,j\}}, \{\{i, j\}\})}.$$

This observation, combined with efficiency, reflects the insurance system: if the worth produced by a pair of players is negative, the relative shares obtained by these players of the worth they produce is the inverse of the relative shares they obtain when they produce a positive worth.

We prove the main result of this article (Theorem 2), which states that the weighted position value is the only allocation rule on $\mathcal{CS}^N$ satisfying the six previous axioms, in two steps. First, we show that the weighted position value is the unique allocation rule satisfying component efficiency, superfluous link property, link decomposability, weak positivity and weak power inversion on the class of communication situations of $\mathcal{CS}^N$ such that the coalitional game is a unanimity game. Second, as each game can be written as a linear combination of unanimity games, the quasi-additivity axiom permits to complete the proof. But as shown in Example 3, there exists communication situations that cannot be written as a sum of comparable unanimity communication situations.

Example 3
Consider $(N,\, v,\, L)$ such that $N = \{1,\, 2,\, 3\}$, $v = 3u_{\{1,2\}} + u_{\{1,3\}} - 2u_{\{1,2,3\}}$ and $L = \{\{1,\, 2\},\, \{2,\, 3\}\}$. The communication situations $(N,\, u_{\{1,3\}},\, L)$ and $(N,\, -2u_{\{1,2,3\}},\, L)$ are not comparable:

$$r^{u_{\{1,3\}}} = \sum_{A \subset L} 0\, u_A + u_L$$
$$r^{-2u_{\{1,2,3\}}} = \sum_{A \subset L} 0\, u_A - 2u_L$$

indeed we have $\alpha_L^{r^{u_{\{1,3\}}}}\, \alpha_L^{r^{-2u_{\{1,2,3\}}}} < 0$. □

In order to complete the proof, we associate to each communication situation in $\mathcal{CS}^N$ a new communication situation, denoted by $(N,\, \eta^v,\, L)$. This new communication situation summarizes all the necessary information to compute $(L,\, r^v)$. It is defined as:

$$\alpha_S^{\eta^v} = \begin{cases} \displaystyle\sum_{\substack{R \in 2^N \setminus \{\emptyset\} \\ H(R)=S}} \alpha_R^v & \text{if } S \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Note that as $(N,\, v,\, L) \in \mathcal{CS}^N$, we know that $\alpha_{\{i\}}^v = 0$ for each $i \in N$. Since $H(S) = \{i\}$ if and only if $S = \{i\}$, we have $\alpha_{\{i\}}^{\eta^v} = \alpha_{\{i\}}^v = 0$. Therefore, we obtain $(N,\, \eta^v,\, L) \in \mathcal{CS}^N$.

Moreover, we can see that for each $S \in 2^N \setminus \{\emptyset\}$, the worth of S in η^v is equal to the worth of $L(S)$ in r^v. Indeed we have:

$$\eta^v(S) = \sum_{\substack{T \subseteq S \\ T \neq \emptyset \\ T = H(T)}} \alpha_T^{\eta^v} = \sum_{\substack{T \subseteq S \\ T \neq \emptyset \\ T = H(T)}} \sum_{\substack{R \in 2^N \setminus \{\emptyset\} \\ H(R)=T}} \alpha_R^v = \sum_{\substack{T \subseteq S \\ T \neq \emptyset \\ T = H(T)}} \sum_{\substack{R \in 2^N \setminus \{\emptyset\} \\ R \in \Delta(L(T))}} \alpha_R^v = \sum_{\substack{T \subseteq S \\ T \neq \emptyset \\ T = H(T)}} \alpha_{L(T)}^{r^v}.$$

To each T such that $T = H(T) \subseteq S$ corresponds a unique $L(T) \subseteq L(S)$ such that $|T/L(T)| = 1$ and conversely, to each $A \subseteq L(S)$ such that $|N(A)/A| = 1$ corresponds a unique $N(A) \subseteq S$ such that $N(A) = A$. Note that $T \subseteq S$ is connected if and only if there exists a unique $A \subseteq L(S)$ such that $N(A)$ is connected. Then:

$$\eta^v(S) = \sum_{\substack{A \subseteq L(S) \\ |N(A)/A|=1}} \alpha_A^{r^v} = r^v(L(S)). \tag{4}$$

It follows by (4) that the communication situation $(N,\, \eta^v,\, L)$ summarizes all the information included in $(N,\, v,\, L)$ that we need to compute r^v: the unanimity coefficients $\alpha_S^{\eta^v}$ such that $H(S) \neq S$ are equal to zero and the unanimity coefficients $\alpha_S^{\eta^v}$ such that $H(S) = S$ contains all the necessary information about the coalitions $R \subseteq S$ such that $H(R) = S$. Note that there is no redundant information because $H(R)$ is unique for each $R \in 2^N \setminus \{\emptyset\}$.

Now we are ready to provide a preliminary result: Lemma 2 states that if an allocation rule satisfies component efficiency, quasi-additivity and link decomposability, the worth of coalitions such that the connected hull is empty are useless for determining the allocations of the players. Moreover, all the necessary information about unanimity coefficients α_S^v such that $H(S) \subseteq R$ can be summarized in a unique unanimity coefficient relative to R.

Lemma 2
If an allocation rule Y satisfies component efficiency, quasi-additivity and link decomposability on $\mathcal{CS}^N$, then for each $(N,\, v,\, L) \in \mathcal{CS}^N$, $Y(N,\, v,\, L) = Y(N,\, \eta^v,\, L)$.

Proof: Consider $(N,\, v,\, L) \in \mathcal{CS}^N$ and $(N,\, w,\, L)$ such that $w = v - \eta^v$. Note that $(N,\, w)$ is zero-normalized. For each $A \subseteq L$, the unanimity coefficients of r^w are given by:

$$\begin{aligned}
\alpha_A^{r^w} &= \sum_{\substack{S \in 2^N \setminus \{\emptyset\} \\ S \in \Delta(A)}} \alpha_S^w \\
&= \sum_{\substack{S \in 2^N \setminus \{\emptyset\} \\ S \in \Delta(A)}} \left(\alpha_S^v - \sum_{\substack{R \in 2^N \setminus \{\emptyset\} \\ S = H(R)}} \alpha_R^v \right) \\
&= \sum_{\substack{S \in 2^N \setminus \{\emptyset\} \\ N(A) = H(S)}} \left(\alpha_S^v - \sum_{\substack{R \in 2^N \setminus \{\emptyset\} \\ S = H(R)}} \alpha_R^v \right),
\end{aligned}$$

where the first equality follows by Lemma 1 and the second equality by the definition of w. Moreover, for each $S \in 2^N \setminus \{\emptyset\}$ such that $N(A) = H(S)$, there is $R \in 2^N \setminus \{\emptyset\}$ such that $S = H(R)$ if and only if $S = H(S)$. This means that $S = N(A)$. Therefore:

$$\begin{aligned}
\alpha_A^{r^w} &= \sum_{\substack{S \in 2^N \setminus \{\emptyset\} \\ N(A) = H(S)}} \alpha_S^v - \sum_{\substack{R \in 2^N \setminus \{\emptyset\} \\ N(A) = H(R)}} \alpha_R^v \\
&= 0.
\end{aligned}$$

From this we obtain that $r^w(A) = 0$ for each $A \subseteq L$. Suppose that $L = \emptyset$. As $(N,\, w)$ is zero normalized, by component efficiency we can easily conclude that $Y_i(N, w, L) = v(\{i\}) = 0$ for each $i \in N$. Now, suppose that $L \neq \emptyset$. As (N, w) is zero normalized, by component efficiency, it follows that $Y_i(N, w, L) = 0$ for each $i \in N \setminus N(L)$. Next, consider $C \in N/L$ such that $|C| > 1$. We have:

$$\begin{aligned} w(C) &= \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\subseteq C}} \alpha_S^v - \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\subseteq C\\ H(S)=S}} \left(\sum_{\substack{R\in 2^N\setminus\{\emptyset\}\\ S=H(R)}} \alpha_R^v \right) \\ &= \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\subseteq C}} \alpha_S^v - \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\subseteq C}} \alpha_S^v \\ &= 0, \end{aligned}$$

where the first equality follows by the definition of w and the fact that $\alpha_S^{\eta^v} \neq 0$ only if $H(S) = S$. Moreover, the communication situation (N, w, L) is trivially link unanimous because $r^w(A) = 0$ for each $A \subseteq L$. These two remarks, combined with link decomposability and component efficiency, give:

$$\begin{aligned} \sum_{i\in C} Y_i(N, w, L) = 0 & \\ &= c \sum_{i\in C} \sum_{l\in L_i} Y_i(N, u_{\{i,j\}}, \{\{i, j\}\}) \\ &= c \sum_{l\in L(C)} \sum_{i\in l} Y_i(N, u_{\{i,j\}}, \{\{i, j\}\}) \\ &= c|L(C)|. \end{aligned}$$

As $L(C) \neq \emptyset$, we have $c = 0$. This is true for each $C \in N/L$ such that $|C| > 1$, so that we obtain $Y_i(N, w, L) = 0$ for each $i \in N(L)$. Finally, we have:

$$Y(N, v, L) = Y(N, v - \eta^v + \eta^v, L) = Y(N, w + \eta^v, L).$$

As $\alpha_A^{r^w} = 0$ for each $A \subseteq L$, we know that (N, w, L) and (N, η^v, L) are comparable. Thus we obtain:

$$Y(N, w + \eta^v, L) = Y(N, w, L) + Y(N, \eta^v, L) = Y(N, \eta^v, L).$$

This yields the desired result: $Y(N, v, L) = Y(N, \eta^v, L)$. ■

Lemma 3 states that in a communication situation where all the links are veto, the payoffs of the players only depend on their relative weights.

Lemma 3
If an allocation rule Y satisfies component efficiency, link decomposability, weak power inversion and weak positivity on $\mathcal{CS}^N$, then there exists a set of weights such that for each link unanimous communication situation $(N,\, v,\, L) \in \mathcal{CS}^N$,

$$Y_i(N,\, v,\, L) = c \sum_{l \in L_i} \frac{\bar{\lambda}_{i,\,l}}{\bar{\lambda}_{i,\,l} + \bar{\lambda}_{j,\,l}}$$

for each $i \in N$, where $\bar{\lambda} = \lambda^+$ if $\sum_{C \in N/L} v(C) \geq 0$ and $\bar{\lambda} = \lambda^-$ if $\sum_{C \in N/L} v(C) < 0$.

Proof: Consider a link unanimous communication situation $(N,\, v,\, L) \in \mathcal{CS}^N$ such that $\sum_{C \in N/L} v(C) \geq 0$. By link decomposability, we have:

$$Y_i(N,\, v,\, L) = c \sum_{\{i,\,j\} \in L_i} Y_i(N,\, u_{\{i,\,j\}},\, \{\{i,\, j\}\})$$

for each $i \in N$. By component efficiency, we know that $Y_i(N,\, u_{\{i,\,j\}},\, \{\{i,\, j\}\}) + Y_j(N,\, u_{\{i,\,j\}},\, \{\{i,\, j\}\}) = 1$. Then we can write:

$$Y_i(N,\, v,\, L) = c \sum_{\{i,\,j\} \in L_i} \frac{Y_i(N,\, u_{\{i,\,j\}},\, \{\{i,\, j\}\})}{Y_i(N,\, u_{\{i,\,j\}},\, \{\{i,\, j\}\}) + Y_j(N,\, u_{\{i,\,j\}},\, \{\{i,\, j\}\})}.$$

For each $i \in N$ and $\{i,\, j\} \in L_i$, set $\mu_{i,\,\{i,\,j\}} = Y_i(N,\, u_{\{i,\,j\}},\, \{\{i,\, j\}\})$. By weak positivity, we know that $\mu_{i,\,\{i,\,j\}} > 0$. We get:

$$Y_i(N,\, v,\, L) = c \sum_{\{i,\,j\} \in L_i} \frac{\mu_{i,\,\{i,\,j\}}}{\mu_{i,\,\{i,\,j\}} + \mu_{j,\,\{i,\,j\}}}. \qquad (5)$$

Similarly, for $\sum_{C \in N/L} v(C) < 0$ we obtain:

$$Y_i(N,\, v,\, L) = c \sum_{\{i,\,j\} \in L_i} \frac{\omega_{i,\,\{i,\,j\}}}{\omega_{i,\,\{i,\,j\}} + \omega_{j,\,\{i,\,j\}}} \qquad (6)$$

where $\omega_{i,\,\{i,\,j\}} = Y_i(N,\, -u_{\{i,\,j\}},\, \{\{i,\, j\}\})$ for each $i \in N$ and $\{i,\, j\} \in L_i$. By weak positivity, we know that $\omega_{i,\,\{i,\,j\}} < 0$. It remains to show that (5) and (6) imply the existence of two sets λ^+ and λ^-. Obviously, we can set:

$$\mu_{i,\,\{i,\,j\}} = \lambda^+_{i,\,\{i,\,j\}}$$

for each $i \in N(L)$ and each $\{i,\, j\} \in L_i$. By weak power inversion, we have:

$$\mu_{i,\,\{i,\,j\}} \omega_{i,\,\{i,\,j\}} = \mu_{j,\,\{i,\,j\}} \omega_{j,\,\{i,\,j\}},$$

which immediately leads to:

$$\omega_{j,\{i,j\}} = \frac{\mu_{i,\{i,j\}}\omega_{i,\{i,j\}}}{\mu_{j,\{i,j\}}}.$$

By component efficiency, we can write:

$$\begin{aligned}
\omega_{i,\{i,j\}} + \omega_{j,\{i,j\}} &= \frac{\mu_{i,\{i,j\}}\omega_{i,\{i,j\}}}{\mu_{j,\{i,j\}}} + \omega_{i,\{i,j\}} \\
&= \frac{\mu_{i,\{i,j\}}\omega_{i,\{i,j\}}}{\mu_{j,\{i,j\}}} + \frac{\mu_{i,\{i,j\}}\omega_{i,\{i,j\}}}{\mu_{i,\{i,j\}}} \\
&= \mu_{i,\{i,j\}}\omega_{i,\{i,j\}}\left[\frac{1}{\mu_{j,\{i,j\}}} + \frac{1}{\mu_{i,\{i,j\}}}\right] \\
&= -1.
\end{aligned}$$

Then we obtain:

$$\omega_{i,\{i,j\}} = -\frac{\dfrac{1}{\mu_{i,\{i,j\}}}}{\dfrac{1}{\mu_{i,\{i,j\}}} + \dfrac{1}{\mu_{j,\{i,j\}}}},$$

and we can define $\lambda^-_{i,\{i,j\}} = 1/\lambda^+_{i,\{i,j\}}$ for each $i \in N(L)$ and each $\{i,j\} \in L_i$. Thus we have :

$$Y_i(N,v,L) = c\sum_{l\in L_i} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l} + \bar{\lambda}_{j,l}}$$

for each $i \in N$, where $\bar{\lambda} = \lambda^+$ if $\sum_{C\in N/L} v(C) \geq 0$ and $\bar{\lambda} = \lambda^-$ if $\sum_{C\in N/L} v(C) < 0$. ■

Now, we have the necessary material to provide a characterization of the weighted position value on $\mathcal{CS}^N$.

Theorem 2
An allocation rule $Y : \mathcal{CS}^N \rightarrow \mathbb{R}^N$ satisfies component efficiency, quasi-additivity, superfluous link property, weak power inversion, weak positivity and link decomposability if and only if there exists a set of weights such that $Y = P^{\lambda^+}$.

Proof: First, we show that the weighted position value satisfies component efficiency. Consider $(N, v, L) \in \mathcal{CS}^N$ and a connected component $C \in N/L$.

We have:

$$\begin{aligned}\sum_{i\in C} P_i^{\lambda^+}(N,\, v,\, L) &= \sum_{i\in C}\sum_{l\in L_i}\frac{\bar\lambda_{i,\,l}}{\bar\lambda_{i,\,l}+\bar\lambda_{j,\,l}} Sh_l(L,\, r^v)\\ &= \sum_{l\in L(C)}\sum_{i\in l}\frac{\bar\lambda_{i,\,l}}{\bar\lambda_{i,\,l}+\bar\lambda_{j,\,l}} Sh_l(L,\, r^v)\\ &= \sum_{l\in L(C)} Sh_l(L,\, r^v)\\ &= \sum_{i\in C} P_i(N,\, v,\, L)\\ &= v(C).\end{aligned}$$

where the third equality follows using that $\sum_{i\in l}(\bar\lambda_{i,\,l}/\bar\lambda_{i,\,l}+\bar\lambda_{j,\,l}) = 1$ for each $l \in L$ and the fifth equality follows since the position value satisfies component efficiency.
Now, we show that the position value satisfies link decomposability. Let $(N,\, v,\, L) \in \mathcal{CS}^N$ be a link unanimous communication situation. The links of L are symmetric players in $(L,\, r^v)$. By the symmetry and efficiency of the Shapley value, it holds that $Sh_l(L,\, r^v) = \sum_{C\in N/L} v(C)/|L|$ for each $l \in L$. Thus, for each $i \in N$,

$$P_i^{\lambda^+}(N,\, v,\, L) = \sum_{\{i,\,j\}\in L_i}\frac{\bar\lambda_{i,\,\{i,\,j\}}}{\bar\lambda_{i,\,\{i,\,j\}}+\bar\lambda_{j,\,\{i,\,j\}}}\frac{\sum_{C\in N/L} v(C)}{|L|},$$

where $\bar\lambda = \lambda^+$ if $\sum_{C\in N/L} v(C) \geq 0$ and $\bar\lambda = \lambda^-$ if $\sum_{C\in N/L} v(C) < 0$. Moreover, it is easy to see that $P_i^{\lambda^+}(N,\, u_{\{i,j\}},\, \{\{i,\, j\}\}) = \lambda^+_{i,\,\{i,j\}}/(\lambda^+_{i,\,\{i,j\}}+\lambda^+_{j,\,\{i,j\}})$ and $P_i^{\lambda^+}(N,\, -u_{\{i,j\}},\, \{\{i,\, j\}\}) = \lambda^-_{i,\,\{i,j\}}/(\lambda^-_{i,\,\{i,j\}}+\lambda^-_{j,\,\{i,j\}})$. Finally, by setting $\sum_{C\in N/L} v(C)/|L| = c$, we obtain:

$$P_i^{\lambda^+}(N,\, v,\, L) = c\sum_{\{i,\,j\}\in L_i} P_i^{\lambda^+}(N,\, u_{\{i,j\}},\, \{\{i,\, j\}\})$$

if $\sum_{C\in N/L} v(C) \geq 0$ and

$$P_i^{\lambda^+}(N,\, v,\, L) = c\sum_{\{i,\,j\}\in L_i} P_i^{\lambda^+}(N,\, -u_{\{i,j\}},\, \{\{i,\, j\}\})$$

if $\sum_{C\in N/L} v(C) < 0$.
In order to see that the weighted position value satisfies the superfluous link

property, consider $(N, v, L) \in \mathcal{CS}^N$ such that $k \in L$ is superfluous. As $r^v(A) - r^v(A \setminus \{k\}) = 0$ for each $A \subseteq L$, we know that k is a null player in (L, r^v). Thus $Sh_k(L, r^v) = 0$. Therefore, for each $i \in N$:

$$\begin{aligned} P_i^{\lambda^+}(N, v, L) &= \sum_{l \in L_i} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l} + \bar{\lambda}_{j,l}} Sh_l(L, r^v) \\ &= \sum_{\substack{l \in L_i \\ l \neq k}} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l} + \bar{\lambda}_{j,l}} Sh_l(L, r^v) \\ &= \sum_{\substack{l \in L_i \\ l \neq k}} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l} + \bar{\lambda}_{j,l}} Sh_l(L \setminus \{k\}, r^v) \\ &= P_i^{\lambda^+}(N, v, L \setminus \{k\}). \end{aligned}$$

The third equality follows from the fact that the Shapley value satisfies the null player out property.
To check weak power inversion, consider $(N, u_{\{i,j\}}, \{\{i, j\}\})$ and $(N, -u_{\{i,j\}}, \{\{i, j\}\})$ in $\mathcal{CS}^N$ such that $i, j \in N$. We have:

$$\begin{aligned} P_i^{\lambda^+}(N, u_{\{i,j\}}, \{\{i, j\}\}) &= \frac{\lambda^+_{i,\{i,j\}}}{\lambda^+_{i,\{i,j\}} + \lambda^+_{j,\{i,j\}}} \\ &= \frac{\lambda^+_{i,\{i,j\}}}{\lambda^+_{j,\{i,j\}}} \frac{\lambda^+_{j,\{i,j\}}}{\lambda^+_{i,\{i,j\}} + \lambda^+_{j,\{i,j\}}} \\ &= \frac{\lambda^+_{i,\{i,j\}}}{\lambda^+_{j,\{i,j\}}} P_j^{\lambda^+}(N, u_{\{i,j\}}, \{\{i, j\}\}). \qquad (7) \end{aligned}$$

Similarly, for $(N, -u_{\{i,j\}}, \{\{i, j\}\})$, we have:

$$P_i^{\lambda^+}(N, -u_{\{i,j\}}, \{\{i, j\}\}) = \frac{\lambda^-_{i,\{i,j\}}}{\lambda^-_{j,\{i,j\}}} P_j^{\lambda^+}(N, -u_{\{i,j\}}, \{\{i, j\}\}). \qquad (8)$$

For all $i \in N$ and all $\{i, j\} \in L_i(N)$, $\lambda^-_{i,\{i,j\}} = 1/\lambda^+_{i,\{i,j\}}$, thus the combination of (7) and (8) gives:

$$\begin{aligned} &P_i^{\lambda^+}(N, u_{\{i,j\}}, \{\{i, j\}\}) P_i^{\lambda^+}(N, -u_{\{i,j\}}, \{\{i, j\}\}) = \\ &\qquad P_j^{\lambda^+}(N, u_{\{i,j\}}, \{\{i, j\}\}) P_j^{\lambda^+}(N, -u_{\{i,j\}}, \{\{i, j\}\}), \end{aligned}$$

the desired result.
It is easy to see that our weighted position value satisfies weak positivity.

Finally, in order to see that the weighted position value is quasi-additive, consider two communication situations $(N,\, v,\, L)$ and $(N,\, w,\, L)$ of $\mathcal{CS}^N$ that are comparable. It holds that:

$$
\begin{aligned}
P_i^{\lambda^+}(N,\, v,\, L) + P_i^{\lambda^+}(N,\, w,\, L) &= \sum_{l \in L_i} \frac{\bar{\lambda}_{i,\,l}}{\bar{\lambda}_{i,\,l} + \bar{\lambda}_{j,\,l}} Sh_l(L,\, r^v) + \sum_{l \in L_i} \frac{\bar{\lambda}_{i,\,l}}{\bar{\lambda}_{i,\,l} + \bar{\lambda}_{j,\,l}} Sh_l(L,\, r^w) \\
&= \sum_{l \in L_i} \frac{\bar{\lambda}_{i,\,l}}{\bar{\lambda}_{i,\,l} + \bar{\lambda}_{j,\,l}} Sh_l(L,\, r^v + r^w) \\
&= \sum_{l \in L_i} \frac{\bar{\lambda}_{i,\,l}}{\bar{\lambda}_{i,\,l} + \bar{\lambda}_{j,\,l}} Sh_l(L,\, r^{v+w}) \\
&= P_i^{\lambda^+}(N,\, v + w,\, L).
\end{aligned}
$$

The third equality follows since $r^v(A) + r^w(A) = \sum_{C \in N/A}[v(C) + w(C)] = \sum_{C \in N/A}(v + w)(C) = r^{v+w}(A)$ for each $A \subseteq L$.
All that is left to prove now is that there is a unique allocation rule Y satisfying these six axioms on $\mathcal{CS}^N$. Consider an allocation rule Y that satisfies the six axioms. Pick $S \in 2^N \setminus \{\emptyset\}$ and consider $(N,\, \alpha u_S,\, L)$ where $\alpha \in \mathbb{R}$. By Lemma 2, we know that $Y(N,\, \alpha u_S,\, L) = Y(N,\, \eta^{\alpha u_S},\, L)$. Consider $(N,\, \eta^{\alpha u_S},\, L) \in \mathcal{CS}^N$ such that $H(S) = \emptyset$. In that case, each link of L is superfluous. By the superfluous link property, component efficiency and zero-normalization of $(N,\, \eta^{\alpha u_S})$, we obtain $Y_i(N,\, \eta^{\alpha u_S},\, L) = Y_i(N,\, \eta^{\alpha u_S},\, \emptyset) = \eta^{\alpha u_S}(\{i\}) = 0$ for each $i \in N$.
Now suppose that $H(S) \neq \emptyset$. The links in $L \setminus L(H(S))$ are superfluous. By the superfluous link property, we know that $Y(N,\, \eta^{\alpha u_S},\, L) = Y(N,\, \eta^{\alpha u_S},\, L(H(S)))$. Note that each player $i \in N \setminus H(S)$ is isolated in graph $(N,\, L(H(S)))$. Using zero normalization of $(N,\, \eta^{\alpha u_S})$ and component efficiency, we obtain $Y_i(N,\, \eta^{\alpha u_S},\, L(H(S))) = \eta^{\alpha u_S}(\{i\}) = 0$. For each $i \in N \setminus H(S)$, $Y_i(N,\, \eta^{\alpha u_S},\, L) = 0 = P_i^{\lambda^+}(N,\, \eta^{\alpha u_S},\, L)$. The link game associated with $(N,\, \eta^{\alpha u_S},\, L)$ is given by:

$$
r^{\eta^{\alpha u_S}}(A) = \begin{cases} \alpha & \text{if } A \supseteq L(H(S)), \\ 0 & \text{otherwise.} \end{cases}
$$

Hence $(N,\, \eta^{\alpha u_S},\, L(H(S)))$ is link unanimous. By Lemma 3, we know that

$$
Y_i(N,\, \eta^{\alpha u_S},\, L(H(S))) = c \sum_{l \in L_i} \frac{\bar{\lambda}_{i,\,l}}{\bar{\lambda}_{i,\,l} + \bar{\lambda}_{j,\,l}},
$$

where $\bar{\lambda} = \lambda^+$ if $\sum_{C \in N/L} v(C) \geq 0$ and $\bar{\lambda} = \lambda^-$ if $\sum_{C \in N/L} v(C) < 0$. Using component efficiency, we obtain:

$$\sum_{i\in H(S)} Y_i(N,\, \eta^{\alpha u_S},\, L(H(S))) = c \sum_{i\in H(S)} \sum_{l\in L_i(H(S))} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l}+\bar{\lambda}_{j,l}} = \eta^{\alpha u_S}(H(S)) = \alpha.$$

This immediately leads to:

$$c = \frac{\alpha}{\displaystyle\sum_{i\in H(S)} \sum_{l\in L_i(H(S))} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l}+\bar{\lambda}_{j,l}}}.$$

By changing the order of the summations and noting that $\sum_{i\in l}(\bar{\lambda}_{i,l}/\bar{\lambda}_{i,l} + \bar{\lambda}_{j,l}) = 1$ for each $l \in L$, we obtain:

$$\sum_{i\in H(S)} \sum_{l\in L_i(H(S))} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l}+\bar{\lambda}_{j,l}} = \sum_{l\in L(H(S))} \sum_{i\in l} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l}+\bar{\lambda}_{j,l}} = |L(H(S))|.$$

Then, for each $i \in H(S)$:

$$Y_i(N,\, \eta^{\alpha u_S},\, L) = Y_i(N,\, \eta^{\alpha u_S},\, L(H(S))) = \frac{\alpha}{|L(H(S))|} \sum_{l\in L_i(H(S))} \frac{\bar{\lambda}_{i,l}}{\bar{\lambda}_{i,l}+\bar{\lambda}_{j,l}}.$$

Now, it remains to show that $Y(N,\, v,\, L)$ is uniquely determined for each $(N,\, v,\, L) \in \mathcal{CS}^N$. By Lemma 2, we know that $Y(N,\, v,\, L) = Y(N,\, \eta^v,\, L)$. We can decompose $(N,\, \eta^v,\, L)$ as a sum of comparable communication situations. Let $L^+ = \{A \subseteq L \,|\, \alpha_A^{r^{\eta^v}} \geq 0\}$ and $L^- = \{A \subseteq L \,|\, \alpha_A^{r^{\eta^v}} < 0\}$. Then:

$$\eta^v = \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S=H(S)\\ L(S)\in L^+}} \alpha_S^{\eta^v} u_S + \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S=H(S)\\ L(S)\in L^-}} \alpha_S^{\eta^v} u_S + \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\neq H(S)}} 0 u_S.$$

Now define $(N,\, \eta^{v^+},\, L)$ and $(N,\, \eta^{v^-},\, L)$ in the following manner:

$$\eta^{v^+} = \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S=H(S)\\ L(S)\in L^+}} \alpha_S^{\eta^v} u_S + \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S=H(S)\\ L(S)\in L^-}} 0\, u_S + \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\neq H(S)}} 0 u_S$$

$$\eta^{v^-} = \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S=H(S)\\ L(S)\in L^+}} 0 u_S + \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S=H(S)\\ L(S)\in L^-}} \alpha_S^{\eta^v} u_S + \sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\neq H(S)}} 0 u_S.$$

Then $(N, \eta^v, L) = (N, \eta^{v^+}, L) + (N, \eta^{v^-}, L)$. By (4), we have:

$$r^{\eta^{v^+}} = \sum_{A \in L^+} \alpha_A^{r^{\eta^v}} u_A + \sum_{A \in L^-} 0 u_A$$
$$r^{\eta^{v^-}} = \sum_{A \in L^+} 0 u_A + \sum_{A \in L^-} \alpha_A^{r^{\eta^v}} u_A.$$

Note that (N, η^{v^+}, L) and (N, η^{v^-}, L) are comparable because $\alpha_A^{r^{\eta^{v^+}}} \alpha_A^{r^{\eta^{v^-}}} = 0$ for each $A \subseteq L$. By quasi-additivity, we can conclude that:

$$Y(N, \eta^v, L) = Y(N, \eta^{v^+}, L) + Y(N, \eta^{v^-}, L).$$

Now we show that the communication situations stemming from the linear decomposition of (N, η^{v^+}, L) are comparable. Consider $(N, \alpha_S^{\eta^{v^+}} u_S, L)$, $S \neq \emptyset$, such that the coalitional game stems from the linear decomposition of η^{v^+}. By Lemma 1, we know that for each $S \subseteq N \setminus \emptyset$ such that $S = H(S)$,

$$\alpha_S^{\eta^v} = \sum_{\substack{R \subseteq N \setminus \emptyset \\ H(R) = S}} \alpha_R^v = \sum_{\substack{R \subseteq N \setminus \emptyset \\ R \in \Delta(L(S))}} \alpha_R^v = \alpha_{L(S)}^{r^v}.$$

Then we have $\alpha_S^{\eta^{v^+}} = \alpha_{L(S)}^{r^{\eta^{v^+}}} \geq 0$ if $S = H(S)$ and $L(S) \in L^+$, $\alpha_S^{\eta^{v^+}} = \alpha_{L(S)}^{r^{\eta^{v^+}}} = 0$ if $S = H(S)$ and $L(S) \in L^-$, and $\alpha_S^{\eta^{v^+}} = 0$ if $S \neq H(S)$. By Lemma 1 we can write $r^{\alpha_S^{\eta^{v^+}} u_S}$ as a linear combination of unanimity games:

$$\begin{aligned} r^{\alpha_S^{\eta^{v^+}} u_S} &= \sum_{A \subseteq L} \alpha_A^{r^{\alpha_S^{\eta^{v^+}} u_S}} u_A \\ &= \sum_{A \subseteq L} \left(\sum_{\substack{T \in 2^N \setminus \{\emptyset\} \\ T \in \Delta(A)}} \alpha_T^{\alpha_S^{\eta^{v^+}} u_S} \right) u_A \\ &= \sum_{\substack{A \subseteq L \\ A \neq L(S)}} 0\, u_A + \alpha_S^{\eta^{v^+}} u_{L(S)}. \end{aligned}$$

The unanimity coefficients of $r^{\alpha_S^{\eta^{v^+}} u_S}$ are all positive or equal to zero. For any $S, R \subseteq N \setminus \emptyset$, $S \neq R$, we obtain $\alpha_A^{r^{\alpha_S^{\eta^{v^+}} u_S}} \alpha_A^{r^{\alpha_R^{\eta^{v^+}} u_R}} \geq 0$ for each $A \subseteq L$. By quasi-additivity:

$$Y(N, \eta^{v^+}, L) = \sum_{S \subseteq N \setminus \{\emptyset\}} Y(N, \alpha_S^{\eta^{v^+}} u_S, L).$$

Similarly, we have:

$$Y(N,\, \eta^{v^-},\, L) = \sum_{S\subseteq N\setminus\{\emptyset\}} Y(N,\, \alpha_S^{\eta^{v^-}} u_S,\, L),$$

which proves that $Y(N,\, v,\, L)$ is uniquely determined for each $(N,\, v,\, L) \in \mathcal{CS}^N$. ■

Note that the property obtained in Lemma 3 can be seen as an axiom, called weighting, that explicitly specifies how the weights of players are used to share the worth of a link unanimous communication situation.

Weighting: an allocation rule Y on $\mathcal{CS}^N$ satisfies the weighting axiom if for each link unanimous communication situation $(N,\, v,\, L) \in \mathcal{CS}^N$ there exists $c \in \mathbb{R}$ such that for each $i \in N$:

$$Y_i(N,\, v,\, L) = c \sum_{\{i,j\}\in L_i} \frac{\bar{\lambda}_{i,\,\{i,j\}}}{\bar{\lambda}_{i,\,\{i,j\}} + \bar{\lambda}_{j,\,\{i,j\}}},$$

where $\bar{\lambda} = \lambda^+$ if $\sum_{C\in N/L} v(C) \geq 0$ and $\bar{\lambda} = \lambda^-$ if $\sum_{C\in N/L} v(C) < 0$.

By Lemma 3, the following result is a corollary of Theorem 2.

Corollary 1
The weighted position value is the unique allocation rule satisfying component efficiency, quasi-additivity, superfluous link property and weighting on $\mathcal{CS}^N$.

4 Conclusion

In this article, we provide a generalization of the position value that allows players to benefit from transfers of worth by investing in communication links. The levels of investment made by players are formalized via a weight scheme similar to the one defined by Haeringer (2006). Our weighted position value can be thought of as an insurance system that protects players who invest the most against loss. We characterize this new allocation rule via six axioms. Component efficiency and superfluous property are satisfied by the Myerson value and the position value. Quasi-additivity is a weak version of additivity. Link decomposability reflects the fact that a player only benefits of transfers of worth from players who share a link with himself. Weak positivity states that if the communicative strength of a link is non null, its adjacent players cannot obtain a null payoff. Finally, weak power inversion reflects the compensation mechanism.

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