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Welfare Bounds in a Growing Population

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Abstract

We study the allocation of collectively owned indivisible goods when monetary transfers are possible. We restrict our attention to incentive compatible mechanisms which allocate the goods efficiently. Among these mechanisms, we characterize those that respect welfare lower bounds. The main characterization involves the *identical-preferences lower-bound*: each agent should be at least as well off as in an hypothetical economy where all agents have the same preference as hers, no agent envies another, and the budget is balanced. This welfare lower-bound grants agents equal rights/responsibilities over the jointly owned resources but insures agents against the heterogeneity in preferences. We also study the implications of imposing variable population axioms together with welfare bounds.
JEL Classifications: C79, D61, D63.
Key words: *collective ownership, allocation of indivisible goods and money, NIMBY problems, imposition of tasks, the Groves mechanisms, the identical-preferences lower-bound, individual rationality, the stand-alone lower-bound, k-fairness, population monotonicity.*

1 Introduction

In several cases in public life, the society needs to allocate resources among its members who own them collectively. We have in mind situations where either all the agents have equal rights over some indivisible goods or all the agents are collectively responsible for the completion of a given set of tasks. We study such cases where a "center" (government, jurisdictional authority, etc.) allocates heterogenous indivisible goods (or bads) among agents whose valuations for the objects are their private information. Since goods or bads to be allocated are indivisible, monetary transfers are allowed to restore fairness.

Many examples fit into this context such as auctions held to allocate water entitlements to farmers and imposition of tasks as in government requisitions and eminent domain proceedings[1] (see Yengin, 2010). Another example is the allocation of indivisible public goods or services to neighborhoods, cities, or states, where all members of the society have collective rights or responsibilities. Assume that there is no question of whether or how much of the public good is to be provided (e.g., building a waste disposal site, siting state capitals). The only question is which localities will provide what public goods and what the compensations are. For instance, consider choosing the locations of desirable facilities or events (state capitals, parks, international airports, etc.) or the siting problem

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[1]Government requisition is the government's demand to use goods and services of the civilians usually in times of national emergency such as natural disasters and wars. Eminent Domain is government's right to seize private property, without the owner's consent, for *public use* (such as to build a road or a public utility over a privately held land), provided owners receive *just compensation*.

of noxious facilities (prisons, hazardous materials facilities such as chemical process facilities, waste disposal sites, nuclear facilities etc.). In the first case, although all members of the society have equal rights, only those in the localities where the desirable facilities are provided derive the benefits. Agents (localities) can overstate their possible benefits from the facility to ensure that it is built at their region. Monetary transfers like taxes and subsidies can provide incentive compatibility and restore fairness by distributing the net benefits over all agents. In the second example, suppose that all localities together use a noxious facility which is hosted by only one of them. All agents are jointly responsible for the construction of hazardous facilities since they all derive the same benefit. However, only the localities where the LULU (locally undesirable land use) facilities are sited incur the costs. Hence, the localities have an incentive to overstate their possible costs in order to avoid the construction of the noxious facility at their place which generates the so called NIMBY (not in my backyard) syndrome. Monetary transfers can ensure truthful reporting and fair compensations.[2]

Several other examples fit in our model such as the following allocation problems: the allocations of social services to the members of society (e.g. municipality child-care in Sweden, Biel, et al, 1997), community housing, charitable goods and money among the needy, fishing or pollution permits, resources in centrally planned economies, commonly owned indivisible goods in cooperative enterprises such as cooperative supported agriculture, inheritance among heirs, landing rights to airlines, job and wage assignments etc. In each of these examples, the resources allocated are indivisible goods or bads over which society has equal rights and side payments such as taxes, subsidies, compensations, etc. are possible.

Without loss of generality, we focus on the allocation of heterogenous tasks among agents based on their reported costs (such as time, money, or effort) of performing the tasks (see also, Porter, Shoham, and Tennenholtz, 2004; Atlamaz, Yengin, 2008; Moulin, 2009). All tasks must be allocated, each task is assigned to only one agent, and an agent may be assigned either no task, a single task, or more than one task. In eminent domain or government requisition problems, a task is imposed on an agent when the property or service of the agent is seized or used by the government. In NIMBY problems, a locality is assigned a task when it has to host a noxious facility. Agents have quasi-linear preferences over the sets of tasks and monetary transfers.[3]

In the above examples, since all agents have equal rights or responsibilities, society is generally concerned with the equity of an allocation and the resulting welfare levels of people. In the literature on fair allocation, several axioms (such as *no-envy* and *egalitarian-equivalence*)[4] are suggested to measure the justness of an allocation. However, a mechanism which satisfies these fairness axioms may still generate welfare levels that are arbitrarily small or large. An allocation which guarantees a minimum level of welfare to all agents may be preferable to an *envy-free* allocation which yields "socially unacceptable" low levels of welfare to some agents. Most societies care about guaranteeing a minimum level of welfare to its members for solidarity and compassionate reasons. Indeed, this minimum guaranteed welfare can be seen as an indicator of the development stage and quality of life of that society.

In the fairness literature, the following thought experiment is generally carried out to determine an equitable welfare bound. First society agrees on a basic set of fairness notions that should be applied in a hypothetical "*reference economy*". These fairness notions determine an allocation and associated

[2] In the literature, several papers study the problem of publicly provided indivisible goods. Unlike most papers, we do not analyze whether the public good will be provided, we take the provision as given and decide who will provide it and what the monetary transfers are. Our paper also differs from others in adopting dominant-strategy implementation, and characterizing mechanisms that respect welfare bounds. Also, our results are applicable to any indivisible goods and money allocation problem.

[3] The utility of each agent is equal to her transfer minus the cost she incurs in performing the tasks assigned to her. The assumption of interpersonally comparable utility is sometimes criticized. See Roemer (1986) for a discussion of why use of utility functions may be justified.

[4] *No-envy* (Foley, 1967) requires that no agent prefers another agent's bundle to her own. *Egalitarian-equivalence* (Pazner and Schmeidler, 1978) requires that each agent should be indifferent between her bundle and a common reference bundle.

welfare levels in this reference economy. Then, these welfare levels are taken as a benchmark for the actual economy.

If all agents have equal rights or responsibilities over the resources and resources are perfectly divisible, then in a reference economy where everyone have the same preference, a Pareto-efficient and fair allocation would distribute resources equally. However, since in the actual economy, agents may have different preferences, equal division is not generally efficient. An allocation which is a Pareto-improvement over equal split would be preferable. In the classical fair division problem, guaranteeing to each agent her utility at equal split of the resources is one of the oldest axioms in the fairness literature and has been well-studied (see for example Steinhaus, 1948; Dubins and Spanier, 1961; and Moulin, 1991).

In economies where indivisible goods and money are allocated, equal division is not well defined. In such economies, an alternative fairness axiom is the *identical-preferences lower-bound,* introduced by Moulin (1990). Pick an agent and consider a reference economy where all agents have preferences identical to hers. Since all agents have equal rights and the same preference, they should enjoy the same welfare. Find the common welfare level enjoyed, if objects are allocated efficiently (*assignment-efficiency*[5]) and the budget is balanced. One can argue that this Pareto-efficient and egalitarian welfare distribution is equitable and should be a benchmark for the actual economy. Since no agent is responsible for the heterogeneity in the preferences in the actual economy, they should all receive at least their benchmark welfare levels. *Identical-preferences lower-bound* requires this be the case.

If a mechanism (that allocates the objects and determine money transfers) respects the *identical-preferences lower-bound*, then the worst-case welfare of an agent is the welfare in her reference economy. In the reference economy, since all preferences are identical to hers, her welfare would only reflect her own preference (for which she is held responsible) and her equal share in the collective right/responsibility of the society over the allocated goods. Suppose in the actual economy, she is guaranteed her reference welfare level. That is, the center insures her against the conditions of the economy for which she is not responsible (i.e. the heterogeneity in preferences) while letting her welfare to reflect the factors for which she has a responsibility or right. Hence, this kind of welfare level could be argued as fair in the liberal-egalitarian theory of distributive justice. According to *liberal-egalitarianism*, fairness calls for eliminating welfare differentials that result from factors for which agents are not held responsible and keeping the welfare differentials that are due to the factors for which they are held responsible (see Fleurbaey, 1995).

To be able to determine an allocation where indivisible goods or bads are assigned efficiently and welfare levels respect the desired lower bound, the center has to extract the private information of the agents about their preferences. Agents can misreport their true preferences in order to manipulate the allocation in their favor. Hence, to ensure *assignment-efficiency* (e.g. in NIMBY problems, constructing the facility in a locality with the lowest actual cost) and fair monetary transfers, inducing the agents to report their preferences truthfully is utmost importance. One of the most appealing incentive compatibility constraints is *strategy-proofness* (truthful reporting of preference is a weakly dominant strategy for all agents).

It is well known that when preferences are unrestricted, by Gibbard (1973) and Satterthwaite (1975), there is no non-dictatorial social choice function that is truthfully implementable in dominant strategies (i.e., *strategy-proof*). Two approaches were designed to overcome this impossibility result. First approach is the implementation theory which weakens the requirement of *strategy-proofness.* However, various equilibrium concepts used in implementation theory require that each agent knows a lot about the preferences of other agents which may not be feasible in real life. On the other hand, *strategy-proofness* has no such requirement which makes this concept very appealing.

The second approach keeps *strategy-proofness* while restricting the domain of preferences. One of the few domain restrictions which allow for non-trivial, non-dictatorial *strategy-proof* mechanisms

[5] That is, if indivisible goods are allocated, then the total value experienced by the agents should be maximal. If indivisible bads (tasks) are allocated, then the total cost incurred by the agents should be minimal.

is the domain of quasi-linear preferences. On this domain, the Vickrey-Clarke-Groves mechanisms (simply referred to as the Groves mechanisms from now on) are the only mechanisms that are *assignment-efficient* and *strategy-proof*. Our goal is three fold, to design Groves mechanisms that respect the *identical-preferences lower-bound*, to make a comparative study of Groves mechanisms that respect different welfare bounds suggested in the literature, and to characterize mechanisms that respect welfare lower bounds in a growing population. To our knowledge, the results presented here are the only ones on these issues.[6]

Our main result (Theorem 1) is the characterization of *assignment-efficient* and *strategy-proof* mechanisms that respect the *identical-preferences lower-bound* and minimize the deficit for each economy. These mechanisms *preserve order* (an agent with lower costs is not worse off then an agent with higher costs). Under some domain restrictions, they are *envy-free*, they have bounded deficits, the deficit is almost zero in large populations, and they generate the minimal deficit among all Groves mechanisms which grant agents welfare levels that are at least as much as the one at a Pareto-efficient and egalitarian allocation (i.e., they are $1-$*fair;* Porter, et al, 2004).

Next, we present the relations between the *identical-preferences lower-bound* and other intuitive welfare-bounds. To be specific, we consider *individual rationality* (that respects the status-quo)*, the stand-alone lower-bound* (that respects the autonomy of agents)*,* and $k-$*fairness* (a welfare bound, based on Rawlsian maxmin criterion, introduced by Porter, et al, 2004). We also study the budget properties of the Groves mechanisms that respect these welfare lower-bounds (Section 6).

Our other important contribution is to study the implications of welfare bounds under population changes. When population grows, the cost of an efficient assignment weakly decreases which is good news for the society. Since no agent is responsible for the population change, solidarity and fairness would require that all agents be weakly better off (*population monotonicity*). Hence, in a way, the welfare levels in the smaller population are taken as lower bounds on the welfares experienced in the larger population. The compatibility of *population monotonicity* with welfare lower bounds such as the *identical-preferences lower-bound* is not always guaranteed in other models. Fortunately, in our model, this is not the case (see Theorem 2).

Section 2 presents the model. In Section 3, we characterize the Groves mechanisms that respect the *identical-preferences lower-bound.* Section 4 presents characterizations with alternative welfare bounds. In Section 5, we characterize *population monotonic* Groves mechanisms respecting welfare bounds. Section 6 presents logical relations and budget properties. All proofs are in the appendix.

2 Model

A finite set of indivisible tasks is to be allocated among a finite set of agents. All tasks must be allocated. An agent can be assigned either no task, a single task, or more than one task. Each task is assigned to only one agent. Let $\mathbb{A}$ be the finite set of tasks, with $|\mathbb{A}| \geq 1$, and α, β be typical elements of $\mathbb{A}$.

There is an infinite set of "potential" agents indexed by the positive natural numbers $\mathbb{N} \equiv \{1, 2, ...\}$. In any given problem, only a finite number of them are present. Let $\mathcal{N}$ be the set of subsets of potential agents with at least two agents. Let $n \geq 2$ and N with $|N| = n$ be a typical element of $\mathcal{N}$. The number of agents may be smaller than, equal to, or greater than the number of tasks.

Let $2^{\mathbb{A}}$ be the set of subsets of $\mathbb{A}$. Each agent i has a cost function $c_i : 2^{\mathbb{A}} \to \mathbb{R}_+$ with $c_i(\emptyset) = 0$.[7] We refer to such a cost function as *unrestricted.* Let $\mathcal{C}_{un}$ be the set of all such functions.

[6] In the same setting as ours, the analysis of the Groves mechanisms from the fairness perspective has also been the object of just a few recent studies (see Atlamaz and Yengin, 2008; Pápai, 2003; Porter, et al, 2004; and Yengin, 2009, 2010).

[7] As usual, $\mathbb{R}_+$ denotes the set of non-negative real numbers.

If for each $A \in (2^{\mathbb{A}} \backslash \emptyset)$, $c_i(A) = \sum_{\alpha \in A} c_i(\{\alpha\})$, then c_i is *additive.* If for each pair $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \cap A' = \emptyset$, $c_i(A \cup A') \leq c_i(A) + c_i(A')$, then c_i is *subadditive*, and if for each $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \cap A' = \emptyset$, $c_i(A \cup A') \geq c_i(A) + c_i(A')$, then c_i is *superadditive.* Let $\mathcal{C}_{ad}, \mathcal{C}_{sub}$, and $\mathcal{C}_{sup}$ be the classes of additive, subadditive, and superadditive cost functions, respectively. Let $\mathcal{C}$ be a generic element of $\{\mathcal{C}_{un}, \mathcal{C}_{ad}, \mathcal{C}_{sub}, \mathcal{C}_{sup}\}$ and $\mathcal{C}^N$ be the $n-$fold Cartesian product of $\mathcal{C}$.[8]

For each $N \in \mathcal{N}$, a *cost profile for* N is a list $c \equiv (c_1, ..., c_n)$. Let $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ be the domain of cost profiles where for each $i \in \mathbb{N}$, $c_i \in \mathcal{C}$. *Unless stated otherwise, the results of this paper hold on any domain.*

A cost profile defines an *economy.* Let $c, c', \widehat{c}$ be typical economies with associated agent sets $N, N', \widehat{N}$. For each $N \in \mathcal{N}$ and each $i \in N$, let c_{-i} be the cost profile of the agents in $N \backslash \{i\}$. For each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subseteq N$ and each $c \in \mathcal{C}^N$, let $c_{N'}$ be the restriction of c to N' : $c_{N'} \equiv (c_i)_{i \in N'}$.

There is a perfectly divisible good we call "money". Let t_i denote agent i's consumption of the good. We call t_i agent *i's transfer*: if $t_i > 0$, it is a transfer from the center to i; if $t_i < 0$, $|t_i|$ is a transfer from i to the center.

We think of a "*center*" that assigns the tasks and determines each agent's transfer. Agent i's utility when she is assigned the set of tasks $A_i \in 2^{\mathbb{A}}$ (note that A_i may be empty) and consumes $t_i \in \mathbb{R}$ is

$$u(A_i, t_i; c_i) = -c_i(A_i) + t_i.$$

For each $A \in 2^{\mathbb{A}}$ and each $N \in \mathcal{N}$, let $\mathcal{A}(A, N) = \{(A'_i)_{i \in N}$: for each $i \in N$, $A_i \in 2^A$, for each pair $\{i, j\} \subseteq N$, $A_i \cap A_j = \emptyset$, and $\bigcup_{i \in N} A'_i = A\}$ be the set of all possible distributions of A among the agents in N. For each $N \in \mathcal{N}$, an *assignment for* N is a list $(A_i)_{i \in N} \in \mathcal{A}(\mathbb{A}, N)$.

A *transfer profile for* N is a list $(t_i)_{i \in N} \in \mathbb{R}^N$. An *allocation for* N is a list $(A_i, t_i)_{i \in N}$ where $(A_i)_{i \in N}$ is an assignment and $(t_i)_{i \in N}$ is a transfer profile for N.

A *mechanism* is a function $\varphi \equiv (A, t)$ defined over the union $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each economy an allocation: for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $\varphi_i(c) \equiv (A_i(c), t_i(c)) \in 2^{\mathbb{A}} \times \mathbb{R}$.

For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, each $i \in N$, and each $\alpha \in \mathbb{A}$, let $c_i^\alpha \equiv c_i(\{\alpha\})$, $c^\alpha \equiv (c_i(\{a\}))_{i \in N}$, and $c_{-i}^\alpha \equiv (c_j(\{a\}))_{j \in N \backslash \{i\}}$. For each $k \leq n$, let $c_{[k]}^\alpha$ be the k-th cost in the ascending order of the costs in $\{c_1(\{\alpha\}), \ldots, c_n(\{\alpha\})\}$ and let $c_{\langle k \rangle}^\alpha \equiv c_{[\min\{k,n\}]}^\alpha$.

For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $A \in 2^{\mathbb{A}}$, let $W(c, A)$ be the minimal total cost generated among all possible distributions of A to the agents in N. That is,

$$W(c, A) = \min \left\{ \sum_{i \in N} c_i(A'_i) : (A'_i)_{i \in N} \in \mathcal{A}(A, N) \right\}.$$

2.1 The Groves Mechanisms

In our model, the utility of each agent is increasing in her transfer. Also, an agent's transfer and total transfer can be of any size. Hence, every allocation is Pareto-dominated by some other allocation with higher transfers. Thus, there is no Pareto-efficient allocation. However, we can define a notion of efficiency restricted to the assignment of the tasks. Since utilities are quasi-linear, given any cost profile c, an allocation that minimizes total cost is Pareto-efficient for c among all allocations with the same, or smaller, total transfer. Our first axiom requires mechanisms to choose only such allocations.

[8] Note that we do not assume monotonicity of the cost functions (c_i is monotonic if for each $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \subset A'$, $c_i(A) \leq c_i(A')$); however, if we imposed this restriction, our results would remain the same.

Assignment-Efficiency: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i \in N} c_i(A_i(c)) = W(c, \mathbb{A})$.

On the *additive* domain, *assignment-efficiency* requires that each task is assigned to an agent who can perform it at the lowest cost: for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N_{ad}$, $\alpha \in A_i(c)$ implies $c_i^\alpha = \min_{j \in N} c_j^\alpha$.

Since costs are private information, an *assignment-efficient* mechanism assigns the tasks so that the actual total cost is minimal only if the agents report their true costs. Truthful reporting is also essential to determine the correct welfare bounds. Then, a desirable property for a mechanism is that no agent should ever benefit by misrepresenting her costs (Gibbard, 1973; Satterthwaite, 1975).

Strategy-proofness[9]: For each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, and each $c'_i \in \mathcal{C}$, $u(\varphi_i(c); c_i) \geq u(\varphi_i(c'_i, c_{-i}); c_i)$.

The Groves mechanisms were introduced by Vickrey (1961), Clarke (1971), and Groves (1973). A Groves mechanism chooses, for each economy, an efficient assignment of the tasks. In the literature, Groves mechanisms are sometimes defined as correspondences that select all the efficient assignments in an economy. We work with single-valued Groves mechanisms and assume that each Groves mechanism is associated with a tie-breaking rule that determines which of the efficient assignments (if there are more than one) is chosen. Let $\mathcal{T}$ be the set of all possible tie-breaking rules and τ be a typical element of this set.

The transfer of each agent determined by a Groves mechanism has two parts. First, each agent pays the total cost incurred by all other agents at the assignment chosen by the mechanism. Second, each agent receives a constant sum of money that does not depend on her own cost. This constant can depend on the cost functions of the other agents or the population size.

For each $i \in \mathbb{N}$, let h_i be a real-valued function defined over the union $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ such that for each $N \in \mathcal{N}$ with $i \in N$ and each $c \in \mathcal{C}^N$, h_i depends only on c_{-i}. Let $h = (h_i)_{i \in \mathbb{N}}$ and $\mathcal{H}$ be the set of all such h.

The Groves mechanism associated with $h \in \mathcal{H}$ and $\tau \in \mathcal{T}$, $\mathbf{G}^{h,\tau}$:
Let $G^{h,\tau} \equiv (A^\tau, t^{h,\tau})$ be such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $A^\tau(c)$ is an efficient assignment for c and for each $i \in N$,

$$\begin{aligned} t_i^{h,\tau}(c) &= -\sum_{j \in N \setminus \{i\}} c_j(A_j^\tau(c)) + h_i(c_{-i}), \\ &= -W(c, \mathbb{A}) + c_i(A_i^\tau(c)) + h_i(c_{-i}). \end{aligned}$$

The following observation will be of much use. For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(G_i^{h,\tau}(c); c_i) = -W(c, \mathbb{A}) + h_i(c_{-i}). \tag{1}$$

By equation (1), for each $h \in \mathcal{H}$, the mechanisms in $\{G^{h,\tau}\}_{\tau \in \mathcal{T}}$ are *Pareto-indifferent*[10]. That is, the particular tie-breaking rule used is irrelevant in the determination of the utilities.

When all types of preferences are allowed and there are at least three alternatives to choose from, by Gibbard (1973) and Satterthwaite (1975), there is no non-dictatorial and *strategy-proof* social choice function. Fortunately, when we restrict our attention to the domain of quasi-linear preferences, this impossibility result disappears as the following theorem indicates.

Theorem A *A mechanism is assignment-efficient and strategy-proof on* $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ *if and only if it is a Groves mechanism.*

Proof: Since for each $N \in \mathcal{N}$, $\mathcal{C}^N$ is convex, the proof follows from Holmström (1979). □

[9] See Thomson (2005) for an extensive survey on strategy-proofness.

[10] Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Allocations $(A_i, t_i)_{i \in N}$ and $(A'_i, t'_i)_{i \in N}$ are *Pareto-indifferent* for c if and only if for each $i \in N$, $u(A_i, t_i; c_i) = u(A'_i, t'_i; c_i)$. The mechanisms φ and φ' are *Pareto-indifferent* if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(\varphi(c); c_i) = u(\varphi'(c); c_i)$.

3 The Identical-Preferences Lower-Bound

In an economy where all agents have the same preference, fairness[11] may require that they all experience the same welfare. Since utilities are quasi-linear, two agents have identical preferences if and only if their cost functions are identical.

For each $N \in \mathcal{N}$, each $i \in N$, and each $c_i \in \mathcal{C}$, let $c^i \in \mathcal{C}^N$ be agent $i's$ "*reference*" economy in which all agents have the same cost function c_i. That is, $c^i \equiv (c^i_j)_{j \in N} \in \mathcal{C}^N$ is such that for each $j \in N$, $c^i_j \equiv c_i$.

In economy c^i, under *assignment-efficiency* and *budget-balance*, a fair allocation would equally distribute among the agents, the cost of an efficient assignment through budget balancing transfers. Then, each agent's utility would be $-\frac{W(c^i,\mathbb{A})}{n}$. This is the common utility at a Pareto-efficient and egalitarian allocation at agent i's reference economy.

Suppose in the actual economy, the preferences differ from agent to agent. No agent is responsible for the preferences of the others. Hence, no one should be worse than she is in her reference economy. The *identical-preferences lower-bound* (Moulin; 1990) requires that this be the case.

The Identical-Preferences Lower-Bound (IPLB): For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq -\frac{W(c^i, \mathbb{A})}{n}.$$

This lower bound has been studied in the problem of allocating indivisible goods when there is a fixed amount of money to be distributed and each agent can be assigned at most one object (see, for instance, Thomson, 2004; Bevia, 1996). In exchange economies and in the problem of allocating a single divisible good over which agents have single-peaked preferences, *IPLB* is equivalent to the *equal-division lower-bound* (no agent's welfare should be less than the one at equal-split of the resources), which is also well-studied (see, for instance, Thomson, 2008).

To characterize the Groves mechanisms that respect *IPLB* on the *unrestricted* domain, we need the following notation:

For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N_{un}$, let $c_i \in \mathcal{C}_{un}$ be such that for each $A \in 2^{\mathbb{A}}$,

$$\overline{c}_i(A) \equiv \max\{0, W(c_{-i}, \mathbb{A}) - W(c_{-i}, \mathbb{A} \backslash A)\}. \tag{2}$$

For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N_{un}$, let $\overline{c}^i \equiv (\overline{c}^i_j)_{j \in N}$ be such that for each $j \in N$, $\overline{c}^i_j \equiv \overline{c}_i$.

Let us explain how $\overline{c}_i$ is calculated. First find the minimum total cost of allocating the objects in $\mathbb{A}$ among the agents in $N \backslash \{i\}$. This cost is $W(c_{-i}, \mathbb{A})$. For each set of tasks A, to find $\overline{c}_i(A)$, we ask the following question: what is the minimum cost agent i could incur for performing the tasks in A, such that if A was assigned to agent i and $\mathbb{A} \backslash A$ was efficiently distributed among the other agents, the total cost achieved by this allocation is not smaller than $W(c_{-i}, \mathbb{A})$. Hence, the cost function $\overline{c}_i$ specifies the minimum cost for each set of tasks A, such that when agent i joins the society $N/\{i\}$, the cost of an efficient assignment does not change, i.e., $W(c_{-i}, \mathbb{A}) = W(c_{-i}, \overline{c}_i, \mathbb{A})$.

The following example demonstrates the calculation of $\overline{c}_i$ in equation (2).

Example 1. Let $\mathbb{A} = \{\alpha, \beta\}$ and $N = \{1, 2, i\}$. Table 1a presents (c_1, c_2) and the calculation of the corresponding $\overline{c}_i$. Table 1b presents (c'_1, c'_2) and the calculation of the corresponding $\overline{c}'_i$.
Table 1a: Here, $W(c_{-i}, \mathbb{A}) = 12$ and is obtained by assigning both tasks to agent 1. Also, $W(c_{-i}, \mathbb{A} \backslash \{\alpha\}) = 17$ (β is assigned to agent 1), $W(c_{-i}, \mathbb{A} \backslash \{\beta\}) = 8$ (α is assigned to agent 2), and of course, $W(c_{-i}, \mathbb{A} \backslash \mathbb{A}) = 0$.

[11] Fairness notions such as no-envy, anonymity, or equal treatment of equals would imply this result.

	$\{\alpha\}$	$\{\beta\}$	$\{\alpha,\beta\}$
c_1	10	17	12
c_2	8	19	15
$\overline{c}_i$	$\min\{0, 12-17\} = 0$	$12-8=4$	$12-0=12$

Table 1 *(a)*

	$\{\alpha\}$	$\{\beta\}$	$\{\alpha,\beta\}$
c'_1	4	7	12
c'_2	8	9	18
$\overline{c}'_i$	$12-7=5$	$12-4=8$	$12-0=12$

Table 1 *(b)*

Table 1: Cost functions in Example 1.

Even if $\{c_1, c_2\} \subset \mathcal{C}_{sub}$, we have $\overline{c}_i \in \mathcal{C}_{sup}$. Note that $W(\overline{c}^i, \mathbb{A}) = 4$ (each task is assigned to a different agent). Also, $W(\overline{c}^i, \mathbb{A}) < W(c_{-i}, \mathbb{A})$.

Table 1b: Here, $W(c_{-i}, \mathbb{A}) = 12$, $W(c_{-i}, \mathbb{A}\backslash\{\alpha\}) = 7$, and $W(c_{-i}, \mathbb{A}\backslash\{\beta\}) = 4$. Even if $\{c_1, c_2\} \subset \mathcal{C}_{sup}$, we have $\overline{c}_i \in \mathcal{C}_{sub}$. Note that $W(\overline{c}^i, \mathbb{A}) = 12$ ($\mathbb{A}$ is assigned to one of the agents). Also, $W(\overline{c}^i, \mathbb{A}) = W(c_{-i}, \mathbb{A})$. ♢

Our first result presents the characterization of the class of Groves mechanisms that respect the *identical-preferences lower-bound* on the *unrestricted,* the *additive,* and the *subadditive* domains.

Proposition 1.

a) *On the unrestricted domain, a Groves mechanism $G^{h,\tau}$ respects the identical-preferences lower-bound if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{un}^N$,*

$$h_i(c_{-i}) \geq W(c_{-i}, \mathbb{A}) - \frac{1}{n} W(\overline{c}^i, \mathbb{A}). \tag{3}$$

b) *Let $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$. A Groves mechanism $G^{h,\tau}$ respects the identical-preferences lower-bound on the domain $\bigcup\limits_{N\in\mathcal{N}} \mathcal{C}^N$ if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq \frac{n-1}{n} W(c_{-i}, \mathbb{A}). \tag{4}$$

Let us give the intuition of the result briefly. By equation (1), $G^{h,\tau}$ respects the *identical-preferences lower-bound* if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A})$. However, checking whether this inequality holds is not enough since agents may lie about their costs. We need to check whether for each $N \in \mathcal{N}$, each $i \in N$, and each $c_{-i} \in \mathcal{C}^{N\backslash\{i\}}$,

$$h_i(c_{-i}) \geq \max_{c_i \in \mathcal{C}} \left\{ W(c_i, c_{-i}, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A}) \right\}. \tag{5}$$

The maximum value of the right-hand side (RHS) of this inequality is equal to the RHS of inequality (3) on the *unrestricted* domain; and inequality (4) on the *additive* or the *subadditive* domain, or when there is a single task to be allocated. Actually, on the *additive* domain, the right-hand-sides of (3) and (4) are identical.[12]

Note that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $W(\overline{c}^i, \mathbb{A}) \leq \overline{c}_i(\mathbb{A}) = W(c_{-i}, \mathbb{A})$. Hence, the RHS of inequality (3) is greater than the RHS of (4).

[12] On the *superadditive* domain, we could not come up with a compact formula for the right-hand side of (5). In general, characterizing Groves mechanisms on the *superadditive* domain is either technically or notationally complex. Hence, papers analyzing Groves mechanisms generally restrict attention to either the single-object case and to the *additive* domain (Porter, et al, 2004; Atlamaz and Yengin, 2008), or to the allocation of homogenous objects where each agent can receive at most one object (Ohseto, 2006, Moulin, 2009), or to the *subadditive* domain (Pápai, 2003).

Many Groves mechanisms respect *IPLB*. An example is the Pivotal mechanisms. These mechanisms are also known by the following names: Vickrey mechanisms, Clarke mechanisms, Vickrey-Clarke-Groves mechanisms, and Second-price sealed-bid auctions. Let $\tau \in \mathcal{T}$. A mechanism P^{τ} is Pivotal if $P^{\tau} \equiv G^{h,\tau}$ where for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in C^N$, $h_i(c_{-i}) = W(c_{-i}, \mathbb{A})$.

For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, the deficit is $\sum_{i \in N} t_i^{h,\tau}(c) = -(n-1)W(c, \mathbb{A}) + \sum_{i \in N} h_i(c_{-i})$. Hence, to minimize deficit (i.e. total transfer) while respecting *IPLB*, the inequalities in Proposition 1 should hold as equalities. Next, we present the mechanisms that are our main interest:

Theorem 1. ***a)*** *On the unrestricted domain, a Groves mechanism generates the minimal budget deficit for each economy among all Groves mechanisms that respect the identical-preferences lower-bound if and only if (3) holds as an equality.*

b) *On the additive or the subadditive domain, or when there is a single task to be allocated, a Groves mechanism generates the minimal budget deficit for each economy among all Groves mechanisms that respect the identical-preferences lower-bound if and only if (4) holds as an equality.*

Pápai (2003) characterizes the *envy-free* (no agent prefers another agent's bundle to her own) Groves mechanisms on the *subadditive* domain. (By Pápai, 2003, there is no *envy-free* Groves mechanism on the *unrestricted* domain.) On the *additive* and the *subadditive* domains, both the Pivotal mechanisms and the mechanisms in Theorem 1b are *envy-free*. Hence, *IPLB* is compatible with this central notion of fairness, *no-envy*.[13] We argue in Yengin (2010) that *no-envy* supports the liberal-egalitarian view of holding agents responsible only for their own preferences but not for the heterogeneity in the resources (the tasks). Hence, the compatibility of *no-envy* with *IPLB* for Groves mechanisms is very good news. An *envy-free* Groves mechanism that respects *IPLB* would ensure that agents' welfare levels do not reflect factors for which they are not responsible, namely the heterogeneity in tasks and in preferences.

When a Pivotal mechanism is used, each agent i's utility is equal to the reduction in the cost of an efficient assignment when she joins the economy (i.e. her positive externality on the economy): for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(P_i^{\tau}(c); c_i) = W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A})$. To achieve this welfare, her transfer is

$$t_i^{\tau}(c) = W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A}) + c_i(A_i^{\tau}(c)). \tag{6}$$

Hence, not only the center covers her cost of performing her assignment $(c_i(A_i^{\tau}(c))$, but also she receives the reduction in total cost when she is assigned $A_i^{\tau}(c)$. The deficit generated by a Pivotal mechanism is

$$\sum_{i \in N} t_i^{\tau}(c) = \sum_{i \in N} [W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A})] + W(c, \mathbb{A}) \geq W(c, \mathbb{A}) \geq 0. \tag{7}$$

That is, the center completely covers the total cost $W(c, \mathbb{A})$ and also pays agents extra money that is equal to their positive externalities. Hence, no agent bears a share of the total cost. Therefore, Pivotal mechanisms are not appealing in situations where all agents are jointly responsible both for the performance of the tasks and for the resulting total cost. In such situations, the mechanisms in Theorem 1 can be used. Although, they can not exactly distribute the total cost among agents (since no Groves mechanism is budget balanced), they still let the total cost be shared by the agents and the center in a "fair" way as explained below.

Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Suppose an agent $i \in N$ reports such a cost function that she is not assigned any task. Then, the cost of an efficient assignment in the actual economy is $W(c_{-i}, \mathbb{A})$. Since i shares the joint responsibility with the other agents for the completion of all the tasks, she should still pay her "fair" share of the total cost. What is this fair share? Suppose the society agrees that

[13] Unless there is an upper bound on the costs an agent may incur, our conjecture is that for Groves mechanisms, *IPLB* is not compatible with *egalitarian-equivalence* or *egalitarianism* (see Yengin 2009, 2010 for the characterizations of the *egalitarian-equivalent* and *egalitarian* Groves mechanisms, respectively).

the welfares should only reflect the factors for which agents are responsible. Each agent is responsible for her own cost function and she shares the joint responsibility of completing the tasks. No agent is responsible for other agents' preferences (cost functions). Hence, an equal share of the cost of an efficient assignment when everyone had the same cost function as agent i (i.e. $\frac{1}{n}W(c^i, \mathbb{A})$) can be argued as agent i's fair share that she should pay. Since i can misrepresent her actual cost function, she would report $\widehat{c}_i$ which makes her payment $\frac{1}{n}W(c^i, \mathbb{A})$ minimal while still allowing her not to be assigned any task. That is,

$$\widehat{c}_i = \arg\min_{c_i \in \mathcal{C}} \{W(c^i, \mathbb{A}) \text{ subject to } W(c_i, c_{-i}, \mathbb{A}) = W(c_{-i}, \mathbb{A})\}. \tag{8}$$

How does $\widehat{c}_i$ look like?

No matter what the domain is, $\widehat{c}_i$ should be such that $\widehat{c}_i(\mathbb{A}) \geq W(c_{-i}, \mathbb{A})$. Otherwise, i would be assigned the whole set of tasks in the economy $(\widehat{c}_i, c_{-i})$ which would cause $W(\widehat{c}_i, c_{-i}, \mathbb{A}) = \widehat{c}_i(\mathbb{A}) < W(c_{-i}, \mathbb{A})$ and contradict (8). To minimize $W(\widehat{c}^i, \mathbb{A})$, she would report $\widehat{c}_i(\mathbb{A}) = W(c_{-i}, \mathbb{A})$.

When the domain is *additive*, if for some $\alpha \in \mathbb{A}$, $\widehat{c}_i^\alpha < (c_{-i}^\alpha)_{[1]}$, then i would be assigned α in $(\widehat{c}_i, c_{-i})$. Hence, by (8), we must have for each $\alpha \in \mathbb{A}$, $\widehat{c}_i^\alpha = (c_{-i}^\alpha)_{[1]}$. Then, $W(\widehat{c}^i, \mathbb{A}) = W(c_{-i}, \mathbb{A})$.

Let the domain be *subadditive*. Since costs are subadditive and all agents have the same costs in $\widehat{c}^i$, then by *assignment-efficiency*, all of the tasks should be assigned to only one agent in $\widehat{c}^i$. Thus, $W(\widehat{c}^i, \mathbb{A}) = \widehat{c}_i(\mathbb{A}) = W(c_{-i}, \mathbb{A})$.

If the domain is *unrestricted*, then for (8) to be satisfied, $\widehat{c}_i = \overline{c}_i$ where $\overline{c}_i$ is as in (2). Then, $W(\overline{c}^i, \mathbb{A}) \leq W(c_{-i}, \mathbb{A}) = W(\overline{c}_i, c_{-i}, \mathbb{A})$.

To sum up, if an agent is not assigned any task, her transfer is $t_i^{h,\tau}((\widehat{c}_i, c_{-i})) = -\frac{1}{n}W(\widehat{c}^i, \mathbb{A})$, and by equation (1), her utility is $-\frac{1}{n}W(\widehat{c}^i, \mathbb{A})$. She can always guarantee this welfare by reporting $\widehat{c}_i$. Suppose that if she reports her actual cost c_i, she would be assigned $A_i^\tau(c)$. If her transfer in this case is $c_i(A_i^\tau(c)) - \frac{1}{n}W(\widehat{c}^i, \mathbb{A})$, then she would be indifferent between reporting $\widehat{c}_i$ or c_i. To give her an added incentive to report the true cost function, the center can pay her the positive externality she would have in the economy by reporting true costs. Then, her transfer and utility would be

$$t_i^{h,\tau}(c) \quad = \quad -\tfrac{1}{n}W(\widehat{c}^i, \mathbb{A}) + c_i(A_i^\tau(c)) + W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A}). \tag{9}$$

$$u(G_i^{h,\tau}(c); c_i) = W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A}) - \frac{1}{n}W(\widehat{c}^i, \mathbb{A}) = u(P_i^\tau(c); c_i) - \frac{1}{n}W(\widehat{c}^i, \mathbb{A}).$$

The transfer in (9) is the transfer prescribed by the mechanisms in Theorem 1. First, each agent i pays her fair share, the share in her reference economy, namely $\frac{1}{n}W(\widehat{c}^i, \mathbb{A})$. Then, if she is assigned any set of tasks, the center covers her cost and pays her the positive externality she generates on the economy. This second part of her transfer is same as what a Pivotal mechanism prescribes. Hence, the transfers specified in (9) differ from the transfers of a Pivotal mechanism by the term $-\frac{1}{n}W((\widehat{c}^i, \mathbb{A})$.

Note that in a large population where the positive externality of an agent is almost zero, the transfer of an agent is such that she pays her share of the total cost in her reference economy (her fair share) and get reimbursed for her own actual cost. Also, on the *additive,* or the *subadditive* domain, for each i, $W(\widehat{c}^i, \mathbb{A}) = W(c_{-i}, \mathbb{A})$. As the number of agents approaches to infinity, $W(c_{-i}, \mathbb{A})$ would approach to $W(c, \mathbb{A})$. Thus, in the limit, each agent's disutility is an equal share of the total cost.

If transfers are as in (9), then total transfer is

$$\sum_{i \in N} t_i^{h,\tau}(c) = \sum_{i \in N} [W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A})] + W(c, \mathbb{A}) - \frac{1}{n} \sum_{i \in N} W(\widehat{c}^i, \mathbb{A}). \tag{10}$$

Note that on the *unrestricted,* the *additive,* or the *subadditive* domain, for each $i \in N$, $W(\widehat{c}^i, \mathbb{A}) \leq W(c_{-i}, \mathbb{A})$. Hence, when transfers are as in (9), $\sum_{i \in N} t_i^{h,\tau}(c) \geq \frac{n-1}{n} \sum_{i \in N} [W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A})]$.

That is, there is budget deficit. But this deficit is smaller than the deficit generated by a Pivotal mechanism (compare (7) with (10)). Also, on the *additive*, or the *subadditive* domain, the deficit is $\frac{n-1}{n}\sum_{i\in N}[W(c_{-i},\mathbb{A})-W(c,\mathbb{A})]$. Since, when the population is large, for each i, the difference $W(c_{-i},\mathbb{A})-W(c,\mathbb{A})$ is negligible, and even zero if agents have replicas, then the budget would be almost balanced in large populations. Such a result does not hold for the Pivotal mechanisms.

When total transfer is as in (10), the center covers the actual total cost and also pays each agent her externality just like in the Pivotal mechanisms. But on top of that, the center collects the average of the total costs in the reference economies of all agents. Hence, the burden of the tasks is shared by the agents and the center, and the center motivates people to report their true costs by paying their externalities. While each agent pays a cost share as if there is no heterogeneity in the preferences, the center covers the difference between these payments and the actual total cost. This way, the center insures the agents against the conditions of the economy for which they are not responsible.

4 Other Welfare Bounds

If participation is voluntary, then a natural requirement from a mechanism is to ensure that no agent experiences a welfare that is less than her status quo welfare when she didn't participate. Even if participation may not be voluntary, the center still may wish to ensure that agents have non-negative utilities. For instance, in the eminent domain proceedings, if the government seizes the property of a civilian, say to build a railroad passing through the location of this property, it is required by law that the government should pay a just compensation. Although there is no consensus on what this "just" compensation should be, most can agree that it should at least ensure a non-negative utility to the civilian. Current laws pay the condemnee, the market price of a property similar to the one that was seized. However, this payment does not ensure incentive compatibility and *assignment-efficiency*. Also, if condemnee's valuation for her property is higher than what other people who traded in the market have for their similar properties, then the condemnee would experience a negative utility. The following property ensures this never be the case.

Individual Rationality (IR)[14]: For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq 0.$$

Although *individual rationality* is a desirable and in many cases, an essential property, there are situations where it may not be required. For instance, when agents who are collectively responsible for the completion of a set of tasks are also jointly responsible for the associated costs, a fair distribution of these costs among agents would lead to utility levels below status quo. As an example, in times of a war or national emergency (such as a natural disaster), all the agents in the society are responsible for the tasks imposed on them. In these cases, government can requisition the goods or services of the people without fully compensating their costs. In general, in imposition problems where agents do not have the right to refuse their task assignments (hence, participation is not necessarily voluntary), agents may end up with negative utilities. Still, society would be concerned with the equity of the distribution of welfare. Hence, welfare lower bounds which are weaker than *IR*, but still guaranteeing a safety net for agents would be required.

One such welfare lower bound is the one that respects agents' autonomy. Imagine, there is only one agent in the society. Since she is the only one who is responsible for the completion of all tasks, she should bear all the cost. However, it would be unfair to tax this agent. Let us call her utility in this reference economy as her stand-alone utility. (For instance, in the siting problem of hazardous facilities, this utility is the welfare when each locality autonomously builds its own facility.) In the actual economy, since all agents are responsible for the tasks, no agent should end up worse than her

[14] (The same property is used by Pápai, 2003, for the allocation of desirable objects and money).

stand-alone utility where she bore all the costs alone. The following welfare bound requires this be the case.

The Stand-Alone Lower-Bound (SALB): For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq -c_i(\mathbb{A}).$$

The following welfare bound, introduced by (Porter et. al., 2004), is a natural implication of Rawlsian maximin criterion when one requires the center to incur no deficit. Suppose there is a single task, say α. In imposition problems, since the agents do not have the option of refusing their task assignments, it may be unfair to hold them responsible for their costs. Hence, fairness may require that agents experience equal utilities unless it is possible to have a Pareto improvement on the equal utility distribution (Rawls' difference principle, also known as the *maximin criterion).* If one requires that the center incurs no budget deficit (total transfer is at most zero), then the maximin criterion implies that the task should be assigned to an agent with the lowest cost and utilities should be equalized through transfers that balance the budget. The resulting allocation is *assignment-efficient, budget-balanced,* and *egalitarian.* A mechanism that always chooses such an allocation would not be *strategy-proof* (Porter et. al; 2004). But one can require the utility achieved at such an allocation to be a welfare lower-bound ($1-$*fairness*). Unfortunately, there is no Groves mechanism which is $1-$*fair*, *strategy-proof* and that guarantees no-deficit (see Corollary 1 of Porter et. al., 2004). To obtain no deficit, we can weaken the fairness property by reducing the lower bound on the utilities.[15] For each $k \geq 1$, consider the hypothetical economy where the task is assigned to an agent with the k-th lowest cost and utilities are equalized through transfers that balance the budget. The resulting utility of each agent is $-\frac{1}{n}c^{\alpha}_{\langle k \rangle}$. We can require this reference utility to be a lower bound on the actual utilities. Porter et. al. (2004) generalize this bound to the multiple task setting as follows (this generalization may be most appealing on the *additive* domain). Let $k \geq 1$.

k-Fairness: For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq -\frac{1}{n}\sum_{\alpha \in \mathbb{A}} c^{\alpha}_{\langle k \rangle}.$$

Note that for each $k \geq 1$, if a mechanism is $k-$*fair*, then it is $(k+1)-$*fair* as well.

Finally, we investigate variations of the IPLB notion where the "identical preferences bound" may be taken as a lower or upper bound on welfare conditional on whether an agent benefits or loses from cooperation with the other agents.

Let $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$. In exchange economies, variety in preferences is typically good news: trade among agents who have different preferences benefit both parties of the trade. But in the problem we study, the heterogeneity in preferences may be good or bad news for agent i since the cost of an efficient assignment in an economy $c \in \mathcal{C}^N$ can be greater or smaller than the one in the reference economy $c^i \in \mathcal{C}^N$. One may argue that it is fair for agent i to benefit from the heterogeneity in preferences whenever it is good news, and lose whenever it is bad news, respectively. The following two axioms represent this idea.

Conditional Identical-Preferences Lower-Bound (CIPLB): For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, if $W(c, \mathbb{A}) \leq W(c^i, \mathbb{A})$, then

$$u(\varphi_i(c); c_i) \geq -\frac{W(c^i, \mathbb{A})}{n}.$$

[15] No deficit is compatible with $k-$*fairness* for $k \geq 3$.

Conditional Identical-Preferences Upper-Bound (CIPUB): For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, if $W(c, \mathbb{A}) \geq W(c^i, \mathbb{A})$, then

$$u(\varphi_i(c); c_i) \leq -\frac{W(c^i, \mathbb{A})}{n}.$$

The next result presents the characterizations of Groves mechanisms that respect the welfare bounds we introduced in this section. Note that parts (b), (d), and (e) are new results whereas part (a) is parallel to Proposition 3 in Pápai (2003) and part (c) follows from Theorem 1 in Atlamaz and Yengin (2008).

Proposition 2.
a) *A Groves mechanism $G^{h,\tau}$ is individually rational if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq W(c_{-i}, \mathbb{A}). \tag{11}$$

b) *A Groves mechanism $G^{h,\tau}$ respects the stand-alone lower-bound if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq 0. \tag{12}$$

c) *Let $k \geq 2$. On the additive domain, a Groves mechanism $G^{h,\tau}$ is $k-$fair if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N_{ad}$,*

$$h_i(c_{-i}) \geq W(c_{-i}, \mathbb{A}) - \frac{1}{n}\sum_{\alpha \in \mathbb{A}} (c^{\alpha}_{-i})_{\langle k-1 \rangle}. \tag{13}$$

d) *Let $\mathcal{C} \in \{\mathcal{C}_{un}, \mathcal{C}_{ad}, \mathcal{C}_{sub}\}$. A Groves mechanism $G^{h,\tau}$ respects the conditional identical-preferences lower-bound on the domain $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq \frac{n-1}{n} W(c_{-i}, \mathbb{A}). \tag{14}$$

e) *A Groves mechanism $G^{h,\tau}$ respects the conditional identical-preferences upper-bound if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \leq 0. \tag{15}$$

Note that inequalities (4) and (14) are the same. Hence, even though the *IPLB* is in general a stronger requirement than *CIPLB*, on the *additive* domain and on the *subadditive* domain, under *assignment-efficiency* and *strategy-proofness*, *IPLB* is equivalent to *CIPLB*. Hence, if we wish that the agents benefit from cooperation whenever cooperation reduces the cost of an efficient assignment, we need to ensure that they benefit from cooperation whether or not cooperation reduces the cost of an efficient assignment.

Note that even though *CIPLB* and *CIPUB* are symmetrical requirements, (14) and (15) are not. Also note that there is no Groves mechanism that respects both *CIPLB* and *CIPUB* since the right-hand sides of (14) and (15) are incompatible in general.[16]

For further analysis of the relationships between the classes of mechanisms characterized in Proposition 2, see Section 6.1.

[16] For Groves mechanisms, *SALB* is compatible with *CIPUB*. On either the *unrestricted*, or the *additive*, or the *subadditive* domain, there is no Groves mechanism that respects both *IPLB* and *CIPUB*. This is because there are cost profiles for which the right-hand-sides of inequalities (3) and (4) are positive, which, by (15), contradicts *CIPUB*. Similarly, on the *additive* domain, for $k \geq 2$, there is no $k-$*fair* Groves mechanism that respects *CIPUB*.

5 Population Changes and Welfare Bounds

When a population increases (or decreases), the resources may need to be reallocated. In doing so, two concerns arise. First, the center may wish to ensure that a welfare bound of its choice is still respected in the new allocation. Second, since the population change is no agent's fault, the center may wish to ensure solidarity. In this section, we investigate whether these two goals are attainable simultaneously. The compatibility of these two goals is not guaranteed in several economic models. Fortunately, this is not the case in our model as our characterization results indicate.

Suppose new agents join some initial population. The cost of an efficient assignment in the larger population is at most as large as the one in the smaller population, which is good news for the society. Since none of the agents in the initial population is responsible for the population growth, solidarity would require that all of them be at least as well off in the larger population as in the smaller one (Thomson, 1983). Hence, the welfare level of an agent in the initial population acts as a welfare lower bound for her in the new population.[17]

Population Monotonicity : For each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each $i \in N'$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_i(c_{N'}); c_i).$$

It is easy to show that[18] a Groves mechanism $G^{h,\tau}$ is population monotonic if and only if for each pair $\{N, N'\} \subset N$ such that $N' \subset N$, each $i \in N'$, and each $c \in C^N$,

$$h_i(c_{-i}) \geq h_i(c_{N' \setminus \{i\}}). \tag{16}$$

Population monotonicity has been studied in several models.[19] In some of these models, it is a rather strong property in the sense that it is incompatible with efficiency and fairness criteria. For instance, *population monotonicity* is incompatible with the *equal-division lower-bound* in exchange economies (Kim, 2004) and in the problem of allocating a single divisible good over which agents have single-peaked preferences (Thomson, 1995b). The following result shows that on any domain, for Groves mechanisms, *population monotonicity* is compatible with *IR*, *IPLB*, *SALB*, and $k-$*fairness* for $k \geq 1$.

Example 2. If for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$h_i(c_{-i}) = \max_{j \in N \setminus \{i\}} \{c_j(\mathbb{A})\},$$

then the Groves mechanism $G^{h,\tau}$ satisfies *population monotonicity, individual rationality*, the *identical-preferences lower-bound*, the *stand-alone lower-bound*, and $k-$*fairness* for $k \geq 1$.

Proof: Let $k \geq 1$, $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$. Since $\max\limits_{j \in N \setminus \{i\}} c_j(\mathbb{A}) \geq W(c, \mathbb{A})$, by equation (1), $u(G_i^{h,\tau}(c); c_i) \geq 0 \geq \max\limits_{c_i \in C} \{-\frac{1}{n} W(c^i, \mathbb{A}),\ -c_i(\mathbb{A}),\ -\frac{1}{n} \sum\limits_{\alpha \in \mathbb{A}} c^{\alpha}_{\langle k \rangle}\}$. Thus, $G^{h,\tau}$ respects *IR, IPLB, SALB,* and *k-fairness.* Since for each $N' \subset N$ with $i \in N'$, $\max\limits_{j \in N \setminus \{i\}} c_j(\mathbb{A}) \geq \max\limits_{j \in N' \setminus \{i\}} c_j(\mathbb{A})$, then by (16), $G^{h,\tau}$ is *population monotonic.* ◊

[17] In the working paper version of our paper, we also consider a strengthening of *population monotonicity* where the minimum utility change under population increases is parameterized. Then, we characterize the Groves mechanisms satisfying this stronger population monotonicity axiom and welfare bounds. The working paper version can be found at "http://www.adelaide.edu.au/directory/duygu.yengin" and "https://economics.adelaide.edu.au/research/papers/".

[18] See Proposition 3 in Yengin (2010).

[19] See Thomson (1995a) for a survey.

It is easy to characterize the class of *population monotonic* Groves mechanisms that respect *SALB*. Unfortunately, if we strengthen the welfare bound to *IR*, or *IPLB*, or $k-fairness$, then the h functions associated with a *population monotonic* Groves mechanism do not have compact formula. On the other hand, in the single-task case, we do obtain characterizations presented below.

Theorem 2:
(a) *A Groves mechanism $G^{h,\tau}$ satisfies population monotonicity and the stand-alone lower-bound if and only if for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each pair $\{i, j\} \subseteq N'$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq h_i(c_{N' \setminus \{i\}}) \text{ and } h_i(c_j) \geq 0. \tag{17}$$

Consider the single-task case. Without loss of generality, let $\mathbb{A} = \{\alpha\}$.
(b) *A Groves mechanism $G^{h,\tau}$ generates the minimal deficit among all Groves mechanisms that satisfy population monotonicity and individual rationality if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = (c^{\alpha}_{-i})_{[n-1]}. \tag{18}$$

(c) *A Groves mechanism $G^{h,\tau}$ generates the minimal deficit among all Groves mechanisms that satisfy population monotonicity and the identical-preferences lower-bound if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = \max_{p \in \{1,2,\ldots,n-1\}} \left\{ \frac{p}{p+1} (c^{\alpha}_{-i})_{[n-p]} \right\}. \tag{19}$$

(d) *Let $k \geq 2$. A Groves mechanism $G^{h,\tau}$ generates the minimal deficit among all Groves mechanisms that satisfy population monotonicity and $k-$fairness if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) = \max_{t \in \{2,3,..,n\}} \max_{s \in \{1,..,n+1-t\}} \left\{ (c^{\alpha}_{-i})_{[s]} - \frac{1}{t} (c^{\alpha}_{-i})_{[s-2+\min(k,t)]} \right\}. \tag{20}$$

Comparison of Theorem 2 with Theorem 1 and Proposition 2 shows that if the center wishes to guarantee that a welfare lower bound among *IR*, *IPLB*, or $k-$*fairness*, is respected in a growing population while maintaining solidarity as the population increases, then the transfer function gets quite complex compared to the transfers of a Groves mechanism respecting that welfare bound in a fixed population. Although, in the fixed-population case, we could provide an intuitive interpretation of the transfers when a Groves mechanism respects a welfare lower-bound such as *IPLB*, such interpretations are hard to come up with for the transfers specified in Theorem 2.

6 Further Results

6.1 Logical Relations

It is easy to see that among the welfare lower bounds we considered so far, *IR* is the strongest one. There are several other logical relations between these conceptually distinct welfare bounds, when one also requires *assignment-efficiency* and *strategy-proofness*. Note that all these welfare lower-bounds are compatible, since the Pivotal mechanisms respect each of them on any domain. To see this, let $\tau \in \mathcal{T}$ and $k \geq 1$. By equation (1), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(P^{\tau}_i(c); c_i) = W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A}) \geq 0 \geq \max_{c_i \in \mathcal{C}} \{-c_i(\mathbb{A}), -\frac{W(c^i, \mathbb{A})}{n}, -\frac{1}{n} \sum_{\alpha \in \mathbb{A}} c^{\alpha}_{\langle k \rangle}\}$, P^{τ} satisfies *IR*, *IPLB*, *SALB*, and $k-$*fairness*.

The following Table 2 presents the logical relations between different welfare lower-bounds for Groves mechanisms. (See Appendix for the proof of these relations).

Domain	**un.**	**ad.**	**sub.**	**sup.**					
(i)	✓	✓	✓	✓	IR	$\Rightarrow$	$IPLB$	$SALB$	$k-fair$ for $k \geq 1$
(ii)	✓	✓	✓	✓	$IPLB$	$\Rightarrow$		$SALB$	
(iii)		✓			$IPLB$	$\Leftrightarrow$			$2-fair$
(iv)		✓	✓		$IPLB$	$\Rightarrow$			$k-fair$ for $k \geq 2$
(v)		✓	✓		$CIPLB$	$\Leftrightarrow$	$IPLB$		
(vi)	✓	✓	✓		$CIPLB$	$\Rightarrow$		$SALB$	
(vii)		✓	✓		$CIPLB$	$\Rightarrow$			$k-fair$ for $k \geq 2$

Table 2: Logical Relations Under Assignment-Efficiency and Strategy-Proofness.

An alternative statement of the results in Table 2 is as follows. For instance, part (ii) can be rephrased as "on any domain, if a Groves mechanism respects the *identical-preferences lower-bound*, then it respects the *stand-alone lower-bound*".

Let us denote the class of Groves mechanisms satisfying an axiom "A" as $\mathcal{G}^A$. Table 3 is based on Table 2 and summarizes the relations between classes of Groves mechanisms that respect different welfare lower bounds.

Domain		
Unrest./Superad.	for $k \geq 2$,	$\mathcal{G}^{IR} \subset \mathcal{G}^{IPLB} \subset \mathcal{G}^{CIPLB} \subset \mathcal{G}^{SALB}$.
Additive	for $k > 2$,	$\mathcal{G}^{IR} \subset \mathcal{G}^{IPLB} \equiv \mathcal{G}^{CIPLB} \equiv \mathcal{G}^{2-fair} \subset (\mathcal{G}^{k-fair} \cap \mathcal{G}^{SALB})$.
Subadditive	for $k \geq 2$,	$\mathcal{G}^{IR} \subset \mathcal{G}^{IPLB} \equiv \mathcal{G}^{CIPLB} \subset (\mathcal{G}^{k-fair} \cap \mathcal{G}^{SALB})$.

Table 3: Relations Between Different Classes of Groves Mechanisms.

It is interesting to note that although, conceptually, *IPLB* and 2−*fairness* have different motivations, on the *additive* domain or when there is a single task, since for $k = 2$, the right hand sides of inequalities (4) and (13) are the same, the class of 2−*fair* Groves mechanisms is same as the class of Groves mechanisms that respect *IPLB* (and *CIPLB*). Hence, by Table 2 (ii), on the *additive* domain, if a Groves mechanism is 2−*fair*, then it respects *SALB*. For $k > 2$, there is no inclusion relationship between $\mathcal{G}^{SALB}$ and $\mathcal{G}^{k-fair}$. This is because for $k > 2$, the RHS of inequality (13) can be negative or positive depending on the cost profile.

Part (iv) of Table 2 is not an "if and only if" statement. To see that consider the following class of mechanisms that are introduced by Porter et. al. (2004). Let $k \geq 2$ and $\tau \in \mathcal{T}$. Let $F^{k,\tau} = G^{h,\tau}$ be such that inequality (13) holds as an equality. Let $\mathcal{F}^k \equiv \{F^{k,\tau}\}_{\tau \in \mathcal{T}}$ be the class of these mechanisms. Note that for each $k \geq 2$, mechanisms in $\mathcal{F}^k$ are $k-$*fair* on any domain. However, for $k > 2$, the mechanisms in $\mathcal{F}^k$ violate *IPLB* on each of the domains we consider in this paper. The mechanisms in $\mathcal{F}^2$ violate *IPLB* on the *unrestricted*, or the *subadditive*, or the *superadditive* domain.

Note that on the *additive* domain or when there is a single task, the mechanisms in Theorem 1b are same as the mechanisms in $\mathcal{F}^2$. These mechanisms have some remarkable properties as shown in Corollary 3 and Theorems 2 and 3, in Atlamaz and Yengin (2008).

Corollary 3 in Atlamaz and Yengin (2008) shows that the mechanisms in $\mathcal{F}^2$ are not only 2−*fair* but also 1−*fair*. Hence, they are the most fair ones in the Rawlsian sense. The deficit generated by these mechanisms is the smallest one among all *k-fair* Groves mechanisms for $k \geq 2$. There is an upper bound on this deficit in any economy. Moulin (2009) states that when there is a single task, the efficiency loss of these mechanisms is the smallest among all Groves mechanisms.

By Theorem 3, in Atlamaz and Yengin (2008), the mechanisms in $\mathcal{F}^2$ also are the only *k-fair* Groves mechanisms for $k \geq 2$, that *preserve order*: for each $N \in \mathcal{N}$, each $\{i, j\} \subseteq N$, each $c \in \mathcal{C}^N$, if $c_i(A) \leq c_j(A)$ for each $A \in 2^{\mathbb{A}}$, then $u(\varphi_i(c); c_i) \geq u(\varphi_j(c); c_j)$. That is, no agent is punished for having lower costs than others.

The following result follows from our Theorem 1 and results in Atlamaz and Yengin (2008). This result reinforces the appeal of our mechanisms presented in Theorem 1.

Remark 1. On the *additive* domain or when there is a single-task, the following statements are equivalent:
(i) a Groves mechanism $G^{h,\tau}$ belongs to $\mathcal{F}^2$,
(ii) $G^{h,\tau}$ generates the minimal deficit for each economy among all Groves mechanisms that respect the *identical-preferences lower-bound* (or *conditional identical-preferences lower-bound*),
(iii) $G^{h,\tau}$ generates the smallest deficit for each economy among all 1-*fair* Groves mechanisms,
(iv) $G^{h,\tau}$ generates the smallest deficit for each economy among all *order-preserving* Groves mechanisms satisfying *k-fairness* for $k \geq 1$,
(v) $G^{h,\tau}$ is a 1-*fair* Groves mechanism such that the deficit is bounded as follows: for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum\limits_{i \in N} t_i(c) \leq \sum\limits_{\alpha \in \mathbb{A}} (c^{\alpha}_{[2]} - c^{\alpha}_{[1]})$.

6.2 Welfare bounds and budget properties

One can argue that among all allocation mechanisms, Groves mechanisms that respect welfare bounds meet several important criteria of the center, namely, efficiency of assignments, incentive compatibility, and guaranteeing that welfare levels are "socially acceptable" or "just". However, the center is also generally concerned with the amount of budget imbalances.

Ideally, when agents are collectively responsible to perform the tasks, they should bear the total cost of doing so, hence, the transfers should add up to zero (*budget-balance*). On the other hand, an important function of the center is to ensure efficiency and fairness in the society which are only possible under *strategy-proofness*. Since, by Green and Laffont (1977), no *assignment-efficient* and *strategy-proof* mechanism balances the budget, in order to fulfill its functions, the center has to allow for budget imbalances. In a way, the imbalance is the price of fulfilling these functions.

Fortunately, there are Groves mechanisms that respect an upper bound on total transfer (budget deficit). In general, the upper bound on the deficit may depend on the economy. Consider an economy where the efficient assignment costs zero, that is no agent incurs any cost to perform the assigned tasks. Then, agents do not need to be compensated, the assignment of tasks does not generate any burden on agents. Suppose the center wishes to incur no deficit in such economies, but agrees to share some of the burden with agents whenever the cost of an efficient assignment is positive. Also, the higher the cost of an efficient assignment, the larger part of it the center agrees to share. Suppose the center wants to impose an upper bound on how much of the burden it would cover.

Let $\mathcal{M}$ be the set of all functions $M : \bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N \to \mathbb{R}$ such that

(i) for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$ with $W(c, \mathbb{A}) = 0$, $M(c) = 0$ and
(ii) for each $N \in \mathcal{N}$ and each pair $\{c, c'\} \subset \mathcal{C}^N$ with $W(c, \mathbb{A}) \leq W(c', \mathbb{A})$, $M(c) \leq M(c')$.

Let $M \in \mathcal{M}$.
$M-$**Bounded-Deficit** **(**$M-$**BD):** For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum\limits_{i \in N} t_i(c) \leq M(c)$.

Note that for each $M \in \mathcal{M}$, each $N \in \mathcal{N}$, and each $c \in \mathcal{C}^N$, $M(c) \geq 0$.

We have seen that *SALB* is a weaker welfare lower bound than *IR, IPLB,* and *CIPLB.* Let us investigate whether *SALB* is compatible with *M-bounded deficit.*

Proposition 3. *Let $M \in \mathcal{M}$. A Groves mechanism $G^{h,\tau}$ satisfies the stand-alone lower-bound and M-bounded deficit if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in C^N$,*

$$h_i(c_{-i}) = 0.$$

If $M : \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \to \mathbb{R}$ is such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $M(c) = 0$, then the total transfer can not be positive. The mechanisms in Proposition 3 are the only ones that respect *SALB* and that generate *no-deficit*.[20]

On the *additive* domain, by Atlamaz and Yengin (2008), $2-$*fair* Groves mechanisms violate *no-deficit*, but if a Groves mechanism is $k-$*fair* for $k \geq 3$, then it generates *no-deficit*. (See Atlamaz and Yengin (2008) for the characterization of $k-$*fair* Groves mechanisms that have bounded deficits.)

By Tables 2 and 3, all *individually rational* Groves mechanisms respect *SALB*. It is easy to see that there are economies for which the RHS of inequality (11) is positive. Hence, by Proposition 3, there is no *individually rational* Groves mechanism that generates *no-deficit* (or respects M-*bounded deficit* for any $M \in \mathcal{M}$). For instance, all Pivotal mechanisms violate *no-deficit*.

By a similar argument, since there are economies for which the RHS of inequalities (3) and (4) are positive, by Table 2 (ii) and Proposition 3, if a Groves mechanism respects *IPLB*, then it violates M-*bounded deficit* for any $M \in \mathcal{M}$. By Table 2 (iv) and Proposition 3, same impossibility holds for *CIPLB* on the *additive*, the *subadditive*, or the *unrestricted* domain. However, remember that on the *additive* or the *subadditive* domain, for the mechanisms in Theorem 1, the deficit approaches to zero as population increases. Also, even if we can not obtain $M-$*bounded deficit*, it is still possible to have a bounded deficit of a different type on the *additive* domain. On this domain, by Remark 1, a Groves mechanism respects the *IPLB* and generates a deficit bounded above by the amount $\sum_{\alpha \in \mathbb{A}} c^{\alpha}_{[2]} - W(c, \mathbb{A})$ if and only if it is as in Theorem 1. When there is a single task, this upper bound on deficit is the difference of the second lowest cost and the lowest cost in the economy.

To sum up, for Groves mechanisms, *no-deficit* (M-*bounded deficit*) is compatible with *SALB* and $k-$*fairness* for $k \geq 3$, on the other hand it is not compatible with the stronger welfare lower-bounds, namely, *CIPLB, IPLB,* and *IR*. Hence, our results confirms the usual trade-off between equity and efficiency. As the lower bound on welfare gets stronger (hence the welfares guaranteed get higher), the deficit incurred by the center gets larger. As we have seen, for Groves mechanisms, *SALB* is weaker than *IPLB*, which is weaker than *IR*. Among all Groves mechanisms that respect *SALB*, the ones in Proposition 3 generate the minimal deficit. Among all the *individually rational* Groves mechanisms, the Pivotal mechanisms generate the minimal deficit. Among all Groves mechanisms that respect *IPLB*, the ones in Theorem 1 generate the minimal deficit. Among these three classes of Groves mechanisms, the deficit generated by Pivotal mechanisms is the highest and the deficit generated by the mechanisms in Proposition 3 is the lowest.

7 Concluding Remarks

In problems where jointly owned resources are allocated, the society cares about the welfare levels attained by its members. The study of welfare lower bounds has been carried out in several models but for the case of Groves mechanisms such an analysis was missing, which our paper intends to provide.

The *identical-preferences lower-bound* has been studied in several papers where objects are allocated and budget is balanced. However, in these models, if non-dictatorial and non-trivial solutions are sought, one can not obtain *strategy-proofness* which is an important property to prevent agents

[20] Note that the mechanisms in Proposition 3 are the only ones that respect *SALB* and *CIPUB*. They are also the ones that generate the minimal budget surplus among all Groves mechanisms that respect *CIPUB*, and they Pareto-dominate all Groves mechanisms that respect *CIPUB*.

from manipulating the mechanism. Hence, we relax the budget balance requirement and restrict our attention to the domain of quasi-linear preferences, and characterize *assignment-efficient* and *strategy-proof* mechanisms that respect welfare bounds.

Our results in Section 6.1 indicate that the *identical-preferences lower-bound* is weaker than *individual rationality* and stronger than the *stand-alone lower-bound* (and $k-$*fairness* on the *additive* domain). *Individual rationality* compensates the agents fully for their costs. The *identical-preferences lower-bound* is appealing in problems where agents are responsible for their own preferences and costs while the center wants to insure them against the effects of the factors for which they are not responsible, namely the heterogeneity in preferences.

In most NIMBY problems, the question is to site a single noxious facility. Our results indicate that the mechanisms in Theorem 1 are very appealing for this case (see Remark 1). They are the only *assignment-efficient* and *strategy-proof* ones that respect *IPLB (CIPLB)* and minimize the deficit for each economy. The deficit is almost zero in big populations. These mechanisms are also the only $1-$*fair* mechanisms with the minimal deficit. They have bounded deficits and satisfy *SALB*, *order-preservation*, and *no-envy*.

We also characterized those Groves mechanisms which are *population monotonic* and respect welfare bounds. However, the transfer functions of a *population monotonic* Groves mechanism get very complicated if we impose any welfare lower bound which is stronger than *SALB*.

8 Appendix

Proof of Proposition 1:

a) For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{un}^N$, let $\{\overline{c}_i, \widetilde{c}_i\} \subset \mathcal{C}_{un}$ be such that for each $A \in 2^{\mathbb{A}}$,

$$\overline{c}_i(A) \equiv \max\{0, W(c_{-i}, \mathbb{A}) - W(c_{-i}, \mathbb{A}\backslash A)\}, \tag{21}$$

$$\widetilde{c}_i(A) \equiv \max\{0, W(c, \mathbb{A}) - W(c_{-i}, \mathbb{A}\backslash A)\}. \tag{22}$$

"If" Part: Let h be as in (3) in Proposition 1. Note that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{un}^N$,

$$W(c, \mathbb{A}) = \min_{A \in 2^{\mathbb{A}}}\{c_i(A) + W(c_{-i}, \mathbb{A}\backslash A)\}. \tag{23}$$

That is, for each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}_{un}^N$, and each $A \in 2^{\mathbb{A}}$, $W(c, \mathbb{A}) \leq c_i(A) + W(c_{-i}, \mathbb{A}\backslash A)$. Hence,

$$c_i(A) \geq \max\{0, W(c, \mathbb{A}) - W(c_{-i}, \mathbb{A}\backslash A)\} \tag{24}$$

with equality for some $A \in 2^{\mathbb{A}}$ (note that A may be empty).

• *Claim 1:* For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{un}^N$,

$$W(c^i, \mathbb{A}) \geq W(\widetilde{c}^i, \mathbb{A}).$$

Proof of Claim 1: Let $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}_{un}^N$. By (22) and (24), for each $A \in 2^{\mathbb{A}}$, $c_i(A) \geq \widetilde{c}_i(A)$. Hence, $W(c^i, \mathbb{A}) \geq W(\widetilde{c}^i, \mathbb{A})$. $\Diamond$

• *Claim 2:* For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{un}^N$,

$$W(c_{-i}, \mathbb{A}) - \frac{1}{n}W(\overline{c}^i, \mathbb{A}) \geq W(c, \mathbb{A}) - \frac{1}{n}W(\widetilde{c}^i, \mathbb{A}). \tag{25}$$

Proof of Claim 2: Let $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}_{un}^N$. Let $W(c, \mathbb{A}) = W(c_{-i}, \mathbb{A}) - r$ for some $r \in [0, W(c_{-i}, \mathbb{A})]$. Note that for each $A \in 2^{\mathbb{A}}$,

$$\max\{0, W(c_{-i}, \mathbb{A}) - r - W(c_{-i}, \mathbb{A}\backslash A)\} \geq \max\{0, W(c_{-i}, \mathbb{A}) - W(c_{-i}, \mathbb{A}\backslash A)\} - r.$$

That is, for each $A \in 2^{\mathbb{A}}$, $\widetilde{c}_i(A) \geq \overline{c}_i(A) - r$.

For each $A \in 2^{\mathbb{A}}$, let $d_i(A) \equiv \overline{c}_i(A) - r$. Note that $\widetilde{c}^i = (\widetilde{c}^i_j)_{j\in N}$ where for each $j \in N$, $\widetilde{c}^i_j = \widetilde{c}_i$. Since for each $A \in 2^{\mathbb{A}}$, $\widetilde{c}_i(A) \geq d_i(A)$, we have

$$W(\widetilde{c}^i, \mathbb{A}) = \min\{\sum_{j\in N} \widetilde{c}_i(A_j) : (A_j)_{j\in N} \in \mathcal{A}(\mathbb{A}, N)\} \geq \min\{\sum_{j\in N} d_i(A_j) : (A_j)_{j\in N} \in \mathcal{A}(\mathbb{A}, N)\}. \quad (26)$$

Let $(A'_j)_{j\in N} \in \arg\min\{\sum_{j\in N} d_i(A_j) : (A_j)_{j\in N} \in \mathcal{A}(\mathbb{A}, N)\}$. Then,

$$\begin{aligned}
\min\{\sum_{j\in N} d_i(A_j) : (A_j)_{j\in N} \in \mathcal{A}(\mathbb{A}, N)\} &= \sum_{j\in N} d_i(A'_j), \\
&= \sum_{j\in N} \overline{c}_i(A'_j) - nr, \\
&\geq \min\{\sum_{j\in N} \overline{c}_i(A_j) : (A_j)_{j\in N} \in \mathcal{A}(\mathbb{A}, N)\} - nr, \\
&= W(\overline{c}^i, \mathbb{A}) - nr. \quad (27)
\end{aligned}$$

Inequalities (26) and (27) together imply $W(\widetilde{c}^i, \mathbb{A}) \geq W(\overline{c}^i, \mathbb{A}) - nr$. Hence,

$$W(c, \mathbb{A}) - \frac{1}{n} W(\overline{c}^i, \mathbb{A}) + r \geq W(c, \mathbb{A}) - \frac{1}{n} W(\widetilde{c}^i, \mathbb{A}). \quad (28)$$

Substituting $W(c, \mathbb{A}) = W(c_{-i}, \mathbb{A}) - r$ into the LHS of (28),

$$W(c_{-i}, \mathbb{A}) - \frac{1}{n} W(\overline{c}^i, \mathbb{A}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(\widetilde{c}^i, \mathbb{A}).$$

Hence, we obtain (25). ◊

By Claims 1 and 2, for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N_{un}$, $W(c_{-i}, \mathbb{A}) - \frac{1}{n} W(\overline{c}^i, \mathbb{A}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A})$. This inequality and (3) together imply that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N_{un}$, $h_i(c_{-i}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A})$. By equation (1), on the *unrestricted* domain, $G^{h,\tau}$ respects *IPLB*. □

"Only If" Part: Let $G^{h,\tau}$ be a Groves mechanism that respects *IPLB* on the *unrestricted* domain. Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N_{un}$ such that

$$h_i(c_{-i}) < W(c_{-i}, \mathbb{A}) - \frac{1}{n} W(\overline{c}^i, \mathbb{A}). \quad (29)$$

Let $\widehat{c} = (\overline{c}_i, c_{-i}) \in \mathcal{C}^N_{un}$. Since $\widehat{c}_{-i} = c_{-i}$, by equation (1) and *IPLB,*

$$h_i(c_{-i}) \geq W(\widehat{c}, \mathbb{A}) - \frac{W(\overline{c}^i, \mathbb{A})}{n}. \quad (30)$$

By (21) and (23), $W(\widehat{c}, \mathbb{A}) = W(c_{-i}, \mathbb{A})$. This equality and (30) together contradict (29). □

b) "If" Part: Let $h \in \mathcal{H}$ be as in (4) in Proposition 1 and $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$.

- *Claim:* For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$\frac{n-1}{n} W(c_{-i}, \mathbb{A}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A}). \quad (31)$$

Proof of the Claim: Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$\frac{n-1}{n} W(c_{-i}, \mathbb{A}) < W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A}). \tag{32}$$

Note that on the *additive* and the *subadditive* domains, for each $c_i \in \mathcal{C}$, $W(c^i, \mathbb{A}) = c_i(\mathbb{A})$. This equality and (32) together imply $\frac{1}{n} c_i(\mathbb{A}) < -(W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A})) + \frac{1}{n} W(c_{-i}, \mathbb{A})$. This inequality and the fact that for each $c_i \in \mathcal{C}$, $W(c, \mathbb{A}) \leq W(c_{-i}, \mathbb{A})$ together imply

$$c_i(\mathbb{A}) < W(c_{-i}, \mathbb{A}). \tag{33}$$

Since $W(c, \mathbb{A}) \leq c_i(\mathbb{A})$ and $W(c^i, \mathbb{A}) = c_i(\mathbb{A})$, then

$$W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A}) \leq \frac{n-1}{n} c_i(\mathbb{A}). \tag{34}$$

Inequalities (32) and (34) together imply $W(c_{-i}, \mathbb{A}) < c_i(\mathbb{A})$, which contradicts (33). $\Diamond$

By (4) and (31), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A})$. By equation (1), $G^{h,\tau}$ respects *IPLB* on the *additive* and the *subadditive* domains. $\square$

"Only If" Part: Let $G^{h,\tau}$ be a Groves mechanism that respects *IPLB* on the *additive* or the *subadditive* domain. Let $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$. Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$h_i(c_{-i}) < \frac{n-1}{n} W(c_{-i}, \mathbb{A}). \tag{35}$$

If $\mathcal{C}^N = \mathcal{C}^N_{ad}$, then let $\widehat{c}_i \in \mathcal{C}_{ad}$ be such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = W(c_{-i}, \mathbb{A}) - W(c_{-i}, \mathbb{A} \backslash A)$. Note that $(\widehat{c}_i, c_{-i}) \in \mathcal{C}^N_{ad}$.
If $\mathcal{C}^N = \mathcal{C}^N_{sub}$, then let $\widehat{c}_i \in \mathcal{C}_{sub}$ be such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = W(c_{-i}, \mathbb{A})$. Note that $(\widehat{c}_i, c_{-i}) \in \mathcal{C}^N_{sub}$.
Let $\widehat{c} = (\widehat{c}_i, c_{-i})$. The fact that $\widehat{c}_{-i} = c_{-i}$, equation (1), and *IPLB* together imply

$$h_i(c_{-i}) \geq W(\widehat{c}_i, c_{-i}, \mathbb{A}) - \frac{1}{n} W(\widehat{c}^{\,i}, \mathbb{A}). \tag{36}$$

Note that $W(\widehat{c}_i, c_{-i}, \mathbb{A}) = W(c_{-i}, \mathbb{A}) = W(\widehat{c}^{\,i}, \mathbb{A})$. These equalities and (36) together imply $h_i(c_{-i}) \geq \frac{n-1}{n} W(c_{-i}, \mathbb{A})$, which contradicts (35). $\square\blacksquare$

Proof of Theorem 1:
Let $G^{h,\tau}$ be as in Theorem 1. Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Since $\sum_{i \in N} t_i^{h,\tau}(c) = -(n-1) W(c, \mathbb{A}) + \sum_{i \in N} h_i(c_{-i})$, for the deficit be minimal, $\sum_{i \in N} h_i(c_{-i})$ should be minimized. For each $i \in N$, by *IPLB*, $h_i(c_{-i})$ has to be as in (3) on the *unrestricted* domain, and as in (4) on the *additive* or the *subadditive* domain. Hence, to minimize $\sum_{i \in N} h_i(c_{-i})$, (3) ((4)) should hold as an equality on the *unrestricted* domain (on the *additive* or the *subadditive* domain). $\blacksquare$

Proof of Proposition 2:
(a) Let $h \in \mathcal{H}$ be as in (11). Then, by equation (1), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(G_i^{h,\tau}(c); c_i) \geq W(c_{-i}, \mathbb{A}) - W(c, \mathbb{A}) \geq 0$. Hence, $G^{h,\tau}$ is *individually rational.*
Conversely, let $G^{h,\tau}$ be *individually rational.* Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$, such that

$$h_i(c_{-i}) < W(c_{-i}, \mathbb{A}). \tag{37}$$

Let $\widehat{c}_i \in \mathcal{C}$ be such that $W(\widehat{c}_i, c_{-i}, \mathbb{A}) = W(c_{-i}, \mathbb{A})$. Let $\widehat{c} = (\widehat{c}_i, c_{-i})$. Since $c_{-i} = \widehat{c}_{-i}$ and $W(c_{-i}, \mathbb{A}) = W(\widehat{c}, \mathbb{A})$, by (1) and (37), $u(G_i^{h,\tau}(\widehat{c}); \widehat{c}_i) < 0$, which contradicts *individual rationality*. □

(b) Let $h \in \mathcal{H}$ be as in (12). Then, by equation (1), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(G_i^{h,\tau}(c); c_i) \geq -W(c, \mathbb{A}) \geq -c_i(\mathbb{A})$. Hence, $G^{h,\tau}$ respects *SALB*.

Conversely, let $G^{h,\tau}$ respect *SALB*. Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$h_i(c_{-i}) < 0. \tag{38}$$

Let $\widehat{c}_i \in \mathcal{C}$ be such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = 0$. Note that $\widehat{c}_i$ is *additive*. Let $\widehat{c} = (\widehat{c}_i, c_{-i}) \in \mathcal{C}^N$. By (1) and *SALB*, $u(G_i^{h,\tau}(\widehat{c}); \widehat{c}_i) = -W(\widehat{c}, \mathbb{A}) + h_i(\widehat{c}_{-i}) \geq -\widehat{c}_i(\mathbb{A})$. Since $W(\widehat{c}, \mathbb{A}) = \widehat{c}_i(\mathbb{A}) = 0$ and $\widehat{c}_{-i} = c_{-i}$, $h_i(c_{-i}) \geq 0$, which contradicts (38). □

(c) Let $k \geq 2$ and h be as in (13). Note that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{ad}^N$, $W(c_{-i}, \mathbb{A}) \geq W(c, \mathbb{A})$, and for each $\alpha \in \mathbb{A}$, $(c_{-i}^{\alpha})_{\langle k-1 \rangle} \leq c_{\langle k \rangle}^{\alpha}$. Hence, for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{ad}^N$, $W(c_{-i}, \mathbb{A}) - \frac{1}{n} \sum\limits_{\alpha \in \mathbb{A}} (c_{-i}^{\alpha})_{\langle k-1 \rangle} \geq W(c, \mathbb{A}) - \frac{1}{n} \sum\limits_{\alpha \in \mathbb{A}} c_{\langle k \rangle}^{\alpha}$. By (1) and (13), $G^{h,\tau}$ is $k-$*fair* on the *additive* domain.

Conversely, for some $k \geq 2$, let $G^{h,\tau}$ be $k-$*fair* on the *additive* domain. Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}_{ad}^N$ such that

$$h_i(c_{-i}) < W(c_{-i}, \mathbb{A}) - \frac{1}{n} \sum_{\alpha \in \mathbb{A}} (c_{-i}^{\alpha})_{\langle k-1 \rangle}. \tag{39}$$

Let $\widehat{c}_i \in \mathcal{C}_{ad}$ be such that for each $\alpha \in \mathbb{A}$, $\widehat{c}_i(\{\alpha\}) = (c_{-i}^{\alpha})_{[1]}$ and $\widehat{c} = (\widehat{c}_i, c_{-i}) \in \mathcal{C}_{ad}^N$. By k-fairness and equation (1),

$$h_i(\widehat{c}_{-i}) \geq W(\widehat{c}, \mathbb{A}) - \frac{1}{n} \sum_{\alpha \in \mathbb{A}} \widehat{c}_{\langle k \rangle}^{\alpha}. \tag{40}$$

Since $\widehat{c}_{-i} = c_{-i}$, by (39) and (40),

$$W(c_{-i}, \mathbb{A}) - W(\widehat{c}, \mathbb{A}) + \frac{1}{n} \left[\sum_{\alpha \in \mathbb{A}} \widehat{c}_{\langle k \rangle}^{\alpha} - \sum_{\alpha \in \mathbb{A}} (c_{-i}^{\alpha})_{\langle k-1 \rangle} \right] > 0. \tag{41}$$

Since for each $\alpha \in \mathbb{A}$, $\widehat{c}_i(\{\alpha\}) = (c_{-i}^{\alpha})_{[1]}$, then $W(\widehat{c}, \mathbb{A}) = W(c_{-i}, \mathbb{A}) = \sum\limits_{\alpha \in \mathbb{A}} (c_{-i}^{\alpha})_{[1]}$ and for each $\alpha \in \mathbb{A}$, $\widehat{c}_{\langle k \rangle}^{\alpha} = (c_{-i}^{\alpha})_{\langle k-1 \rangle}$. These equalities together contradict (41). □

(d) Let $h \in \mathcal{H}$ be as in (14) and $\mathcal{C} \in \{\mathcal{C}_{un}, \mathcal{C}_{ad}, \mathcal{C}_{sub}\}$.

• *Claim:* For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$ such that $W(c, \mathbb{A}) \leq W(c^i, \mathbb{A})$,

$$\frac{n-1}{n} W(c_{-i}, \mathbb{A}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A}). \tag{42}$$

Proof of the Claim: Let $N \in \mathcal{N}$, $i \in N$, and $c_{-i} \in \mathcal{C}^{N \setminus \{i\}}$. Since for each $c_i \in \mathcal{C}$, $W(c, \mathbb{A}) \leq W(c_{-i}, \mathbb{A})$, then $\frac{n-1}{n} W(c_{-i}, \mathbb{A}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c, \mathbb{A})$. This inequality and the fact that $W(c, \mathbb{A}) \leq W(c^i, \mathbb{A})$ together imply (42). ◊

By (14) and (42), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$ such that $W(c, \mathbb{A}) \leq W(c^i, \mathbb{A})$, $h_i(c_{-i}) \geq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A})$. By equation (1), $G^{h,\tau}$ respects *CIPLB* on the domain $\bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N$.

Conversely, let $G^{h,\tau}$ respect *CIPLB* on the domain $\bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N$. Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$h_i(c_{-i}) < \frac{n-1}{n} W(c_{-i}, \mathbb{A}). \tag{43}$$

If $\mathcal{C}^N = \mathcal{C}^N_{ad}$, then let $\widehat{c}_i \in \mathcal{C}_{ad}$ be such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = W(c_{-i}, \mathbb{A}) - W(c_{-i}, \mathbb{A}\backslash A)$.
If $\mathcal{C}^N \in \{\mathcal{C}^N_{un}, \mathcal{C}^N_{sub}\}$, then let $\widehat{c}_i \in \mathcal{C}$ be such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = W(c_{-i}, \mathbb{A})$.

Let $\widehat{c} = (\widehat{c}_i, c_{-i}) \in \mathcal{C}^N$. Since $W(\widehat{c}_i, c_{-i}, \mathbb{A}) = W(\widehat{c}^{\,i}, \mathbb{A}) = W(c_{-i}, \mathbb{A})$, we have

$$W(\widehat{c}, \mathbb{A}) - \frac{1}{n} W(\widehat{c}^{\,i}, \mathbb{A}) = \frac{n-1}{n} W(c_{-i}, \mathbb{A}). \tag{44}$$

Since $W(\widehat{c}_i, c_{-i}, \mathbb{A}) \leq W(\widehat{c}^{\,i}, \mathbb{A})$, by (1) and *CIPLB*, $h_i(\widehat{c}_{-i}) \geq W(\widehat{c}, \mathbb{A}) - \frac{1}{n} W(\widehat{c}^{\,i}, \mathbb{A})$. This inequality, (44), and the fact that $\widehat{c}_{-i} = c_{-i}$ together contradict (43). □

(e) Let $h \in \mathcal{H}$ be as in (15). Note that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$ such that $W(c, \mathbb{A}) \geq W(c^i, \mathbb{A})$,

$$W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A}) \geq 0. \tag{45}$$

By (15) and (45), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$ such that $W(c, \mathbb{A}) \geq W(c^i, \mathbb{A})$, $h_i(c_{-i}) \leq W(c, \mathbb{A}) - \frac{1}{n} W(c^i, \mathbb{A})$. By equation (1), $G^{h,\tau}$ respects *CIPUB*.

Conversely, let $G^{h,\tau}$ respect *CIPUB*. Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$h_i(c_{-i}) > 0. \tag{46}$$

Let $\widehat{c}_i \in \mathcal{C}$ be such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = 0$ and $\widehat{c} = (\widehat{c}_i, c_{-i}) \in \mathcal{C}^N$. Note that $W(\widehat{c}, \mathbb{A}) = W(\widehat{c}^{\,i}, \mathbb{A}) = 0$. Since $W(\widehat{c}, \mathbb{A}) \geq W(\widehat{c}^{\,i}, \mathbb{A})$, by (1) and *CIPUB*, $h_i(\widehat{c}_{-i}) \leq W(\widehat{c}, \mathbb{A}) - \frac{1}{n} W(\widehat{c}^{\,i}, \mathbb{A}) = 0$. This inequality and the fact that $\widehat{c}_{-i} = c_{-i}$ together contradict (46). □■

For the next proofs, we need the following notation: Let $c, \widehat{c}, \widetilde{c}, \ldots$ denote typical economies associated with the agent sets $N, \widehat{N}, \widetilde{N}, \ldots$, respectively. For each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, and each $r \in \{2, 3, ..., n\}$, let $\mathcal{C}^r$ be the set of economies that has r number of agents and $\mathcal{D}^r(N, i, c)$ be the set of all economies obtained by removing the cost functions of any $(n - r)$ number of agents from c_{-i} :

Notation 1. For each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, and each $r \in \{2, 3, ..., n\}$,

$$\mathcal{D}^r(N, i, c) \equiv \{c' \in \mathcal{C}^r\text{: there exists } N' \subseteq N \text{ with } i \in N' \text{ and } |N'| = r \text{ such that } c' = c_{N'}\}.$$

Proof of Theorem 2:
(a) Let $h \in \mathcal{H}$ be as in (17). By inequality (16), $G^{h,\tau}$ satisfies *population monotonicity*. Assume, by contradiction, that $G^{h,\tau}$ does not respect *SALB*. Then, by Proposition 2b, there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that

$$h_i(c_{-i}) < 0. \tag{47}$$

Let $N' \subseteq N$ be such that $N' = \{i, j\}$ for some $j \in N$. By (17), $h_i(c_j) \geq 0$ and by *population monotonicity*, $h_i(c_{-i}) \geq h_i(c_j)$. Altogether, $h_i(c_{-i}) \geq 0$, which contradicts (47).

Conversely, let $G^{h,\tau}$ satisfy *population monotonicity* and *SALB*. By inequality (12) and (16), h is as in (17). □

The proof for the rest of the parts is constructive.

Consider the single-task case. Let $\mathbb{A} = \{\alpha\}$. Let $G^{h,\tau}$ be a Groves mechanism that generates the minimal deficit in each economy among all Groves mechanisms that satisfy *population monotonicity* and *Axiom A**, where A^* is *IR* in part (b), *IPLB* in part (c), and $k-$*fairness* with $k \geq 2$ in part (d).

Note that (16) can be rephrased as follows: A Groves mechanism $G^{h,\tau}$ is population monotonic if and only if for each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq \max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} \{h_i(\widetilde{c}_{-i})\}. \tag{48}$$

Note that *population monotonicity* doesn't impose any restriction on economies with two agents, so we take $r \geq 3$.

As an example, let $N = \{1, 2, 3, 4\}$ and $c \in \mathcal{C}^N$. Let $G^{h,\tau}$ be population monotonic. Then, (48) should be true for N, c, and $i = 1$. That is,

· for $r = n = 4$, $h_1(c_2, c_3, c_4) \geq \max\{h_1(c_2, c_3), h_1(c_2, c_4), h_1(c_3, c_4)\}$, and

· for $r = 3$, for each pair $\{j, k\} \subset \{2, 3, 4\}$, $h_1(c_j, c_k) \geq \max\{h_1(c_j), h_1(c_k)\}$.

Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Note that since $\mathbb{A} = \{\alpha\}$, for each $i \in N$, $W(c_{-i}, \mathbb{A}) = (c^{\alpha}_{-i})_{[1]}$.
Since $\sum_{i \in N} t_i^{h,\tau}(c) = -(n-1)W(c, \mathbb{A}) + \sum_{i \in N} h_i(c_{-i})$, to minimize the deficit we need to minimize $\sum_{i \in N} h_i(c_{-i})$. For each $i \in N$, *population monotonicity* and *Axiom A** restrict the minimal value that $h_i(c_{-i})$ can take, which we investigate in the rest of the proof.

(b) By (48) and (11), *population monotonicity* and *IR* together imply, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^2(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq (\widehat{c}^{\alpha}_{-i})_{[1]}, \tag{49}$$

(note that *population monotonicity* doesn't impose any restriction on economies with two agents, hence the only restriction on economies in $\mathcal{D}^2(N, i, c)$ is by *IR*),
and for each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq \max\left(\max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} \{h_i(\widetilde{c}_{-i})\},\ (\widehat{c}^{\alpha}_{-i})_{[1]}\right). \tag{50}$$

To minimize the deficit, (49) and for $r = 3$, (50) should hold as an equality. Then, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c} \in \mathcal{D}^2(\widehat{N}, i, \widehat{c})} \left\{(\widetilde{c}^{\alpha}_{-i})_{[1]}\right\},\ (\widehat{c}^{\alpha}_{-i})_{[1]}\right). \tag{51}$$

Note that for each $i \in N$, each $r \in \{3, ..., n\}$, each $\widehat{c} \in \mathcal{D}^r(N, i, c)$, and each $s \in \{1, 2..., r-2\}$,

$$\max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} (\widetilde{c}^{\alpha}_{-i})_{[s]} = (\widehat{c}^{\alpha}_{-i})_{[s+1]}. \tag{52}$$

(Let c' be a maximizer of the LHS of (52). Then, c' is obtained from $\widehat{c}$ by removing the cost function of an agent with the lowest cost in $(\widehat{c}^{\alpha}_{-i})$. That is, $c' = \widehat{c}_{-j}$ where $\widehat{c}^{\alpha}_j = (\widehat{c}^{\alpha}_{-i})_{[1]}$.)

By (51) and (52), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max\left((\widehat{c}^{\alpha}_{-i})_{[2]},\ (\widehat{c}^{\alpha}_{-i})_{[1]}\right) = (\widehat{c}^{\alpha}_{-i})_{[2]}. \tag{53}$$

Similarly, to minimize the deficit, for $r = 4$, (50) should hold as an equality. Then, by (52) and (53), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^4(N, i, c)$,

$$\begin{aligned} h_i(\widehat{c}_{-i}) &= \max\left(\max_{\widetilde{c} \in \mathcal{D}^3(\widehat{N}, i, \widehat{c})} \left\{\max(\widetilde{c}^{\alpha}_{-i})_{[2]}\right\},\ (\widehat{c}^{\alpha}_{-i})_{[1]}\right), \\ &= \max\left((\widehat{c}^{\alpha}_{-i})_{[3]},\ (\widehat{c}^{\alpha}_{-i})_{[1]}\right) = (\widehat{c}^{\alpha}_{-i})_{[3]}. \end{aligned}$$

By recursive substitution, at each step applying (52) and minimizing the deficit (i.e., (50) holding as an equality for each $r \in \{3, ..., n\}$), we obtain, for each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max\left((\widehat{c}^{\alpha}_{-i})_{[r-1]},\ (\widehat{c}^{\alpha}_{-i})_{[1]}\right) = (\widehat{c}^{\alpha}_{-i})_{[r-1]}.$$

Note that $\widehat{c} \in \mathcal{D}^n(N, i, c)$ if and only if $\widehat{c} = c$. Hence, for $r = n$, we obtain the h function in (18). □

(c) By (48) and (4), *population monotonicity* and *IPLB* together imply, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^2(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq \frac{1}{2}(\widehat{c}^{\alpha}_{-i})_{[1]}, \tag{54}$$

and for each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq \max\left(\max_{\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})} \{h_i(\widetilde{c}_{-i})\},\ \frac{r-1}{r}(\widehat{c}^{\alpha}_{-i})_{[1]}\right). \tag{55}$$

To minimize the deficit, (54) and for $r = 3$, (55) should hold as an equality. Then, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c} \in \mathcal{D}^2(\widehat{N}, i, \widehat{c})} \left\{\frac{1}{2}(\widetilde{c}^{\alpha}_{-i})_{[1]}\right\},\ \frac{2}{3}(\widehat{c}^{\alpha}_{-i})_{[1]}\right). \tag{56}$$

By (56) and (52), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\frac{1}{2}(\widehat{c}^{\alpha}_{-i})_{[2]},\ \frac{2}{3}(\widehat{c}^{\alpha}_{-i})_{[1]}\right). \tag{57}$$

Similarly, to minimize the deficit, for $r = 4$, (55) should hold as an equality. Then, by (57) and a similar argument to (52), for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^4(N, i, c)$,

$$\begin{aligned} h_i(\widehat{c}_{-i}) &= \max\left(\max_{\widetilde{c} \in \mathcal{D}^3(\widehat{N}, i, \widehat{c})} \left\{\max\left(\frac{1}{2}(\widetilde{c}^{\alpha}_{-i})_{[2]},\ \frac{2}{3}(\widetilde{c}^{\alpha}_{-i})_{[1]}\right)\right\},\ \frac{3}{4}(\widehat{c}^{\alpha}_{-i})_{[1]}\right), \\ &= \max\left(\frac{1}{2}(\widehat{c}^{\alpha}_{-i})_{[3]},\ \frac{2}{3}(\widehat{c}^{\alpha}_{-i})_{[2]},\ \frac{3}{4}(\widehat{c}^{\alpha}_{-i})_{[1]}\right). \end{aligned}$$

By recursive substitution, at each step applying (52) and minimizing the deficit (i.e., (55) holding as an equality for each $r \in \{3, ..., n\}$), we obtain, for each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$\begin{aligned} h_i(\widehat{c}_{-i}) &= \max\left(\frac{1}{2}(\widehat{c}^{\alpha}_{-i})_{[r-1]},\ \frac{2}{3}(\widehat{c}^{\alpha}_{-i})_{[r-2]},\ \frac{3}{4}(\widehat{c}^{\alpha}_{-i})_{[r-3]}, ...,\ \frac{r-1}{r}(\widehat{c}^{\alpha}_{-i})_{[1]}\right), \\ &= \max_{p \in \{1,2,...,r-1\}} \left\{\frac{p}{p+1}(\widehat{c}^{\alpha}_{-i})_{[r-p]}\right\}. \end{aligned}$$

Note that $\widehat{c} \in \mathcal{D}^n(N, i, c)$ if and only if $\widehat{c} = c$. Hence, for $r = n$, we obtain the h function in (19). □

(d) By (48) and (13), *population monotonicity* and *k−fairness* together imply, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^2(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq (\widehat{c}^{\alpha}_{-i})_{[1]} - \frac{1}{2}(\widehat{c}^{\alpha}_{-i})_{\langle k-1 \rangle}, \tag{58}$$

and for each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$h_i(\widehat{c}_{-i}) \geq \max\left(\max_{\widetilde{c}\in\mathcal{D}^{r-1}(\widehat{N},i,\widehat{c})}\{h_i(\widetilde{c}_{-i})\},\ (\widehat{c}^{\alpha}_{-i})_{[1]} - \frac{1}{r}(\widehat{c}^{\alpha}_{-i})_{\langle k-1\rangle}\right). \tag{59}$$

To minimize the deficit, (58) and for $r = 3$, (59) should hold as an equality. Then, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N, i, c)$,

$$h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c}\in\mathcal{D}^{2}(\widehat{N},i,\widehat{c})}\left\{(\widetilde{c}^{\alpha}_{-i})_{[1]} - \frac{1}{2}(\widetilde{c}^{\alpha}_{-i})_{\langle k-1\rangle}\right\}, (\widehat{c}^{\alpha}_{-i})_{[1]} - \frac{1}{3}(\widehat{c}^{\alpha}_{-i})_{\langle k-1\rangle}\right). \tag{60}$$

Observation 1: For each $i \in N$, each $r \in \{2, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$(\widehat{c}^{\alpha}_{-i})_{[1]} - \frac{1}{r}(\widehat{c}^{\alpha}_{-i})_{\langle k-1\rangle} \equiv (\widehat{c}^{\alpha}_{-i})_{[1]} - \frac{1}{r}(\widehat{c}^{\alpha}_{-i})_{[\min(k-1,r-1)]} \equiv \max_{t\in\{r\}}\max_{s\in\{1,..,r+1-t\}}\left\{(\widehat{c}^{\alpha}_{-i})_{[s]} - \frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\right\}.$$

Proof: Note that there are $r - 1 = |\widehat{N}\backslash\{i\}|$ agents in $\widehat{c}_{-i}$. Hence, $(\widehat{c}^{\alpha}_{-i})_{\langle k-1\rangle} = (\widehat{c}^{\alpha}_{-i})_{[\min(k-1,r-1)]}$. $\Diamond$

Let $r \in \{3, ..., n\}$ and $t \in \{2, 3, ..., r - 1\}$.

Observation 2: For each $i \in N$, each $\widehat{c} \in \mathcal{D}^r(N, i, c)$, and each $m \in \{1, 2, ..., r - t\}$,

$$\max_{\widetilde{c}\in\mathcal{D}^{r-1}(\widehat{N},i,\widehat{c})}\left\{\max_{s\in\{m\}}\{(\widetilde{c}^{\alpha}_{-i})_{[s]} - \frac{1}{t}(\widetilde{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\}\right\} = \max_{s\in\{m,m+1\}}\left\{(\widehat{c}^{\alpha}_{-i})_{[s]} - \frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\right\}. \tag{61}$$

Proof: Let $\widetilde{c} \in \mathcal{D}^{r-1}(\widehat{N}, i, \widehat{c})$ be a maximizer of (61). Since $|\widetilde{N}| = |\widehat{N}| - 1 = r - 1$ and $i \in \widetilde{N}$, then $\widetilde{c} = \widehat{c}_{-j}$ for some $j \in \widehat{N}\backslash\{i\}$. That is, we obtain $\widetilde{c}_{-i}$ by removing the cost function of one agent from $\widehat{c}_{-i}$. Hence, either (i) $(\widetilde{c}^{\alpha}_{-i})_{[m]} = (\widehat{c}^{\alpha}_{-i})_{[m]}$ or (ii) $(\widetilde{c}^{\alpha}_{-i})_{[m]} = (\widehat{c}^{\alpha}_{-i})_{[m+1]}$.
In order to maximize (61), given $(\widetilde{c}^{\alpha}_{-i})_{[s]}$, we need $(\widetilde{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}$ as small as possible. Note that since $\min(k, t) \geq 2$, then $(\widetilde{c}^{\alpha}_{-i})_{[s]} \leq (\widetilde{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}$. Thus, if (i) holds, then $\widetilde{c}_{-i}$ includes the first $r - 2$ smallest costs in $\widehat{c}_{-i}$, i.e., $\widetilde{c}_{-i} = ((\widehat{c}^{\alpha}_{-i})_{[1]}, (\widehat{c}^{\alpha}_{-i})_{[2]}, ..., (\widehat{c}^{\alpha}_{-i})_{[\widehat{n}-2]})$. In this case, $(\widetilde{c}^{\alpha}_{-i})_{[m-2+\min(k,t)]} = (\widehat{c}^{\alpha}_{-i})_{[m-2+\min(k,t)]}$. Similarly, if (ii) holds, then $\widetilde{c}_{-i}$ includes the last $r - 2$ smallest costs in $\widehat{c}_{-i}$, i.e., $\widetilde{c}_{-i} = ((\widehat{c}^{\alpha}_{-i})_{[2]}, (\widehat{c}^{\alpha}_{-i})_{[3]}, ..., (\widehat{c}^{\alpha}_{-i})_{[\widehat{n}-1]})$ which implies $(\widetilde{c}^{\alpha}_{-i})_{[m-2+\min(k,t)]} = (\widehat{c}^{\alpha}_{-i})_{[m-1+\min(k,t)]}$. $\Diamond$

Observation 3: For each $i \in N$, each $r \in \{3, ..., n\}$, and each $\widehat{c} \in \mathcal{D}^r(N, i, c)$,

$$\max_{\widetilde{c}\in\mathcal{D}^{r-1}(\widehat{N},i,\widehat{c})}\left\{\max_{t\in\{2,..,r-1\}}\max_{s\in\{1,..,r-t\}}\{(\widetilde{c}^{\alpha}_{-i})_{[s]} - \tfrac{1}{t}(\widetilde{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\}\right\}$$
$$= \max_{t\in\{2,..,r-1\}}\max_{\widetilde{c}\in\mathcal{D}^{r-1}(\widehat{N},i,\widehat{c})}\left\{\max_{s\in\{1,..,r-t\}}\{(\widetilde{c}^{\alpha}_{-i})_{[s]} - \tfrac{1}{t}(\widetilde{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\}\right\},$$
$$= \max_{t\in\{2,..,r-1\}}\max_{s\in\{1,...,r+1-t\}}\left\{(\widehat{c}^{\alpha}_{-i})_{[s]} - \tfrac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\right\}.$$

Proof: The second equality follows from Observation 2. $\Diamond$

By (60) and Observations 1 and 3, for each $i \in N$ and each $\widehat{c} \in \mathcal{D}^3(N, i, c)$,

$$
\begin{aligned}
h_i(\widehat{c}_{-i}) &= \max\left(\max_{\widetilde{c}\in\mathcal{D}^2(\widehat{N},i,\widehat{c})}\{\max_{t\in\{2\}}\max_{s\in\{1,..,3-t\}}\{(\widetilde{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widetilde{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\}\},(\widehat{c}^{\alpha}_{-i})_{[1]}-\frac{1}{3}(\widehat{c}^{\alpha}_{-i})_{\langle k-1\rangle}\right),\\
&= \max\left(\max_{t\in\{2\}}\max_{s\in\{1,..,4-t\}}\{(\widehat{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\},\ (\widehat{c}^{\alpha}_{-i})_{[1]}-\frac{1}{3}(\widehat{c}^{\alpha}_{-i})_{\langle k-1\rangle}\right),\\
&= \max\left(\max_{t\in\{2\}}\max_{s\in\{1,..,4-t\}}\{(\widehat{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\},\max_{t\in\{3\}}\max_{s\in\{1,..,4-t\}}\{(\widehat{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\}\right)\\
&= \max_{t\in\{2,3\}}\max_{s\in\{1,..,4-t\}}\left\{(\widehat{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\right\}. \qquad (62)
\end{aligned}
$$

Similarly, to minimize the deficit, for $r=4$, (59) should hold as an equality. Then, by (62), for each $i\in N$ and each $\widehat{c}\in\mathcal{D}^4(N,i,c)$,

$$
h_i(\widehat{c}_{-i}) = \max\left(\max_{\widetilde{c}\in\mathcal{D}^3(\widehat{N},i,\widehat{c})}\left\{\max_{t\in\{2,3\}}\max_{s\in\{1,..,4-t\}}\{(\widetilde{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widetilde{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\}\right\},(\widehat{c}^{\alpha}_{-i})_{[1]}-\frac{1}{4}(\widehat{c}^{\alpha}_{-i})_{\langle k-1\rangle}\right).
$$

This equation and Observations 1 and 3 together imply, for each $i\in N$ and each $\widehat{c}\in\mathcal{D}^4(N,i,c)$,

$$
h_i(\widehat{c}_{-i}) = \max_{t\in\{2,3,4\}}\max_{s\in\{1,..,5-t\}}\left\{(\widehat{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\right\}.
$$

By recursive substitution, at each step applying Observations 1 and 3, and minimizing the deficit (i.e., (59) holding as an equality for each $r\in\{3,...,n\}$), we obtain, for each $i\in N$, each $r\in\{3,...,n\}$, and each $\widehat{c}\in\mathcal{D}^r(N,i,c)$,

$$
h_i(\widehat{c}_{-i}) = \max_{t\in\{2,3,..,r\}}\max_{s\in\{1,..,r+1-t\}}\left\{(\widehat{c}^{\alpha}_{-i})_{[s]}-\frac{1}{t}(\widehat{c}^{\alpha}_{-i})_{[s-2+\min(k,t)]}\right\}.
$$

Note that $\widehat{c}\in\mathcal{D}^n(N,i,c)$ if and only if $\widehat{c}=c$. Hence, for $r=n$, we obtain the h function in (20).□■

Proof of Table 2:

(i) Let $G^{h,\tau}$ be *individually rational*. Then, for each $N\in\mathcal{N}$, each $i\in N$, and each $c\in\mathcal{C}^N$, $u(G^{h,\tau}_i(c);c_i)\geq 0$. Since $0\geq\max_{c_i\in\mathcal{C}}\{-\frac{W(c^i,\mathbb{A})}{n},-c_i(\mathbb{A}),-\frac{1}{n}\sum_{\alpha\in\mathbb{A}}c^{\alpha}_{\langle k\rangle}\}$, $G^{h,\tau}$ respects *IPLB*, *SALB*, and $k-$*fairness* for $k\geq 1$.

(ii) Let $G^{h,\tau}$ be a Groves mechanism that respects *IPLB*. Then, for each $N\in\mathcal{N}$, each $i\in N$, and each $c\in\mathcal{C}^N$, $u(G^{h,\tau}_i(c);c_i)\geq -\frac{1}{n}W(c^i,\mathbb{A})$. Since $\frac{1}{n}W(c^i,\mathbb{A})\leq W(c^i,\mathbb{A})\leq c_i(\mathbb{A})$, then $G^{h,\tau}$ respects *SALB*.

(iii) On the *additive* domain, for each $N\in\mathcal{N}$, each $i\in N$, and each $c\in\mathcal{C}^N$, $\sum_{\alpha\in\mathbb{A}}(c^{\alpha}_{-i})_{[1]}=W(c_{-i},\mathbb{A})$. Hence, the result follows from the fact that on the *additive* domain, inequalities (4) and (13) are the same for $k=2$.

(iv) Let $\mathcal{C}\in\{\mathcal{C}_{ad},\mathcal{C}_{sub}\}$. Let $G^{h,\tau}$ be a Groves mechanism that respects *IPLB* on the domain $\bigcup_{N\in\mathcal{N}}\mathcal{C}^N$. Let $N\in\mathcal{N}$, $i\in N$, and $c\in\mathcal{C}^N$. By Proposition 1b and equation (1), $u(G^{h,\tau}_i(c);c_i)\geq W(c_{-i},\mathbb{A})-W(c,\mathbb{A})-\frac{1}{n}W(c_{-i},\mathbb{A})$. Note that $W(c_{-i},\mathbb{A})\geq W(c,\mathbb{A})$, for each $k\geq 2$ and each $\alpha\in\mathbb{A}$, $(c^{\alpha}_{-i})_{\langle k-1\rangle}\leq c^{\alpha}_{\langle k\rangle}$, and $W(c_{-i},\mathbb{A})\leq\sum_{\alpha\in\mathbb{A}}(c^{\alpha}_{-i})_{\langle k-1\rangle}$. Altogether, for each $k\geq 2$, $u(G^{h,\tau}_i(c);c_i)\geq -\frac{1}{n}\sum_{\alpha\in\mathbb{A}}c^{\alpha}_{\langle k\rangle}$, that is, $G^{h,\tau}$ is $k-$*fair* on the domain $\bigcup_{N\in\mathcal{N}}\mathcal{C}^N$.

(v) Follows from Proposition 1b and Proposition 2d.

(vi) Let $\mathcal{C} \in \{\mathcal{C}_{un}, \mathcal{C}_{ad}, \mathcal{C}_{sub}\}$. Let $G^{h,\tau}$ be a Groves mechanism that respects *CIPLB* on the domain $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$. Then, on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, h is as in (14). The result follows from the fact that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, the RHS of inequality (14) is greater than the RHS of (12).

(vii) The result follows from (iv) and (v). ■

Proof of Proposition 3:

Let $h \in \mathcal{H}$ be as in Proposition 3. Then, by equation (1), for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(G_i^{h,\tau}(c); c_i) = -W(c, \mathbb{A}) \geq -c_i(\mathbb{A})$. Hence, $G^{h,\tau}$ satisfies *SALB*.

Note that for each $M \in \mathcal{M}$, each $N \in \mathcal{N}$, and each $c \in \mathcal{C}^N$, $M(c) \geq 0$. Since for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{j \in N} t_j^{h,\tau}(c) = -(n-1)W(c, \mathbb{A}) \leq 0$, $G^{h,\tau}$ respects *M-bounded deficit* for any $M \in \mathcal{M}$.

Conversely, let $M \in \mathcal{M}$ and $G^{h,\tau}$ satisfy *SALB* and *M-bounded deficit*. Assume, by contradiction, that there are $N \in \mathcal{N}$, $i \in N$, and $c \in \mathcal{C}^N$ such that $h_i(c_{-i}) \neq 0$. By *SALB* and (12), $h_i(c_{-i}) > 0$ and $\sum_{j \in N} h_j(c_{-j}) > 0$. Now, let $\widehat{c}_i \in \mathcal{C}$ be such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = 0$ and $\widehat{c} = (\widehat{c}_i, c_{-i})$. Since $\widehat{c}_{-i} = c_{-i}$, we have $h_i(\widehat{c}_{-i}) = h_i(c_{-i}) > 0$. By *SALB* and (12), $\sum_{j \in N} h_j(\widehat{c}_{-j}) > 0$. This inequality and *M-bounded deficit* together imply

$$\begin{aligned} -(n-1)W(\widehat{c}, \mathbb{A}) &< -(n-1)W(\widehat{c}, \mathbb{A}) + \sum_{j \in N} h_j(\widehat{c}_{-j}), \\ &= \sum_{j \in N} t_j^{h,\tau}(\widehat{c}), \\ &\leq M(\widehat{c}). \end{aligned} \tag{63}$$

Since $W(\widehat{c}, \mathbb{A}) = \widehat{c}_i(\mathbb{A}) = 0$, we have $M(\widehat{c}) = 0$. Then, inequality (63) is $0 < 0$ which is a contradiction.
■

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