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Long-run equilibria, dominated strategies, and local interactions

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Long-run equilibria, dominated strategies, and local interactions*

Simon Weidenholzer[†]

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Abstract

The present note revisits a result by Kim and Wong (2010) showing that any strict Nash equilibrium of a coordination game can be supported as a long run equilibrium by properly adding dominated strategies. We show that in the circular city model of local interactions the selection of $\frac{1}{2}$-dominant strategies remains when adding strictly dominated strategies if interaction is "decentral". Conversely, if the local interaction structure is "central" by adding properly suited dominated strategies any equilibrium strategy of the original game can be supported as long run equilibrium.

Keywords: Dominated Strategies, Local Interactions, Learning.

JEL Classification Numbers: C72, D83.

1 Introduction

There are many situations where people can benefit from coordinating on the same action, e.g. a joint technology standard or a common social norm, thus giving rise to coordination game with multiple strict Nash equilibria. Traditional game theory stays however silent on the question which of these equilibria will emerge, as no equilibrium refinement concept can discard a strict Nash equilibrium.

Starting with the seminal works of Kandori, Mailath, and Rob (1993) and Young (1993) a strand of literature has emerged that can provide equilibrium predictions in the presence of multiple strict Nash equilibria. The basic idea behind this approach is to consider a population of boundedly rational agents who recurrently play a game against each other and decide upon their strategies using simple behavioral rules, as e.g. myopic best response learning or imitation learning. Within this setting a strategy configuration is then stochastically stable or long run equilibrium if it is more difficult to leave to other states than to enter from other states by the mean of independent mistakes. For coordination games, in a setting where everybody interacts with everybody else, this basically translates into the selection of $\frac{1}{2}$-dominant strategies, i.e. strategies that perform well in a world of uncertainty on the behavior of others. The reason for the selection of $\frac{1}{2}$-dominant strategies is that from any other state less than one half of the population has to be shifted (to the $\frac{1}{2}$-dominant strategy) for the $\frac{1}{2}$-dominant convention to be established. On the contrary, to upset the state where everybody plays the $\frac{1}{2}$-dominant strategy more than half of the population have to adopt a different strategy.

In a recent paper Kim and Wong (2010) challenge the validity of this approach by adding dominated strategies to the original game. Surprisingly, under best response learning this may change the prediction for the long run outcome. In particular, any Nash equilibrium of the base game can be supported by

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adding just one strategy that is dominated by all other strategies. The basic idea is that for any Nash equilibrium of a game one can construct a dominated strategy that is such that an agent will choose that Nash equilibrium strategy once only a "very small" fraction of her opponents choose the dominated strategy. This essentially implies that in a (properly) extended game one agent changing to the dominated strategy is enough to move into the basin of attraction of any Nash equilibrium strategy. Thus, by adding dominated strategies to a game the long run prediction can be reversed in a setting where interaction is global.

In the present note we demonstrate that this critique does not necessarily go through in Ellison's (1993) local interactions model where the population of agents is arranged around a circle and each agent interacts with a subset of the population only. In particular, we show that if the original game has a $\frac{1}{2}$-dominant strategy then the state where all agents choose the $\frac{1}{2}$-dominant strategy is the unique long run equilibrium in any extended game, that is obtained by adding strictly dominated strategies to the original game, provided that the interaction structure is sufficiently "decentral". The main idea behind this result is that $\frac{1}{2}$-dominant strategies may still spread out contagiously from an initially small subset of the population. Thus, the number of mutations required to move into the basin of attraction of the $\frac{1}{2}$-dominant convention is independent of the population size. Conversely, even in the presence of dominated strategies the effect of mutations in the presence of $\frac{1}{2}$-dominant strategies is local and hence depends on the population size. This implies that for a sufficiently large population the selection of $\frac{1}{2}$-dominant strategies remains. However, we find that the critical population size required for the selection of a $\frac{1}{2}$-dominant strategy is larger in the extended game than in the original game. We exploit this observation to show that if the population size is relatively small, meaning that interaction is rather "central", we can always find an extended game such that any equilibrium strategy of the original game is selected.

It is interesting to note that the robustness of Kandori, Mailath, and Rob's (1993) model of global interactions has been challenged in the following three ways: i) Kandori, Mailath, and Rob (1993) themselves have argued that the low speed of convergence renders the model's predictions irrelevant for large populations, ii) Bergin and Lipman (1996) have shown that the model's predictions are not independent of the underlying model of noise, and just recently iii) Kim and Wong (2010) have argued that the model is not robust to the addition of strictly dominated strategies. In Ellison's (1993) model of local interactions risk dominant strategies are able to spread out contagiously implying a high speed of convergence for even large populations, thereby essentially challenging the first critique. Further, Lee, Szeidl, and Valentinyi (2003) argue that this contagious spread essentially also implies that the prediction in a local interactions model will be independent of the underlying model of noise for a sufficiently large population. The present note shows that for a sufficiently large population the local interaction model is also robust to the addition (and thus also elimination) of strictly dominated strategies.

The remainder of the note is organized in the following way. Section 2 presents the model and discusses the main techniques used. Section 3 spells out our main results and Section 4 concludes.

2 The model

2.1 Local interaction games

We consider a population of m agents who recurrently play a symmetric two player $n \times n$ game against each other. We denote by S the set of pure strategies. We denote by Σ the set of mixed strategies over S and by $\sigma(s)$ the probability put on the pure strategy s under a mixed strategy $\sigma \in \Sigma$. We denote the payoff of the pure strategy s against the pure strategy s' by $u(s,s')$ and against a mixed strategy $\sigma \in \Sigma$ by $u(s,\sigma)$. We let

$$E = \{s \in S | u(s,s) > u(s',s) \text{ for all } s' \neq s\}$$

denote the set of pure strategy (symmetric) strict Nash equilibria of the game, and denote a typical element of E by e. Further we denote by

$$BR(\sigma) = \{s \in S | u(s,\sigma) \geq u(s',\sigma) \text{ for all } s' \neq s\}$$

the set of pure strategy best responses to a mixed strategy $\sigma \in \Sigma$.

As in Ellison's (1993) model of local interactions we assume that our population is arranged around a circle, so that a given agent i has agents $i-1$ and $i+1$ (mod m) as immediate neighbors (see Figure 1). Each agent plays the game against his $2k$ closest neighbors (with $k < \frac{m-1}{2}$) in discrete time $t = 0, 1, 2, \ldots$. We refer to k as the interaction radius. We denote by $K(i) = \{i-k, \ldots, i-1, i+1, \ldots, i+k\}$ the set of agents who interact with agent i and call this set the neighbors of agent i. If $\omega(t) = (s_1, \ldots, s_m)$ is the profile of strategies adopted by players at time t, the payoff for player i under the $2k$-neighbor matching rule is

$$U(i,\omega(t)) = \sum_{j \in K(i)}^{k} u(s_i, s_j).$$

We denote the monomorphic states $\omega = (s, s, \ldots, s)$ where all agents adopt the same strategy s as $\overrightarrow{s}$.

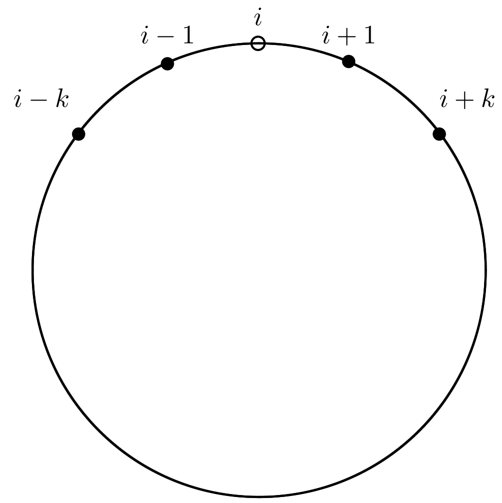

Figure 1: The circular city model of local interaction.

2.2 Learning

Each period, with exogenous probability $\eta \in (0,1)$ an agent might receive the opportunity to revise her strategy.[1] We assume that when such a revision opportunity arises she switches to a myopic best response, i.e. she plays a best response to the distribution of play in her neighborhood in the previous period. More formally, we consider a *myopic best reply dynamics*, where at time $t+1$ player i chooses

$$s_i(t+1) \in \arg\max U\left(i, \omega(t)\right)$$

given the state $\omega(t)$ at t. If a player has several alternative best replies, we assume that she randomly adopts one of them, assigning positive (exogenously given) probability to each. Further, with small

[1] I.e. we consider a model of positive inertia where agents may not adjust their strategy with certainty in each period. We remark however that considering a model with no inertia ($\eta = 1$) will not qualitatively change our results.

probability ϵ the agent ignores her prescription and chooses a strategy at random, i.e. she makes a mistake or mutates.

The adjustment process described gives rise to a Markov process with state space $\Omega = S^m$ and transition matrix $P(\epsilon)$, for which the standard techniques apply. In particular, we will be using Ellison's (2000) Radius–Coradius Theorem to identify the set of long run equilibria, corresponding to states in the support of the limit invariant distribution as the mutation probability goes to zero.

2.3 Review of Techniques

We refer to the process without mistakes ($\epsilon = 0$) as the *unperturbed* process and call the process with mistakes ($\epsilon > 0$) the *perturbed* process. Since $P(\epsilon)$ is strictly positive for $\epsilon > 0$, the perturbed process always has a unique strictly positive invariant distribution $\mu(\epsilon) \in \Delta(\Omega)$.

The *limit invariant distribution* (as the rate of experimentations tends to zero) $\mu^* = \lim_{\epsilon \to 0} \mu(\epsilon)$ exists and is an invariant distribution *of the unperturbed process* (see e.g. Freidlin and Wentzell (1988), Kandori, Mailath, and Rob (1993), Young (1993), or Ellison (2000)). It singles out a stable prediction of the original process, in the sense that, for any ϵ small enough, the play approximates that described by μ^* in the long run. The states in the support of μ^*, $\{\omega \in \Omega \mid \mu^*(\omega) > 0\}$ are called *Long Run Equilibria (LRE)* or stochastically stable states. Ellison (2000) presents a powerful method to determine the set of LRE which is based on a characterization by Freidlin and Wentzell (1988). Let X and Y be two absorbing sets of the unperturbed process and let $c(X, Y) > 0$ be the minimal number of mistakes needed for a direct transition from X to Y (i.e. the cost of transition). Define a path P of length $\ell(P)$ from X to Y as a finite sequence of absorbing sets $P = \{X = S_0, S_1, ..., S_{\ell(P)} = Y\}$ and let $S(X, Y)$ be the set of all paths from X to Y. The *cost* of the path is given by the sum of its transition costs

$$c(P) = \sum_{k=1}^{l(P)} c(S_{k-1}, S_k).$$

The minimal number of mistakes required for a (possibly indirect) transition from X to Y is

$$C(X, Y) = \min_{P \in S(X,Y)} c(P).$$

The *Radius* of an absorbing set X is defined as

$$R(X) = \min\{C(X, Y) | Y \text{ is an absorbing set, } Y \neq X\},$$

i.e. the minimal number of mistakes needed to leave X. The *coradius* of X is defined as

$$CR(X) = \max\{C(Y, X) | Y \text{ is an absorbing set, } Y \neq X\},$$

i.e. the maximal number of mistakes needed to reach X. Ellison (2000) shows that

Lemma 1. *(Ellison 2000). If $R(X) > CR(X)$ the only long run equilibria (LRE) are contained in X.*

Note that $R(X) > CR(X)$ simply expresses the idea that for X to be LRE X should be easier to reach than to leave by simultaneous mutations.

3 Dominated strategies and local interactions

3.1 An example

In a first step, we present an example proposed by Kim and Wong (2010) to illustrate their critique and to show why their critique may not bite in a local interactions framework. To this end, consider the

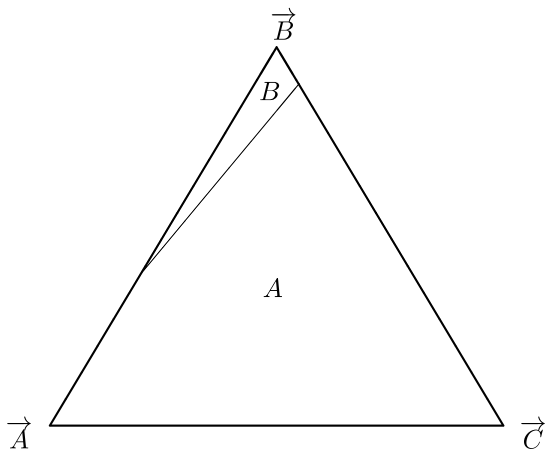

Figure 2: Best response regions of the extended game $\tilde{G}$ for large W.

following 2×2-coordination game.

$G =$

	A	B
A	8, 8	0, 4
B	4, 0	6, 6

We have two Nash equilibria in pure strategies, (A, A) and (B, B). Furthermore, B is a best response whenever at least a fraction of $\frac{2}{5}$ of the population adopts it. Hence, B is risk dominant in the sense of Harsanyi and Selten (1988), i.e. it is a best response against a mixed strategy that assigns probability $\frac{1}{2}$ to both actions. In the setup of Kandori, Mailath, and Rob (1993) where each agent interacts with each other agent in the population, for a sufficiently large population, it takes approximately $\frac{2}{5}$ of the population to switch strategies to move from the convention $\overrightarrow{A}$ to the convention $\overrightarrow{B}$. The opposite transition takes approximately $\frac{3}{5}$ of the population to mutate. Hence we have $CR(\overrightarrow{B}) \simeq \frac{2m}{5}$ and $R(\overrightarrow{B}) \simeq \frac{3m}{5}$, implying that $\overrightarrow{B}$ is LRE in a sufficiently large population.

Kim and Wong add a third strategy to obtain the following extended game.

$\tilde{G} =$

	A	B	C
A	8, 8	0, 4	$-W, -3W$
B	4, 0	6, 6	$-2W, -3W$
C	$-3W, -W$	$-3W, -2W$	$-3W, -3W$

Note that if W is chosen large enough we have that A is a best response whenever only one agent chooses C. Figure 2 underscores this point by plotting the best response regions of the extended game. Hence, in the extended game we can move with one mutation from $\overrightarrow{B}$ to $\overrightarrow{A}$, implying $CR(\overrightarrow{A}) = 1$. For a large enough population, $\overrightarrow{A}$ can however not be left with one mutation, establishing $R(\overrightarrow{A}) > 1$.

Let us now consider the circular city model of local interactions and assume that each player only interacts with her two closest neighbors, i.e. $k = 1$. As B is risk dominant each agent will have B as her best response if one of her neighbors adopts it. Consider the convention $\overrightarrow{A}$ and assume that one agent mutates to B:

$$\ldots AABAA \ldots$$

With positive probability the B-agent does not receive a revision opportunity whereas the boundary A-agents receive a revision opportunity and we move to the state:

$$\ldots ABBBA \ldots$$

Iterating this argument, we arrive at the risk dominant convention $\overrightarrow{B}$. Thus, establishing that $CR(\overrightarrow{A}) = 1$, in both the original game G and the extended game $\tilde{G}$.

In the original game G we have that any state where two adjacent players choose B lies in the basin of $\overrightarrow{B}$. Hence, if $m \geq 3$ we need more than one mutation in order to move out of the basin of attraction of $\overrightarrow{B}$, establishing $R(\overrightarrow{B}) > 1$, whenever $m \geq 3$.

Consider now the extended game $\tilde{G}$ and the risk dominant convention $\overrightarrow{B}$. Assume that one agent mutates to C:

$$\ldots BBCBB \ldots$$

With positive probability the C-player does not adjust her strategy whereas the B-players switch to A and we reach the state

$$\ldots BACAB \ldots \rightarrow \ldots BAAAB \ldots$$

Unless, there is no or only one B-agent left, we will for sure move back to the risk dominant convention, establishing that $R(\overrightarrow{B}) > 1$, whenever $m \geq 5$.

Thus, in the circular city model the selection of the risk dominant convention $\overrightarrow{B}$ remains for a sufficiently large population. In the following section, we will discuss this idea in more detail.

3.2 p-dominance and extended games

We first introduce the concepts of p-dominance and extended games which will be key to our analysis.

Morris, Rob, and Shin (1995) define a strategy s to be p-dominant if s is the unique best response against any mixed strategy σ such that $\sigma(s) \geq p$. Note that in 2×2 games the concept of $\frac{1}{2}$-dominance coincides with Harsanyi and Selten's (1988) concept of risk dominance. Further note that a p-dominant strategy is a strict Nash equilibrium whenever $p < 1$.

We adopt the following definition given by Kim and Wong (2010) to introduce the concept of an extended game.

Definition 1. We say that a finite symmetric game $\tilde{G} = (\tilde{S}, \tilde{u})$ extends another finite symmetric game $G = (S, u)$ if

(a) (Enlargement) $S \subseteq \tilde{S}$, and $\tilde{u}|_{S \times S} = u$

(b) (Dominance) $\tilde{u}(\tilde{s}, s) < \tilde{u}(s, s')$ for all $s \in S$, all $\tilde{s} \in \tilde{S} \backslash S$ and all $s' \in \tilde{S}$

That is an extended game $\tilde{G}$ of a game G is obtained by adding strategies to the game G which are all strictly dominated by all strategies of the original game. Consequently, the set of Nash equilibria remains unaffected by the extension, i.e. $\tilde{E} = E$. Note however that if a strategy $s^* \in S$ is p-dominant in the original game it need not be p-dominant in the extended game.

Kim and Wong (2010) have shown that for each equilibrium strategy $e \in E$ of the original game G there exists an extended game $\tilde{G}$ with $\tilde{S} = S \cup d$ such that $BR(\sigma) = e$ for all $\sigma \in \Sigma$ with $\sigma(d) > x$ where x can be chosen arbitrarily close to zero.

The basic idea behind this extension is the following. Strategy d earns a payoff of $-3W$ against all strategies, strategy e earns a payoff of $-W$ against d, and all other strategies different to e and s earn a payoff of $-2W$ against d. So, by choosing W sufficiently large e can be made a best response to any mixed strategy that only puts a very small weight on d. Under global interactions this however implies that one can always find an extended game such that each agent will have e as a best response as soon as only one of his interaction partners adopts d. Thus, to move into the basin of attraction of the state where everybody chooses e one mutation is sufficient. This in turn implies that for each equilibrium strategy e of a game G one can find an extended game $\tilde{G}$ such that the long run prediction changes to e. In the next section we will establish that this manipulation of the original game G might not necessarily change the long run prediction.

3.3 Main results

In a first step, as a benchmark scenario, we consider a game G with a $\frac{1}{2}$-dominant strategy. This essentially replicates the analysis of Ellison (1993, 2000) with the main difference that we have a model with positive inertia whereas Ellison's model features strategy adjustment in each round. Although the proof essentially follows Ellison (1993, 2000) we report it here for the reader's convenience.

Proposition 2. *Suppose the game G has a $\frac{1}{2}$-dominant strategy s^*. Then the state $\overrightarrow{s}^*$ is the unique LRE provided that*

$$m > (k+1)^2. \tag{1}$$

Proof. First, observe that each agent will have s^* as her best response as soon as at least k of her $2k$ neighbors adopts it. Consider, now any absorbing state $\omega \neq \overrightarrow{s}$. If there is a cluster of k adjacent s^*-players, players at the boundary of this cluster will have s^* as their best response and when given revision opportunity will switch to s^*. In a next step, agents at the new boundary might receive revision opportunity and will also switch to s^*. In this manner the $\frac{1}{2}$-dominant strategy will spread out contagiously and we reach the convention $\overrightarrow{s}^*$, thus establishing that $CR(\overrightarrow{s}^*) \leq k$.

Second, let us assess how difficult it is to leave $\overrightarrow{s}^*$ by the mean of independent mutations. First, note that if we have a cluster of $k+1$ adjacent s^*-players the dynamics will for sure move back to $\overrightarrow{s}^*$. For, (i) each of the agents in the cluster has k neighbors choosing s^* and thus will never switch, and (ii) agents at the boundary of such a cluster will switch to s^* whenever given revision opportunity.[2] Hence, in order to leave the basin of attraction of $\overrightarrow{s}^*$ we at least need one mutation per each $k+1$ agents, establishing $R(\overrightarrow{s}^*) \geq \lceil \frac{m}{k+1} \rceil$.[3] Thus, $\overrightarrow{s}^*$ is LRE if (1) holds. □

Let us now consider an extended game $\tilde{G}$. It is important now that if the game G has a $\frac{1}{2}$-dominant strategy s^* this strategy will not necessarily be $\frac{1}{2}$-dominant in the extended game $\tilde{G}$. For instance, in the introductory example B was $\frac{1}{2}$-dominant in the original game but not in the extended game. Nevertheless, we find that if the original game has a $\frac{1}{2}$ strategy under local interactions there does not exist an extension that can reverse the long run prediction provided that the population size is large compared to the interaction radius. This is the content of the next proposition.

Proposition 3. *Suppose the game G has a $\frac{1}{2}$-dominant strategy s^*. Then the state $\overrightarrow{s}^*$ is the unique LRE in any extended game $\tilde{G}$ provided that*

$$m > (k+1)(3k+1). \tag{2}$$

Proof. In a first step, recall that all strategies $\tilde{s} \in \tilde{S} \backslash S$ are strictly dominated by all strategies $s \in S$. Thus, under our best response dynamic there can not be an absorbing state where some player chooses a strategy $\tilde{s} \in \tilde{S} \backslash S$. Now consider any state $\omega \neq \overrightarrow{s}^*$. As above, s^* can spread out contagiously from any cluster of adjacent s^*-players of at least size k, establishing $CR(\overrightarrow{s}^*) \leq k$.

[2]Note that if there are less than $k+1$ adjacent s^* agents those agents in the cluster have less than k of their neighbors choosing s^* and thus when given revision opportunity may switch to some other strategy $s^* \neq s$.

[3]Where $\lceil x \rceil$ denotes the smallest integer larger than x.

Now consider the $\frac{1}{2}$-dominant convention $\overrightarrow{s}^*$ and assume that one agent mutates to a strategy $\tilde{s} \in \tilde{S}\backslash S$:

$$\ldots s^* s^* \underbrace{s^* \ldots s^*}_{k} \tilde{s} \underbrace{s^* \ldots s^*}_{k} s^* s^* \ldots$$

In the worst case, in the extended game each player will have some $s \in S$ with $s \neq s'$ as her best response if only one of her $2k$-neighbors chooses a strictly dominated strategy $\tilde{s} \in \tilde{S}\backslash S$. Thus, it might be the case that after only one mutation a cluster of $2k+1$ agents shifts to some other strategy $s \in S$. Note that if after this event there is still a cluster of s^*-players of size $k+1$ left we will for sure go back to the state $\overrightarrow{s}^*$. Hence, the most efficient way to move out of the basin of attraction of $\overrightarrow{s}^*$ is by seeding one mutation per $3k+1$ agents:

$$\ldots \underbrace{s \ldots s}_{2k+1} \underbrace{s^* \ldots s^*}_{<k+1} \underbrace{s \ldots s}_{2k+1} \underbrace{s^* \ldots s^*}_{<k+1} \ldots$$

This implies that we need at least $\lceil \frac{m}{3k+1} \rceil$ mutations to move out of the basin of attraction of $\overrightarrow{s}^*$, establishing $R(\overrightarrow{s}^*) \geq \lceil \frac{m}{3k+1} \rceil$. Hence, $\overrightarrow{s}^*$ is LRE in the extended game if (2) holds. □

Proposition 3 implies that the selection of $\frac{1}{2}$-dominant strategies remains in the circular city model of local interaction in the extended game $\tilde{G}$ provided that the population size is large relative to the interaction radius. Thus, if the interaction structure is rather "decentral" the selection of $\frac{1}{2}$- dominant can not be reverted by adding dominated strategies. The main reason behind this result is that a $\frac{1}{2}$-dominant strategy may still spread out contagiously from an initially small subgroup in the extended game. In other words, the number of mistakes to move into the $\frac{1}{2}$-dominant convention is independent of the population size. Conversely, the number of mistakes needed to upset the $\frac{1}{2}$-dominant convention, despite being increased in the extended game, still depends on the population size. Thus, for a sufficiently large population the first effect always dominates the second effect and the $\frac{1}{2}$-dominant convention will be LRE.

It is interesting to note that the threshold for the extended game $\tilde{G}$ identified in (2) is stronger than the threshold for the original game G identified in (1). Thus, for smaller population sizes adding dominated strategies might influence the long run prediction. We will study this issue in more detail in the next proposition. Before that we will however introduce the following definition.

Definition 2. We define the *p-potential* of a strategy s to be smallest p for which s is p-dominant and denote the p-potential of a strategy s by p^s.

Note that all strict Nash equilibrium strategies of a game have a p-potential strictly smaller than one. Further, note that if strategy s has a p-potential of p^s no other strategy can have a p-potential smaller than $1 - p^s$. For instance, in the introductory example we had $p^A = \frac{3}{5}$ and $p^B = \frac{2}{5}$. We are now able to state the following result.

Proposition 4. *For each equilibrium strategy $e \in E$ of a symmetric game G there exists an extended game $\tilde{G}$ with $\tilde{S} = S \cup d$ such that the state $\overrightarrow{e}$ is the unique LRE provided that*

$$m < (2k - 2kp^e - 1)(2k + 1). \tag{3}$$

Proof. First, note that as (e, e) is a Nash equilibrium it follows that the state $\overrightarrow{e}$ is absorbing. Now, consider any state $\omega \neq \overrightarrow{e}$ and assume that one agent mutates to $d \in \tilde{S}\backslash S$.

$$\ldots ss \underbrace{s \ldots s}_{k} d \underbrace{s \ldots s}_{k} ss \ldots$$

As there exists an extended game in which e is a best response for any player who has only one of his neighbors playing d. Thus, with positive probability all agents who interact with the d-player will switch and the d-player will remain at her strategy and we reach the state.

$$\dots ss\underbrace{e\dots e}_{k}d\underbrace{e\dots e}_{k}ss\dots$$

When given revision opportunity the d-player will also switch to e, implying that with one mutation we can directly shift $2k+1$ agents to e. Further, note that if there is one agent left playing some strategy $s \neq e$, then this player will always switch to e, as e is a strict Nash equilibrium. This implies that with (at most) $\lceil \frac{m}{2k+2} \rceil$ mutations we can move into the state $\overrightarrow{e}$, establishing $CR(\overrightarrow{e}) \leq \lceil \frac{m}{2k+2} \rceil$.

Now consider the state $\overrightarrow{e}$. As e has a p-potential of p^e, a player will only switch to some other strategy if more than $\lceil 2k(1-p^e) \rceil$ of her neighbors choose some strategies $s \neq e$. Hence, to move out of the basin of attraction of $\overrightarrow{e}$ we need more than $\lceil 2k(1-p^e) \rceil$ agents to mutate to some other strategy, establishing $R(\overrightarrow{e}) \geq \lceil 2k(1-p^e) \rceil$. Thus, if (3) holds $\overrightarrow{e}$ is LRE. □

First, note that if $p^e > \frac{1}{2}$ the threshold (3) is weaker than the threshold (2), implying that Propositions 3 and 4 are consistent with each other. Let us now assess whether the threshold (3) identified in Proposition 4 is compatible with a local interaction system, i.e. a situation where $m > 2k+1$ holds. This translates into the condition

$$(2k - 2kp^e - 1)(2k+1) - 2k + 1 > 0.$$

Note that as $p^e < 1$ one can always find some k such that the previous inequality holds. Thus, for rather "central" interaction structures one can indeed find dominated strategies such that any equilibrium strategy of the original game is the long run equilibrium in a local interaction model. We highlight our results of Proposition 3 and 4 by plotting local interactions systems (i.e. interaction radii and population sizes) under which the prediction of the introductionary example can be reversed and the local interaction systems where this is never possible.

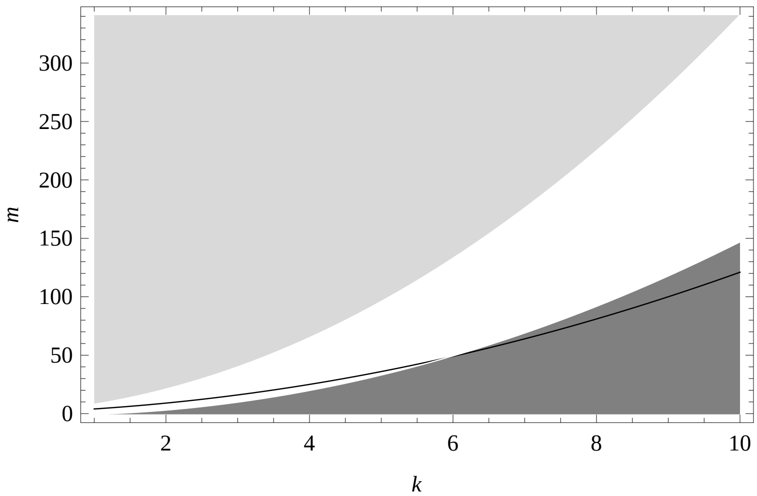

Figure 3: Equilibrium selection regions for the introductionary example $\tilde{G}$ for sufficiently large W. The light gray region indicates local interaction systems where in light of Proposition 3 the selection of $\vec{B}$ can not be reversed. The dark grey region indicates local interaction systems where Proposition 4 bites and $\vec{A}$ is selected. The black line represents the threshold for selection of $\vec{B}$ in the original game G, identified in Proposition 2. Thus, in the dark gray area above the black line the long run prediction is reversed in the extended game.

4 Conclusion

The main reason why the local interaction model is robust to the critique by Kim and Wong (2010) (but also why it has a high speed of convergence and its predictions are largely independent of the underlying model of noise) is the fact that a $\frac{1}{2}$-dominant strategy may spread out contagiously in the circular city model. Thus, one might be able to show that this nice feature of the circular city model might carry over to more general interaction structures if there is a strategy that is contagious in the sense of Morris (2000).[4] One might be tempted to use the nice properties of the local interaction model to defend results of global interaction models. Note, however, that the examples in Ellison (1993) and Alós-Ferrer and Weidenholzer (2007) demonstrate that local and global interactions models may display different long run predictions, once one moves beyond the class of 2×2 games. Thus, one has to be very cautious when justifying the results of a global model by a local model.

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[4]The original notion of contagion by Morris (2000) covers only infinite populations. See e.g. Alós-Ferrer and Weidenholzer (2008) for an adaption to finite populations.