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On constrained set-valued optimization*

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Abstract

The set-valued optimization problem $\min_C F(x)$, $G(x)\cap(-K) \neq \emptyset$, is considered, where $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ and $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ are set-valued functions, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones. Two type of solutions, called w-minimizers (weakly efficient points) and i-minimizers (isolated minimizers), are treated. In terms of the Dini set-valued directional derivative first-order necessary conditions for a point to be a w-minimizer, and first-order sufficient conditions for a point to be an i-minimizer are established, both in primal and dual form.

Key words: Set-valued optimization, First-order optimality conditions, Dini derivatives.

Math. Subject Classification: 49J53, 49J52, 90C29, 90C30, 90C46.

1 Introduction

The constrained set-valued optimization problem (svp)

$$\min_C F(x), \quad G(x) \cap (-K) \neq \emptyset\,, \tag{1}$$

is considered, where $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ and $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ are set-valued functions (svf) with non-empty values, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones. First order optimality conditions in terms of the Dini set-valued directional derivative are derived. The obtained results generalize those of [3] from vector to set-valued problem, and of [1] from unconstrained to constrained problem. Recently optimality conditions for svp are studied mainly by means of epiderivatives, e. g. in [4], [5] and [2]. We consider the optimality conditions based on directional derivatives as certain alternative of those based on epiderivatives. Some comparison of the two methods is done in [1].

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2 Preliminaries

The space $\mathbb{R}^k$ is considered with the usual topology. It is generated by an arbitrary norm. Since all the norms in $\mathbb{R}^k$ are equivalent, we make use of this sometimes choosing a special norm. The dual pairing in $\mathbb{R}^k$ is a denoted $\langle \cdot, \cdot \rangle$. It is a mapping $\mathbb{R}^k \times (\mathbb{R}^k)^* \to \mathbb{R}$, where $(\mathbb{R}^k)^*$ stands for the dual of $\mathbb{R}^k$. In fact $(\mathbb{R}^k)^*$ can be identified with $\mathbb{R}^k$ as linear and topological spaces, but when $\mathbb{R}^k$ is considered with a norm, the dual space $(\mathbb{R}^k)^*$ is supplied with the dual norm. When $\mathbb{R}^k$ is supplied with an Euclidean norm, then $(\mathbb{R}^k)^*$ can be identified with $\mathbb{R}^k$ also as a norm space. The notations B_k and $\bar{B}_k$ are used for the open and closed unit balls, and $B_k(x^0)$ and $\bar{B}_k(x^0)$ for the open and closed unit balls with center x^0.
For a given closed convex cone $M \subset \mathbb{R}^k$ its positive polar cone is defined by $M' = \{\xi \in (\mathbb{R}^k)^* \mid \langle \xi, x \rangle \geq 0 \text{ for all } y \in M\}$. When $x^0 \in M$ we put $M'[x^0] = \{\xi \in M' \mid \langle \xi, x^0 \rangle = 0\}$ and $M[x^0] = (M'[x^0]))'$. It holds $M \subset M[x^0]$.
When $\mathbb{R}^k$ is considered with a concrete norm, the distance from a point $x \in \mathbb{R}^k$ to a set $A \subset \mathbb{R}^k$ is given by $d(x, A) = \inf\{\|x - y\| \mid a \in A\}$. The oriented distance from x to A is defined by $D(x, A) = d(x, A) - d(x, \mathbb{R}^k \setminus A)$. When $M \subset \mathbb{R}^k$ is a proper closed convex cone, then $D(x, -M) = \sup\{\langle \xi, x \rangle \mid \xi \in M',\ \|\xi\| = 1\}$. We define the oriented distance $D(P, A)$ from a set $P \subset \mathbb{R}^k$ to the set $A \subset \mathbb{R}^k$ putting $D(P, A) = \inf\{D(x, A) \mid x \in P\}$.
Using the oriented distance we introduce the following notation. Let $M \subset \mathbb{R}^k$ be a cone and let a be a real number. Then we put $M(a) = \{x \in \mathbb{R}^k \mid D(x, M) \leq a\,\|x\|\}$. The weakly efficient frontier (w-frontier) $w\text{-Min}_M A$ and the properly efficient frontier (p-frontier) $p\text{-Min}_M A$ of A are defined respectively by $w\text{-Min}_M A = \{x \in A \mid A \cap (x - \operatorname{int} M) = \emptyset\}$ and $p\text{-Min}_M A = \{x \in A \mid \exists\, a \in (0, 1) : A \cap (x - M(a)) = \{x\}\}$.
The set of the feasible points of svp (1) is defined by $\mathcal{G} = \{x \in \mathbb{R}^n \mid G(x) \cap (-K) \neq \emptyset\}$. Further $\mathcal{N}(x^0)$ denotes the family of the neighbourhoods of x^0. We deal with local solutions of (1), which in any case are pairs (x^0, y^0), $y^0 \in F(x^0)$, with x^0 feasible. Here we use the following concepts of solutions for problem (1). The pair (x^0, y^0), $x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, is said a w-minimizer (weakly efficient point) if there exists $U \in \mathcal{N}(x^0)$ such that $x \in U \cap \mathcal{G}$ implies $F(x) \cap (y^0 - \operatorname{int} C) = \emptyset$ (then necessary $y^0 \in w\text{-}Min_C F(x^0)$). The pair (x^0, y^0) is said an i-minimizer (isolated minimizer) if there exists $U \in \mathcal{N}(x^0)$ and a constant $A > 0$ such that $D(F(x) - y^0, -C) \geq A\,\|x - x^0\|$ and $y^0 \in p\text{-Min}_C F(x^0)$ for $x \in U \cap \mathcal{G}$ (the concept of i-minimizer is norm-independent, since all norms in finite-dimensional spaces are equivalent).
The svf $\Phi : \mathbb{R}^n \rightsquigarrow \mathbb{R}^k$ is said locally Lipschitz at $x^0 \in \mathbb{R}^n$, if there exists $U \in \mathcal{N}(x^0)$ and a constant $L > 0$, such that for x^1, $x^2 \in U$ it holds $\Phi(x^2) \subset \Phi(x^1) + L\,\|x^2 - x^1\|\,\bar{B}_k$. The svf Φ is said locally Lipschitz, if it is locally Lipschitz at each $x^0 \in \mathbb{R}^n$. Given a cone $M \subset \mathbb{R}^k$, we say that Φ is locally M-Lipschitz at x^0 if the svf $x \rightsquigarrow \Phi(x) + M$ is locally Lipschitz at x^0. The svf Φ is said locally M-Lipschitz, if it is locally M-Lipschitz at each point x^0.
The cone $M \subset \mathbb{R}^k$ is said pointed if $(-M) \cap M = \{0\}$, and M is contained in a half-space of $\mathbb{R}^k$. If $\mathbb{R}^k$ is supplied with an Euclidean norm, then the cone M is said non-obtuse, if $\langle x^1, x^2 \rangle \geq 0$ for all $x^1, x^2 \in M$. Each non-obtuse cone is pointed. The following result converts in some sense this statement.

Lemma 1 ([1]) *If $M \subset \mathbb{R}^k$ is a pointed closed convex cone, then there exists an Euclidean norm in $\mathbb{R}^k$ with respect to which M is a non-obtuse cone.*

The next lemma is essential for the proof of the optimality conditions for (1).

Lemma 2 ([1]) *Let $M \subset \mathbb{R}^k$ be a non-obtuse closed convex cone and $M \setminus \{0\} \neq \emptyset$. Let the svf $\Phi : \mathbb{R}^n \rightsquigarrow \mathbb{R}^k$ be C-Lipschitz with constant L in $U \in \mathcal{N}(x^0)$ and $y^0 \in \Phi(x^0)$. Suppose that for some $\sigma \in (0,\, 1/2)$ it holds $\Phi(x^0) \cap (y^0 - M(2\sigma)) = \{y^0\}$. Then for each $x \in U$ and each $y \in \Phi(x) \cap (y^0 - M(\sigma))$ it holds*

$$\|y - y^0\| \leq \frac{L(1+\sigma)}{\sigma} \, \|x - x^0\| \, .$$

Our aim is to obtain optimality conditions for svp (1) in terms of Dini derivatives. For the svf $\Phi : \mathbb{R}^n \rightsquigarrow \mathbb{R}^k$ the Dini derivative of Φ at (x^0, y^0), $y^0 \in \Phi(x^0)$, in direction $u \in \mathbb{R}^n$ is defined as the upper limit

$$\Phi'(x^0, y^0; u) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t} \left(\Phi(x^0 + tu) - y^0 \right) \, .$$

3 First-order optimality conditions

Theorem 1 (Necessary Conditions, w-minimizers) *Consider svp (1) with $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ closed convex cones, and $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ and $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ svf. Let the pair (x^0, y^0), $x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, be a w-minimizer of svp (1), and let $z^0 \in G(x^0) \cap (-K)$. Then*

$$\forall\, u \in \mathbb{R}^m : (F \times G)'(x^0, (y^0, z^0); u) \cap (-(\operatorname{int} C \times \operatorname{int} K[-z^0]) = \emptyset \, . \qquad (2)$$

Proof. Suppose the contrary, that there exists $(\bar{y}^0, \bar{z}^0) \in (F \times G)'(x^0, (y^0, z^0); u^0)$, such that $\bar{y}^0 \in -\operatorname{int} C$, $\bar{z}^0 \in -\operatorname{int} K[-z^0]$. Let $\bar{y}^0 = \lim_k (1/t_k)(y^k - y^0)$, $\bar{z}^0 = \lim_k (1/t_k)(z^k - z^0)$, where $y^k \in F(x^0 + t_k u^0)$ and $z^k \in G(x^0 + t_k u^0)$ for some $t_k \to 0^+$ and $u^0 \in \mathbb{R}^n$. These equalities imply that $y^k \to y^0$ and $z^k \to z^0$, and the boundedness of the sequences $\{y^k\}$ and $\{z^k\}$.
Let $\bar{\eta} \in K'$, $\|\bar{\eta}\| = 1$. We show that there exists a positive integer $k(\bar{\eta})$ and a neighbourhood $V(\bar{\eta})$ of $\bar{\eta}$, such that $\langle \eta, z^k \rangle < 0$ for $k > k(\bar{\eta})$ and $\eta \in V(\bar{\eta})$. For this purpose we consider the following cases:
1^0. $\bar{\eta} \in K'[-z^0]$. Since $\bar{z}^0 \in -\operatorname{int} K[-z^0]$, we have

$$\lim_k \frac{1}{t_k} \langle \bar{\eta}, z^k - z^0 \rangle = \langle \bar{\eta}, \bar{z}^0 \rangle < 0 \, .$$

Therefore there exists $k(\bar{\eta})$, such that for all $k > k(\bar{\eta})$ it holds $\langle \bar{\eta}, z^k \rangle < \langle \bar{\eta}, z^0 \rangle = 0$. Now the boundedness of the sequence $\{z^k\}$ implies the existence of a neighbourhood $V(\bar{\eta})$ of $\bar{\eta}$, such that $\langle \bar{\eta}, z^k \rangle < 0$ for all $k > k(\bar{\eta})$.
2^0. $\bar{\eta} \in K' \setminus K'[-z^0]$. We have $\langle \bar{\eta}, z^0 \rangle < 0$, whence $\langle \bar{\eta}, z^k \rangle < 0$ for all $k > k(\bar{\eta})$ with suitable $k(\bar{\eta})$. This implies as above $\langle \eta, z^k \rangle < 0$ for all $k > k(\bar{\eta})$ and $\eta \in V(\bar{\eta})$ with suitable neighbourhood $V(\bar{\eta})$ of $\bar{\eta}$.
The set $\Gamma = \{\eta \in K' \mid \|\eta\| = 1\}$ is compact, whence $\Gamma \subset V(\bar{\eta}^1) \cup \ldots \cup V(\bar{\eta}^s)$ for some $\bar{\eta}^1, \ldots, \bar{\eta}^s \in \Gamma$. Let $k_0 = \max(k(\bar{\eta}^1), \ldots, k(\bar{\eta}^s))$. Take $k > k_0$. Then $\langle \eta, z^k \rangle < 0$ for all $\eta \in \Gamma$, and hence for all $\eta \in K'$. Therefore $z^k \in= \operatorname{int} K \subset -K$. In other words, the points $x^0 + t_k u^0$ are feasible.

According to the made assumption $\bar{y}^0 = \lim_k (1/t_k)(y^k - y^0) \in -\operatorname{int} C$. Therefore $y^k - y^0 \in -\operatorname{int} C$ for all sufficiently large k, a contradiction to the hypothesis that (x^0, y^0) is a w-minimizer of (1). □

Remark 1 *Condition (2), which can be referred to as primal form condition, can be substituted by the equivalent dual form condition*

$$\begin{array}{c} \forall\, u \in \mathbb{R}^n \setminus \{0\} : \forall\, (\bar{y}^0, \bar{z}^0) \in (F \times G)'(x^0, (y^0, z^0); u) : \\ \exists\, (\xi, \eta) \in C' \times K'[-z^0], (\xi, \eta) \neq (0, 0) : \langle \xi, \bar{y}^0 \rangle + \langle \eta, \bar{z}^0 \rangle \geq 0\,. \end{array} \tag{3}$$

Theorem 2 (Sufficient Conditions, i-minimizers) *Consider svp (1) with $C \subset \mathbb{R}^m$ pointed closed convex cone, $K \subset \mathbb{R}^p$ closed convex cone, $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ locally C-Lipschitz svf, and $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ locally Lipschitz svf. Suppose that the pair (x^0, y^0), $x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, is such that $y^0 \in p\text{-}\mathrm{Min}_C F(x^0)$, and there exists $z^0 \in G(x^0)$ for which*

$$\forall\, u \in \mathbb{R}^n \setminus \{0\} : (F \times G)'(x^0, (y^0, z^0); u) \cap (-(C \times K[-z^0])) = \emptyset\,. \tag{4}$$

Suppose also that the svf G satisfies the following condition:

$$\mathbb{G}(x^0, z^0) : \quad \begin{array}{c} \exists\, U \in \mathcal{N}(x^0) : \exists\, \ell > 0 : \forall\, x \in U : \\ G(x) \cap (-K) \neq \emptyset \Rightarrow G(x) \cap \ell\, \|x - x^0\|\, \bar{B}_p(z^0) \cap (-K) \neq \emptyset \end{array}$$

Then (x^0, y^0) is an i-minimizer of svp (1).

Proof. We can assume without loss of generality that $\mathbb{R}^m$ (the image space of F) is supplied with an Euclidean norm, with respect to which the cone C is non-obtuse. We may assume that F is C-Lipschitz with constant $L > 0$ on $\bar{B}_n(x^0)$. Suppose that (x^0, y^0) is not an i-minimizer. Fix a sequence $\varepsilon_k \to 0^+$. According to the assumption, there exist sequences $t_k \to 0^+$, and $u^k \in \mathbb{R}^n$, $\|u^k\| = 1$, such that:
1^0. $G(x^0 + t_k u^k) \cap (-K) \neq \emptyset$,
2^0. $D(F(x^0 + t_k u^k) - y^0, -C) < \varepsilon_k t_k$.
Passing to a subsequence, we may assume that $u^k \to u^0$, and $0 < t_k < r$.
By the C-Lipschitz property of F we have

$$D(\frac{1}{t_k}\left(F(x^0 + t_k u^0) - y^0\right), -C) < \varepsilon_k + L\, \|u^k - u^0\|\,.$$

Let $y^k \in F(x^0 + t_k u^0)$ be such that $D(\bar{y}^k, -C) < \varepsilon_k + L\|u^k - u^0\|$, where $\bar{y}^k = (1/t_k)(y^k - y^0)$. The sequence $\{\bar{y}^k\}$ is bounded, which follows from the following reasoning. Since $y^0 \in p\text{-}\mathrm{Min}_C F(x^0)$, there exists σ, $0 < \sigma < 1/2$, such that $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. Let k be such that $\varepsilon_k + L\|u^k - u^0\| < L$, whence $D(y^k - y^0, -C) < L t_k$. Then it holds $\|\bar{y}^k\| \leq L\,(1 + 1/\sigma)$. Indeed, assume on the contrary, that $\|\bar{y}^k\| > L\,(1 + 1/\sigma)$, or equivalently $\|y^k - y^0\| > L\,(1 + 1/\sigma)\, t_k$. We have

$$D(y^k - y^0, -C) < L t_k\, \frac{\sigma}{L t_k (1 + \sigma)}\, \|y^k - y^0\| < \sigma\, \|y^k - y^0\|\,.$$

This inequality shows that $y^k - y^0 \in -C(\sigma)$, whence, from Lemma 2 we get

$$\|y^k - y^0\| \leq \frac{L\,(1 + \sigma)}{\sigma}\, \|(x^0 + t_k u^0) - x^0\| = L\, \left(1 + \frac{1}{\sigma}\right)\, t_k\,,$$

a contradiction.

We proved that the sequence $\{\bar{y}^k\}$ is bounded and $\|\bar{y}^k\| \leq L\,(1 + 1/\sigma)$ for all sufficiently large k. Passing to a subsequence, we may assume that $\bar{y}^k \to \bar{y}^0$, whence $\|\bar{y}^0\| \leq L\,(1 + 1/\sigma)$ and $\bar{y}^0 \in F'(x^0, y^0; u^0)$. Taking a limit in the inequality $D(\bar{y}^k, -C) < \varepsilon_k + L\|u^k - u^0\|$ we get $D(\bar{y}^0, -C) \leq 0$. Since C is closed, this inequality gives $\bar{y}^0 \in -C$.

The hypothesis $\mathbb{G}(x^0, z^0)$ together with the condition $G(x^0 + t_k u^k) \cap (-K) \neq \emptyset$ give that $G_0(x^0 + t_k u^k) \cap (-K) \neq \emptyset$, where $G_0(x) = G(x) \cap \ell\,\|x - x^0\|\,\bar{B}_p(z^0)$. The local Lipschitz property of G gives that there exists a point $z^k \in G(x^0 + t_k u^0)$ such that $D(z^k, G_0(x^0 + t_k u^k)) \leq L t_k\,\|u^k - u^0\|$ (here we suppose that G is locally Lipschitz with constant L on $r\,\bar{B}_n$). From the triangle inequality we get $\|z^k - z^0\| \leq (\ell + L\|u^k - u^0\|)\,t_k$. Putting $\bar{z}^k = (1/t_k)(z^k - z^0)$, we have $\|\bar{z}^k\| \leq (\ell + L\|u^k - u^0\|) \leq \ell + 2L$. Therefore the sequence $\bar{z}^k$ is bounded. Passing to a subsequence we may assume that $\bar{z}^k \to \bar{z}^0$.

The construction of z^k yields the existence of $\tilde{z}^k \in G_0(x^0 + t_k u_k) \cap (-K)$, such that $z^k \in \tilde{z}^k + L t_k\,\|u^k - u^0\|\,\bar{B}_p$, whence for arbitrary $\eta \in K'[-z^0]$, $\|\eta\| = 1$, we have

$$\langle \eta, \bar{z}^k \rangle = \frac{1}{t_k}\,\langle \eta, z^k \rangle \leq \frac{1}{t_k}\,\langle \eta, \tilde{z}^k \rangle + L\,\|u^k - u^0\| \leq L\,\|u^k - u^0\|\,.$$

Here we have used $\langle \eta, \tilde{z}^k \rangle \leq 0$, a consequence of $\tilde{z}^k \in -K$. Taking the limit in the above inequality, we get $\langle \eta, \bar{z}^0 \rangle \leq 0$, whence

$$D(\bar{z}^0, -K[-z^0]) = \sup\{\langle \eta, \bar{z}^0 \rangle \mid \eta \in K'[-z^0],\ \|\eta\| = 1\} \leq 0\,.$$

Regarding that $K'[-z^0]$ is closed, this gives $\bar{z}^0 \in -K[-z^0]$.

We have used the same sequence $t_k \to 0^+$ to construct both $\bar{y}^0$ and $\bar{z}^0$, hence we have $(\bar{y}^0, \bar{z}^0) \in (F \times G)'(x^0, (y^0, z^0); u^0)$. So far we have proved that $(\bar{y}^0, \bar{z}^0) \in -(C \times K[-z^0])$. On the other hand condition (4) gives $(\bar{y}^0, \bar{z}^0) \notin -(C \times K[-z^0])$, a contradiction. □

Remark 2 *Condition (4), which can be referred to as primal form condition, can be substituted by the equivalent dual form condition*

$$\begin{array}{c} \forall\, u \in \mathbb{R}^n \setminus \{0\} : \forall, (\bar{y}^0, \bar{z}^0) \in (F \times G)'(x^0, (y^0, z^0); u) : \\ \exists\,(\xi, \eta) \in C' \times K'[-z^0], (\xi, \eta) \neq (0, 0) : \langle \xi, \bar{y}^0 \rangle + \langle \eta, \bar{z}^0 \rangle > 0\,. \end{array} \tag{5}$$

The next example shows that without condition $\mathbb{G}(x^0, x^0)$ Theorem 2 is not true.

Example 1 *Consider problem (1) with $n = 1$, $m = 1$, $p = 2$, $C = \mathbb{R}_+$, $K = \mathbb{R}^2_+$, $F : \mathbb{R} \to \mathbb{R}$ arbitrary single-valued differentiable function, and $G : \mathbb{R} \rightsquigarrow \mathbb{R}^2$ given by $G(x) = [(|x|, -1), (-|x|, 0)]$. Let $x^0 = 0$, $y^0 = F(x^0)$, $z^0 = (0, -1)$. All conditions of Theorem 2, with exception of $\mathbb{G}(x^0, z^0)$, are satisfied, independently on the concrete function F. In particular $K[-z^0] = \mathbb{R}_+ \times \mathbb{R}$ and $(F \times G)'(x^0, (y^0, z^0); u) = (F'(0)u, G'(x^0, z^0; u))$, where $G'(x^0, z^0; u) = \{|u|\} \times \mathbb{R}_+$, which verifies condition (4). Since any point $x \in \mathbb{R}$ is feasible, problem (1) is equivalent to the optimization problem* $\min F(x)$*, $x \in \mathbb{R}$. But, if for instance $F(x) = -x^2$, the point x^0 is not an i-minimizer.*

The following theorem is a modification of Theorem 2 and is proved similarly.

Theorem 3 *Consider svp (1) with* $C \subset \mathbb{R}^m$ *and* $K \subset \mathbb{R}^p$ *pointed closed convex cones, and* $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ *and* $G : \mathbb{R}^n \rightsquigarrow \mathbb{R}^p$ *respectively locally* C*-Lipschitz and locally* K*-Lipschitz svf. Suppose that the pair* (x^0, y^0)*,* $x^0 \in \mathbb{R}^n$*,* $y^0 \in F(x^0)$*, is such that* $y^0 \in p\text{-Min}_C F(x^0)$*, and there exists* $z^0 \in G(x^0)$ *for which* $z^0 \in p\text{-Min}_K G(x^0)$ *and condition (4) holds. Suppose also that the svf* G *satisfies condition* $\mathbb{G}(x^0, z^0)$*. Then* (x^0, y^0) *is an i-minimizer of svp (1).*

When the functions F and G are single-valued, then problem (1) transforms into the vector optimization problem $\min_C F(x)$, $G(x) \in -K$, and Theorems 1 and 2 reduce to those proved in [3]. Then the conditions $y^0 \in p\text{-Min}_C F(x^0)$ and $\mathbb{G}(x^0, z^0)$ are automatically satisfied.

Though condition $\mathbb{G}(x^0, z^0)$ does not appear in Theorem 1, the interesting applications of this theorem could be those in which $G(x) \cap (-K)$ possess points near z^0. Indeed, suppose that $\forall\, \ell > 0 : \exists U \in \mathcal{N}(x^0) : \forall\, x \in U : G(x) \cap (-K) \neq \emptyset \Rightarrow G(x) \cap \ell \, \|x - x^0\| \, \bar{B}_p(x^0) \cap (-K) = \emptyset$. Then $(F \times G)'(x^0, (y^0, z^0); u) = \emptyset$ for all $u \in \mathbb{R}^n$, and condition (2) is satisfied for arbitrary svf F.

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