M. Papalia, M. Rocca

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Well-posedness in vector optimization and scalarization results

Melania Papalia *   Matteo Rocca †

Abstract

In this paper, we give a survey on well-posedness notions of Tykhonov's type for vector optimization problems and the links between them with respect to the classification proposed by Miglierina, Molho and Rocca in [33]. We consider also the notions of extended well-posedness introduced by X.X. Huang ([19],[20]) in the nonparametric case to complete the hierchical structure characterizing these concepts.
Finally we propose a review of some theoretical results in vector optimization mainly related to different notions of scalarizing functions, linear and nonlinear, introduced in the last decades, to simplify the study of various well-posedness properties.

1 Introduction

In scalar optimization the theory of well-posedness is described through two fundamental approaches. The first one, based on existence and uniqueness of the optimal solution together with continuous dependence from problem's data, extends to optimization problems the classical idea of well-posedness for problems in mathematical physics, due to the French mathematician J. Hadamard [16].
The second one, introduced by A.N. Tykhonov [35] in the early sixties, imposes, besides existence and uniqueness of the global minimun point, a stability requirement tested by the convergence of every minimizing sequence.
Hence, well-posedness properties are deeply linked to the behaviour of the objective function with respect to a particular notion of minimizing sequence. Moreover, every method building up sequences converging to some minimizer of the optimization problem corresponds to approximately computing of the solution. For these reasons well-posedness properties play an important role both from a theoretical point of view and as useful tool in the convergence analysis of some algorithms.

These concepts have been widely studied in order to establish links between them and to formulate some scalar extensions considering, for example, the relaxation of

*Università dell'Insubria, Department of Economics, via Ravasi 2, 21100 Varese, Italia. email:melaniapa@libero.it

†Università dell'Insubria, Department of Economics, via Ravasi 2, 21100 Varese, Italia. email:mrocca@eco.uninsubria.it

the uniqueness requirement (see e.g. the monographs [11],[29] for a review on this topic).
In 1996 T. Zolezzi [37] introduced a notion of extended well-posedness, which is a combination of Hadamard and Tykhonov ideas. The original problem is embedded in a family of perturbed ones depending on a parameter and it is called well-posed in the extended sense if every asymptotically minimizing sequence converges to some solution. In this way the notion considers the behaviour of appropriate minimizing sequences and, at the same time, realizes a continuous dependence of the solutions on the parameter.

In vector optimization the image space is generally characterized by a partial order endowed by a closed, convex, pointed cone and this caused some difficulties to the generalization of well-posedness concepts. The concept of minimal value is not uniquely determined, so one can distinguish between different degrees of minimality ([36]) as for example minimal value and weakly minimal value and, as a consequence, the uniqueness requirement is too restrictive for vector functions; moreover, it is possible to choose several concepts of minimizing sequence ([26], [28]). So, while in the scalar case the stability concerns only the behaviour of minimizing sequences as link, through the objective function, between domain and image set, in vector optimization the choice of a minimizing sequence concept leads to different formulations of well-posedness that, implicitly, impose some geometrical features of the solutions in the image space ([32],[33]).

The first attempts to generalize Tykhonov's idea to vector optimization problems can be found in [1],[28]. In the following years various concepts of well-posedness were introduced and, as pointed out in [32] and [33], they can be listed into two main classes characterizing different levels of analysis: pointwise and global. In the first case the well-posedness notions are referred to a fixed solution point in the image set or in the domain of the function, see for instance [4],[26],[10], for the concepts of Tykhonov's type and [21] with regards to the properties in the extended sense. In the second set there are those definitions which involve the efficient frontier as a whole, for example [4], [7], [13], [19],[20], [32]. Moreover Miglierina, Molho and Rocca in [33] establish a hierchical structure in both classes. As further generalization, in order to increase the number of problems satisfing a well-posedness property, in the last years global notions based on the weakly efficient frontier appeared (for example see [6] and [9]).
As recent contribution on this topic, in [34] the authors underline that extended well-posedness notions are part of those global on the ground of the classification proposed by Miglierina, Molho and Rocca when the parameter is fixed, hence they investigate the links between global notions that involve the weakly efficient solutions set.

To simplify the study of the well-posedness in the vector case, Miglierina, Molho and Rocca ([33]) proposed a new method based on a scalarization procedure with a nonlinear function. They establish the relationships between the well-posedness of a vector optimization problem and the well-posedness of an appropriate scalarized one. In this way they also construct a stronger link between the original ideas and their generalizations. Some links between well-posedness of vector problems and of linearly scalarized ones have been investigated by Deng in [9], under convexity assumptions.

In this survey, after a review of well-posedness properties where we take a particular care with regards to the global notions of Tykhonov's type, we focus our attention on linear and nonlinear scalarizing procedures that preserve well-posedness of the different classes of notions listed above. The outline of the work is the following. Section 2 is devoted to the problem formulation and the notation we refer through the paper. In Section 3, following the hierchical structure of well-posedness notions proposed by Miglierina, Molho and Rocca ([33]), we derive some examples to stress the main differences between the generalizations and finally we show that for strictly quasiconvex functions all the well-posedness notions coincide. Finally, in Section 4 we focus our attention on scalarization results, both nonlinear and linear.

2 Problem setting and notation

Consider a continuous vector function $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$, let $X \subseteq \mathbb{R}^m$ be a closed set of admissible points, $Y = f(X) \subseteq \mathbb{R}^l$ be the image set and $C \subseteq \mathbb{R}^l$ a closed convex pointed cone, with nonempty interior, inducing an order relation on $\mathbb{R}^l$. So,

$$y \leq_C w \iff w - y \in C,$$

$$y <_C w \iff w - y \in \operatorname{int} C.$$

In the following, B denotes the unit ball both in $\mathbb{R}^m$ and in $\mathbb{R}^l$, (from the context will be clear to which space we refer) and C^+ the positive polar cone of C, defined by

$$C^+ = \{v \in \mathbb{R}^l : \langle v, c \rangle \geq 0, \ \forall c \in C\}.$$

Let (X, f) the vector optimization problem given by

$$\min f(x), \quad x \in X.$$

A point $\bar{x} \in X$ is called *efficient solution* for problem (X, f) when

$$(f(X) - f(\bar{x})) \cap (-C) = \{0\}.$$

We denote by $\operatorname{Eff}(X, f)$ the set of all efficient solutions of the problem (X, f) and by $\operatorname{Min}(X, f)$ the set of all *minimal points*, i.e. the image of $\operatorname{Eff}(X, f)$ through the objective function f.
A point $\bar{x} \in X$ is called *weakly efficient solution* for problem (X, f) when

$$(f(X) - f(\bar{x})) \cap (-\operatorname{int} C) = \emptyset.$$

We denote by $\operatorname{WEff}(X, f)$ the set of weakly efficient solutions of the problem (X, f) and by $\operatorname{WMin}(X, f)$ the set of all *weakly minimal points*. We stress that in the scalar case the notions of minimal point and weakly minimal point coincide.

We recall also (see e.g. [36]) that a point $\bar{y} \in f(X)$ is called *strictly minimal point* for problem (X, f) when for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$(f(X) - \bar{y}) \cap (\delta B - C) \subseteq \varepsilon B.$$

We denote by $\mathrm{StMin}\,(X, f)$ the set of strictly minimal points for problem (X, f). It is easy to see that $\mathrm{StMin}\,(X, f) \subseteq \mathrm{Min}\,(X, f)$ but the converse is not true in general. The next result emphasizes the geometrical features of strictly minimal points.

Proposition 2.1. *(see [5]) Let $\bar{y} \in f(X)$. Then $\bar{y} \in \mathrm{StMin}\,(X, f)$ if and only if for every sequences $\{z^n\}$, $\{y^n\}$ with $\{z^n\} \subseteq f(X)$, $y^n \in z^n + C$ and $y^n \to \bar{y}$, it holds $z^n \to \bar{y}$.*

3 Well-posedness for vector optimization problems

In this section we review the main notions of Tykhonov's type, pointwise and global, considering also extended well-posedness in the nonparametric case. Among various mathematical tools thanks to which it is possible to introduce the stability conditions ([11]), we choose the minimizing sequences of points.

3.1 Pointwise well-posedness

Under the label pointwise well-posedness, Miglierina, Molho and Rocca ([33]) classify those notions in which a minimal point or an efficient solution is fixed. In this way pointwise concepts avoid uniqueness assumption but investigate a local condition of stability, with reference to a single element.
E. Bednarczuk ([4]) and P. Loridan ([26]) consider a fixed minimal value, while D. Dentcheva and S. Helbig ([10]) and X.X. Huang ([21]) a fixed efficient solution as the original Tykhonov's idea.

Definition 3.1. *(see [26]) Let $\bar{y} \in \mathrm{Min}\,(X, f)$. A sequence $\{x^n\} \subseteq X$ is called $\bar{y}-$minimizing sequence for problem (X, f), when there exists a sequence $\{\varepsilon^n\} \subseteq C$, $\varepsilon^n \to 0$, such that $f(x^n) \leq_C \bar{y} + \varepsilon^n$.*

Definition 3.2. *(see [4]) Let $\bar{y} \in \mathrm{Min}\,(X, f)$. Problem (X, f) is said to be B–$\bar{y}$ well-posed if and only if every $\bar{y}-$minimizing sequence $\{x^n\} \subseteq X \setminus f^{-1}(\bar{y})$ admits a subsequence $\{x^{n_k}\}$ such that $x^{n_k} \to \bar{x} \in f^{-1}(\bar{y})$.*

Definition 3.3. *(see [26]) Let $\bar{y} \in \mathrm{Min}\,(X, f)$. Problem (X, f) is said to be L–$\bar{y}$ well-posed if and only if every $\bar{y}-$minimizing sequence admits a subsequence converging to an element of $f^{-1}(\bar{y})$.*

The next definition is formulated considering the diameter of the level sets of the function f as it is possible in the scalar case to identify Tykhonov well-posedness.

Definition 3.4. *(see [10]) Let $\bar{x} \in \mathrm{Eff}\,(X, f)$. Problem (X, f) is said to be DH–$\bar{x}$ well-posed if and only if*

$$\inf_{\alpha>0} \operatorname{diam} L(\bar{x}, c, \alpha) = 0, \quad \forall c \in C,$$

where $L(\bar{x}, c, \alpha) = \{x \in X : f(x) \leq_C f(\bar{x}) + \alpha c\}$.

Definition 3.5. *(see [21]) Let $\bar{x} \in \mathrm{Eff}\,(X, f)$. Problem (X, f) is said to be H–$\bar{x}$ well-posed if and only if $\forall \{x^n\} \subseteq X$ such that $f(x^n) \to f(\bar{x}), \quad x^n \to \bar{x}$.*

The pointwise notions introduced by X.X. Huang ([21]) considering a perturbation of the objective function coincide, in the nonparametric case, with the above definition.
We know that DH–$\bar{x}$ well-posedness implies H–$\bar{x}$ well-posedness but the converse is not true as shown in Example 2.1 in [21]. The same example shows that H–$\bar{x}$ well-posedness implies neither L–$\bar{y}$ well-posedness nor B–$\bar{y}$ well-posedness.

Proposition 3.1. *Let $\bar{x} \in \mathrm{Eff}\,(X, f)(X, f)$ and assume that $\forall \varepsilon > 0, \ \exists \delta > 0$ s.t.*

$$(f(X) - f(\bar{x})) \cap (\delta B - C) \subseteq \varepsilon B. \tag{1}$$

Then (X, f) is H–$\bar{x}$ well-posed if and only if it is DH–$\bar{x}$ well posed.

Proof: We must show only one direction, i.e. H–$\bar{x}$ $\Rightarrow$ DH–$\bar{x}$.
By contradiction (X, f) is not DH–$\bar{x}$ well-posed. Since the problem is H–$\bar{x}$ wel-posed, there exist a sequence $\{x^n\} \subseteq X$ and $\bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}, \ \alpha > 0, \ c \in C$

$$f(x^n) \leq_C f(\bar{x}) + \alpha c, \qquad \text{but} \qquad f(x^n) \not\to f(\bar{x}),$$

and

$$x^n \in L(\bar{x}, c, \alpha) \qquad \text{but} \qquad x^n \not\to \bar{x}.$$

This means that $\exists \varepsilon > 0$ s.t. $f(x^n) \notin f(\bar{x}) + \varepsilon B$, i.e. $f(x^n) - f(\bar{x}) \notin \varepsilon B$, and thus $\nexists \delta(\varepsilon), \ \delta > 0$ s.t. $(f(X) - f(\bar{x})) \cap (\delta B - C) \subseteq \varepsilon B$ contradicting the assumption. $\Box$

Miglierina, Molho and Rocca in [33] complete the relationships between pointwise notions with theoretical results and examples showing the main geometrical features of each definition and summarize their work on this topic as in the following scheme where, for simplicity, we omit the term well-posedness.

$$\begin{array}{rcl}
 & \text{H–}\bar{x} & \\
(1)\Downarrow & & \Uparrow \\
 & \text{DH–}\bar{x} & \\
\Downarrow & & \Uparrow f^{-1}(\bar{y}) = \bar{x} \\
 & \text{L–}\bar{y} & \\
\Downarrow & & \Uparrow f^{-1}(\bar{y}) \text{ compact} \\
 & \text{B–}\bar{y} &
\end{array}$$

3.2 Global well-posedness

In this subsection we review the main notions of global well-posedness, i.e. the notions where the stability condition is investigated with reference to the whole set of solutions. To get a tidy and complete comparison between the properties of this class, we divide them in two groups: first we consider global well-posedness and efficient solutions, then global well-posedness and weakly efficient solutions where we present before the notions introduced by X.X. Huang rewritten in the nonparametric case and then the other concepts. In each subsection we compare the notions between them, we underline, when it is possible, the relationships with the previous concepts and finally we stress the geometrical features characterizing the image set through an example that permit us to achieve another well-posedness property which enlarge, in some way, the class of well-posed problems. Then we compare the notions belonging to the different groups in order to draw a final outline that will be our start point to study well-posedness under generalized convexity assumptions.

3.2.1 Global well-posedness and efficient solutions

The passage from pointwise to global notions is traced by E. Bednarczuk ([4]), considering the concept of B–minimizing sequence and the generalization of the stability condition.

Definition 3.6. *A sequence $\{x^n\} \subseteq X$ is called B–minimizing for problem (X, f), when for each $n \in \mathbb{N}$ there exists $\varepsilon^n \in C$ and $y^n \in \mathrm{Min}\,(X, f)$ such that $f(x^n) \leq_C y^n + \varepsilon^n, \quad \varepsilon^n \to 0$.*

Definition 3.7. *(see [4]) Problem (X, f) is said to be B–well-posed if and only if*

i) $\mathrm{Min}\,(X, f) \neq \emptyset$;

ii) every B–minimizing sequence $\{x^n\} \subseteq X \setminus \mathrm{Eff}\,(X, f)$ admits a subsequence converging to some element of $\mathrm{Eff}\,(X, f)$.

Proposition 3.2. *(see [4]) Let $\mathrm{Min}\,(X, f)$ a compact set. If problem (X, f) is B–$\bar{y}$ well-posed for every $\bar{y} \in \mathrm{Min}\,(X, f)$, then (X, f) is B–well-posed.*

Example 3.1. *Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1, x_1)$ with $X = C = \mathbb{R}^2_+$. The only minimal value is $(0, 0)$, while $\mathrm{Eff}\,(X, f) = \{(0, x_2) : x_2 \geq 0\}$. The problem (X, f) is not B–well-posed as for example the B–minimizing sequence $x^n = \left(\frac{1}{n}, n\right)$ doesn't admit any subsequence converging to some efficient solution.*

To enlarge the class of well-posed problem, E. Bednarczuk proposed a new definition in which the stability condition is based on the distance, in the norm sense, from the efficient solution set instead of the convergence of appropriate minimizing subsequences.

Definition 3.8. *(see [4]) Problem (X, f) is said to be Bw–well-posed if and only if* *i)* $\text{Min}(X, f) \neq \emptyset$;

ii) for every B–minimizing sequence $\{x^n\} \subseteq X, \quad d(x^n, \text{Eff}(X, f)) \to 0$.

It is clear that B–well-posedness implies Bw–well-posedness, while the converse is in general not true. The equivalence can be stated under the compactness of the efficient set.

Proposition 3.3. *Let* $\text{Eff}(X, f)$ *be a nonempty compact set. If* (X, f) *is Bw–well-posed then it is also B–well-posed.*

Proof. We distinguish two cases:

1. $\text{Eff}(X, f) = X$ means that there aren't B–minimizing sequences out of the efficient set and hence (X, f) is B–well-posed;

2. $\text{Eff}(X, f) \subset X$. By compactness assumption one has $d(x^n, \text{Eff}(X, f)) \to 0$ if and only if $\exists x^{n_k} \to \bar{x} \in \text{Eff}(X, f)$ for every B–minimizing sequence.

□

The notion of Bw–well-posedness fails when the image set and the ordering cone have some asymptote in common as the following example shows.

Example 3.2. *Let* $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1, x_2)$ *with* $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0 \text{ or } x_2 \geq -x_1\}$ *and* $C = \mathbb{R}^2_+$. *The problem* (X, f) *is not Bw–well-posed as for example the B–minimizing sequence* $x^n = (-n, 0)$ *doesn't satisfy the stability condition.*

To avoid this difficulty, Miglierina and Molho proposed to relax the requirement of convergence of the minimizing sequences also in the feasible region.

Definition 3.9. *(see [32]) Problem* (X, f) *is said to be M–well-posed when for every* $\{x^n\} \subseteq X$ *such that* $d(f(x^n), \text{Min}(X, f)) \to 0$, *one has* $d(x^n, \text{Eff}(X, f)) \to 0$.

Miglierina and Molho in [32] proved that Bw–well-posedness implies M–well-posedness while the converse is not true in general as it is showed in Example 3.2. The converse is proved under an additional hyphotesis implying that all minimal points are also strict minimal points [33]. For the reader convenience, we recall this result.

Theorem 3.1. *If* (X, f) *is M–well-posed and for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$(f(X) - \text{Min}(X, f)) \cap (\delta B - C) \subseteq \varepsilon B, \tag{2}$$

then it is also Bw–well-posed.

Note that Definition 3.9 is always satisfied when the efficient solution set is empty and in this particular case there is no attention to the structure of the optimization problem with reference to a weak concept of minimal points identified by the given ordering cone. In the next subsections we focus on global definitions seeking a weaker concept of efficient solution.
Till now, we can trace the following scheme:

$$\begin{array}{ccc} & \mathrm{B} & \\ \Downarrow & & \Uparrow \mathrm{Eff}\,(X,f) \text{ compact} \\ & \mathrm{Bw} & \\ \Downarrow & & \Uparrow (2) \\ & \mathrm{M} & \end{array}$$

3.2.2 Global well-posedness and weakly efficient solutions

We introduce three notions due to X.X. Huang in [19] as generalization of the previous concepts and keeping attention to the original idea published by Zolezzi ([37]).

Definition 3.10. *A sequence $\{x^n\} \subseteq X$ is called Hs–minimizing for problem (X, f), if there exists $c \in \operatorname{int} C$, $\;t_n > 0, t_n \to 0$ such that $f(X) - f(x^n) + t_n c \notin -C$.*

Definition 3.11. *Problem (X, f) is said to be Hs–well-posed if and only if*

i) $\mathrm{WEff}\,(X, f) \neq \emptyset$;

ii) every Hs–minimizing sequence $\{x^n\} \subseteq X$ admits a subsequence converging to some element of $\mathrm{WEff}\,(X, f)$.

Example 3.3. *Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1, x_2)$ with $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq x_1 e^{-x_1},\ x_1 \geq 0\}$ and $C = \mathbb{R}^2_+$. The problem (X, f) is not Hs–well-posed as for example the Hs–minimizing sequence $x^n = (n, ne^{-n})$ doesn't admit any subsequence converging to some weakly efficient solution.*

The following notion of well-posedness is a generalization of B–well-posedness and it has the detail to consider the set of minimal points in the formulation of minimizing sequence while the stability condition is referred to the weakly efficient solutions set.

Definition 3.12. *A sequence $\{x^n\} \subseteq X$ is called H–minimizing for problem (X, f) if there exist $c \in \operatorname{int} C$, $\alpha_n > 0$, $\alpha_n \to 0$, and $y^n \in \mathrm{Min}\,(X, f)$ such that $f(x^n) \leq_C y^n + \alpha_n c$.*

Definition 3.13. *Problem (X, f) is said to be H–well-posed if and only if*

i) $\mathrm{WEff}\,(X, f) \neq \emptyset$;

ii) every H–minimizing sequence $\{x^n\} \subseteq X$ admits a subsequence converging to some element of $\mathrm{WEff}\,(X, f)$.

X.X. Huang in [19] pointed out that Hs $\Rightarrow$ H, but the converse is not true in general. In fact, the problem in Example 3.3 is H–well-posed but not Hs–well-posed. The geometrical feature of the image set of a problem that doesn't satisfy Definition 3.13 is the same we have already met considering Bw–well-posedness, i.e. the presence of some asymptote in common with the ordering cone.

Example 3.4. *Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1, x_2)$ with $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq -x_1 e^{x_1},\ x_1 \leq 0\}$ and $C = \mathbb{R}^2_+$. The problem (X, f) is not H–well-posed as for example the H–minimizing sequence $x^n = (-n, ne^n)$ doesn't admit any subsequence converging to some weakly efficient solution.*

To extend the class of well-posedness problems, Huang used the same trick of distance instead of convergence of every minimizing sequence in the image set.

Definition 3.14. *Problem (X, f) is said to be Hw–well-posed when for every $\{x^n\} \subseteq X$ such that $d(f(x^n), \mathrm{WMin}\,(X, f)) \to 0$, there exists a subsequence converging to a weakly efficient solution.*

As in the parametric case, the following implications hold:

$$\text{Hs} \Rightarrow \text{H} \Rightarrow \text{Hw}.$$

Problem in Example 3.4 is Hw–well-posedness, while the problem in Example 3.2 is neither H–well-posed nor Hw–well-posedness, as for example the H-minimizing sequence $x^n = (-n, 0)$ doesn't admit any subsequence converging to some weakly efficient solution.
As pointed out by Huang ([19]), any one of Definitions 3.11, 3.13, 3.14 implies that $\mathrm{WEff}\,(X, f)$ is compact; so to establish the equivalence of these three notions with a property in which there isn't a requirement of convergence in the domain, it is necessary to assume $\mathrm{WEff}\,(X, f)$ compact. A comparison with the notions based on the efficient solutions, under the assumption $\mathrm{Eff}\,(X, f) = \mathrm{WEff}\,(X, f)$, gives the following outline

$$\begin{array}{ccc} & & \text{Hs} \\ & & \Downarrow \\ \text{Bw} & = & \text{H} \\ \Downarrow & & \Downarrow \\ \text{M} & \underset{(a)}{=} & \text{Hw} \end{array}$$

where $(a) = \mathrm{WEff}\,(X, f)$ compact.
A proof of the equality between Bw–well-posedness and H–well-posedness can be found in [34], while the link between M–well-posedness and Hw–well-posedness follows directly from definitions.
We note that Hw–well-posedness implies H–well-posedness under the same assumptions for which M–well-posedness implies Bw–well-posedness.

Remark 3.1. Huang and Yang ([23]) introduced six different types of generalized well-posedness in the extended sense inspired by the scalar notion due to Levitin-Polyak ([25]) and the scalar generalization in [22] where the constraint is specified by a function. It is worth noting that in our framework, i.e. the stability condition is investigated with reference to an appropriate notion of minimizing sequence when it belongs to the feasible region, the several notions presented in [23] coincide with Hw, H and Hs–well-posedness.

In [6] the authors introduced a new notion of well-posedness of a vector optimization problem and established a link with the same property of a vector variational inequality of differential type, under generalized convexity assumption.

Definition 3.15. *A sequence $\{x^n\} \subseteq X$ is called CGR–minimizing for problem (X, f), when there exist $c^0 \in \operatorname{int} C$, $\varepsilon_n \geq 0$, $\varepsilon_n \to 0$ such that $f(x) - f(x^n) + \varepsilon_n c^0 \notin -\operatorname{int} C$, $\forall x \in X$.*

Definition 3.16. *Problem (X, f) is said to be CGR–well-posed if and only if*

i) $\operatorname{WEff}(X, f) \neq \emptyset$;

ii) for every CGR–minimizing sequence $d(x^n, \operatorname{WEff}(X, f)) \to 0$ as $n \to +\infty$.

The problem in Example 3.4 is not CGR–well-posed.

Remark 3.2. The vector well-posedness in the extended sense introduced in [7] coincides with CGR–well-posedness in the nonparametric case.

In [9] the author showed that coercivity implies well-posedness without any convexity assumptions on problem data.

Definition 3.17. *(see [9]) A sequence $\{x^n\} \subseteq X$ is called D–minimizing for problem (X, f), when $d(f(x^n), \operatorname{WMin}(X, f)) \to 0$.*

Definition 3.18. *Problem (X, f) is said to be D–well-posed if and only if*

i) $\operatorname{WMin}(X, f)$ is closed;

ii) for every D–minimizing sequence $d(x^n, \operatorname{WEff}(X, f)) \to 0$ as $n \to +\infty$.

We underline that S. Deng introduced the previous definition in a particular case, namely that the ordering convex pointed cone is always the paretian one, i.e. $C = \mathbb{R}^l_+$. In this work we consider the general case in which the cone satisfies the requirements as specified in Section 2.

Theorem 3.2. *Let $\operatorname{WMin}(X, f)$ be closed. If Problem (X, f) is CGR–well-posed, then it is D–well-posed.*

Proof: To prove the statement is equivalent to show that

$$\{x^n \in X : d(f(x^n), \operatorname{WMin}(X, f)) \to 0\} \subseteq \{x^n \in X : f(X) - f(x^n) + \varepsilon_n c^0 \notin -\operatorname{int} C\}.$$

By definition of minimal point, $f(x) - f(\bar{x}) \notin -\operatorname{int} C$, $\forall x \in X$.
If $d(x^n, \operatorname{WEff}(X, f)) \to 0$, one can find a sequence $\varepsilon_n \geq 0$, $\varepsilon_n \to 0$ and a vector $c^0 \in \operatorname{int} C$ such that definition 3.15 is satisfied. □

The assumption WMin(X,f) closed in Theorem 3.2 cannot be avoided.

Example 3.5. *Let* $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1,x_2) = (x_1^2, e^{x_2})$, $X = \mathbb{R}^2$, $C = \mathbb{R}^2_+$. *The problem* (X,f) *is CGR–well-posed but not D, since* WMin(X,f) *is not closed.*

The converse of Theorem 3.2 is not true in general, for instance problem in Example 3.4 is not CGR–well-posed but it is D–well-posed.

Theorem 3.3. *If problem* (X,f) *is D–well-posed and for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$(f(X) - \mathrm{WMin}\,(X,f)) \cap (\delta B - C) \subseteq \varepsilon B, \tag{3}$$

then it is also CGR–well-posed.

Proof: Suppose, to the contrary, that problem (X,f) is not CGR–well-posed, i.e. $\exists\{x^n\} \subseteq X$ that satisfies the following two properties:

1. $\{x^n\}$ is CGR–minimizing, i.e. $\exists \varepsilon_n > 0$, $\varepsilon_n \to 0$, $c^0 \in \mathrm{int}\, C$ such that $f(X) - f(x^n) + \varepsilon_n c^0 \notin -\mathrm{int}\, C$;

2. $\exists \alpha > 0$ such that $x^n \in [\mathrm{WEff}\,(X,f) + \alpha B]^c$ for all n large enough.

Either of the two following cases occur:

i) x^n is such that $d(f(x^n), \mathrm{WMin}\,(X,f)) \to 0$. In this case x^n is also a D-minimizing sequence and hence by the assumption problem (X,f) is D–well-posed, it follows it is also CGR–well-posed as we ontradict the previous point 2.

ii) $\exists \delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$f(x^n) \in [\mathrm{WMin}\,(X,f) + \delta B]^c\,, \quad \forall n > n_0. \tag{4}$$

Since $\{x^n\}$ is CGR–minimizing

$$f(X) - f(x^n) + \varepsilon c^0 \notin -\mathrm{int}\, C$$

$$f(x^n) - f(X) - \varepsilon c^0 \notin \mathrm{int}\, C$$

$$f(x^n) \in \left[f(X) + \varepsilon c^0 + \mathrm{int}\, C\right]^c.$$

So, $\forall \bar{y} \in \mathrm{WMin}\,(X,f)$ one has $\bar{y} + \varepsilon c^0 \in \bar{y} + \delta B$, $\forall n > n_1$ and hence

$$f(x^n) \in [\bar{y} + \delta B + \mathrm{int}\, C]^c$$

and also

$$f(x^n) \in f(X) \cap [\bar{y} + \delta B + \mathrm{int}\, C]^c\,.$$

Recalling (4), we have a contradiction to the assumption $(f(X) - \mathrm{WMin}\,(X,f)) \cap (\delta B - C) \subseteq \varepsilon B$.

□

Now, we compare CGR–well-posedness and D–well-posedness with the previous notions, in particular with reference to the work of Huang we have the following result.

Theorem 3.4. *(see [34]) Let* $\mathrm{WEff}\,(X, f)$ *a compact set. Problem* (X, f) *is CGR–well-posed if and only if it is Hs–well-posed.*

The compactness assumption is fundamental only to show that CGR–well-posedness implies Hs–well-posedness.

Example 3.6. *Let* $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0$ *or* $x_2 \geq -x_1\}$, $C = \mathbb{R}^2_+$. *The problem* (X, f) *is CGR–well-posed, but not Hs, as, for example, the Hs–minimizing sequence* $x^n = \left(n, -n + \frac{1}{n}\right)$ *doesn't admit any subsequence converging to a weak efficient solution.*

The final outline, completed with all global definitions, is based on the following assumptions

(-) $\mathrm{Eff}\,(X, f) = \mathrm{WEff}\,(X, f)$

(+) $\mathrm{Eff}\,(X, f)$ compact

(*) $\mathrm{WEff}\,(X, f)$ compact

(**) $\mathrm{WMin}\,(X, f)$ closed

(***) $\mathrm{WEff}\,(X, f)$ compact and $\mathrm{WMin}\,(X, f)$ closed.

					(*)			
	B			Hs	=		CGR	
(+) $\Uparrow$		$\Downarrow$	(-)	$\Downarrow$				
	Bw		=	H		(**)$\Downarrow$		(3) $\Uparrow$
(2) $\Uparrow$		$\Downarrow$		$\Downarrow$				
	M		=	Hw	=		D	
			(*,-)		(***)			

3.3 Generalized convex functions

In the previous subsection we point out that any new notion of well-posedness enlarges, in some way, the class of problems characterized by that property, but an interesting investigation about well-posedness properties consists to identify which classes of fuctions satisfy, for sure, a given concept. It is known that, under appropriate generalized convexity assumptions, some well-posedness properties are satified (see for instance [6], [31], [32], [33]).
We recall some basic concepts of generalized convex functions.

Definition 3.19. *(see [27]) A function* $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$, X *convex, is said to be:*

i) $C-$convex if

$$f(tx + (1-t)z) - tf(x) - (1-t)f(z) \in -C$$

for every $x, z \in \mathbb{R}^m$ and $t \in [0,1]$;

ii) $C-$quasiconvex if for every $y \in \mathbb{R}^l$ the level sets

$$\operatorname{Lev}(f, y, X) := \{x \in \mathbb{R}^m : f(x) \in y - C\}$$

are either empty or convex;

iii) strictly $C-$quasiconvex when for every $y \in \mathbb{R}^l$ and $x, z \in X,\ x \neq z$,

$$f(x), f(z) \in y - C$$

implies $f(tx + (1-t)z) \in y - \operatorname{int} C$ for every $t \in (0,1)$.

The next proposition gives some characterizations of the strictly $C-$quasiconvex functions.

Proposition 3.4. *(see [7],[27]) Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be continuous and strictly $C-$quasiconvex. Then*

i) $\operatorname{WEff}(X, f) = \operatorname{Eff}(X, f)$;

ii) for every $y \in \operatorname{Min}(X, f)$, $f^{-1}(y)$ is a singleton.

Theorem 3.5. *Assume $\operatorname{WEff}(X, f)$ be nonempty and bounded. If $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be continuous and strictly $C-$quasiconvex then all global well-posedness notions coincide.*

Proof: The proof follows from Proposition 3.4 and from the CGR–well-posedness of $C-$quasiconvex functions, under compactness of $\operatorname{WEff}(X, f)$. $\square$

Remark 3.3. One can easy check that in Theorem 3.5 the strictly $C-$quasiconvexity cannot be replaced, without further assumptions, with the requirement that f is $C-$convex as shows problem (X, f) in Example 3.3.

4 Well-posedness of scalarized problems

In this section we deal with the relationships between the well-posedness of a vector optimization problem and the well-posedness of associated scalar ones. For this subject we need to recall some scalar notions of well-posedness ([11],[26]).
Consider the scalar minimization problem (X, h) given by

$$\min h(x), \quad x \in X$$

where $h : X \to \mathbb{R}$, the feasible region X is a closed subset of $\mathbb{R}^m$ and denote by $\arg\min(X, h)$ the solution set of problem (X, h).

Definition 4.1. *A sequence* $\{x^n\} \subseteq X$ *is called minimizing for problem* (X, h), *when* $h(x^n) \to \inf_X h$.

Definition 4.2. *(see [35]) Problem* (X, h) *is said to be Tykhonov well-posed if and only if*

i) $\arg\min(X, h) = \{\bar{x}\}$;

ii) every minimizing sequence converges to $\bar{x}$.

Towards vector optimization, the scalar generalizations of this notion relaxed the requirement of uniqueness of the solution. In this direction, we meet the notion of generalized Tykhonov well-posedness introduced by M. Furi and A. Vignoli in 1970 ([15]).

Definition 4.3. *Problem* (X, h) *is said to be generalized Tykhonov well-posed if and only if*

i) $\arg\min(X, h)$ *is a nonempty compact set;*

ii) every minimizing sequence admits a subsequence converging to some element of $\arg\min(X, h)$.

As further generalization, E. Bednarczuck and J.P. Penot in 1992 remove any assumption on the efficient set and formulate two stability concepts, respectively topologically well-setness ([2]) and metrically well-setness ([3]).

Definition 4.4. *Problem* (X, h) *is said to be topologically well-set when*

i) $\arg\min(X, h) \neq \emptyset$;

ii) every minimizing sequence contained in $X \setminus \arg\min(X, h)$ *has a cluster point in* $\arg\min(X, h)$.

Definition 4.5. *Problem* (X, h) *is said to be metrically well-set when*

i) $\arg\min(X, h) \neq \emptyset$;

ii) for every minimizing sequence $\{x^n\} \subseteq X, \quad d(x^n, \arg\min(X, h)) \to 0$.

4.1 Nonlinear scalarization and pointwise well-posedness

In [33] the authors use a scalarization procedure based on the so-called oriented distance function from a point to a set. This function has been introduced by J.B. Hiriart-Urruty ([17],[18]) to analyse the geometry of nonsmooth optimization problems, to derive necessary optimality conditions and then it has been applied, in several papers, to characterize the different types of efficient points as degrees of minimality in a particular scalarized problem (see for example [30],[36]).

Definition 4.6. *Let A be a subset of a normed vector space Y. The oriented distance function for A is $\Delta_A(y) : Y \to \mathbb{R} \cup \pm\{\infty\}$ defined as*

$$\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y)$$

where $d_A(y) = \inf_{x \in A} \|y - x\|$.

The main properties of function Δ_A are gathered in the following proposition ([36]).

Proposition 4.1. *1. If $A \neq \emptyset$ and $A \neq Y$ then Δ_A is real valued;*

2. Δ_A is 1-Lipschitzian;

3. $\Delta_A < 0, \quad \forall y \in \operatorname{int} A, \quad \Delta_A = 0, \quad \forall y \in \partial A$ and $\Delta_A > 0, \quad \forall y \in \operatorname{int} A^c$;

4. if A is convex, then Δ_A is convex;

5. if A is a cone, then Δ_A is positively homogeneous;

6. if A is a closed convex cone, then Δ_A is nonincreasing with respect to the ordering relation induced on Y, if $y_1, y_2 \in Y$ then

$$y - z \in A \quad \Rightarrow \quad \Delta_A(y) \leq \Delta_A(z)$$

if A has nonempty interior, then

$$y - z \in \operatorname{int} A \quad \Rightarrow \quad \Delta_A(y) < \Delta_A(z).$$

We use the notation ∂A to denote the frontier of the set A and A^c for the complementary of set A.
Now, consider the scalar problem (X, Δ_{-C}) associated to the vector problem (X, f) given by

$$\min \Delta_{-C}(f(x) - p), \quad x \in X$$

where $p \in Y = f(X)$. Using this scalar problem Miglierina, Molho, Rocca ([33]) derive the following results as link with the pointwise well-posedness of the vector problem (X, f).

Theorem 4.1. *Let $\bar{y} \in \operatorname{Min}(X, f)$. Problem (X, Δ_{-C}) with $p = \bar{y}$ is topologically well-set (according to definition 4.4), if and only if problem (X, f) is B-well-posed at $\bar{y}$ (definition 3.2).*

We observe that no assumption of generalized convexity or monotonicity is required and that function Δ_{-C} doesn't imply any boundedness assumption on the feasible region X.

Corollary 4.1. *Let $\bar{y} \in \operatorname{Min}(X, f)$. Problem (X, Δ_{-C}) with $p = \bar{y}$ is generalized Tykhonov well-posed (according to definition 4.3), if and only if problem (X, f) is L-well-posed at $\bar{y}$ (definition 3.3).*

Corollary 4.2. *Let $\bar{x} \in \mathrm{Eff}\,(X, f)$. Problem (X, Δ_{-C}) with $p = \bar{y}$ is Tykhonov well-posed (according to definition 4.2), if and only if problem (X, f) is DH-well-posed at $\bar{x}$ (definition 3.4).*

Remark 4.1. A direct link between nonlinear scalarization and H–$\bar{x}$ well-posedness (Definition 3.5) is established in [12].

Thanks to the scalarization with oriented distance function, the results of this subsection are equivalence, i.e. the pointwise well-posedness of the vector problem is completely represented in a scalar model.

4.2 Nonlinear scalarization and global well-posedness

In Section 3 we distinguish global well-posedness notions involving the efficient frontier from those considering the weakly efficient solutions. As we have already pointed out, in scalar case efficient and weakly efficient points coincide, in other words, all global notions of well-posedness generalizes the weak concept of well-setness. Hence, to get a scalarized procedure for global notions in which the efficient frontier does not include also weakly efficient solutions, we need to separate in some way the two concepts, maybe checking different properties of the scalarizing function. Only after a scalar characterization of solutions set, one can seek a scalar well-setness condition.

In [34] the authors propose a scalar problem associated to the vector one and study the CGR-well-posedness. They used the approach based on the oriented distance function Δ_{-C}, considering the whole image set and not only a fixed minimal point.
Let (X, h) the scalar problem defined as

$$\min h(x), \quad x \in X,$$

where $h(x) = -\inf_{z \in X} \Delta_{-C}(f(z) - f(x))$.
The weak solutions of (X, f) can be completely characterized by the solutions of the scalar problem (X, h).

Theorem 4.2. *(see [34]) Let $\bar{x} \in X$. Then $\bar{x} \in \mathrm{WEff}\,(X, f)$ if and only if $h(\bar{x}) = 0$ (and hence $\bar{x} \in \mathrm{Eff}\,(X, f)$).*

Theorem 4.3. *(see [34]) Problem (X, h) is metrically well-set (according to definition 4.5) if and only if problem (X, f) is CGR-well-posed (definition 3.16).*

Because of the summary of the links between the different global notions,

i) Let $\mathrm{WMin}\,(X, f)$ be closed. (X, h) well-set $\iff$ (X, f) CGR–well-posed $\Rightarrow$ (X, f) D–well-posed.

ii) Let $\mathrm{WEff}\,(X, f)$ be compact. (X, h) well-set $\iff$ (X, f) CGR–well-posed $\iff$ (X, f) Hs–well-posed $\Rightarrow$ (X, f) H–well-posed $\Rightarrow$ (X, f) Hs–well-posed.

4.3 Linear scalarization and pointwise well-posedness

The links between well-posedness of a linearly scalarized problem and well-posedness of a vector problem, are proved under convexity or generalized convexity assumptions.
Consider the scalar problem (X, g_λ), associated to the vector problem (X, f), given by

$$\min g_\lambda(x), \quad x \in X,$$

where $g_\lambda(x) = \langle \lambda, f(x) - p \rangle$ in which $\lambda \in C^+ \cap \partial B$ and $p \in Y = f(X)$.

Theorem 4.4. *Let $\bar{y} \in \mathrm{Min}\,(X, f)$. If there exists $\bar{\lambda} \in C^+ \cap \partial B$ such that problem $(X, g_{\bar{\lambda}})$ with $p = \bar{y}$ is topologically well-set (according to definition 4.4) and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$*, then (X, f) is B-well-posed in $\bar{y}$ (definition 3.2).*

Proof: Recalling Theorem 4.1, if ab absurdo problem (X, f) is not B-well-posed in $\bar{y}$, then

$$\exists x^n \in X \setminus \arg\min(X, \Delta_{-C}) \text{ such that } \Delta_{-C}(f(x^n) - \bar{y}) \to 0,$$

$$\text{but } \nexists x^{n_k} \text{ such that } x^{n_k} \to \bar{x} \in \arg\min(X, \Delta_{-C}).$$

Since $\Delta_{-C}(f(x^n) - \bar{y}) = \max\{\langle \lambda, f(x^n) - \bar{y} \rangle : \lambda \in C^+ \cap \partial B\}$ (see [33]), it follows

$$0 \leq \left\langle \bar{\lambda}, f(x^n) - \bar{y} \right\rangle \leq \Delta_{-C}(f(x^n) - \bar{y})$$

and recalling the assumptions, $\arg\min(X, \Delta_{-C}) = \arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$. But this means $g_{\bar{\lambda}}(x^n) \to 0$, a contradiction with topologically well-setness of (X, g_λ). $\square$

The assumption $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$ cannot be avoided as the following example shows.

Example 4.1. *Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (x, 0)$ with $X = C = \mathbb{R}^2_+$. Let $\bar{\lambda} = (0, 1)$ and $g_{\bar{\lambda}}(x, y) = 0$. The set $\mathrm{Min}\,(X, f) = \{(0, 0)\}$, all the assumptions of Therorem 4.4 are satisfied except one:* $\arg\min(X, g_{\bar{\lambda}}) = X \quad \neq \quad f^{-1}(0, 0) = \{(x, y) : x = 0, \ y \geq 0\}$*. Problem $(X, g_{\bar{\lambda}})$ is topologically well-set, but problem (X, f) is not B-well-posed, for instance the B-minimizing sequence $\{x^n\} = \left(\frac{1}{n}, n\right)$ does not converge.*

Corollary 4.3. *Let $\bar{y} \in \mathrm{Min}\,(X, f)$. If there exists $\bar{\lambda} \in C^+ \cap \partial B$ such that problem $(X, g_{\bar{\lambda}})$ with $p = \bar{y}$ is generalized Tykhonov well-posed (according to definition 4.3) and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$*, then (X, f) is L-well-posed in $\bar{y}$ (definition 3.3).*

Corollary 4.4. *Let $\bar{y} \in \mathrm{Min}\,(X, f)$. If there exists $\bar{\lambda} \in C^+ \cap \partial B$ such that problem $(X, g_{\bar{\lambda}})$ with $p = \bar{y}$ is Tykhonov well-posed (according to definition 4.2) and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$*, then (X, f) is DH-well-posed in $\bar{x}$ (definition 3.4).*

Remark 4.2. The existence of a linear scalarized satisfing a well-posed notion is a sufficient condition in order that problem (X, f) is pointwise well-posed. If the ordering cone C satisfies the geometrical requirement $C \subseteq \mathbb{R}^l_+$, the sufficient condition can be tested by a single scalar problem (X, f_i) where f_i is a component of the vector objective function f. In the most interesting case, i.e. the investigation of DH-well-posedness in $\bar{x}$, it is possible to prove that if there exists at least one problem (X, f_i) Tykhonov well-posed, then problem (X, f) is DH-well-posed in $\bar{x}$.

Remark 4.3. The previous results permit us to identify a class of well-posed vector problems which satisfy a further regularity condition, since there exists a vector $\bar{\lambda}$ such that the scalarized problem $(X, g_{\bar{\lambda}})$ is well-posed. We call this property $\bar{\lambda}-$well-posedness.

The link between well-posedness of a linearly scalarized problem and well-posedness of the original one is weaker than the relation involving nonlinear scalarization, since the results are only in one direction, but we haven't yet imposed convexity or generalized convexity requirements. In order to find a class of functions for which the equivalence is valid we recall the notion of $*-$quasiconvexity, a particular subset of $C-$quasiconvex functions ([14], [24]).

Definition 4.7. *(see [14], [24]) A function $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ is said to be $*-$quasiconvex if and only if $\forall \lambda \in C^+$ function $\langle \lambda, f(\cdot) \rangle : X \to \mathbb{R}$ is quasiconvex.*

The next result identifies the class of $*-$quasiconvex function as satisfing a $\bar{\lambda}-$well-posedness; for these functions it is possible to replace the well-posedness analysis of the vector problem with both nonlinear and linear scalarization.

Theorem 4.5. *Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be $*-$quasiconvex and $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$ be a bounded set. Then, problem (X, f) is B-well-posed in $\bar{y}$ if and only if $\exists \bar{\lambda} \in C^+ \cap \partial B$ such that $\left\langle \bar{\lambda}, y - \bar{y} \right\rangle \geq 0,\ \forall y \in f(X)$ and $(X, g_{\bar{\lambda}})$ is topologically well-set.*

Proof: Recalling Theorem 4.4 we only need to prove one direction. As function f is $*-$quasiconvex, the set $(f(X) + C)$ is convex ([14]) and thanks to a classical separation theorem, every $\bar{y} \in \operatorname{Min}(X, f)$ is unique solution of a scalarized problem. Function $g_{\bar{\lambda}}(x)$ is quasiconvex as linear combination of continuous functions and hence problem $(X, g_{\bar{\lambda}})$ is topologically well-set since $\arg\min(X, g_{\bar{\lambda}})$ is bounded. $\square$

Corollary 4.5. *Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be $*-$quasiconvex and $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$ be a bounded set. Then, problem (X, f) is L-well-posed in $\bar{y}$ if and only if $\exists \bar{\lambda} \in C^+ \cap \partial B$ such that $\left\langle \bar{\lambda}, y - \bar{y} \right\rangle \geq 0,\ \forall y \in f(X)$ and $(X, g_{\bar{\lambda}})$ is generalized Tykhonov well-posed.*

Corollary 4.6. *Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be $*-$quasiconvex and $\arg\min(X, g_{\bar{\lambda}}) = \bar{x}$. Then, problem (X, f) is DH-well-posed in $\bar{x}$ if and only if $\exists \bar{\lambda} \in C^+ \cap \partial B$ such that $\left\langle \bar{\lambda}, y - \bar{y} \right\rangle \geq 0,\ \forall y \in f(X)$ and $(X, g_{\bar{\lambda}})$ is Tykhonov well-posed.*

We observe that Corollary 4.6 cannot be improved considering $\arg\min(X, g_{\bar{\lambda}})$ unbounded or the larger class of C-quasiconvex functions as it is possible to see in the following examples.

Example 4.2. *Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, the identity function on $X = \{(x, y) \in \mathbb{R}^2 : y \geq -x\}$ and $C = \mathbb{R}^2_+$. Function f is $*-$quasiconvex, problem (X, f) is DH-well-posed in $(0, 0)$, but it is not $\bar{\lambda}$-well-posed for any $\bar{\lambda} \in C^+ \cap \partial B$.*

Example 4.3. *Let $f : X \subseteq \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, -x^3)$ with $X = \mathbb{R}$ and $C = \mathbb{R}^2_+$. Function f is C-quasiconvex but not $*-$quasiconvex. Problem (X, f) is DH-well-posed in 0, but it is not $\bar{\lambda}$-well-posed for any $\bar{\lambda} \in C^+ \cap \partial B$.*

4.4 Linear scalarization and global well-posedness

Consider a convex vector optimization problem, i.e. assume that X is convex and f is $C-$convex. We recall that the functions $g_\lambda(x) = \langle \lambda, f(x) \rangle$ with $\lambda \in C^+ \setminus \{0\}$ are convex when f is $C-$convex ([27]). Consider the family of parametric scalar problems (X, g_λ) given by

$$\min g_\lambda(x) = \langle \lambda, f(x) \rangle \,, \quad x \in X,$$

where $\lambda \in C^+ \cap \partial B$.
By convexity assumptions follows that a point $x \in X$ is a weakly efficient solution for vector problem (X, f) if and only if it is an optimal solution for a scalar problem (X, g_λ).

Theorem 4.6. *(see [8]) Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be $C-$convex on the convex set X and assume* $\mathrm{WMin}\,(X, f)$ *is closed.*
If problems (X, g_λ) are metrically well-set (according to definition 4.5) for every $\lambda \in C^+ \cap \partial B$, then problem (X, f) is D-well-posed.

Proof: We know that an asymptotically minimizing sequence for problem (X, f), is always asymptotically minimizing for problem (X, h) defined in the previous subsection.
Let x^n be an asymptotically minimizing sequence for problem (X, f). Then $h(x^n) \to 0$ and by the compactness of $C^+ \cap \partial B$, there exists a sequence $\lambda^n \to \lambda^* \in C^+ \cap \partial B$ such that

$$\min_{\lambda \in C^+ \cap \partial B} \langle \lambda, f(x^n) - f(x) \rangle = \langle \lambda^n, f(x^n) - f(x) \rangle \,,$$

and hence

$$\sup_{x \in X} \langle \lambda^n, f(x^n) - f(x) \rangle \to 0,$$

i.e.

$$\langle \lambda^n, f(x^n) \rangle - \inf_{x \in X} \langle \lambda^n, f(x) \rangle \to 0.$$

We observe that $g_\lambda(x)$ is a convex function for every $\lambda \in C^+ \cap \partial B$ (see [27]) and since $\lambda^n \to \lambda^*$, it follows $\langle \lambda^n, f\rangle \to \langle \lambda^*, f\rangle$. Hence (see e.g. [29]),

$$g_{\lambda^n}(x^n) = \langle \lambda^n, f(x^n)\rangle \to \inf_{x\in X} \langle \lambda^*, f(x)\rangle = \inf_{x\in X} g_{\lambda^*}(x).$$

We claim that $g_{\lambda^*}(x^n) \to \inf_{x\in X} g_{\lambda^*}(x)$.
Since $\lambda^n \to \lambda^*$, $\forall \varepsilon > 0$, $\exists \bar{n}$ such that $\forall n > \bar{n}$

$$|\langle \lambda^*, f(x^n)\rangle - \langle \lambda^n, f(x^n)\rangle| < \frac{\varepsilon}{2},$$

i.e. $|\langle \lambda^* - \lambda^n, f(x^n)\rangle| < \frac{\varepsilon}{2}$. Hence, $\forall n > \bar{n}$

$$\begin{aligned} 0 &\leq \langle \lambda^*, f(x^n)\rangle - \inf_{x\in X} \langle \lambda^*, f(x)\rangle \\ &= g_{\lambda^*}(x^n) - \inf_{x\in X} \langle \lambda^*, f(x)\rangle \\ &= \langle \lambda^n, f(x^n)\rangle - \inf_{x\in X} \langle \lambda^*, f(x)\rangle + \langle \lambda^* - \lambda^n, f(x^n)\rangle \\ &\leq \langle \lambda^n, f(x^n)\rangle - \inf_{x\in X} \langle \lambda^*, f(x)\rangle + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\langle \lambda^n, f(x^n)\rangle \to \inf_{x\in X} \langle \lambda^*, f(x)\rangle$ and ε is arbitrary, we prove the claim. Hence recalling the assumption of metrically well-setness on (X, g_λ) the proof is completed. $\square$

In general, the reverse of the Theorem is not true as the following example shows.

Example 4.4. *Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x_1, x_2) = \left(\frac{x_1^2}{x_2}, x_1\right)$, $C = \mathbb{R}^2_+$ and $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 1\}$. The objective function is $C-$convex,* $\mathrm{WMin}\,(X, f) = \{(0, 0)\}$, $\mathrm{WEff}\,(X, f) = \{(0, x_2) : x_2 \geq 1\}$ *the problem is D-well-posed since every D-minimizing sequence is identified when x_1 tends to zero, but the scalar problem (X, g_λ) with $\lambda = (1, 0)$ is not metrically well-set.*

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