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Asymptotic results for a generalized Pólya urn and applications to clinical trials

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Abstract

In this paper a new Pólya urn model is introduced and studied; in particular, a strong law of large numbers and two central limit theorems are proven. This urn generalizes a model studied in Berti et al. (2004), May et al. (2005) and in Crimaldi (2007) and it has natural applications in clinical trials. Indeed, the model include both delayed and missing (or null) responses. Moreover, a connection with the conditional identity in distribution of Berti et al. (2004) is given.

1 Introduction

We consider the following experiment. An urn contains $b \in \mathbb{N}^*$ black and $r \in \mathbb{N}^*$ red balls. Let us suppose given two sequences $(r_i)_{i\geq 0}$ and $(u_i)_{i\geq 0}$ of integers such that

$$r_0 = u_0 = 0 < r_1 \leq u_1 < r_2 \leq u_2 < r_3 \leq u_3 < \ldots .$$

At each time $n \geq 1$, a ball is drawn from the urn and then it is put again in the urn. Moreover, at each time u_i the urn is updated in the following way: for each j with $u_{i-1}+1 \leq j \leq r_i$, we put in the urn other N_j balls of the same color as the ball drawn at time j. The numbers N_j are *randomly* chosen in $\mathbb{N}^*$. The way in which the number N_j is chosen may depend on j but it must be suitably independent of the results of the choices for the preceding numbers and of the preceding drawings (see sec. 2). The special case in which $r_i = u_i = i$ for all i is just the case of the generalized Pólya urn studied in Berti et al. (2004) and in Crimaldi (2007). Moreover, if we take $r_i = u_i = i$ for all i and the random variables N_j identically distributed, then we fall in the case considered in May et al. (2005).

In clinical trials this urn can be used to allocate patients to two different treatments. The black balls represent the first treatment, while the red balls represent the second; at each time $n \geq 1$ a patient is allocated to a treatment by picking a ball and observing its color. The introductions N_j represent, according to the interpretation of May et al. (2005), the responses. At time u_i, a part of these responses, precisely those associated to an index j with $u_{i-1}+1 \leq j \leq r_i$, arrives with delay. The responses associated to an index j with $r_i + 1 \leq j \leq u_i$ are considered null or missing because of various facts: for example, decease of

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the patient for reasons that we can't connect to the treatments, responses that are missed from the analysis laboratory, or responses that the doctor considers irrelevant for future allocations. Hu-Zhang (2004) and Zhang et al. (2007) have introduced interesting urn models with delayed responses which differs from ours in the structure and in the mechanism of updating so that they can be applied to different situations. On the contrary, the purposes and the type of the given results are similar. In the last section of this paper, the reader can find another experiment that can be formalized by the model we study.

Let us denote by Y_n the indicator function of the event {black ball at time n}, that is, in the language of clinical trials, the indicator function of the event {first treatment to patient n}, then the random variable $C_n = \sum_{i=1}^n Y_i$ counts the number of patients assigned to the first treatment in the first n trials. In Section 3, we prove a Strong law of large numbers for $(Y_n)_{n\geq 1}$, i.e.

$$\frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{a.s.} V,$$

where V is also the almost sure limit of $V_n = \mathrm{E}[Y_{n+1}|\mathcal{F}_n]$ (with $(\mathcal{F}_n)_{n\geq 0}$ the natural filtration associated with the model). In the language of clinical trials, this means

$$\frac{C_n}{n} \xrightarrow{a.s.} V. \tag{1}$$

Moreover, we prove two central limit theorems: precisely, under suitable conditions, we obtain that

$$\sqrt{n}\big(\mathrm{E}[Y_{n+1}\,|\,\mathcal{G}_n] - V\big) \xrightarrow{\mathcal{D}} \nu_1, \tag{2}$$

and

$$\sqrt{r_{i(n)}}\big(\frac{C_{r_{i(n)}}}{r_{i(n)}} - V_n\big) \xrightarrow{\mathcal{D}} \nu_2, \tag{3}$$

where $\mathcal{D}$ means "convergence in distribution" and ν_1 and ν_2 are suitable " mixtures" of Gaussian distributions that are formally defined in Sections 4 and 5. Actually, we show that stronger convergences hold for the two above sequences: almost sure conditional convergence (in the sense of Crimaldi, 2007) for the first sequence and stable convergence (see, for instance, Jacod-Memin, 1981) for the second one. The proof of (2) is based on a limit theorem for martingales which has been proved in Crimaldi (2007) and it employs the same technique used in that paper; while, in order to prove (3), we apply a classical result regarding the stable convergence. Moreover, for the first central limit theorem, we illustrate also an example; while, for the second one, we give for the particular case $r_i = u_i$ (but not necessarily equal to i) for all i, a set of conditions, which are less difficult to be verified in practice than the general conditions of the theorem.

Finally we can note that, if we consider the proposed model by a more deeper theoretical point of view, then we can say that, with respect to a suitable filtration $(\mathcal{G}_n)_{n\geq 0}$, the sequence $(Y_n)_{n\geq 1}$ has all the property of conditionally identically distributed (cid, abbreviated) sequences, introduced in Berti et al. (2004), except the adaption to the filtration. The study of non adapted sequences of random variables is very interesting because sometimes the request of adaptation can be restrictive. Thus it could be a fertile ground for further researches. To the best of our knowledge the only paper on this argument is Jayte (2002), which deals with non adapted martingale.

The literature on urn models is very wide. For instance, in addition to the above cited papers, the reader may look at Hill et al. (1980), Gouet (1993), Dirienzo (2000), Kotz et al. (2000), Janson (2006), Muliere et al. (2006).

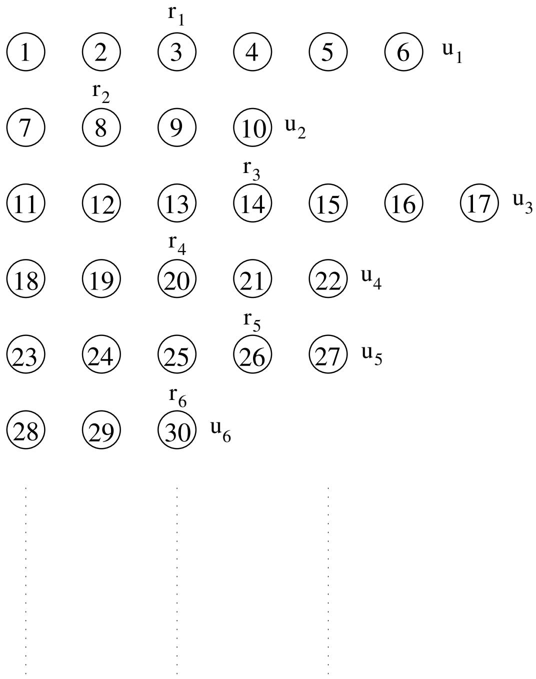

Figure 1: An example of sequences (r_i), (u_i)

2 The model and preliminary results

Let us set

$$l_i = r_i - u_{i-1} \qquad \text{for each } i \geq 1,$$
$$i(n) = \sup\{i \geq 0 : u_i \leq n\} \qquad \text{for each } n \geq 0.$$

Given a sequence $(\mu_i)_{i\geq 1}$ of probability measures on $(\mathbb{N}^*)^{\otimes l_i}$, it is possible to build a probability space $(\Omega, \mathcal{A}, P)$ and, on it, a sequence $(Y_n)_{n\geq 1}$ of random variables with values in $\{0, 1\}$ and a sequence $(L_i)_{i\geq 1}$ of random vectors of the form

$$L_i = [N_j : \ u_{i-1} + 1 \leq j \leq r_i]$$

such that the following conditions are satisfied:

(a) For each $n \geq 0$, a version of the conditional distribution of Y_{n+1} given the

σ-field

$$\mathcal{F}_n := \sigma(Y_1,\dots,Y_{u_1},L_1,Y_{u_1+1},\dots,Y_{u_2},L_2,\dots,Y_{u_{i(n)-1}+1},\dots,Y_{u_{i(n)}},L_{i(n)},Y_{u_{i(n)}+1},\dots Y_n)$$
$$= \sigma(Y_1,\dots,Y_n)\vee\sigma(L_i:\ 1\le i\le i(n)) \qquad (\text{where } \mathcal{F}_0 := \{\emptyset,\Omega\})$$

is the kernel $\big(\mathcal{B}(1,V_n(\omega))\big)_{\omega\in\Omega}$, where $\mathcal{B}(1,p)$ denotes the Bernoulli distribution with parameter p and V_n is the random variable defined by[1]

$$V_n := \Big(b+\textstyle\sum_{i=1}^{i(n)}\sum_{j=u_{i-1}+1}^{r_i} Y_jN_j\Big)\Big(b+r+\textstyle\sum_{i=1}^{i(n)}\sum_{j=u_{i-1}+1}^{r_i} N_j\Big)^{-1}.$$

(b) For each $i\ge 1$, the random vector $L_i=[N_j:\ u_{i-1}+1\le j\le r_i]$ has distribution μ_i and it is independent of the sub-σ-field

$$\mathcal{F}_{u_{i-1}}\vee\sigma(Y_{u_{i-1}+1},\dots,Y_{u_i}).$$

With this formalization, for each $n\ge 1$, the random variable Y_n denotes the indicator function of the event $\{$black ball at time $n\}$ and the random variable V_n represents the proportion of black ball in the urn at time n.

By condition (a), we have $\mathrm{E}[Y_{n+1}|\mathcal{F}_n]=V_n$ for each $n\ge 0$. Moreover, if we set

$$\mathcal{H}_n := \sigma(Y_j:\ 1\le j\le r_{i(n)})\vee\sigma(L_i:\ 1\le i\le i(n)) \qquad (\text{where } \mathcal{H}_0 := \{\emptyset,\Omega\}),$$

we also have $\mathrm{E}[Y_{n+1}|\mathcal{H}_n]=V_n$. Finally, by this equality and condition (b), if we set

$$\mathcal{G}_n := \mathcal{H}_n\vee\sigma(L_{i(n)+1}),$$

we also have $\mathrm{E}[Y_{n+1}|\mathcal{G}_n]=V_n$. Indeed, for each $n\ge 0$, only the two following cases are possible:

1) $i(n+1)=i(n)$ and so $n+1<u_{i(n)+1}$;

2) $i(n+1)=i(n)+1$ and so $n+1=u_{i(n)+1}$.

In both cases, since $i(u_{i(n)})=i(n)$, the sub-σ-field $\mathcal{H}_n\vee\sigma(Y_{n+1})$ is contained in the sub-σ-field

$$\mathcal{F}_{u_{i(n)}}\vee\sigma(Y_{u_{i(n)}+1},\dots,Y_{u_{i(n)+1}}).$$

Thus, by assumption (b), the random variable $L_{i(n)+1}$ is independent of the sub-σ-field $\mathcal{H}_n\vee\sigma(Y_{n+1})$.

Proposition 2.1. *The sequence $(V_n)_{n\ge 0}$ is a martingale with respect to the filtration $\mathcal{G}=(\mathcal{G}_n)_{n\ge 0}$ (and the filtration $\mathcal{H}=(\mathcal{H}_n)_{n\ge 0}$).*

Proof. Since $(V_n)_n$ is $\mathcal{H}$-adapted and $\mathcal{H}_n\subset\mathcal{G}_n$ for each n, then it suffices to prove that $(V_n)_n$ is a $\mathcal{G}$-martingale. To this end, we observe as above that, for each $n\ge 0$, only the two following cases are possible:

1) $i(n+1)=i(n)$;

2) $i(n+1)=i(n)+1$.

[1]Throughout this paper we use the convention that $\sum_a^b=0$ if $b<a$.

In the first case, we have $V_{n+1} = V_n$ and so $\mathrm{E}[V_{n+1}|\,\mathcal{G}_n] = V_n$. In the second case, if we set

$$S_n := (b + r + \textstyle\sum_{i=1}^{i(n)} \sum_{j=u_{i-1}+1}^{r_i} N_j), \tag{4}$$

then we can write

$$\begin{aligned} V_{n+1} &= S_{n+1}^{-1}(V_n S_n + \textstyle\sum_{j=u_{i(n+1)-1}+1}^{r_{i(n+1)}} Y_j N_j) \\ &= S_{n+1}^{-1}(V_n S_n + \textstyle\sum_{j=u_{i(n)}+1}^{r_{i(n)+1}} Y_j N_j). \end{aligned}$$

It follows that

$$\mathrm{E}[V_{n+1}\,|\,\mathcal{G}_n] = S_{n+1}^{-1}\big(V_n S_n + \textstyle\sum_{j=u_{i(n)}+1}^{r_{i(n)+1}} N_j \mathrm{E}[Y_j\,|\,\mathcal{G}_n]\big).$$

On the other hand, for each j with $u_{i(n)}+1 \le j \le r_{i(n)+1}$, we have $i(j-1) = i(n)$ and so we have

$$\mathrm{E}[Y_j\,|\,\mathcal{G}_n] = \mathrm{E}[Y_j\,|\,\mathcal{G}_{j-1}] = V_{j-1} = V_n.$$

Thus, we obtain $\mathrm{E}[V_{n+1}|\mathcal{G}_n] = V_n$. □

Remark 2.2. Since each random variable Y_n takes values in $\{0,1\}$, the above proposition implies that, for each real function f on $\{0,1\}$, the sequence of conditional expectations $(\mathrm{E}[f(Y_{n+1})|\mathcal{G}_n])_{n\ge 0}$ is a $\mathcal{G}$-martingale. However, we can not conclude that the sequence $(Y_n)_{n\ge 1}$ is $\mathcal{G}$-conditionally identically distributed in the sense of Berti et al. (2004) because it is generally not $\mathcal{G}$-adapted. On the other hand, the sequence $(Y_n)_{n\ge 1}$ is adapted with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\ge 0}$ but $(V_n)_{n\ge 0}$ can not be an $\mathcal{F}$-martingale. For example, if we consider the particular case in which the random variables N_j are deterministic, we have

$$\begin{aligned} \mathrm{E}[V_{u_k}\,|\,\mathcal{F}_{u_k-1}] &= S_{u_k}^{-1}\big(V_{u_k-1}S_{u_k-1} + \textstyle\sum_{j=u_{k-1}+1}^{u_k-1} Y_j N_j + \sum_{j=u_k}^{r_k} N_j \mathrm{E}[Y_j\,|\,\mathcal{F}_{u_k-1}]\big) \\ &= S_{u_k}^{-1}\big(V_{u_k-1}S_{u_k-1} + \textstyle\sum_{j=u_{k-1}+1}^{u_k-1} Y_j N_j + \sum_{j=u_k}^{r_k} N_j V_{u_k-1}\big) \end{aligned}$$

which is equal to V_{u_k-1} if and only if $u_{k-1}+1 = u_k = r_k$, that is $u_k = r_k = k$ for all $k \ge 0$. This is the case of the generalized Pólya urn studied in Berti et al. (2004) and in Crimaldi (2007).

3 The strong law of large numbers

The sequence $(V_n)_{n\ge 0}$ is a uniformly bounded martingale and so it converges almost surely and in L^1 to a bounded random variable V. This random variable V is also the limit of the sequence of the empirical means

$$M_n = \frac{C_n}{n} = \frac{1}{n}\textstyle\sum_{j=1}^{n} Y_j.$$

More precisely, we have the following proposition.

Proposition 3.1. *The sequence $(M_n)_{n\ge 1}$ converges in L^1 and almost surely to the random variable V.*

Proof. The sequence $(M_n)_n$ is uniformly bounded and so it suffices to prove only the almost sure convergence. To this end, we start with observing that, by definition, we have $V_n = \mathrm{E}[Y_{n+1}\,|\,\mathcal{F}_n]$ and the sequence

$$Z_n = \textstyle\sum_{j=1}^{n} j^{-1}\,(Y_j - V_{j-1}) = \sum_{j=1}^{n} j^{-1}\,\big(Y_j - \mathrm{E}[Y_j\,|\,\mathcal{F}_{j-1}]\big)$$

is obviously an $\mathcal{F}$-martingale. Moreover, since each random variable Y_j takes values in $\{0,1\}$, we have $\sup_n \mathrm{E}[Z_n^2] < \infty$. Hence, the martingale $(Z_n)_n$ converges almost surely. Kronecker's lemma ensures that

$$\tfrac{1}{n}\sum_{j=1}^{n}(Y_j - V_{j-1}) \xrightarrow{a.s.} 0.$$

Now, we recall that, if $(a_n)_n$ and $(b_n)_n$ are any real sequences, then $\frac{1}{n}\sum_{k=1}^{n} a_k b_k \to ab$ whenever $a_n \geq 0$ for each n, $\frac{1}{n}\sum_{k=1}^{n} a_k \to a$ and $b_n \to b$. Therefore, since V_{j-1} converges almost surely to V, we obtain

$$\tfrac{1}{n}\sum_{j=1}^{n} V_{j-1} \xrightarrow{a.s.} V$$

and so

$$M_n = \tfrac{1}{n}\sum_{j=1}^{n}(Y_j - V_{j-1}) + \tfrac{1}{n}\sum_{j=1}^{n} V_{j-1} \xrightarrow{a.s.} V.$$

□

Remark 3.2. Since each random variable Y_n takes values in $\{0,1\}$, the above result implies that, for each real function f on $\{0,1\}$, the sequence

$$\tfrac{1}{n}\sum_{j=1}^{n} f(Y_j)$$

converges in L^1 and almost surely to the random variable $V_f = f(0)(1-V) + f(1)V$. Indeed, we have

$$\begin{aligned}\tfrac{1}{n}\sum_{j=1}^{n} f(Y_j) &= \tfrac{1}{n}\sum_{j=1}^{n} f(0)(1-Y_j) + \tfrac{1}{n}\sum_{j=1}^{n} f(1)Y_j \\ &= f(0)\big(1 - \tfrac{1}{n}\sum_{j=1}^{n} Y_j\big) + \tfrac{f(1)}{n}\sum_{j=1}^{n} Y_j.\end{aligned}$$

4 A central limit theorem

We are going to prove the following limit theorem.

Theorem 4.1. *Let us set*

$$Q_k := \begin{cases} 0 & \text{if } 0 \leq k < u_1 - 1 \\ \big(\sum_{i=1}^{i(k+1)}\sum_{h=u_{i-1}+1}^{r_i} N_h\big)^{-1}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} N_j & \text{if } k \geq u_1 - 1 \end{cases}$$

and

$$Q_{k,j} := \begin{cases} 0 & \text{if } 0 \leq k < u_1 - 1 \\ N_j\,\big(\sum_{i=1}^{i(k+1)}\sum_{h=u_{i-1}+1}^{r_i} N_h\big)^{-1} & \text{if } k \geq u_1 - 1. \end{cases}$$

Moreover, let us set

$$W_n := \sqrt{n}(V_n - V).$$

Further, let us denote by K_n a version of the conditional distribution of W_n given $\mathcal{G}_n$.

Suppose that the following conditions are satisfied:

(i) $n\sum_{k\geq n} Q_k^2 \xrightarrow{a.s.} H$, where H is a positive real random variable.

(ii) $\sum_{k\geq 0} k^2\,\mathrm{E}[Q_k^4] < \infty$.

(iii) $n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j}\sum_{h=u_{i(k)}+1}^{j-1} Q_{k,h} \xrightarrow{a.s.} 0$.

Then, for almost every ω in Ω, the sequence $\big(K_n(\omega,\cdot)\big)_n$ of probability measures converges weakly to the Gaussian distribution

$$\mathcal{N}\big(0, H(\omega)(V(\omega)-V^2(\omega))\big).$$

In other words, for each bounded continuous function f on $\mathbb{R}$, the conditional expectation $\mathrm{E}[f(W_n)|\mathcal{G}_n]$ *converges almost surely to the random variable*

$$\omega \mapsto \int f(x)\,\mathcal{N}\big(0, H(\omega)(V(\omega)-V^2(\omega))\big)(\mathrm{d}x).$$

More briefly, the statement of the above theorem can be so reformulated: with respect to the conditioning system $\mathcal{G}=(\mathcal{G}_n)_n$, the sequence $(W_n)_n$ converges to the Gaussian kernel

$$\mathcal{N}(0,H(V-V^2)) = \big(\mathcal{N}(0,H(\omega)(V(\omega)-V^2(\omega)))\,\big)_{\omega\in\Omega}$$

in the sense of the *almost sure conditional convergence* (see Crimaldi, 2007, Sec. 2). In particular, it follows that the sequence $(W_n)_n$ converges $\mathcal{A}$-stably to the kernel $\mathcal{N}(0,H(V-V^2))$. It is well known that this fact implies that the sequence $(W_n)_n$ converges in distribution to the probability measure ν_1 on $\mathbb{R}$ defined by

$$\nu_1(B) = \int \mathcal{N}\big(0, H(\omega)(V(\omega)-V^2(\omega))\big)(B)\;P(\mathrm{d}\omega).$$

Remark 4.2. Note that the random variables Q_k have been defined in such a way that $Q_k=0$ when $i(k)=i(k+1)$.

Remark 4.3. If we are in the case $u_{i-1}+1=r_i$ for all i, then assumption (iii) is obviously satisfied since the third sum is zero.

Proof. It will be sufficient to prove that the $\mathcal{G}$-martingale $(V_n)_n$ satisfies conditions (a) and (b) of Proposition 2.2 in Crimaldi (2007) with $U=H(V-V^2)$ (see the appendix). To this end, we recall firstly that we can have only two cases $i(k+1)=i(k)$ or $i(k+1)=i(k)+1$. Then, after some calculations, we get

$$\begin{aligned}&V_k - V_{k+1} = \\ &\Big(V_k \textstyle\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} N_j - \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j N_j\Big)\ (b+r+\sum_{i=1}^{i(k+1)}\sum_{h=u_{i-1}+1}^{r_i} N_h)^{-1}.\end{aligned} \tag{5}$$

Moreover, it is immediate to verify that

$$\textstyle\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j} = Q_k \tag{6}$$

(Note that, if $i(k+1)=i(k)$, then $r_{i(k+1)}<u_{i(k)}+1$ and the sums in the above relations are equal to zero. On the contrary, if $i(k+1)=i(k)+1$, then $r_{i(k+1)}=r_{i(k)+1}\geq u_{i(k)}+1$.) Thus, from (5) and (6), we have

$$\begin{aligned}|V_k - V_{k+1}| &= \Big|\textstyle\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} (V_k - Y_j)N_j\Big|\ (b+r+\sum_{i=1}^{i(k+1)}\sum_{h=u_{i-1}+1}^{r_i} N_h)^{-1} \\ &\leq \textstyle\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} |V_k - Y_j| Q_{k,j} \\ &\leq \textstyle\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j} = Q_k,\end{aligned}$$

and so, using assumption (ii), we find

$$\sup_k k^2\,|V_k - V_{k+1}|^4 \leq \sum_{k\geq 0} k^2 Q_k^4 \in L^1.$$

Furthermore, we have

$$\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} N_j\left(b+r+\sum_{i=1}^{i(k+1)}\sum_{h=u_{i-1}+1}^{r_i} N_h\right)^{-1} \sim Q_k \qquad \text{for } \ k\to+\infty,$$

$$N_j\left(b+r+\sum_{i=1}^{i(k+1)}\sum_{h=u_{i-1}+1}^{r_i} N_h\right)^{-1} \sim Q_{k,j} \qquad \text{for } \ k\to+\infty,$$

and hence, by (5),

$$\sum_{k\geq n}(V_k - V_{k+1})^2 \sim \sum_{k\geq n}\left(V_k Q_k - \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j Q_{k,j}\right)^2 \qquad \text{for } \ n\to+\infty.$$

Therefore, in order to complete the proof, it suffices to prove, for $n\to+\infty$, the following convergence:

$$n\sum_{k\geq n}\left(V_k Q_k - \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j Q_{k,j}\right)^2 \xrightarrow{a.s.} H(V-V^2).$$

Since we have $Y_j^2 = Y_j$, the above sum can be rewritten as

$$n\sum_{k\geq n}\left[V_k^2Q_k^2 + (\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j})^2 - 2V_kQ_k\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j}\right] =$$
$$n\sum_{k\geq n} V_k^2Q_k^2 + n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j}^2 +$$
$$2n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j}\sum_{h=u_{i(k)}+1}^{j-1} Y_hQ_{k,h} - 2n\sum_{k\geq n} V_kQ_k\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j}.$$

Now, by assumption (i) and the almost sure convergence of $(V_k)_k$ to V, we have

$$n\sum_{k\geq n} V_k^2Q_k^2 \xrightarrow{a.s.} V^2H. \tag{7}$$

In the sequel, we are going to prove the following convergences for $n\to+\infty$:

(c1) $n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j}^2 \xrightarrow{a.s.} VH$;

(c2) $n\sum_{k\geq n} V_kQ_k\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j} \xrightarrow{a.s.} V^2H$;

(c3) $n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_jQ_{k,j}\sum_{h=u_{i(k)}+1}^{j-1} Y_hQ_{k,h} \xrightarrow{a.s.} 0$.

Let us start with convergence (c1). By assumptions (i) and (iii), we have

$$\begin{aligned} &n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j}^2 \\ =&n\sum_{k\geq n}\left(\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j}\right)^2 - 2n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j}\sum_{h=u_{i(k)}+1}^{j-1} Q_{k,h} \\ =&n\sum_{k\geq n} Q_k^2 - 2n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j}\sum_{h=u_{i(k)}+1}^{j-1} Q_{k,h} \xrightarrow{a.s.} H \end{aligned} \tag{8}$$

Thus, by the almost sure convergence of $(V_j)_j$ to V, we have

$$n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} V_{j-1}Q_{k,j}^2 \xrightarrow{a.s.} VH \tag{9}$$

Therefore, it will be enough to prove the following convergence:

$$n\sum_{k\geq n}\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} (Y_j - V_{j-1})Q_{k,j}^2 \xrightarrow{a.s.} 0. \tag{10}$$

Indeed, from this and (9), we obtain convergence (c1).

In order to prove (10), we consider the process $(Z_n)_{n\in\mathbb{N}}$ defined by

$$Z_n := \sum_{k=0}^{n-1} k \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} (Y_j - V_{j-1}) Q_{k,j}^2.$$

The random variable Z_n is $\mathcal{G}_n$-measurable and we have

$$Z_{n+1} = \begin{cases} Z_n & \text{if } i(n+1) = i(n) \\ Z_n + n \sum_{j=u_{i(n)}+1}^{r_{i(n)+1}} (Y_j - V_{j-1}) Q_{n,j}^2 & \text{if } i(n+1) = i(n)+1 \end{cases}$$

where

$$\mathrm{E}[(Y_j - V_{j-1}) Q_{n,j}^2 \,|\, \mathcal{G}_n] = \mathrm{E}[(Y_j - V_{j-1}) \,|\, \mathcal{G}_n]\, Q_{n,j}^2 = 0.$$

Indeed, for $u_{i(n)} + 1 \leq j \leq r_{i(n)+1}$, we have $\mathcal{G}_{j-1} = \mathcal{G}_n$ and so

$$\mathrm{E}[(Y_j - V_{j-1}) \,|\, \mathcal{G}_n] = \mathrm{E}[Y_j \,|\, \mathcal{G}_{j-1}] - V_{j-1} = 0.$$

We have so proved that $(Z_n)_n$ is a martingale with respect to the filtration $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$. Moreover, by assumption (ii), we have

$$\begin{aligned} \mathrm{E}[Z_n^2] &= \sum_{k=0}^{n-1} k^2\, \mathrm{E}\left[\left(\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} (Y_j - V_{j-1}) Q_{k,j}^2 \right)^2 \right] \\ &\leq \sum_{k=0}^{n-1} k^2\, \mathrm{E}\left[\left(\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j}^2 \right)^2 \right] \\ &\leq \sum_{k=0}^{n-1} k^2\, \mathrm{E}\left[\left(\sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j} \right)^4 \right] = \sum_{k=0}^{n-1} k^2\, \mathrm{E}[Q_k^4] \\ &\leq \sum_{k\geq 0} k^2\, \mathrm{E}[Q_k^4] < \infty. \end{aligned}$$

Hence, the martingale $(Z_n)_n$ is bounded in L^2 and so it converges almost surely; that is, the series

$$\sum_{k\geq 0} k \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} (Y_j - V_{j-1}) Q_{k,j}^2$$

is almost surely convergent. On the other hand, by a well-known Abel's result, the convergence of a series $\sum_k a_k$, with $a_k \in \mathbb{R}$, implies the convergence of the series $\sum_k k^{-1} a_k$ and the relation $n \sum_{k\geq n} k^{-1} a_k \to 0$ for $n \to +\infty$. Applying this result, we find (10).

From (c1), we obtain (c2). Indeed, we have

$$\begin{aligned} & n \sum_{k\geq n} V_k Q_k \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j Q_{k,j} \\ =& n \sum_{k\geq n} V_k \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j Q_{k,j}^2 + n \sum_{k\geq n} V_k \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j (Q_k - Q_{k,j}) Q_{k,j}. \end{aligned}$$

From (c1) and the almost sure convergence of (V_k) to V, we get that

$$n \sum_{k\geq n} V_k \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j Q_{k,j}^2 \xrightarrow{a.s.} V^2 H.$$

Moreover, from (6), (8) and (i), we get

$$\begin{aligned} & n \sum_{k\geq n} V_k \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j (Q_k - Q_{k,j}) Q_{k,j} \leq \\ & n \sum_{k\geq n} \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} (Q_k - Q_{k,j}) Q_{k,j} = \\ & n \sum_{k\geq n} Q_k^2 - n \sum_{k\geq n} \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j}^2 \xrightarrow{a.s} 0. \end{aligned}$$

Finally, we observe that, by assumption (iii), we have

$$n \sum_{k\geq n} \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Y_j Q_{k,j} \sum_{h=u_{i(k)}+1}^{j-1} Y_h Q_{k,h} \leq$$
$$n \sum_{k\geq n} \sum_{j=u_{i(k)}+1}^{r_{i(k+1)}} Q_{k,j} \sum_{h=u_{i(k)}+1}^{j-1} Q_{k,h} \xrightarrow{a.s.} 0.$$

The proof is so concluded. $\square$

Example 4.4. Let us suppose that $r_i = 2i-1$ and $u_i = 2i$ for each $i \geq 1$. Then we have

$$l_i = 1 \quad \text{and} \quad L_i = N_{2i-1} \qquad \text{for each } i \geq 1.$$

Let us assume that the random variables N_i are identically distributed (that is $\mu_i = \mu$ for each $i \geq 1$) with $\mathrm{E}[N_i^4] < +\infty$. If we set

$$m := \mathrm{E}[N_i], \qquad \delta := \mathrm{E}[N_i^2], \qquad h := \frac{\delta}{m^2},$$

then, using the same notation as in the previous theorem, we get that W_n converges $\mathcal{G}$-stably in the strong sense to the Gaussian kernel $\mathcal{N}(0, 2h(V - V^2))$.

In order to prove this fact, we have to verify conditions (i), (ii) and (iii) of the previous theorem. We firstly observe that $u_{i-1} + 1 = r_i$ for all $i \geq 1$ and so assumption (iii) is obviously fulfilled. Moreover, for each $k \geq 0$, we have $i(k) = [k/2]$ where the simbol $[\cdot]$ denotes the integer part. Therefore, we have

$$Q_k := \begin{cases} 0 & \text{if } k \text{ is even} \\ (\sum_{i=1}^{i(k+1)} N_{2i-1})^{-1} N_{2i(k+1)-1} = (\sum_{i=1}^{i(k)+1} N_{2i-1})^{-1} N_{2i(k)+1} & \text{if } k \text{ is odd} \end{cases}$$

and so

$$\sum_{k\geq 0} k^2 \mathrm{E}[Q_k^4] \leq \sum_{j\geq 1} (2j-1)^2 \mathrm{E}[Q_{2j-1}^4]$$
$$\leq \mathrm{E}[N_1^4] \sum_{j\geq 1} (2j-1)^2 j^{-4} \leq 4\mathrm{E}[N_1^4] \sum_{j\geq 1} j^{-2} < +\infty.$$

Further, we have for $n \to +\infty$

$$n \sum_{k\geq n} Q_k^2 = n \sum_{j\in\mathbb{N},\, j\geq (n+1)/2} Q_{2j-1}^2 \sim 2n \sum_{j\geq n} Q_{2j-1}^2$$
$$= 2n \sum_{j\geq n} N_{2j-1}^2 (\textstyle\sum_{i=1}^{j} N_{2i-1})^{-2}.$$

Since the random variables N_i are independent, identically distributed and integrable, then, by the strong law of large numbers, we get

$$\sum_{i=1}^{j} N_{2i-1} \overset{a.s}{\sim} jm \qquad \text{for } j \to +\infty$$

and so we obtain

$$n \sum_{k\geq n} Q_k^2 \overset{a.s}{\sim} 2m^{-2} n \sum_{j\geq n} j^{-2} N_{2j-1}^2.$$

Now, for each $j \geq 1$, let us set

$$X_j := \frac{(N_{2j-1}^2 - \delta)}{j}.$$

The random variables X_j are independent, with mean equal to zero and variance $\mathrm{Var}[X_j] = j^{-2}\,\mathrm{Var}[N_1^2]$. Thus, the series $\sum_{j\geq 1} X_j$ converges almost surely and so we obtain

$$n \sum_{j\geq n} j^{-1} X_j \xrightarrow{a.s.} 0.$$

This implies that

$$n \sum_{j \geq n} j^{-2} N_j \overset{a.s.}{\sim} \delta n \sum_{j \geq n} j^{-2} \to \delta$$

and we can conclude that assumption (i) is satisfied with $H = 2h$.

5 Another central limit theorem

We have the following result.

Theorem 5.1. *For each $n \geq u_1$, let us set*

$$S_n = \sqrt{r_{i(n)}} (M_{r_{i(n)}} - V_n)$$

and

$$X_{n,j} = \frac{1}{\sqrt{r_{i(n)}}} \Big(\sum_{k=r_{i(j-1)}+1}^{r_{i(j)}} Y_k + (u_{i(j)} - r_{i(j)}) V_{u_{i(j)-1}} - \min(r_{i(n)}, u_{i(j)})\, V_{u_{i(j)}} \\ - (u_{i(j-1)} - r_{i(j-1)}) V_{u_{i(j-1)-1}} + \min(r_{i(n)}, u_{i(j-1)})\, V_{u_{i(j-1)}} \Big)$$

for $1 \leq j \leq n$. Suppose:

(a) $U_n = \sum_{j=1}^n X_{n,j}^2 \xrightarrow{P} U$.

(b) $X_n^* = \sup_{1 \leq j \leq n} |X_{n,j}| \xrightarrow{L^1} 0$.

Then the sequence $(S_n)_{n \geq 1}$ converges $\mathcal{A}$-stably to the Gaussian kernel $\mathcal{N}(0, U)$.

In particular, condition (a) and (b) are satisfied if the following conditions hold:

(a1) $r_i = u_i$ *for all* i *and* $\frac{r_{i(n)-1}}{r_{i(n)}} \to 1$ *for* $n \to +\infty$.

(b1) $U_n = \sum_{j=1}^n X_{n,j}^2 \xrightarrow{a.s.} U$.

(c1) $\sup_n \mathrm{E}[S_n^2] < +\infty$.

As we have already recalled, the $\mathcal{A}$-stable convergence of $(S_n)_n$ to the Gaussian kernel $\mathcal{N}(0, U)$ implies that $(S_n)_n$ converges in distribution to the probability mesure ν_2 on $\mathbb{R}$ defined by

$$\nu_2(B) = \int \mathcal{N}\big(0, U(\omega)\big)(B)\ P(\mathrm{d}\omega).$$

Remark 5.2. It is worthwhile to note that, for each n, we have $X_{n,j} = 0$ when $i(j-1) = i(j)$.

Remark 5.3. If $r_j = u_j = j$, the above conditions become the same conditions as in Berti et al. (2004) or in Berti et al. (2005).

Proof. We will use Theorem A.1 in appendix. For each $n \geq u_1$, let us set

$$D_n = \sqrt{r_{i(n)}} (M_{r_{i(n)}} - V),$$

and for $0 \leq j \leq n$

$$L_{n,j} = \mathrm{E}[D_n \,|\, \mathcal{G}_j] \qquad \mathcal{F}_{n,j} = \mathcal{G}_j.$$

Then, for each $n \geq u_1$, the sequence $(L_{n,j})_{0 \leq j \leq n}$ is a martingale with respect to $(\mathcal{F}_{n,j})_{0 \leq j \leq n}$ such that $L_{n,0} = \mathrm{E}[D_n | \mathcal{G}_0] = 0$ and

$$L_{n,j} - L_{n,j-1} = \mathrm{E}[D_n \,|\, \mathcal{G}_j] - \mathrm{E}[D_n \,|\, \mathcal{G}_{j-1}] = X_{n,j} \qquad \text{for } 1 \leq j \leq n.$$

Indeed we have

$$
\begin{aligned}
&\mathrm{E}[D_n \,|\, \mathcal{G}_j] - \mathrm{E}[D_n \,|\, \mathcal{G}_{j-1}] \\
=&\tfrac{1}{\sqrt{r_{i(n)}}}\big(\textstyle\sum_{k=1}^{r_{i(j)}} Y_k + \sum_{k=r_{i(j)}+1}^{u_{i(j)}} \mathrm{E}[Y_k|\mathcal{G}_j] + \sum_{k=u_{i(j)}+1}^{r_{i(n)}} \mathrm{E}[Y_k|\mathcal{G}_j] - r_{i(n)} V_j \\
&- \textstyle\sum_{k=1}^{r_{i(j-1)}} Y_k - \sum_{k=r_{i(j-1)}+1}^{u_{i(j-1)}} \mathrm{E}[Y_k|\mathcal{G}_{j-1}] - \sum_{k=u_{i(j-1)}+1}^{r_{i(n)}} \mathrm{E}[Y_k|\mathcal{G}_{j-1}] + r_{i(n)} V_{j-1}\big) \\
=&\tfrac{1}{\sqrt{r_{i(n)}}}\big(\textstyle\sum_{k=r_{i(j-1)}+1}^{r_{i(j)}} Y_k + \sum_{k=r_{i(j)}+1}^{u_{i(j)}} V_{u_{i(j)-1}} + \sum_{k=u_{i(j)}+1}^{r_{i(n)}} V_{u_{i(j)}} - r_{i(n)} V_{u_{i(j)}} \\
&- \textstyle\sum_{k=r_{i(j-1)}+1}^{u_{i(j-1)}} V_{u_{i(j-1)-1}} - \sum_{k=u_{i(j-1)}+1}^{r_{i(n)}} V_{u_{i(j-1)}} + r_{i(n)} V_{u_{i(j-1)}}\big) \\
=&\tfrac{1}{\sqrt{r_{i(n)}}}\big(\textstyle\sum_{k=r_{i(j-1)}+1}^{r_{i(j)}} Y_k + (u_{i(j)} - r_{i(j)}) V_{u_{i(j)-1}} + +(r_{i(n)} - u_{i(j)})^+ V_{u_{i(j)}} - r_{i(n)} V_{u_{i(j)}} \\
&-(u_{i(j-1)} - r_{i(j-1)}) V_{u_{i(j-1)-1}} - (r_{i(n)} - u_{i(j-1)})^+ V_{u_{i(j-1)}} + r_{i(n)} V_{u_{i(j-1)}}\big) \\
=&\tfrac{1}{\sqrt{r_{i(n)}}}\big(\textstyle\sum_{k=r_{i(j-1)}+1}^{r_{i(j)}} Y_k + (u_{i(j)} - r_{i(j)}) V_{u_{i(j)-1}} - \min(r_{i(n)}, u_{i(j)})\, V_{u_{i(j)}} \\
&-(u_{i(j-1)} - r_{i(j-1)}) V_{u_{i(j-1)-1}} + \min(r_{i(n)}, u_{i(j-1)})\, V_{u_{i(j-1)}}\big) \\
=&X_{n,j}.
\end{aligned}
$$

Moreover, we have

$$S_n = \mathrm{E}[D_n \,|\, \mathcal{G}_n] = L_{n,n} = \textstyle\sum_{j=1}^n X_{n,j}.$$

Finally, if $\mathcal{N}$ denotes the sub-σ-field generated by the P-negligible events, then

$$\mathcal{V}_j = \liminf_n \mathcal{F}_{n,j\wedge n} = \liminf_n \mathcal{G}_{j\wedge n} = \mathcal{G}_j$$

and

$$\mathcal{V} = \mathcal{N} \vee \textstyle\bigvee_{j\geq 0} \mathcal{V}_j = \mathcal{N} \vee \bigvee_{j\geq 0} \mathcal{G}_j$$

and so the random variable U is measurable with respect to the σ-field $\mathcal{V}$. At this point we can apply Theorem A.1 together with Remark A.2 and the proof of the first assertion is concluded.

If conditions (a1) and (b1) hold, then condition (a) is obviously verified and we have

$$X_{n,j} = \tfrac{1}{\sqrt{r_{i(n)}}} Z_j$$

where

$$Z_j = \textstyle\sum_{k=r_{i(j-1)}+1}^{r_{i(j)}} Y_k - u_{i(j)}\, V_{u_{i(j)}} + u_{i(j-1)}\, V_{u_{i(j-1)}}.$$

Therefore, since $i(u_{i(n)}) = i(n)$ and $i(u_{i(n)} - 1) = i(n) - 1$, we can write

$$
\begin{aligned}
\tfrac{1}{r_{i(n)}} Z_{r_{i(n)}}^2 &= X_{u_{i(n)},u_{i(n)}}^2 = \textstyle\sum_{j=1}^{u_{i(n)}} X_{u_{i(n)},j}^2 - \sum_{j=1}^{u_{i(n)}-1} X_{u_{i(n)},j}^2 \\
&= \textstyle\sum_{j=1}^{u_{i(n)}} X_{u_{i(n)},j}^2 - \tfrac{1}{r_{i(n)}} \sum_{j=1}^{u_{i(n)}-1} Z_j^2 \\
&= U_{u_{i(n)}} - \frac{r_{i(n)-1}}{r_{i(n)}} U_{u_{i(n)}-1} \xrightarrow{a.s.} 0,
\end{aligned}
$$

This fact implies that

$$X_n^* = \sup_{1\leq j\leq n} |X_{n,j}| \xrightarrow{a.s.} 0,$$

Indeed,

$$\begin{aligned}\sup_{0\le j\le n} X_{n,j}^2 &= \sup_{0\le j\le n} \tfrac{1}{r_{i(n)}} Z_j^2\\ &= \sup_{0\le j\le r_{i(n)}} \tfrac{1}{r_{i(n)}} Z_j^2 \xrightarrow{a.s.} 0.\end{aligned}$$

(Note that the second equality holds because $Z_j = 0$ for $r_{i(n)} = u_{i(n)} < j \le n$ since $i(j-1) = i(j)$.) Further, we have

$$\begin{aligned}\mathrm{E}[(X_n^*)^2] &= \mathrm{E}[\sup_{1\le j\le n} X_{n,j}^2] \le \textstyle\sum_{j=1}^n \mathrm{E}[X_{n,j}^2]\\ &= \textstyle\sum_{j=1}^n \mathrm{E}\big[\big(L_{n,j} - L_{n,j-1}\big)^2\big]\\ &= \textstyle\sum_{j=1}^n \mathrm{E}\big[L_{n,j}^2 - L_{n,j-1}^2\big]\\ &= \mathrm{E}\big[L_{n,n}^2\big] = \mathrm{E}[S_n^2].\end{aligned}$$

From (c1) we obtain that the sequence (X_n^*) is bounded in L^2 and so we get condition (b). □

6 Other interpretation

The proposed model can be employed also for the following experiment. At time 0 an urn contains $b \in \mathbb{N}^*$ black and $r \in \mathbb{N}^*$ red balls. At each time $i \ge 1$, a sample of $u_i - u_{i-1}$ patients are assigned to a treatment by this procedure: for each patient we pick a ball from the urn, we observe its color and we put it again in the urn. Then $r_i - u_{i-1}$ "significant" responses arrive, we give for convenience number $j = u_{i-1}+1, \dots, r_i$ to the corresponding patients and the urn is so updated: for each $j = u_{i-1}+1, \dots, r_i$, we add N_j balls of the color corresponding to the treatment assigned to patient j ($Y_j = 1$ means black ball and first treatment and $Y_j = 0$ means red ball and second treatment). In this context, the random variable $C_{u_i} = \sum_{k=1}^{u_i} Y_k$ represents the number of patients allocated to the first treatment until time i.

A Appendix

For the reader's convenience, we state some results used above. For more details on the stable convergence or on the amost sure conditional convergence, we refer to Jacod-Memin (1981) and Crimaldi (2007), respectively.

Theorem A.1. *Let $(l_n)_{n\ge1}$ be a sequence of strictly positive integers. On a probability space $(\Omega, \mathcal{A}, P)$, for each $n \ge 1$, let $(\mathcal{F}_{n,j})_{0\le j\le l_n}$ be a filtration and $(L_{n,j})_{n\ge1, 0\le j\le l_n}$ be a triangular array of real random variables on $(\Omega, \mathcal{A}, P)$ with values such that, for each n, the family $(L_{n,j})_{0\le j\le l_n}$ is a martingale with respect to $(\mathcal{F}_{n,j})_{0\le j\le l_n}$ and $L_{n,0} = 0$. For each pair (n,j), with $n \ge 1$, $1 \le j \le l_n$, let us set $X_{n,j} = L_{n,j} - L_{n,j-1}$ and*

$$S_n = \textstyle\sum_{j=1}^{l_n} X_{n,j} = L_{n,l_n}, \qquad U_n = \sum_{j=1}^{l_n} X_{n,j}^2, \qquad X_n^* = \sup_{1\le j\le l_n} |X_{n,j}|.$$

Let us suppose that the sequence $(U_n)_{n\ge1}$ converges in probability to a positive random variable U. Further, let us suppose $X_n^* \xrightarrow{L^1} 0$. Finally, let $\mathcal{N}$ be the *sub-σ-field generated by the P-negligible events and let us set*

$$\mathcal{V}_j = \liminf_n \mathcal{F}_{n,j\wedge l_n} \quad \textit{for } j \geq 0, \qquad \mathcal{V} = \mathcal{N} \vee \bigvee_{j\geq 0} \mathcal{V}_j.$$

If U is measurable with respect to the σ-field $\mathcal{V}$, then $(S_n)_{n\geq 1}$ converges $\mathcal{V}$-stably to the Gaussian kernel $\mathcal{N}(0, U)$.

Remark A.2. We recall that, if the random variable S_n is $\mathcal{V}$-measurable for each n, then the $\mathcal{V}$-stable convergence implies the $\mathcal{A}$-stable convergence.

For a proof of this theorem, the reader may look at Th. 5 and Cor. 7 in sec. 7 of Crimaldi et al. (2007). It maybe worthwhile to note that in Crimaldi et al. (2007) there exists a stronger version of the previous result and so also Theorem 5.1 could be enunciated in a stronger way.

Proposition A.3. *(see Prop. 2.2 in Crimaldi (2007))*
On a probability space $(\Omega, \mathcal{A}, P)$, let $(V_n)_{n\in\mathbb{N}}$ be a real martingale with respect to a filtration $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$. Suppose that $(V_n)_n$ converges in L^1 to a random variable V. Moreover, setting

$$U_n := n \sum_{k\geq n} (V_k - V_{k+1})^2, \qquad Y := \sup_k \sqrt{k}\, |V_k - V_{k+1}|,$$

assume that the following conditions hold:

(a) The random variable Y is integrable.

(b) The sequence $(U_n)_{n\geq 1}$ converges almost surely to a positive real random variable U.

Then, with respect to $\mathcal{G}$, the sequence $(W_n)_{n\geq 1}$ defined by

$$W_n := \sqrt{n}(V_n - V)$$

converges to the Gaussian kernel $\mathcal{N}(0, U)$ in the sense of the almost sure conditional convergence.

In particular, the sequence $(W_n)_n$ converges $\mathcal{A}$-stably to $\mathcal{N}(0, U)$.

References

Berti, P., Pratelli,L. and Rigo, P. (2004). Limit Theorems for a Class of Identically Distributed Random Variables. Ann. Probab. 32(3A), pp. 2029-2052.
Berti, P., Leisen, F., Pratelli, L. and Rigo P. (2005). Limit Theorems for empirical processes based on dependent data. Transactions of the XXV International Seminar on Stability Problems for stochastic Models, pp. 47-53.
Crimaldi, I., Letta G., Pratelli, L. (2007). A strong form of stable convergence. Séminaire de Probabilités XL, to appear.
Crimaldi, I. (2007). Almost sure conditional convergence for a generalized Pólya urn. Submitted.
Dirienzo, A.G. (2000). Using urn models for the design of clinical trials. Sankhya Series B 62(1), pp. 43-69.
Gouet, R. (1993). Martingale Functional Central Limit Theorems for a Generalized Pólya Urn. Ann. Probab. 21, No. 3, pp. 1624-1639.
Hill, B., Lane, D. and Sudderth, W. (1980). A strong law for some generalized

urn processes. Ann. Prob., 8, pp. 214-226.
Hu, F. and Zhang L.X. (2004). Asymptotic normality of urn models for clinical trials with delayed response. Bernoulli 10, no. 3, pp. 447-463.
Jacod, J. and Memin, J. (1981). Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilités. Sem. de Prob. XV, Lecture Notes in Math. 850, pp. 529-546.
Jajte, R. and Paszkiewicz, A. (1999). Pseudo martingales. Probab. Math. Statist. 19, pp. 181-201.
Janson, S. (2006). Limit theorems for triangular urn schemes. Probab. Theory Related Fields 134, no. 3, pp. 417-452.
Kotz, S., Mahmoud, H. and Robert, P. (2000). On generalized Pólya urn models. Statistics & Probability Letters 49, pp. 163-173.
May, C., Paganoni, A.M. and Secchi, P. (2005). On a two-color generalized Pólya urn. Metron 63, No. 1, pp. 115-134.
Muliere, P., Paganoni, A.M. and Secchi, P. (2006). A randomly reinforced urn. Journal of Statistical Planning and Inference 136, pp. 1853-1874.
Zhang L.X., Chan W.S., Cheung S.H. and Hu F. (2007). A generalized drop-the-loser urn for clinical trials with delayed response. Statistica Sinica, vol. 17, no. 1, pp. 387-406.