Sequential coalition formation and the core in the presence of externalities*

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Abstract

The sequential coalition formation model of Bloch (1996) to solve cooperative games with externalities exhibits some anomalies when related to classical concepts. We elaborate on these problems, define a modification of Bloch's model and show that its order-independent equilibria coincide with the (pessimistic) recursive core (Kóczy, 2007).

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1 Introduction

Solving cooperative games with externalities remains an open problem in game theory although numerous suggestions have been made. One school tries to generalise solutions of TU games, the other seeks equilibria of noncooperative coalition formation games. Our aim is to bridge the gap between the two schools.

On the one side we have the sequential model of coalition formation (Bloch, 1996) or rather, a modification, as the original model relates poorly even to the most inclusive core concept, the α-core (Bloch, 1996); on the other hand we

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use the pessimistic recursive core (Kóczy, 2007) and show that this coincides with set of order-independent equilibrium coalition structures as produced by Bloch's modified model.

Studying the relation of noncooperative models and the core is not new (Chatterjee et al., 1993; Lagunoff, 1994; Perry and Reny, 1994). Most of the work is, however, focussed on TU-games and therefore ignore externalities. Moldovanu and Winter (1995) study NTU-games, which often exhibit similarities with games with externalities; their work is both the source and motivation for this paper, where we extend their results to games in discrete partition form (Lucas and Macelli, 1978). Huang and Sjöström (2006) independently established a similar, or, rather, the reverse result modifying the model of Perry and Reny (1994). Here the order of players is not fixed, order independence is not an issue, the focus is on stationary subgame perfect equilibria and show equivalence to the r-core (Huang and Sjöström, 2003) for totally r-balanced games (a condition that corresponds to our assumption of non-empty residual cores).

The structure of the paper is as follows. After the introduction of Bloch's model, we explain the two aforementioned objections. The third section contains our modified model. Then we present the recursive core and finally our main results.

2 Preliminaries

Let N be a set of players. Subsets are called *coalitions*. A partition π of N is a splitting of N into disjoint coalitions. Π_S is the set of partitions of $S \subseteq N$ with π_S denoting a typical element. The game (N, v) is given by the player set N and a *discrete partition function* (DPF, Lucas and Macelli, 1978) $v : \Pi(N) \longmapsto \mathbb{R}^N$, where $v_i(\pi)$ denotes the payoff for player i in case partition π formed. For vectors $x, y \in \mathbb{R}^N$ we write $x >_S y$ if $x_i \geq y_i$ for all $i \in S \subset N$ and there exists $j \in S$ such that $x_j > y_j$.

A *rule of order* ρ is a strict ordering of the players. Let $\rho(S)$ denote the player ranked first in the set S.

3 Sequential coalition formation

A *game of sequential coalition formation* (SCF) (v, ρ) (Bloch, 1996) is defined by a DPF v and the rule of order ρ. It is played as follows.

1. Start at the highest ranked player.
2. The current player makes a *proposal*. A proposal affects a coalition S of players.
3. The following player in S gets the word. He can either reject the proposal, become the next proposer and the game continues at step 2. Alternatively he can accept the proposal and the step is repeated.
4. When all players in S approve, the coalition forms and these players exit. If all players exit the game terminates, otherwise return to Step 1.

In the original model a proposal is a coalition of players: when all accept the invitation, the coalition forms and leaves the game. In the following we slightly modify the game and allow a proposal to be a *partition* of S, that is a set of coalitions. This model represents an improvement over the original one in at least two respects. Bloch (1996) studies stationary perfect equilibria (See Definition 3) and stationary equilibrium coalition structures (SECS) as the results of those equilibria. It turns out that SECS are unrelated even to the most inclusive cooperative core concept for games with externalities: the α-core $C\alpha(N, v)$ moreover SECS may be inefficient while efficient coalition structures may be unsupportable by stationary perfect equilibria[1]

In the following we formalise the model.

A *history* $h^t = (\hat{K}(h^t), \pi_{\hat{K}(h^t)}, \hat{T}(h^t), S, i)$ at date t is a list of offers, acceptances and rejections up to period t, where $\hat{K}(h^t) \subset N$ is the set of players who have already left the game, $\pi_{\hat{K}(h^t)} \in \Pi(\hat{K}(h^t))$ is the set of coalitions they have formed, $\hat{T}(h^t)$ is the ongoing proposal, $S \subset N$ who have already accepted the proposal, finally $i \in N$ active at time t. The collection of histories at which i is active is denoted H_i.

[1] For an illustration see Kóczy (2006, Example 1).

Strategy σ_i of player i is a mapping from H_i to his set of actions:

$$\sigma_i(h^t) \in \begin{cases} \{\text{Yes}, \text{No}\} & \text{if } \hat{\tau}(h^t) \neq \emptyset \\ \mathrm{T}(i, \hat{K}(h^t)) & \text{if } \hat{\tau}(h^t) = \emptyset \end{cases} \tag{3.1}$$

where $\mathrm{T}(i, \hat{K}(h^t)) = \left\{\tau \in \Pi(T), T \subseteq N \setminus \hat{K}(h^t), i \in T\right\}$, the set of partitions that i can form with the remaining set of players.

We are interested in stationary strategies:

Definition 1. A strategy is *stationary* if it does not depend on the history, but only on the current state $s = (K, \pi_K, \tau)$.

Definition 2. A *subgame-perfect equilibrium* σ^* is a strategy profile such that $\forall$ players $i \in N$, $\forall$ histories $h^t \in H_i$ and $\forall$ strategies σ_i of player i we have

$$v_i(\pi(\sigma_i^*(h^t), \sigma_{-i}^*)) \geq v_i(\pi(\sigma_i(h^t), \sigma_{-i}^*))$$

Definition 3. A *stationary perfect equilibrium* is an subgame-perfect equilibrium profile that is also stationary.

The coalition structures that emerge as subgame-perfect equilibria in stationary strategies are denoted by $SECS(v, \rho)$. Then we have the following:

Lemma 1. *For any modified SCF game* (v, ρ) *we have* $SECS(v, \rho) \subseteq C\alpha(N, v)$.

Proof. In case $SECS(v, \rho) = \emptyset$ the result is trivial.
Otherwise let $\pi \in SECS(v, \rho)$ and assume that $\pi \notin C\alpha(N, v)$. Then there exists a coalition S such that for some partition $\pi_S \in \Pi(S)$ we have

$$v_i(\pi_S \cup \pi_{N \setminus S}) > v_i(\pi) \text{ for all } i \in S \text{ and all partitions } \pi_{N \setminus S} \in \Pi(N \setminus S). \tag{3.2}$$

Now consider the corresponding deviation for the stationary perfect strategy profile σ^* for which we have $\pi = \pi(\sigma^*)$. Consider a deviation by the member i of S who has the first chance to speak: if the set of players who have already left the game is K, then $K \cap S = \emptyset$. Suppose that this player rejects any ongoing proposal and suggests π_S to form. As before, $\pi(\pi_S)$ denotes the coalition structure formed *in case* the proposal π_S is accepted. We write $\pi(\pi_S) = \pi_S \cup \pi^*_{N \setminus S}$, where $\pi^*_{N \setminus S}$ denotes the partition that remaining players form following σ^* together with the partition π_K that has already formed. Then the payoff of player $i \in S$ would be $v_i(\pi_S \cup \pi^*_{N\setminus S}) > v_i(\pi)$ by Equation 3.2. Therefore the deviation is accepted by other players in S and hence $\pi \notin SECS(v, \rho)$. Contradiction.

Hence $\pi \in C\alpha(N, v)$. □

It is easy to verify that core stability $CC(N, v)$ introduced by Shenoy (1979) is equivalent to the following: The coalition structure π is not core stable if $\exists$ a coalition $S \subset N$ and partitions $\pi_S \in \Pi(S)$ and $\pi_{N\setminus S} \in \Pi(N \setminus S)$ such that

$$v_i(\pi_S \cup \pi_{N\setminus S}) > v_i(\pi) \text{ for all } i \in S. \tag{3.3}$$

Therefore core stability is neutral to allowing players to propose partitions.

Proposition 2. *Let v be a valuation such that $CC(N, v) \neq \emptyset$ and for all restrictions v' of v to $R \subset N$ we also have $CC(R, v') \neq \emptyset$. Then for any rule of order ρ we have $CC(N, v) \subseteq SECS(v, \rho)$.*

The proof is analogous to the proof of Bloch (1996, Cor. 3.5). The following corollary gives a similar sufficiency condition to lemma 3.4 by Bloch (1996) on the nonemptiness of the set of stationary equilibrium coalition structures.

Corollary 3. *Stationary equilibrium coalition structures exist if $CC(N, v) \neq \emptyset$ and for all restrictions v' of v to $R \subset N$ we also have $CC(R, v') \neq \emptyset$.*

Corollary 4. *Let v be a valuation with $CC(N, v) \neq \emptyset$ and for all restrictions v' of v to $R \subset N$ we also have $CC(R, v') \neq \emptyset$. Then for any rule of order ρ*

$$CC(N, v) \subseteq SECS(v, \rho) \subseteq C\alpha(N, v).$$

4 Recursive core

Kóczy (2007) generalises the core to partition function games; here we adapt his pessimistic version (and hence drop the adjective in the sequel) to DPF games.

Definition 4 (Residual Game)**.** Let R be a subset of N and π_S a partition of its complement S. The residual game (R, v_{π_S}) is the DPF form game over the player set R and with the DPF $v_{\pi_S} : \Pi(R) \to \mathbb{R}^R$, where $v_{\pi_S}(\pi_R) = v(\pi_R \cup \pi_S)$.

The residual game is a discrete partition function form game and in the recursive core the same solution is used to solve this game as the original one.

Before defining the core, please note that as the partition uniquely determines payoffs, instead of imputations or payoffs, the core consists of *partitions.*

Definition 5 (Recursive core)**.** The definition consists of four steps.

1. *Trivial game.* $C(\{1\}, v) = \{\{1\}\}$.

2. *Inductive assumption.* Given the definition of the core $C(R, v)$ for every game with $|R| < k$ players we define dominance for a game of k players. Let $A(R, v)$ denote the *assumption about the game* (R, v). If $C(R, v) \neq \varnothing$ then $A(R, v) = C(R, v)$, otherwise $A(R, v) = \Pi(R)$, the set of partitions.

3. *Dominance.* The partition π is *dominated via the coalition S forming partition π_S* if for all assumptions $\pi_R \in A(N \setminus S, v_{\pi_s})$ we have $v(\pi_S \cup \pi_R) >_S v(\pi)$.
   The partition π is *dominated* if it is dominated via a coalition.

4. *Core.* The *core* of a game of k players is the set of undominated partitions and we denote it by $C(N, v)$.

For a discussion about the various properties see Kóczy (2007).

5 Results

The core is a static concept: *once* a core partition is attained, it is never abandoned. It does not, however, offer a recipe, or even a proof of the possibility to attain such a partition. In this section we establish the relationship between the core and equilibrium coalition structures of the modified version of Bloch's noncooperative game of coalition formation.

5.1 Stationary equilibrium coalition structures

First we relax the sufficiency condition for the nonemptiness of the SECS.

Proposition 5. *Let (N, v) be a DPF form game such that $C(N \setminus S, v_{\pi_S}) \neq \varnothing$ for all residual games $(N \setminus S, v_{\pi_S})$. Then $C(N, v) \subseteq SECS(v, \rho)$ for all ρ.*

Proof. The proof is inspired by that of Bloch (1996, Proposition 3.2) in part, and is by construction. We show that if $\tilde{\pi} \in C(N, v)$ there exists a stationary perfect strategy profile $\tilde{\sigma} = \tilde{\sigma}(K, \pi_K, \tau)$ such that $\pi(\tilde{\sigma}) = \tilde{\pi}$.

Let $\pi(\tau)$ denote the partition that the acceptance of a proposal τ ultimately produces. In the DPF form game π_K, as a deviation defines a residual game $(N \setminus K, v_{\pi_K})$. The "harsh response" to π_K is an element $\tilde{\pi}_{N\setminus K}$ of the (by assumption non-empty) residual core $C(N \setminus K, v_{\pi_K})$ ensuring that the deviation π_K is not profitable. That is, $\tilde{\pi}_{N\setminus K}$ satisfies

$$\exists i \in S: \quad v_i(\pi_K, \tilde{\pi}_{N\setminus K}) < v_i(\tilde{\pi}), \text{ or} \tag{5.1}$$

$$\forall i \in S: \quad v_i(\pi_K, \tilde{\pi}_{N\setminus K}) = v_i(\tilde{\pi}). \tag{5.2}$$

As $\tilde{\pi} \in C(N, v)$ such a $\tilde{\pi}_{N\setminus K}$ exists for all deviations π_K.

The stationary strategy $\tilde{\sigma}_i$ for player i is then constructed as follows:

$$\begin{aligned} \text{If } \pi_K = \varnothing, \qquad \tilde{\sigma}_i(K, \pi_K, \varnothing) &= \tilde{\pi} \\ \tilde{\sigma}_i(K, \pi_K, \tilde{\pi}) &= \text{Yes} \\ \tilde{\sigma}_i(K, \pi_K, \tau) &= \begin{cases} \text{Yes} & \text{if } v_i(\pi(\tau)) \geq v_i(\tilde{\pi}) \\ \text{No} & \text{otherwise.} \end{cases} \end{aligned} \tag{5.3}$$

$$\begin{aligned} \text{If } \pi_K \neq \varnothing, \qquad \tilde{\sigma}_i(K, \pi_K, \varnothing) &= \tilde{\pi}_{N\setminus K} \\ \tilde{\sigma}_i(K, \pi_K, \tilde{\pi}_{N\setminus K}) &= \text{Yes} \\ \tilde{\sigma}_i(K, \pi_K, \tau) &= \begin{cases} \text{Yes} & \text{if } v_i(\pi(\tau)) \geq v_i(\pi_K, \tilde{\pi}_{N\setminus K}) \\ \text{No} & \text{otherwise.} \end{cases} \end{aligned} \tag{5.4}$$

In equilibrium $\pi(\tilde{\sigma}) = \tilde{\pi}$ and the strategy is stationary by construction so we only need subgame perfection. We show this by induction. As subgame-perfection holds for a trivial game we may assume that it holds for all games of size less than $|N|$.

Now consider game (N, v) and observe the following. If a set of players K have left the game to form π_K the subgame is simply a coalition formation game with less players. Moreover, the proposed strategy exhibits the same similarity property: in equilibrium the core partition is proposed and accepted, while residual cores form off-equilibrium. The minimality condition then ensures that the off-equilibrium path is subgame perfect so we only need to check whether a deviation τ is ever accepted. This deviation corresponds to a deviation in the DPF game. Since $\tilde{\pi} \in C(N, v)$, by the construction of $\tilde{\pi}_{N\setminus K}$ we know that there exists a player in S for whom the deviation τ is not profitable. Finally observe that we do not use a particular rule of order. □

Lemma 1 and this result provide an upper and lower bound (in terms of set inclusion) on the modified stationary equilibrium coalition structures.

Corollary 6. *Let (N, v) be a DPF form game such that $C(N \setminus S, v_{\pi_S}) \neq \varnothing$ for all residual games $(N \setminus S, v_{\pi_S})$. Then*

$$C(N, v) \subseteq SECS(v, \rho) \subseteq C\alpha(N, v) \text{ for all } \rho. \tag{5.5}$$

This result has the following consequence:

Corollary 7. *Let (N, v) be a DPF form game such that $C(N, v) \neq \varnothing$ and $C(N \setminus S, v_{\pi_S}) \neq \varnothing$ for all residual games $(N \setminus S, v_{\pi_S})$. Then for any rule of order ρ, $SECS(v, \rho) \neq \varnothing$.*

As $C_+(N, v) \subseteq C(N, v)$ this corollary weakens the condition in Corollary 3.

5.2 Order-independent equilibria

We show that the order independent equilibria (OIE, Moldovanu and Winter, 1995, p.27) coincide with the recursive core.

Definition 6. (Moldovanu and Winter, 1995, p.27) A strategy profile σ is an *order-independent equilibrium* for the SCF game (v, ρ) if for any rule of order ρ

1. σ is a stationary, subgame perfect equilibrium in (v, ρ)
2. If σ is played in (v, ρ), the payoff vector $v(\pi(\sigma))$ is independent of ρ.

We denote the set of order-independent equilibria by $\mathrm{OIE}(N, v)$ and partitions resulting from playing such equilibrium strategies by $\mathrm{OIP}(N, v)$.

Theorem 8. *Let (N, v) be a DPF form game such that $C(N \setminus S, v_{\pi_S}) \neq \varnothing$ for all residual games $(N \setminus S, v_{\pi_S})$. Then $C(N, v) = \mathrm{OIP}(N, v)$.*

Before proving this theorem we prove two auxiliary results.

Lemma 9. *If Theorem 8 holds for all games with up to $k-1$ players,* $\mathrm{OIP}(N, v) \subseteq C(N, v)$ *for all k-player games with nonempty residual cores.*

Proof. The proof is based on the proof of Proposition A by Moldovanu and Winter (1995) and is by contradiction.

Assume that $\pi = \pi(\sigma) \in \text{OIP}(N, v)$, but $\pi \notin C(N, v)$. Then there exists a coalition S such that a deviation $\pi_S \in \Pi(S)$ is profitable in the cooperative game for all assumptions about the residual game $(N \setminus S, v_{\pi_S})$. In this game of nonempty residual cores this implies $v(\pi_S \cup \pi_{N \setminus S}) >_S v(\pi)$ for all $\pi_{N \setminus S} \in C(N \setminus S, v_{\pi_S})$. In particular let $i \in S$ be such that $v_i(\pi_S \cup \pi_{N \setminus S}) > v_i(\pi)$. As $|N \setminus S| < k$, by Theorem 8, $C(N \setminus S, v_{\pi_S}) = \text{OIP}(N \setminus S, v_{\pi_S})$. The restriction of an OIE to a subgame is also an OIE, which, by our assumption belongs to the recursive core of the corresponding cooperative game. Therefore if deviation π_S forms in the noncooperative game, the resulting coalition structure is $\pi(\pi_S) = \pi_S \cup \pi_{N \setminus S}$, where $\pi_{N \setminus S}$ belongs to $C(N \setminus S, v_{\pi_S}) = \text{OIP}(N \setminus S, v_{\pi_S})$. By our arguments for the cooperative game

$$v(\pi(\pi_S)) >_S v(\pi). \tag{5.6}$$

Without loss of generality let ρ be such that $\rho(N) = i$. Consider strategy σ_i' for i: "when no players have left the game, and it is i's turn to propose a partition, propose π_S otherwise play σ_i." We show that this deviation from σ_i is profitable for i and hence $\pi(\sigma) \notin \text{OIP}(N, v)$, which is a contradiction.

Consider another $j \in S$ and assume that after j's approval of π_S the partition forms and S leaves the game (either j is the last player to accept or the rest is known to approve). We show that it is optimal for j to approve. A rejection by j makes her a proposer and the strategy profile (σ_i', σ_{-i}) is played.

If, from here, no coalition ever forms, but the game goes on forever, the payoff for j is 0, which is clearly inferior as $0 < v_j(\pi(\pi_S))$. Then assume that coalition T leaves the game first, forming partition π_T. Consider the part of the game from j's proposal until T's departure. If i becomes the proposer again, by stationarity, the game goes on forever without a coalition forming, contradicting our assumption that π_T forms. Therefore i is never a proposer. But then i's deviation is never played and playing (σ_i', σ_{-i}) in (v, ρ) is identical to playing (σ) in (v, ρ') with $\rho'(N) = j$. By the assumption that σ is an OIE playing it in (v, ρ) or (v, ρ') results in the same payoffs, which, by Inequality 5.6 are inferior to accepting proposal π_S.

Finally note that $i's$ deviation is limited to the game while all players participate. The game after the departure of a coalition is unaffected; in particular if $j \notin T$, j's payoff is unaffected by i's deviation giving the same conclusion.

We have discussed all cases and found that j's refusal is never optimal. By backward induction the proposal is accepted by all players in S and then, by Inequality 5.6, i benefits from the deviation. □

Lemma 10. *If Theorem 8 holds for all games with less than k players, then* $\mathrm{OIP}(N, v) \supseteq C(N, v)$ *for all k-player games with nonempty residual cores.*

Proof. The stationary-perfect equilibrium constructed in the proof of Proposition 5 is unconditional on any rule of order ρ. On the other hand, it produces the same coalition structure, $\tilde{\pi}$ for each rule of order ρ. It is therefore also an OIE. Such an OIE is constructed for each $\tilde{\pi}$ and therefore the result follows. □

Proof of Theorem 8. The proof is by induction. The result holds for trivial, single-player games. Assuming that the result holds for all $k-1$ player games, the result for k-player games is a corollary of Lemmata 9 & 10. □

6 Conclusion

Theorem 8 states that the core coincides with the order-independent equilibria of the SCF game. This result is not so surprising considering that a similar relation has already been established for characteristic function form games without transferable utility (Moldovanu and Winter, 1995, Corollary 2.). Huang and Sjöström (2006) show the converse result for their very similar r-core concept (Huang and Sjöström, 2003) with a modification of the continuous-time coalition formation process of Perry and Reny (1994): While order-independence is part of this process by definition, establishing stationary subgame-perfect equilibrium coalition structures is no easy task.

While these results bridge the gap between the cooperative and noncooperative approaches, one question remains, which is the relation of equilibrium strategies and equilibrium coalition structures. Here we have shown that coalition structures produced by order-independent equilibria coincide with the recursive core. Whether the same would hold for partitions that can be produced by equilibria for any rule of order, remains an open question.

Seemingly a small technical detail, it has some rather profound effects on the game. When the equilibrium coalition structure belongs to the recursive core it is *also* supported by a strategy profile where the initial proposer proposes the entire coalition structure and the others accept. Therefore “sequential” is reserved for off-equilibrium behaviour. For SECSs outside the core such a proposal may be rejected.

Finally note that this model does not punish for delay: if a player is “disinclined” to make the equilibrium proposal one can define a stationary equilibrium profile where the right to initiate is passed around a bit before the equilibrium partition is proposed. Delay can of course arise in more general forms, too.

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