TÜSİAD-KOÇ UNIVERSITY ECONOMIC RESEARCH FORUM
WORKING PAPER SERIES

DELEGATION WITH INCOMPLETE AND RENEGOTIABLE CONTRACTS

Emanuele Gerratana
Levent Koçkesen

Working Paper 0803
April 2008

TÜSİAD-KOÇ UNIVERSITY ECONOMIC RESEARCH FORUM
Rumeli Feneri Yolu 34450 Sarıyer/Istanbul

Delegation with Incomplete and Renegotiable Contracts*

Emanuele Gerratana
Koç University

Levent Koçkesen†
Koç University

This version: April 2008

Abstract

It is well known that delegating the play of a game to an agent via incentive contracts may serve as a commitment device and hence provide a strategic advantage. Previous literature has shown that any Nash equilibrium outcome of an extensive-form principals-only game can be supported as a sequential equilibrium outcome of the induced delegation game when contracts are unobservable and non-renegotiable. In this paper we characterize equilibrium outcomes of delegation games with unobservable and incomplete contracts with and without renegotiation opportunities under the assumption that the principal cannot observe every history in the game when played by her agent. We show that incompleteness of the contracts restricts the set of outcomes to a subset of Nash equilibrium outcomes and renegotiation imposes further constraints. Yet, there is a large class of games in which non-subgame perfect equilibrium outcomes of the principals-only game can be supported even with renegotiable contracts, and hence delegation still has a bite.

JEL Classification: C72, C78, D86, L13.
Keywords: Strategic Delegation, Incomplete Contracts, Renegotiation.

*This research has been supported by TUBITAK Grant No. 106K317.

†Corresponding author: Department of Economics, Koç University, Rumelifeneri Yolu, Sariyer 34450, Istanbul, Turkey. E-mail: lkockesen@ku.edu.tr.

1 Introduction

The main idea behind strategic delegation is that contracts with third parties may serve as a commitment device and hence provide a strategic advantage (Schelling (1960)). Under the assumption of observable contracts, previous literature has illustrated this possibility in many settings.[1] If contracts are unobservable, however, Katz (1991) showed that the Nash equilibrium outcomes of a game without delegation and those of the same game played between agents are identical. However, if the goal is to understand the role of delegation in an extensive form game, it is more appropriate to compare the set of equilibrium outcomes, with and without delegation, when some form of sequential rationality is imposed.

Recently, Koçkesen and Ok (2004) and Koçkesen (2007) addressed this question within the context of finite two-person extensive form (principals-only) games. They showed that any Nash equilibrium outcome of the principals-only game in which the principals receive more than their individually rational (minmax) payoffs can be supported as a sequential equilibrium outcome of the delegation game.

Delegation with unobservable contracts yields equilibrium outcomes that differ from the subgame perfect equilibrium outcome of the principals-only game by making the agent behave in a sequentially irrational manner, from the perspective of the principal, at certain points in the game. These points in the game must be off the equilibrium path since otherwise the principal-agent pair could increase the total surplus available to them by inducing the agent to act sequentially rational. Therefore, if the game ever reaches such a point, the principal and the agent will have an incentive to renegotiate the existing contract and write a new one that makes the agent act sequentially rationally. This implies that if contracts can be renegotiated without any friction at any point in the game, then the agent must play sequentially rationally from the perspective of the principal at every point in the game. Thus, delegation under renegotiable contracts cannot yield any equilibrium outcome different from the subgame perfect equilibrium outcome of the principals-only game.

Therefore, the question becomes interesting only when there are frictions in the renegotiation process. In this paper we analyze an environment in which such a friction arises quite naturally: we assume that the principal cannot observe every history in the game, and hence can contract only on a partition of the set of outcomes. In this case, after certain histories, there may arise a disagreement between the agent and the principal regarding what is a Pareto improving contract. If, furthermore, the agent cannot credibly signal the existence of such Pareto improving opportunities, his renegotiation attempt may fail. Motivated by this observation we ask and answer the following question in the current paper: *Which outcomes can be supported in a delegation game with renegotiable contracts if the principal cannot observe every history of the original game when it is played by her agent?*

We limit our analysis to finite two-stage principals-only games, in which player 1 moves first by choosing an action $a_1 \in A_1$, and after observing a_1, player 2 chooses an action $a_2 \in A_2$. In the induced delegation game, player 2 (the principal) offers a contract $f : A_2 \to \mathbb{R}$ to her agent, which specifies a transfer from the agent to the principal as a function of the agent's action. In essence, we assume that the principal cannot observe a_1 and hence contracts are incomplete in the sense that they specify a transfer as a function of a_2, rather than (a_1, a_2). The contract is unobservable to player 1, who chooses

[1] See Fershtman and Judd (1987), Gatsios and Karp (1991), and Persson and Tabellini (1993), among others, for various applications. Fershtman et al. (1991), Polo and Tedeschi (2000), and Katz (2006) prove different "folk theorems" for some classes of delegation games under observable and non-renegotiable contracts.

an action a_1, after which the agent decides whether to end the game by choosing an action a_2 or offer a new contract g to the principal. The principal has to decide whether to accept g or not, without being informed about a_1. Our objective is to characterize the set of outcomes of the principals-only game that can be supported in a perfect Bayesian equilibrium (PBE) of the induced delegation game.

Therefore, in our setting, where the only friction in the renegotiation process is the inability of the principal to observe every history, contract incompleteness is a necessary condition for supporting outcomes that are not subgame perfect equilibrium outcomes of the principals-only game. However, contract incompleteness itself brings about interesting issues that are independent of the existence of renegotiation opportunities. Supporting an outcome in a PBE of the delegation game depends on the ability of writing a contract that gives proper incentives to the agent to play certain strategies. When contracts are complete, as in Koçkesen and Ok (2004) and Koçkesen (2007), finding such contracts is relatively easy, as incentive compatibility does not arise as a binding constraint. When contracts are incomplete, however, only incentive compatible strategies of the agent can be supported. We analyze this question in section 4.1 and show that, if payoff functions exhibit increasing differences, then only the Nash equilibria of the principals-only game in which the agent plays an increasing strategy can be supported.

As we show in section 4.2, renegotiation imposes further constraints on outcomes that can be supported. In that section, we completely characterize contract-strategy pairs that are renegotiation proof and give necessary and sufficient conditions for a strategy of the agent to be renegotiation proof. In section 5 we apply our results to an environment that is common to many economically relevant games, such as the Stackelberg and ultimatum bargaining games, and completely characterize the set of outcomes that can be supported with incomplete and renegotiable contracts.

Previous literature has identified two scenarios, which are complementary to ours, in which renegotiable contracts may have a commitment value: (1) games in which there is exogenous asymmetric information between the principal and the agent (Dewatripont (1988) and Caillaud et al. (1995)); and (2) two-stage games with nontransferable utilities (Bensaid and Gary-Bobo (1993)).

Dewatripont (1988) analyzes an entry-deterrence game in which the incumbent signs a contract with the union before the game begins. This contract is observable by the potential entrant, who chooses whether to enter or not. Renegotiation takes place after the entry decision is made, during which the union offers a new contract to the incumbent, who has by this time received a payoff relevant private information. He shows that commitment effects exist in such a model and may deter entry. This is similar to our model in that the principals-only game is a two-stage game and renegotiation happens after the outside party chooses his action. However, in his model the friction in the renegotiation process arises from an exogenously given asymmetric information, whereas in ours it comes from the inability of the principal to observe the outside party's move. Furthermore, unlike Dewatripont, we analyze arbitrary two-stage games, which enables us to identify conditions on the supportable outcomes in terms of the primitives of the principals-only game.

In Caillaud et al. (1995), unlike in our model, the principals-only game is a simultaneous move game. The delegation game begins by the principal offering a publicly observable contract, which may be renegotiated secretly afterwards. After the renegotiation stage, the agent receives a payoff relevant information, after which he may decide to quit. If he does not quit, the agent and the outside party (which is another principal-agent pair) simultaneously choose their actions and the game ends. Their main question is whether *publicly* announced contracts, which may be secretly renegotiated afterwards, can have a commitment value. They show that the answer to this question depends on whether the principals-only game exhibits strategic complementarity or substitutability and whether there are positive or negative externalities.

Bensaid and Gary-Bobo (1993) also analyze a model in which the principals-only game is a two-stage game and the initial contract can be renegotiated after player 1 chooses an action. However, in their model player 1's action is contractible and observable, but utility is not transferable between the principal and the agent. They show that, in a certain class of games, contracts with third parties has a commitment effect, even when they are renegotiable.

In the next section we provide two simple games, one of which illustrates that non-subgame perfect outcomes can be supported while the other one shows that this is not true in general. Therefore, characterization of PBE outcomes that can supported with renegotiable contracts is an interesting and, as we will show, non-trivial matter. Sections 4 and 5 deal with this question in general two-stage principals-only games and Section 6 does the same using intuitive criterion (Cho and Kreps (1987)) as the equilibrium concept. Section 7 concludes with some remarks and open questions, while section 8 contains the proofs of our results.

2 Examples and Motivation

In this section we analyze two simple games, an ultimatum bargaining and a sequential battle-of-the-sexes game, each of which has a unique subgame perfect equilibrium. We will show that renegotiable contracts can support a Nash equilibrium outcome that is not perfect in the bargaining game, while only the subgame perfect equilibrium outcome can be supported in the battle-of-the-sexes game.

ULTIMATUM BARGAINING

Consider a simple ultimatum bargaining game in which player 1 moves first, by choosing the action L or R, after which player 2 moves by choosing l or r. The payoffs corresponding to each outcome is given in the game tree in Figure 1, where the first number is player 1's payoff and the second number player 2's.

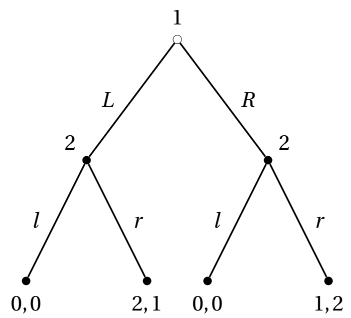

Figure 1: Ultimatum Bargaining Game

In the unique subgame perfect equilibrium (SPE) of this game player 2 chooses r after each action and player 1 chooses L. There is another Nash equilibrium of this game in which player 2 chooses l after L, and r after R, while player 1 plays R. This Nash equilibrium gives player 2 a higher payoff than does the subgame perfect equilibrium, and hence if she could commit to such a strategy in a credible way she would want to do so.

Now consider the following delegation game. Player 2 offers a contract to a neutral agent whose outside option is $\delta \geq 0$. A contract specifies a transfer between player 2 and the agent, as a function of the contractible outcomes of the game. If the agent accepts the contract, player 1 and the agent play the game, otherwise players 1 and 2 play the game and the agent receives his outside option. Let us assume that δ is small enough so that the cost delegation is low. The set of perfect Bayesian equilibrium outcomes of the delegation game differs depending upon the characteristics of the contracts.

If contracts are observable, non-renegotiable, and complete, in the sense that the transfers can be made conditional on the entire set of outcomes, then the unique PBE outcome of the delegation game is (R, r). A contract that pays the agent his outside option if the outcome is (L, l) or (R, r) and pays zero otherwise is a possible equilibrium contract that achieves this outcome. This is nothing but another illustration of the commitment value of observable, non-renegotiable, and complete contracts.

If contracts are unobservable, then the SPE outcome of the original game, i.e., (L, r), is also an equilibrium outcome of the game, in addition to (R, r). This is an example illustrating the main results in Koçkesen and Ok (2004) and Koçkesen (2007) which state that all Nash equilibrium outcomes can be supported with unobservable (but complete and non-renegotiable) contracts.

If contracts can be renegotiated after the game begins, but they are complete, then the unique equilibrium outcome of the delegation game is the SPE outcome of the original game, irrespective of whether contracts are observable or unobservable. The reason is simple: The only way a non-SPE outcome can be supported is through the agent playing l after player 1 plays L, which is sequentially irrational from the perspective of player 2's preferences in the principals-only game. Therefore, if player 1 plays L, player 2 and the agent have an incentive to renegotiate the contract so that under the new contract the agent plays r. In other words, in any PBE, the agent must play r after any action choice of player 1, and hence player 1 must play L.

The conclusion is entirely different if player 2 can observe her agent's action but not that of player 1. This implies that feasible contracts are incomplete, i.e., they can specify transfers conditional on only the agent's actions but not player 1's actions. We will show that the non-SPE outcome (R, r) is an equilibrium outcome of the delegation game even if contracts can be renegotiated.[2] To this end let us specify the renegotiation process as an explicit game form: after player 1 plays, the agent decides whether to renegotiate, by offering a new contract to player 2, or not, in which case he chooses an action and the game ends. If he offers a new contract, then player 2 either accepts or rejects it, after which the agent chooses an action and the game ends. If the new contract offer is accepted by player 2, then the payoffs are determined according to the new contract while if rejected they are determined according to the old contract. The crucial assumption is that the principal cannot observe player 1's action at any time.

For the simplicity of exposition assume that $\delta = 0$. Then, the following is a PBE of this game. Player 2 offers the contract that transfers 2 from the agent to the principal if the agent plays r, and transfers 0.5 if the agent plays l. The agent accepts any contract that gives him an expected payoff of at least zero; player 1's beliefs put probability 1 on this contract and she plays R; the agent chooses not to renegotiate and plays l following L and r following R. In the event of an out-of-equilibrium renegotiation offer by the agent, player 2 believes that player 1 has played R and rejects any contract that transfers him less than 2.

[2]This has been first shown by Katz (1991) for the ultimatum bargaining game, which provided the initial motivation for this research.

SEQUENTIAL BATTLE-OF-THE-SEXES

Consider now the sequential battle-of-the-sexes game given in Figure 2. This game is very similar to the ultimatum bargaining game analyzed above. It has a unique SPE in which player 2 plays l after L and r after R while player 1 plays L. There is another Nash equilibrium in which player 2 plays r after both actions and player 1 plays R. It can be shown easily that the unique equilibrium outcome of the induced delegation game is (R, r) if the contracts are observable, non-renegotiable, and complete, whereas the SPE outcome (L, l) can also be supported if contracts are unobservable. If contracts are complete and renegotiable, then only the SPE outcome can be supported. All these observations are in line with those made for the ultimatum bargaining game.

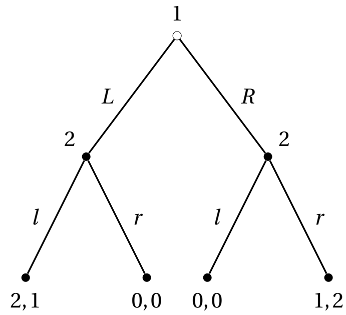

Figure 2: Battle-of-the-Sexes Game

However, the conclusion differs drastically from that in the ultimatum bargaining example if we assume that contracts are renegotiable and incomplete. In this game only the SPE outcome can be supported, while in ultimatum bargaining non-SPE outcomes could also be supported. Let us prove that the Nash equilibrium outcome (R, r) cannot be supported by renegotiable contracts. Suppose, for contradiction, that there exists a PBE of the delegation game that supports this outcome. Let $f : \{l, r\} \to \mathbb{R}$ be the equilibrium contract that specifies the transfer to be made from the agent to player 2. For this outcome to be supported, the agent must be playing r after both actions and hence the following incentive compatibility constraints must be satisfied.

$$0 - f(r) \geq 1 - f(l)$$
$$2 - f(r) \geq 0 - f(l)$$

Also, in equilibrium, player 2 must be extracting all the surplus from the agent, and hence $f(r) = 2$. Together with incentive compatibility conditions we therefore have $f(l) \geq 3$. Now consider the renegotiation offer by the agent $g(l) = g(r) = 2.5$ after player 1 plays L. Note that player 2 does not know which action has been played by player 1 when faced with this renegotiation offer. If she accepts g she will receive a payoff of 2.5 irrespective of player 1's action. If, on the other hand, she rejects it, she believes that the agent will play r after any action by player 1 and hence she will receive a payoff of 2. Therefore, whatever her beliefs are, she has an incentive to accept this renegotiation offer. Furthermore, the agent has an incentive to make such an offer after player 1 plays L since under f his expected payoff is -2, whereas under g his expected payoff is -1.5. This establishes that there is no PBE that supports the outcome (R, r) with renegotiable contracts. Indeed, the unique outcome that can be supported in this case is the SPE outcome of the original game, i.e., (L, l).

In this section we presented two games that are superficially similar, for which delegation with renegotiable contracts gives completely different results. In the rest of the paper we will provide an answer to why this is the case and characterize outcomes that can be supported with renegotiable contracts in arbitrary two-stage extensive form games.

3 The Model

3.1 Preliminaries

An *extensive form game* with perfect recall is a collection $\Upsilon = [N, H, P, (\mathscr{I}_i, u_i)_{i\in N}]$, where N denotes a finite *set of players* and H stands for a set of sequences interpreted as the *set of histories*. The initial history is denoted $\emptyset$ and we assume that for any integer $k \geq 1$, $(a^1, ..., a^k) \in H$ whenever $(a^1, ..., a^{k+1}) \in H$. An history h is said to be *terminal* if it is infinite or $(h, a) \notin H$ for any a and the set of all terminal histories (also called outcomes) is denoted Z. The function $u_i : Z \to \mathbb{R}$ is the *payoff function* of player i, and the function $P : H\backslash Z \to N$ is the *player function*. If $P(h) = i$, we understand that i moves immediately after history h and chooses an action from the set $A(h) \equiv \{a : (h, a) \in H\}$. For each i, $\mathscr{I}_i$ is a partition of $H(i) \equiv \{h \in H : P(h) = i\}$ such that $A(h) = A(h')$ whenever $h, h' \in I \in \mathscr{I}_i$. Consequently, without ambiguity, we may write $A(I)$ ($P(I)$, resp.) instead of $A(h)$ ($P(h)$, resp.) for any $h \in I$. Any member of $\mathscr{I}_i$ is called an *information set* for player i. If all information sets in Υ are singletons, we say that this game is with *perfect information*, and omit information partitions in its definition. The subgames of Υ are defined in the usual way.

A *behavior strategy* for player i is defined as a set of independent probability measures $\beta_i \equiv \{\beta_i[I] : I \in \mathscr{I}_i\}$ where $\beta_i[I]$ is defined on $A(I)$. One may write $\beta_i[h]$ for $\beta_i[I]$ for any $h \in I$ with the understanding that for any h and h' that belong to the same information set, we have $\beta_i[h] = \beta_i[h']$. If $\beta_i[I](a) = 1$ for some $a \in A(I)$ for all I, we call it a *pure (behavior) strategy* and write $\beta_i[I] = a$. The set of all behavior strategies of player i is denoted $\mathscr{B}_i(\Upsilon)$, whereas $\mathscr{B}(\Upsilon) \equiv \times_{i\in N}\mathscr{B}_i(\Upsilon)$ is the set of all behavior strategy profiles. We denote the set of all *Nash* and *subgame perfect* equilibria of Υ in behavior strategies by $NE(\Upsilon)$ and $SPE(\Upsilon)$, respectively.

By a *system of beliefs*, we mean a set $\mu \equiv \{\mu[I] : I \in \mathscr{I}_i \text{ for some } i\}$, where $\mu[I]$ is a probability measure on I. We denote the set of all systems of beliefs by $\mathscr{M}(\Upsilon)$. A 2-tuple $(\beta, \mu) \in \mathscr{B}(\Upsilon) \times \mathscr{M}(\Upsilon)$ is called an *assessment*. An assessment (β, μ) is said to be a *perfect Bayesian equilibrium* (PBE) if (1) each player's strategy is optimal at every information set given her beliefs and the other players' strategies; and (2) beliefs at every information set are consistent with observed histories and strategies.[3] We denote the set of all such assessments as $PBE(\Upsilon)$.

3.2 Delegation Environments

Our objective is to start with a two-player game, called the *principals-only game*, and characterize the set of equilibrium outcomes of the delegation game induced by it. Delegation takes place by one of the players (the principal) of the principals-only game delegating the play of the game to an agent. The agent acts under a contract that maps a partition of the set of outcomes into monetary transfers between the principal and the agent. We are particulary interested in whether the induced delegation game has equilibrium outcomes that are not equilibrium outcomes in the principals-only game, i.e., whether delegation "matters".

[3] See Fudenberg and Tirole (1991) for a precise definition of perfect Bayesian equilibrium.

As we will see in the sequel, the nature of the contract space and whether we allow for renegotiation of the contracts during the play of the game is crucial in our query. Previous literature has analyzed this question under the assumption of complete and non-renegotiable contracts, which may be observable or unobservable by third parties. Our focus, in contrast, is on contracts that can be renegotiated at any point in a costless and secret manner. This immediately implies that contracts are unobservable, since they can be renegotiated before the agent starts to play. If we also assume that contracts are complete, i.e., the contract space is the set of all functions $f : Z \to \mathbb{R}$, and there are no frictions in the renegotiation process, such as asymmetric information between the principal and the agent, delegation cannot "alter" the set of equilibrium outcomes of the original game. Therefore, one has to introduce some sort of friction into the renegotiation process to make the analysis interesting.

As we mentioned before, previous literature has analyzed two particular forms of frictions in the renegotiation process: (1) exogenous asymmetric information between the principal and the agent and (2) nontransferable utilities.

We analyze a model in which the friction arises from the assumption that the principal cannot observe all the histories of the principals-only game when it is played by her agent. We believe that this is a natural environment to consider, for otherwise we would have to assume that the principal constantly and perfectly monitors the play of the game, including the actions of the players other than her agent. In any case, we think that the resulting model is quite rich and introduces new dimensions into the analysis of contracts in strategic settings, such as contract incompleteness and moral hazard.

The assumption that the principal cannot observe every history in the game implies that monetary transfers cannot be conditioned on every terminal history of the game and hence contracts must be incomplete. Furthermore, if the principal cannot observe some of her agent's actions, then moral hazard becomes an issue in contract design. In this paper we assume that the principal can observe all of her agent's actions and focus on incompleteness, leaving the analysis of issues associated with moral hazard for future work.

Contract incompleteness in our setting, therefore, is a necessary condition for obtaining nontrivial results regarding the effects of renegotiation in delegation. However, incompleteness introduces novel issues into the analysis and is interesting in itself. The set of equilibrium outcomes that can be attained in a delegation game depends on the ability of the contracts to give the agent incentives to play certain actions. Incentive compatibility is satisfied in a trivial way in models with complete contracts (such as the one in Koçkesen and Ok (2004)). However, as we will see later on, incentive compatibility becomes a binding constraint in a model with incomplete contracts and obtaining sharp results requires imposing further structure on the model, such as assuming that payoff functions exhibit increasing differences.

It is easiest to prove and understand our results in a particularly simple model in which the principals-only game has only two stages: Player 1 moves first and player 2 second. Limiting player 1's move to only the first stage makes formulating the model, e.g., introducing an order structure on the set of histories in the game and defining increasing differences, much easier and renders the results more transparent. Since the main intuition behind our results is best seen in this simple model, we conduct the entire analysis for two-stage games.

Limiting the analysis to two-stage games simplifies the analysis further as we may, without loss of generality, assume that only the second mover has the option to delegate. Delegation introduces equilibrium outcomes that are not equilibrium outcomes in the principals-only game by making the agent play in a sequentially irrational manner (from the perspective of the principal's preferences in the principals-only game) at information sets that are not reached in equilibrium. Since, when contracts are unobservable, the first mover has only one information set, which is reached in every equilibrium, allowing him to delegate would not change the set of equilibrium outcomes at all.[4]

In light of these observations, we define the *principals-only game* as a two-player finite extensive form game with perfect information:

$$G = (\{1,2\}, H, P, (u_1, u_2))$$

We assume that this game is composed of two stages: Player 1 chooses $a_1 \in A_1$, and player 2, after observing a_1, chooses $a_2 \in A_2$, where A_1 and A_2 are finite sets. Therefore $H = \{\emptyset\} \cup A_1 \cup A_1 \times A_2$ and $P(\emptyset) = 1$, $P(a_1) = 2$, for all $a_1 \in A_1$. Payoff function of player i is given by $u_i : A \to \mathbb{R}$, where $A = A_1 \times A_2$.

The *delegation game with incomplete and non-renegotiable contracts* $\Gamma(G)$ is a three player extensive form game described by the following sequence of events:

Stage I. Player 2, whom we sometimes call the principal, offers a contract $f : A_2 \to \mathbb{R}$ to the agent. In other words, we assume that the contract space is given by $\mathscr{C} = \mathbb{R}^{A_2}$.

Stage II. The agent accepts (denoted y) or rejects (denoted n) the contract.

1. In case of rejection the game ends, the agent receives his outside option which we normalize to be zero, and player 1 and 2 receive $-\infty$.[5]
2. If the agent accepts, the game goes to Stage III.

Stage III. Player 1 chooses an action $a_1 \in A_1$ (without observing the contract), and the agent observes a_1.

Stage IV. The agent chooses an action $a_2 \in A_2$.

More precisely

$$\Gamma(G) = [N, H, P, (\mathscr{I}_i, v_i)_{i \in N}].$$

The set of players is $N = \{1,2,3\}$, where player 3 is the agent. The set of histories is given by

$$H = \{\emptyset\} \cup \mathscr{C} \cup \mathscr{C} \times \{y,n\} \cup \mathscr{C} \times \{y\} \times A_1 \cup \mathscr{C} \times \{y\} \times A_1 \times A_2$$

whereas the set of outcomes is

$$Z = \mathscr{C} \times \{n\} \cup \mathscr{C} \times \{y\} \times A_1 \times A_2.$$

The player function is defined by

$$P(\emptyset) = 2, P(f) = 3, P(f,y) = 1, P(f,y,a_1) = 3, \text{ for all } f \in \mathscr{C} \text{ and } a_1 \in A_1.$$

All information sets are singletons except those of player 1, whose information partition is given by $\mathscr{I}_1 = \{\mathscr{C}\}$.

[4]Of course, as it was shown in Koçkesen (2007), in games with more than two stages this is not the case.

[5]Alternatively, we could assume that if the agent rejects an offer, then the principal plays the game. However, this assumption introduces additional notation and technical details without changing any of the results in any substantive way.

Since we assume that if the contract offer is rejected by the agent the game ends and players 1 and 2 receive very small payoffs, the contract offer is accepted in all equilibria. Therefore, we will, for the sake of notational simplicity, denote the set of outcomes as $Z = \mathscr{C} \times A$. For any outcome $\left(f, a\right) \in Z$ the payoff functions are given by

$$\begin{aligned} v_1\left(f, a\right) &= u_1\left(a_1, a_2\right) \\ v_2\left(f, a\right) &= f\left(a_2\right) \\ v_3\left(f, a\right) &= u_2\left(a_1, a_2\right) - f\left(a_2\right) \end{aligned}$$

This completes the definition of the *delegation game with incomplete and non-renegotiable contracts*.

The delegation game is with renegotiable contracts if the agent and the principal can renegotiate the contract after Stage III and before Stage IV. We assume that renegotiation can be initiated only by the agent. However, as it will become apparent after we introduce our concept of renegotiation-proofness, the results remain intact if the renegotiation process is initiated by player 2. The following sequence of events describe the renegotiation process after any history $\left(f, a_1\right)$.

Stage III(i). The agent either offers a (renegotiation) contract $g \in \mathscr{C}$ to the principal or chooses an action a_2. In the latter case the game ends and the outcome is $\left(f, a\right)$.

Stage III(ii). If the agent offers a contract, the principal (without observing a_1) either accepts (denoted y) or rejects (denoted n) the offer.

If the principal rejects the renegotiation offer g, then the agent chooses $a_2 \in A_2$ and the outcome is payoff equivalent to $\left(f, a\right)$. If she accepts, then the agent chooses $a_2 \in A_2$ and the outcome is payoff equivalent to $\left(g, a\right)$. All the information sets are singletons except that of player 1, whose information partition is given by $\mathscr{I}_1 = \{\mathscr{C}\}$, and those of player 2 following a renegotiation offer by the agent. Let $I_2(f, g) = \{(f, a_1, g) : a_1 \in A_1\}$ for any $(f, g) \in \mathscr{C}^2$. Player 2's information partition is given by

$$\mathscr{I}_2 = \{\emptyset\} \cup \bigcup_{f,g} \{I_2(f, g)\}$$

This completes the description of the *delegation game with incomplete and renegotiable contracts*, which we denote as $\Gamma_R(G)$.

3.3 The Query

We will limit our analysis to pure behavioral strategies since considering mixed strategies does not add anything in substance but brings additional notational and technical complexity to our presentation and proofs. Therefore, a strategy profile of the principals-only game G is given by $(b_1, b_2) \in A_1 \times A_2^{A_1}$.

Fix a behavior strategy profile $(b_1, b_2) \in A_1 \times A_2^{A_1}$ in G. We say that a pure strategy assessment (β, μ) in $\Gamma(G)$ *induces* (b_1, b_2) if $\beta_1[\mathscr{I}_1] = b_1$ and $\beta_3[\beta_2[\emptyset], a_1] = b_2(a_1)$, for any $a_1 \in A_1$. In $\Gamma_R(G)$, the agent may choose an action $a_2 \in A_2$ either without renegotiating the initial contract $\beta_2[\emptyset]$, i.e., $\beta_3[\beta_2[\emptyset], a_1] \in A_2$ for all $a_1 \in A_1$, or after attempting renegotiation, i.e., $\beta_3[\beta_2[\emptyset], a_1] \in \mathscr{C}$ for some $a_1 \in A_1$. Accordingly, we say that a pure strategy assessment (β, μ) in $\Gamma_R(G)$ induces a behavior strategy profile $(b_1, b_2) \in A_1 \times A_2^{A_1}$ in G if $\beta_1[\mathscr{I}_1] = b_1$ and

$$b_2(a_1) = \begin{cases} \beta_3[\beta_2[\emptyset], a_1], & \text{if } \beta_3[\beta_2[\emptyset], a_1] \in A_2 \\ \beta_3[\beta_2[\emptyset], a_1, \beta_3[\beta_2[\emptyset], a_1], \beta_2[I_2(\beta_2[\emptyset], \beta_3[\beta_2[\emptyset], a_1])]], & \text{if } \beta_3[\beta_2[\emptyset], a_1] \in \mathscr{C} \end{cases}$$

for all $a_1 \in A_1$.

We restrict our attention to equilibria in which the equilibrium contract is not renegotiated. As Beaudry and Poitevin (1995) point out, this is necessary for renegotiation to have any bite, as one can always replicate an equilibrium outcome of the game without renegotiation by making the principal offer an initial contract that is accepted only because it is going to be renegotiated later on. This leads to the following definition.

Definition 1 (Renegotiation Proof Equilibria)**.** A perfect Bayesian equilibrium (β^*, μ^*) of $\Gamma_R(G)$ is *renegotiation proof* if $\beta_3^*[\beta_2^*[\emptyset], a_1] \in A_2$ for all $a_1 \in A_1$, i.e., if the equilibrium contract $\beta_2^*[\emptyset]$ is not renegotiated after any choice of player 1.

Note that the set of renegotiation proof equilibria is actually a subset of perfect Bayesian equilibria in which the equilibrium contract is not renegotiated. The latter would be defined so that the equilibrium contract is not renegotiated after any action of player 1 that gives her a higher payoff under a renegotiated contract than the equilibrium payoff. However, working with this weaker notion of renegotiation proofness would only introduce additional complexity into our presentation without changing the main results in an interesting way.

Definition 2. A strategy profile (b_1, b_2) of the principals-only game G can be *supported with incomplete and non-renegotiable contracts* if there exists a perfect Bayesian equilibrium of $\Gamma(G)$ that induces (b_1, b_2).

Similarly, a strategy profile (b_1, b_2) of the principals-only game G can be *supported with incomplete and renegotiable contracts* if there exists a renegotiation proof perfect Bayesian equilibrium of $\Gamma_R(G)$ that induces (b_1, b_2).

Our main query can therefore be phrased as follows:

> *Which outcomes of a given principals-only game can be supported with incomplete and renegotiable (or non-renegotiable) contracts?*

Clearly, if an outcome can be supported with renegotiable contracts, it can also be supported with non-renegotiable contracts. Therefore, we start by characterizing the set of outcomes that can be supported with non-renegotiable contracts before we analyze the restrictions imposed by renegotiation. We should emphasize that $\Gamma(G)$ is a delegation game with unobservable but *incomplete* contracts. The results provided in Koçkesen and Ok (2004) are valid only for delegation games with *complete contracts* and hence do not provide the relevant starting point for our analysis. Applied to our setting, Koçkesen and Ok (2004) implies that every Nash equilibrium outcome can be supported with complete contracts whereas, as we will see in the next section, only a subset of these can be supported when the contracts are incomplete.

4 Main Results

In this section we will provide an answer to our main query for two-stage principals-only games, first for incomplete and non-renegotiable contracts and then for renegotiable contracts.

4.1 Incomplete and non-Renegotiable Contracts

Let G be an arbitrary principals-only game and $\Gamma(G)$ be the delegation game with incomplete and non-renegotiable contracts. We first prove the following.

Proposition 1. *A strategy profile (b_1^*, b_2^*) of G can be supported with incomplete and non-renegotiable contracts if and only if*

1. *(b_1^*, b_2^*) is a Nash equilibrium of G*

   and there exists an $f' \in \mathscr{C}$ such that

2. *$f'(b_2^*(b_1^*)) = u_2(b_1^*, b_2^*(b_1^*))$,*

3. *$u_2(a_1, b_2^*(a_1)) - f'(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f'(b_2^*(a_1'))$, for all $a_1, a_1' \in A_1$.*

Proposition 1 provides necessary and sufficient conditions for an outcome of an arbitrary principals-only game to be supported with incomplete and non-renegotiable contracts. Condition 1 states that only Nash equilibrium outcomes can be supported, which is in line with Koçkesen and Ok (2004). Condition 2 simply states that the agent does not receive rents in equilibrium, whereas condition 3 is the incentive compatibility constraint imposed by the incompleteness of contracts.

Although Proposition 1 provides a complete characterization, it falls short of precisely identifying the supportable outcomes in terms of the primitives of the principals-only game. As it is standard in adverse selection models, we can obtain a much sharper characterization if we impose an order structure on A_1 and A_2 and assume that the agent's payoff function exhibits increasing differences. Given the definition of the payoff function of the agent, this is equivalent to assuming that u_2 has increasing differences. To this end, let $\succsim_1$ be a linear order on A_1 and $\succsim_2$ a linear order on A_2, and denote their asymmetric parts by $\succ_1$ and $\succ_2$, respectively.

Definition 3 (Increasing Differences). $u_2 : A_1 \times A_2 \to \mathbb{R}$ is said to have *increasing differences* in $(\succsim_1, \succsim_2)$ if $a_1 \succsim_1 a_1'$ and $a_2 \succsim_2 a_2'$ imply that $u_2(a_1, a_2) - u_2(a_1, a_2') \geq u_2(a_1', a_2) - u_2(a_1', a_2')$. It is said to have *strictly increasing differences* if $a_1 \succ_1 a_1'$ and $a_2 \succ_2 a_2'$ imply that $u_2(a_1, a_2) - u_2(a_1, a_2') > u_2(a_1', a_2) - u_2(a_1', a_2')$.

Definition 4 (Increasing Strategies). $b_2 : A_1 \to A_2$ is called *increasing* in $(\succsim_1, \succsim_2)$ if $a_1 \succsim_1 a_1'$ implies that $b_2(a_1) \succsim_2 b_2(a_1')$.

From now on, we restrict our analysis to principals-only games in which there exists a linear order $\succsim_1$ on A_1 and a linear order $\succsim_2$ on A_2 such that u_2 has strictly increasing differences in $(\succsim_1, \succsim_2)$. We have the following result.

Theorem 1. *A strategy profile (b_1^*, b_2^*) of G can be supported with incomplete and non-renegotiable contracts if and only if (b_1^*, b_2^*) is a Nash equilibrium of G and b_2^* is increasing.*

This result completely characterizes the strategy profiles that can be supported with incomplete contracts and precisely identifies the restrictions imposed by incompleteness. While earlier papers showed that any Nash equilibrium of the principals-only game can be supported in a delegation game with unobservable and *complete* contracts, this result shows that only the subset of Nash equilibria in which the second player plays an increasing strategy can be supported if, instead, contracts are incomplete.

The reason why only increasing strategies of the second player can be supported is very similar to the reason why only increasing strategies of the agent can be supported in standard adverse selection models: If the payoff function of the agent exhibits increasing differences, then incentive compatibility is equivalent to increasing strategies. The set of actions of player 1, A_1, plays the role of the type set of the agent in standard models. The fact that contracts cannot be conditioned on A_1 transforms the model into an adverse selection model, which, combined with increasing differences exhibited by $u_2(a_1,a_2)-f(a_2)$, necessitates increasing strategies to satisfy incentive compatibility, i.e., condition 3 of Proposition 1. We prove sufficiency by using a theorem of the alternative.

As we noted before, if contracts are renegotiable and complete, then the only equilibrium that can be supported is the subgame perfect equilibrium of the principals-only game. Therefore, for renegotiable contracts to have any effect on the outcome of the game, they must be incomplete. However, as we have just seen, contract incompleteness also acts as a restriction on the set of supportable outcomes. Therefore, our query to identify outcomes that can be supported with *renegotiable and incomplete contracts* is an interesting and a non-trivial one. The next section attacks precisely this problem.

4.2 Incomplete and Renegotiable Contracts

Let G be an arbitrary principals-only game and $\Gamma_R(G)$ be the delegation game with incomplete and renegotiable contracts. As stated before we would like to identify the set of outcomes of G that can be supported by renegotiation proof perfect Bayesian equilibria of $\Gamma_R(G)$.

When faced with a renegotiation offer, player 2 has to form beliefs regarding how the agent would play under the new contract and compare her payoffs from the old and the new contracts to decide whether to accept it or not. As we have seen in section 4.1, contract incompleteness imposes incentive compatibility constraints on the strategy of the agent, and therefore player 2 has to restrict her beliefs to strategies that are incentive compatible under the new contract. For future reference, let us first define incentive compatibility as a property of any contract-strategy pair $(f,b_f)\in\mathscr{C}\times A_2^{A_1}$.

Definition 5 (Incentive Compatibility). $(f,b_f)\in\mathscr{C}\times A_2^{A_1}$ is *incentive compatible* if

$$u_2(a_1,b_f(a_1))-f(b_f(a_1))\geq u_2(a_1,b_f(a_1'))-f(b_f(a_1'))\text{ for all }a_1,a_1'\in A_1.$$

To understand the constraints imposed by renegotiation proofness suppose that (β,μ) is a renegotiation proof PBE of $\Gamma_R(G)$ and define $f=\beta_2[\emptyset]$, $b_{2,f}(a_1)=\beta_3[f,a_1]$ for all $a_1\in A_1$. Now suppose that for a particular choice of action by player 1, say a_1', there exists an incentive compatible $(g,b_{2,g})\in\mathscr{C}\times A_2^{A_1}$ such that $u_2(a_1',b_{2,g}(a_1'))-g(b_{2,g}(a_1'))>u_2(a_1',b_{2,f}(a_1'))-f(b_{2,f}(a_1'))$ and $g(b_{2,g}(a_1))>f(b_{2,f}(a_1))$ for all $a_1\in A_1$. This implies that, after a_1' is played, the agent will have an incentive to renegotiate and offer g and the principal will have an incentive to accept it. This would contradict that (β,μ) is a renegotiation proof PBE of $\Gamma_R(G)$. This leads to the following definition.

Definition 6 (Renegotiation Proofness)**.** We say that $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is *renegotiation proof* if for all $a_1 \in A_1$ for which there exists an incentive compatible $(g, b_{2,g}) \in \mathscr{C} \times A_2^{A_1}$ such that

$$u_2(a_1, b_{2,g}(a_1)) - g(b_{2,g}(a_1)) > u_2(a_1, b_{2,f}(a_1)) - f(b_{2,f}(a_1)) \tag{1}$$

and

$$g(b_{2,g}(a_1)) > f(b_{2,f}(a_1)) \tag{2}$$

there exists an $a_1' \in A_1$ such that

$$f(b_{2,f}(a_1')) \geq g(b_{2,g}(a_1')). \tag{3}$$

Again, the intuition behind this definition is clear: Whenever there is an agent (i.e., a_1) for whom there is a contract, g, and an incentive compatible continuation play $b_{2,g}$ such that both the agent and the principal prefer g over f (i.e., (1) and (2) hold), there exists a belief of the principal under which it is optimal to reject g, which is implied by (3).[6]

Let the number of elements in A_1 be equal to n and order its elements so that $a_1^n \succsim_1 a_1^{n-1} \succsim_1 \cdots a_1^2 \succsim_1 a_1^1$. The following result completely characterizes renegotiation proof contract-strategy pairs.

Theorem 2. *$(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is renegotiation proof if and only if for any $i \in \{1, 2, \ldots, n\}$ and increasing $b_{2,g} \in A_2^{A_1}$ such that $u_2(a_1^i, b_{2,g}(a_1^i)) > u_2(a_1^i, b_{2,f}(a_1^i))$ there exists a $k \in \{1, 2, \ldots, i-1\}$ such that*

$$u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i)) + \sum_{j=k}^{i-1} \Big(u_2(a_1^j, b_{2,g}(a_1^j)) - u_2(a_1^j, b_{2,g}(a_1^{j+1})) \Big) \leq f(b_{2,f}(a_1^k)) - f(b_{2,f}(a_1^i)) \tag{4}$$

or there exists an $l \in \{i+1, i+2, \ldots, n\}$ such that

$$u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i)) + \sum_{j=i+1}^{l} \Big(u_2(a_1^j, b_{2,g}(a_1^j)) - u_2(a_1^j, b_{2,g}(a_1^{j-1})) \Big) \leq f(b_{2,f}(a_1^l)) - f(b_{2,f}(a_1^i)) \tag{5}$$

Let us illustrate the proof of the sufficiency part of the theorem when $A_1 = \{a_1^1, a_1^2\}$ and $b_{2,f}(a_1^1) \in \operatorname{argmax}_{a_2} u_2(a_1^1, a_2)$. Suppose that there exists an incentive compatible $(g, b_{2,g}) \in \mathscr{C} \times A_2^{A_1}$ such that (1) and (2) hold for a_1^2. This implies that $b_{2,g}$ is increasing and $u_2(a_1^2, b_{2,g}(a_1^2)) > u_2(a_1^2, b_{2,f}(a_1^2)$, and thus, by (4),

$$f(b_{2,f}(a_1^1)) \geq u_2(a_1^1, b_{2,g}(a_1^1)) - u_2(a_1^1, b_{2,g}(a_1^2)) + u_2(a_1^2, b_{2,g}(a_1^2)) - u_2(a_1^2, b_{2,f}(a_1^2)) + f(b_{2,f}(a_1^2)). \tag{6}$$

Therefore, (1) and incentive compatibility of $(g, b_{2,g})$ imply

$$f(b_{2,f}(a_1^1)) > u_2(a_1^1, b_{2,g}(a_1^1)) - u_2(a_1^1, b_{2,g}(a_1^2)) + g(b_{2,g}(a_1^2)) \geq g(b_{2,g}(a_1^1))$$

so that (3) is satisfied and we conclude that $(f, b_{2,f})$ is renegotiation proof. The proof of the theorem for the general case uses a theorem of the alternative to show that the condition stated in the theorem is necessary and sufficient.

[6] One may find this definition too weak as it allows the beliefs to be arbitrary following an off-the-equilibrium renegotiation offer. A more reasonable alternative could be to require the beliefs to satisfy intuitive criterion. In Section 6 we show that our results go through with minor modifications when we adopt this stronger version of renegotiation proofness.

In order to apply this theorem directly to a given game and a strategy b_2 one would first identify the set of contracts under which the agent has an incentive to play b_2, and then check if any of those contracts satisfies the conditions of the theorem. It is best to illustrate this using the examples introduced in Section 2. For both the ultimatum bargaining and sequential battle-of-the-sexes games, define $\succsim_1$ and $\succsim_2$ so that $R \succ_1 L$ and $r \succ_2 l$ and note that u_2 has strictly increasing differences in $(\succsim_1, \succsim_2)$.

ULTIMATUM BARGAINING

There are three Nash equilibria of the game (L, rl), (L, rr), and (R, lr), where, for example, (L, rl) denotes the strategy profile in which player 1 plays L and player 2 plays r after L and l after R. The second one is the unique SPE and it has the same outcome as the first. The third one is not subgame perfect. Notice that the last two equilibria have increasing b_2 and hence, by Theorem 1, can be supported with incomplete and non-renegotiable contracts. Since the SPE can be supported with renegotiable contracts as well, the question is whether (R, lr) can be supported with incomplete and renegotiable contracts.[7]

Any equilibrium contract f that supports (R, lr) must satisfy the incentive compatibility constraint given by $1 \le f(r) - f(l) \le 2$. Since agent R is already best responding, the only candidate for renegotiation is agent L and we must have $b_{2,g}(L) = r$. Incentive compatibility implies that $b_{2,g}$ is non-decreasing, and therefore, $b_{2,g}(R) = r$. From Theorem 2, $(f, b_{2,f})$ is renegotiation proof if and only if

$$[u_2(L, b_{2,g}(L)) - u_2(L, b_{2,f}(L))] + [u_2(R, b_{2,g}(R)) - u_2(R, b_{2,g}(L))] \le f(b_{2,f}(R)) - f(b_{2,f}(L))$$

Substituting for $b_{2,f}$ and $b_{2,g}$, this is equivalent to $1 \le f(r) - f(l)$. Since incentive compatibility holds if $1 \le f(r) - f(l) \le 2$, we conclude that $b_{2,f} = lr$ can be supported with a renegotiation proof contract and hence (R, lr) can be supported with incomplete and renegotiable contracts.

SEQUENTIAL BATTLE-OF-THE-SEXES

There are three Nash equilibria of the game: (L, ll), (L, lr), and (R, rr). The second one is the unique SPE and it has the same outcome as the first. The third one is not subgame perfect. All of these equilibria have an increasing b_2 and hence can be supported with incomplete and non-renegotiable contracts. The question again is whether the (non-subgame perfect) Nash equilibrium (R, rr) can be supported with incomplete and renegotiable contracts.

First note that incentive compatibility implies $f(l) - f(r) \ge 1$. The only candidate for renegotiation is agent L and we must have $b_{2,g}(L) = l$. Theorem 2 implies that if $(f, b_{2,f})$ is renegotiation proof then

$$[u_2(L, b_{2,g}(L)) - u_2(L, b_{2,f}(L))] + [u_2(R, b_{2,g}(R)) - u_2(R, b_{2,g}(L))] \le f(b_{2,f}(R)) - f(b_{2,f}(L))$$

or $u_2(R, b_{2,g}(R)) + 1 \le 0$, which is impossible since $u_2(R, b_{2,g}(R)) \ge 0$. We conclude that it is not possible to support (R, rr) with incomplete and renegotiable contracts.

Although Theorem 2 is quite powerful in applications, it would still be desirable to obtain general results that involve only the primitives of the principals-only game. In particular, we would like to ob-

[7] Clearly if a contract supports a SPE, it is renegotiation-proof as there is no $a_1 \in A_1$ such that (1) and (2) hold.

tain conditions for a strategy $b_2 \in A_2^{A_1}$ in the principals-only game to be supportable with incomplete and renegotiable contracts. We call such a strategy *renegotiation proof.* More formally,

Definition 7 (Renegotiation Proof Strategy)**.** A strategy $b_2 \in A_2^{A_1}$ is *renegotiation proof* if there exists an $f \in \mathscr{C}$ such that (f, b_2) is incentive compatible and renegotiation proof.

We then have the following result, which follows easily from the definition of renegotiation proofness.

Proposition 2. *A strategy profile (b_1^*, b_2^*) of G can be supported with incomplete and renegotiable contracts if and only if (b_1^*, b_2^*) is a Nash equilibrium of G and b_2^* is increasing and renegotiation proof.*

In order to facilitate the statement of next results we first introduce some definitions. For any $i \in \{1, 2, \dots, n\}$ and $b_2^* \in A_2^{A_1}$ let

$$\mathfrak{B}(i, b_2^*) = \{b_2 \in A_2^{A_1} : b_2 \text{ is increasing and } u_2(a_1^i, b_2(a_1^i)) > u_2(a_1^i, b_2^*(a_1^i))\}.$$

In other words, $\mathfrak{B}(i, b_2^*)$ is the set of strategies that are incentive compatible and, following a_1^i, give a higher surplus to the agent and the principal than does b_2^*.

Definition 8. For any $i = 1, \dots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ we say that $m(b_2^i) \in \{1, 2, \dots, n\}$ is a *blocking action* if

$$u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=m(b_2^i)}^{i-1} \Big(u_2(a_1^j, b_2^i(a_1^j)) - u_2(a_1^j, b_2^i(a_1^{j+1})) \Big)$$
$$\leq \sum_{j=m(b_2^i)}^{i-1} \Big(u_2(a_1^j, b_2^*(a_1^j)) - u_2(a_1^j, b_2^*(a_1^{j+1})) \Big) \quad (7)$$

or

$$u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{m(b_2^i)} \Big(u_2(a_1^j, b_2^i(a_1^j)) - u_2(a_1^j, b_2^i(a_1^{j-1})) \Big)$$
$$\leq \sum_{j=i+1}^{m(b_2^i)} \Big(u_2(a_1^j, b_2^*(a_1^j)) - u_2(a_1^j, b_2^*(a_1^{j-1})) \Big) \quad (8)$$

The following proposition provides a necessary condition for a strategy to be renegotiation proof.

Proposition 3. *A strategy $b_2^* \in A_2^{A_1}$ is renegotiation proof only if for any $i = 1, \dots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ there is a blocking action.*

However, this condition is not sufficient for renegotiation proofness and becomes sufficient with an additional condition on the blocking actions for different a_1's. More precisely,

Proposition 4. *A strategy $b_2^* \in A_2^{A_1}$ is renegotiation proof if for any $i = 1, \dots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ there is a blocking action $m(b_2^i)$ such that $i < i'$, $m(b_2^i) > i$, and $m(b_2^{i'}) < i'$ implies $m(b_2^i) \leq m(b_2^{i'})$.*

The conditions given in Propositions 3 and 4 coincide for the case of $A_1 = \{a_1^1, a_1^2\}$, which we will use to give a rough intuition for these results. Suppose that $b_2^*(a_1^1) \in \operatorname{argmax}_{a_2} u_2(a_1^1, a_2)$ and hence

$\mathfrak{B}(1, b_2^*)$ is empty. By Theorem 2, b_2^* is renegotiation proof if and only if for any $b_2 \in \mathfrak{B}(2, b_2^*)$ there exists an $f \in \mathscr{C}$ such that (f, b_2^*) is incentive compatible and

$$[u_2(a_1^2, b_2(a_1^2)) - u_2(a_1^2, b_2^*(a_1^2))] + [u_2(a_1^1, b_2(a_1^1)) - u_2(a_1^1, b_2(a_1^2))] \leq f(b_2^*(a_1^1)) - f(b_2^*(a_1^2)). \tag{9}$$

Incentive compatibility implies that

$$f(b_2^*(a_1^1)) - f(b_2^*(a_1^2)) \leq u_2(a_1^1, b_2^*(a_1^1)) - u_2(a_1^1, b_2^*(a_1^2)). \tag{10}$$

Therefore, b_2^* is renegotiation proof only if

$$[u_2(a_1^2, b_2(a_1^2)) - u_2(a_1^2, b_2^*(a_1^2))] + [u_2(a_1^1, b_2(a_1^1)) - u_2(a_1^1, b_2(a_1^2))] \leq u_2(a_1^1, b_2^*(a_1^1)) - u_2(a_1^1, b_2^*(a_1^2)), \tag{11}$$

i.e., only if there is a blocking action, which, in this case, would have to be given by $m(b_2^1) = 1$. Conversely, if there is a blocking action, i.e., (11) is satisfied, then we can set $f(b_2^*(a_1^1)) = u_2(a_1^1, b_2^*(a_1^1))$ and $f(b_2^*(a_1^2)) = u_2(a_1^1, b_2^*(a_1^2))$, so that (9) and (10) are satisfied. Therefore, by Theorem 2, b_2^* is renegotiation proof. However, this method does not directly carry over to the general case to prove even necessity, for which we again use a theorem of the alternative.

To apply Propositions 3 and 4 one would have to check whether there is a proper blocking action for every possible renegotiation opportunity in $\mathfrak{B}(i, b_2^*)$. However, we can use Proposition 4 to derive a sufficient condition that can be directly applied to a given strategy b_2. To this end, let us first introduce some definitions.

Definition 9. For any $b_2 \in A_2^{A_1}$ we say that $i \in \{1, 2, \ldots, n\}$ has *right (left) deviation* at b_2 if there exists an $a_2 \in A_2$ such that $a_2 \succsim_2 (\precsim_2) b_2(a_1^i)$ and $u_2(a_1^i, a_2) > u_2(a_1^i, b_2(a_1^i))$. Otherwise, we say that i has *no right (left) deviation* at b_2.

Let $BR_j(a_{-j}) = \operatorname{argmax}_{a_j} u_j(a_j, a_{-j})$, for $j = 1, 2$. For any $b_2 \in A_2^{A_1}$ and $i \in \{1, \ldots, n\}$ that has right deviation at b_2 define

$$R(i) = \{k > i : b_2(a_1^k) \in BR_2(a_1^k) \text{ and } i < j < k \text{ implies that } j \text{ has no left deviation at } b_2\}.$$

In other words, for any action a_1^i that has a right deviation at b_2, a_1^k belongs to $R(i)$ if $a_1^k \succ_1 a_1^i$, $b_2(a_1^k)$ is a best response, and there are no actions with left deviations in between them. Let us call such actions "right blocking action" for easy reference. Similarly, for $i \in \{1, \ldots, n\}$ that has left deviation at b_2 let

$$L(i) = \{k < i : b_2(a_1^k) \in BR_2(a_1^k) \text{ and } k < j < i \text{ implies that } j \text{ has no right deviation at } b_2\},$$

and call any member of $L(i)$ a "left blocking action." We are now ready to state a sufficient condition that is particularly easy to apply.

Proposition 5. *b_2^* is renegotiation proof if for any i_1 (i_2) that has right (left) deviation at b_2^*, $R(i_1) \neq \emptyset$ ($L(i_2) \neq \emptyset$), and $i_1 < i_2$ implies $R(i_1) \cap L(i_2) \neq \emptyset$.*

In other words, one needs to check if for any action a_1 that has a right deviation, there is a right blocking action, and for any that has a left deviation, there is a left blocking action. Furthermore, it must be checked that for any action with a right deviation and a "larger" action with a left deviation, there is a common blocking action in between.

Although, we do not have a full characterization of renegotiation proof b_2 for general principals-only games, Propositions 3 and 4 enable us to do so in more special environments, as the next section shows.

5 A Special Environment and Applications

In this section we analyze a class of games that includes many economic models, among which are certain Stackelberg games, sequential Bertrand games with differentiated products, and ultimatum bargaining. To define this class of games, take any principals-only game G and consider the strategic form game $S(G) = (\{1,2\},(A_1,A_2),(u_1,u_2))$, i.e., $S(G)$ is the simultaneous move version of G. Let br_i denote a selection from the best-response correspondence of player i in $S(G)$, i.e., $br_i(a_{-i}) \in BR_i(a_{-i})$ for all $a_{-i} \in A_{-i}$.

Definition 10. u_1 has *positive externality* in $\succsim_2$ if $a_2 \succsim_2 a_2'$ implies $u_1(a_1,a_2) \geq u_1(a_1,a_2')$ for all $a_1 \in A_1$.

Definition 11. u_1 is *single-peaked* in $\succsim_1$ if for all $br_1 \in BR_1$ and $a_2 \in A_2$, $br_1(a_2) \succsim_1 a_1' \succsim_1 a_1$ implies $u_1(a_1',a_2) \geq u_1(a_1,a_2)$ and $a_1 \succsim_1 a_1' \succsim_1 br_1(a_2)$ implies $u_1(a_1',a_2) \geq u_1(a_1,a_2)$. Define single-peaked u_2 in a similar manner.

Let $\mathscr{G}$ denote the class of principals-only games in which u_1 and u_2 have strictly increasing differences in $(\succsim_1,\succsim_2)$ and are single-peaked, and u_1 has positive externality. Note that $S(G)$ is a supermodular game for any $G \in \mathscr{G}$ and hence it has a smallest and largest (in the given orders) pure strategy Nash equilibria (Topkis (1979)). Denote the smallest Nash equilibrium by $\underline{a}^{NE}$ and the largest by $\overline{a}^{NE}$. Also, let $\overline{a}_i = \max_{\succsim_i} A_i$ and $\underline{a}_i = \min_{\succsim_i} A_i$. The following result provides necessary and sufficient conditions for an outcome to be supported with incomplete and renegotiable contracts.

Proposition 6. *Let $G \in \mathscr{G}$. An outcome (a_1^*, a_2^*) of G can be supported with incomplete and renegotiable contracts if (only if, resp.) $a_1^* \succsim_1 \overline{a}_1^{NE}$ ($a_1^* \succsim_1 \underline{a}_1^{NE}$, resp.), $a_2^* \in BR_2(a_1^*)$, and*

$$u_1(a_1^*, br_2(a_1^*)) \geq \max\left\{u_1(br_1(\underline{a}_2),\underline{a}_2), u_1(\overline{a}_1, br_2(\overline{a}_1))\right\}, \tag{12}$$

for some selection $(br_1, br_2) \in BR_1 \times BR_2$.

In other words, in this environment only those outcomes in which player 1 plays an action that is "greater" than his smallest Nash equilibrium action (in the simultaneous move version of the principals-only game) can be supported. Conversely, any outcome in which player 1's action is greater than his largest Nash equilibrium action can be supported, as long as player 2 best responds to that action and condition (12) is satisfied. Also note that, if $S(G)$ has a unique Nash equilibrium, then the above proposition provides a full characterization. In many games condition (12) is trivially satisfied, which implies that, in this case, an outcome can be supported if and only if player 1's action is greater than his Nash equilibrium action in $S(G)$ and player 2's action is a best response to that.

For example, consider a *Stackelberg game* in which firm 1 moves first by choosing an output level $q_1 \in Q_1$ and firm 2, after observing q_1, chooses its own output level $q_2 \in Q_2$. We assume that Q_i,

$i = 1,2$, is a finite subset of $\mathbb{R}_+$ and includes 0. Let $p : \mathbb{R}_+^2 \to \mathbb{R}_+$ be the market inverse demand function and $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the firm i's cost function. We assume that c_i is increasing, with $c_i(0) = 0$, p is decreasing, and $p(q_1, q_2) = 0$, if $q_1 = \max Q_1$ or $q_2 = \max Q_2$. Profit function of firm i is given by $\pi_i(q_1, q_2) = p(q_1, q_2)q_i - c_i(q_i)$ and both firms are profit maximizers.

Define the principals-only game G_S as follows: Let $A_1 = Q_1$ and $A_2 = \{-q_2 : q_2 \in Q_2\}$ and define $\succsim_i$ on A_i as $a_i \succsim_i a'_i \Leftrightarrow a_i \geq a'_i$. Let the payoff functions be equal to the profit functions, that is

$$u_1(a_1, a_2) = p(a_1, -a_2)a_1 - c_1(a_1)$$
$$u_2(a_1, a_2) = p(a_1, -a_2)(-a_2) - c_2(-a_2)$$

for any $(a_1, a_2) \in A_1 \times A_2$. This principals-only game is equivalent to the Stackelberg game defined in the previous paragraph, and u_1 has positive externality. If we further assume that the payoff functions are single-peaked and have strictly increasing differences, then $G_S \in \mathcal{G}$, and hence we can apply Proposition 6.[8] Also note that, under our assumptions,

$$\max\{u_1(br_1(\underline{a}_2), \underline{a}_2), u_1(\overline{a}_1, br_2(\overline{a}_1))\} = 0.$$

Therefore, if there is a unique Nash equilibrium of the corresponding Cournot game, then an outcome can be supported if and only if firm 1 obtains non-negative profit, its output is at least as high as its Cournot Nash equilibrium output, and the follower's output is a best response to that. In such a game, therefore, delegation may benefit firm 2, even if the contracts are renegotiable.

As another example, consider an *ultimatum bargaining game* in which the set of possible offers is $A_1 = \{1, 2, \ldots, n\}$, for some integer $n > 1$, and $A_2 = \{Y, N\}$. Let $a_1 \succsim_1 a'_1$ if and only if $a_1 \geq a'_1$ and $Y \succ_2 N$. Suppose that if the responder (player 2) accepts an offer a_1, i.e., chooses Y, then the proposer's (player 1) payoff is $n - a_1$ and that of the responder is a_1, while if the responder rejects, i.e., chooses N, they both get zero payoff. This game satisfies all the assumptions required for Proposition 6, its simultaneous move version has a unique Nash equilibrium given by $(1, Y)$, and condition (12) is trivially satisfied. Therefore, every offer can be supported with incomplete and renegotiable contracts, a result first proved by Katz (1991).

6 Strong Renegotiation Proofness

One may object to our definition of renegotiation proof perfect Bayesian equilibrium on the basis that off-the-equilibrium beliefs during the renegotiation process are left free. In particular, after the equilibrium initial contract f and faced with an (off-the-equilibrium) renegotiation offer g, our definition allows the principal's beliefs to assign positive probability to any action a_1. This enables us to construct a PBE in the proof of Proposition 2 in which the initial contract f is not renegotiated as long as $(f, b_{2,f})$ is renegotiation proof as defined in Definition 6. A plausible way to strengthen our definition of renegotiation proof equilibrium is to require that it survives the Intuitive Criterion as defined by Cho and Kreps (1987). When applied to our setting this criterion requires that the principal's beliefs put positive probability only on actions for which it is not sub-optimal to offer g, i.e., only on

[8] A sufficient condition for u_i to satisfy strictly increasing differences is $q_1 > q'_1$ and $q_2 > q'_2$ imply $p(q_1, q'_2) - p(q_1, q_2) > p(q'_1, q'_2) - p(q'_1, q_2)$

those actions a_1' for which $u_2(a_1', b_{2,g}(a_1')) - g(b_{2,g}(a_1')) \geq u_2(a_1', b_{2,f}(a_1')) - f(b_{2,f}(a_1'))$. This leads to the following definition.

Definition 12 (Strong Renegotiation Proofness)**.** We say that $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is *strongly renegotiation proof* if for all $a_1 \in A_1$ for which there exists an incentive compatible $(g, b_{2,g}) \in \mathscr{C} \times A_2^{A_1}$ such that

$$u_2(a_1, b_{2,g}(a_1)) - g(b_{2,g}(a_1)) > u_2(a_1, b_{2,f}(a_1)) - f(b_{2,f}(a_1)) \tag{13}$$

and

$$g(b_{2,g}(a_1)) > f(b_{2,f}(a_1)) \tag{14}$$

there exists an $a_1' \in A_1$ such that

$$f(b_{2,f}(a_1')) \geq g(b_{2,g}(a_1')). \tag{15}$$

$$u_2(a_1', b_{2,g}(a_1')) - g(b_{2,g}(a_1')) \geq u_2(a_1', b_{2,f}(a_1')) - f(b_{2,f}(a_1')) \tag{16}$$

When we work with this definition, Theorem 2 needs to be modified as follows.

Theorem 3. *$(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is strongly renegotiation proof if and only if for any $i \in \{1, 2, \ldots, n\}$ and increasing $b_{2,g} \in A_2^{A_1}$ such that $u_2(a_1^i, b_{2,g}(a_1^i)) > u_2(a_1^i, b_{2,f}(a_1^i))$ there exists a $k \in \{1, 2, \ldots, i-1\}$ such that*

$$u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i)) + \sum_{j=k}^{i-1} u_2(a_1^j, b_{2,g}(a_1^j)) - u_2(a_1^j, b_{2,g}(a_1^{j+1})) \\ - \min\{0, u_2(a_1^k, b_{2,g}(a_1^k)) - u_2(a_1^k, b_{2,f}(a_1^k))\} \leq f(b_{2,f}(a_1^k)) - f(b_{2,f}(a_1^i)) \tag{17}$$

or there exists an $l \in \{i+1, i+2, \ldots, n\}$ such that

$$u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i)) + \sum_{j=i+1}^{l} u_2(a_1^j, b_{2,g}(a_1^j)) - u_2(a_1^j, b_{2,g}(a_1^{j-1})) \\ - \min\{0, u_2(a_1^l, b_{2,g}(a_1^l)) - u_2(a_1^l, b_{2,f}(a_1^l))\} \leq f(b_{2,f}(a_1^l)) - f(b_{2,f}(a_1^i)) \tag{18}$$

Note that (17) and (18) are identical to their counterparts in Theorem 2 if $u_2(a_1^k, b_{2,g}(a_1^k)) \geq u_2(a_1^k, b_{2,f}(a_1^k))$ and $u_2(a_1^l, b_{2,g}(a_1^l)) \geq u_2(a_1^l, b_{2,f}(a_1^l))$. In this case, $f(b_{2,f}(a_1^k)) \geq g(b_{2,g}(a_1^k))$ and $f(b_{2,f}(a_1^l)) \geq g(b_{2,g}(a_1^l))$ imply that

$$u_2(a_1^k, b_{2,g}(a_1^k)) - g(b_{2,g}(a_1^k)) \geq u_2(a_1^k, b_{2,f}(a_1^k)) - f(b_{2,f}(a_1^k))$$

and

$$u_2(a_1^l, b_{2,g}(a_1^l)) - g(b_{2,g}(a_1^l)) \geq u_2(a_1^l, b_{2,f}(a_1^l)) - f(b_{2,f}(a_1^l)).$$

Therefore, in this case a renegotiation proof $(f, b_{2,f})$ is also strongly renegotiation proof. If, however, there exists no $j \neq i$ such that $u_2(a_1^j, b_{2,g}(a_1^j)) \geq u_2(a_1^j, b_{2,f}(a_1^j))$, then a renegotiation proof $(f, b_{2,f})$ might not be strongly renegotiation proof.

Also, Proposition 2 goes through when "renegotiation proof" is replaced with "strongly renegotiation proof." When $-\min\{0, u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)}))\}$ is added to the left hand side of the inequalities (7) and (8) in Definition 8, Propositions 3 and 4 go through with "renegotiation proof" replaced by "strongly renegotiation proof." [9] Finally, Propositions 5 and 6 go through as well

[9] In the proof of Proposition 3 one needs to simply add $-\min\{0, u_2(a_1^j, b_2^i(a_1^j)) - u_2(a_1^j, b_2^*(a_1^j))\}$ to the definition of w_j, $j =$

when "renegotiation proof" is replaced with "strongly renegotiation proof." [10]

7 Concluding Remarks

In this paper we characterized outcomes that can be supported in delegation games with incomplete and non-renegotiable as well as renegotiable contracts. We have seen that (Theorem 1) incompleteness of the contracts restricts the outcomes that can be supported, in a natural way, to those in which the second mover's strategy is increasing. Renegotiation imposes further constraints on these outcomes (Theorem 2) that limit them to subgame perfect equilibrium outcomes in some games. Yet, there is a large class of games in which non-subgame perfect equilibrium outcomes can be supported even with renegotiable contracts, and hence delegation still has a bite. In particular, in an environment common to many economic models, such as the Stackelberg and ultimatum bargaining games, any outcome in which player 1 plays an action that is larger than his Nash equilibrium action in the simultaneous move version of the game and player 2 plays a best response can be supported with incomplete and renegotiable contracts.

There are several directions along which the current work can be extended in interesting ways. The most obvious of them is to consider more general information structures and contract spaces. One interesting possibility is to assume that the principal can observe only an outcome in some arbitrary outcome space Q and that only Q is contractible. The model is closed by assuming that there is a stochastic function $p : A_1 \times A_2 \to Q$ such that $p(q|a_1, a_2)$ is the probability of outcome q when (a_1, a_2) is played in the game. This introduces moral hazard issues into the model and might change our results in non-trivial ways. Another extension along similar lines would be a model in which the agent has some payoff relevant information that is not available to the principal. This is closer to a standard adverse selection model but is embedded in a strategic environment.[11] Characterization of renegotiation proof outcomes in either of these models is left for future work.

Throughout the analysis we assumed that the principals-only game is a finite two-stage game in which the second mover's set of actions A_2 is the same after any choice a_1 by the first mover. This allowed us to formulate incentive compatibility and renegotiation proofness as sets of linear inequalities, which were relatively easy to manipulate and apply theorems of the alternative. A more technical extension of our work would be to consider arbitrary two-player finite extensive form games. Although, we expect similar results in such a framework, adapting the methods we used in the proofs to arbitrary games is not a trivial matter.

The results of this paper and the methods used to derive them can also be applied to contractual settings other than pure delegation, such as debt contracts, franchising agreements, etc. In such models, the agent's payoff depends directly on the outcome (a_1, a_2) in addition to the transfers between him and the principal, but we expect the analysis of the effects of contract incompleteness and renegotiation to remain similar to the one presented in the current paper.

---

k, l, and in the proof of Proposition 4, $-\min\{0, u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)}))\}$ to the definition of $w_{b_2^i}$.

[10] To see this, note that inequalities (42) and (43) in the proof of Proposition 5 hold when $\min\{0, u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)}))\}$ is added to their left hand sides.

[11] As we mentioned before, Dewatripont (1988) analyzes an example of such a model and shows that contracts can have a commitment value even under renegotiation.

8 Proofs

Since $\Gamma(G)$ is infinite (because of the contract space), we first start with defining what we mean by a perfect Bayesian equilibrium.

Definition 13 (Perfect Bayesian Equilibrium)**.** An assessment (β^*, μ^*) of $\Gamma(G)$ is a perfect Bayesian equilibrium if

$$\beta_2^*[\emptyset] \in \operatorname*{argmax}_{f \in \mathscr{C}} f(\beta_3^*[f, \beta_1^*[\mathscr{C}]]) \tag{19}$$

$$u_2(\beta_1^*[\mathscr{C}], \beta_3^*[\beta_2^*[\emptyset], \beta_1^*[\mathscr{C}]]) \geq \beta_2^*[\emptyset](\beta_3^*[\beta_2^*[\emptyset], \beta_1^*[\mathscr{C}]]) \tag{20}$$

$$\beta_1^*[\mathscr{C}] \in \operatorname*{argmax}_{a_1 \in A_1} u_1(a_1, \beta_3^*[\beta_2^*[\emptyset], a_1]) \tag{21}$$

$$\beta_3^*[f, a_1] \in \operatorname*{argmax}_{a_2 \in A_2} u_2(a_1, a_2) - f(a_2), \text{ for all } f \in \mathscr{C} \text{ and } a_1 \in A_1 \tag{22}$$

$$\mu^*[\mathscr{C}](\beta_2^*[\emptyset]) = 1 \tag{23}$$

In the above definition, (19) through (22) are the sequential rationality and whereas (23) is the consistency conditions.

Proof of Proposition 1. **[If]** Let (b_1^*, b_2^*) be a Nash equilibrium of G and f' satisfy the conditions of the proposition. For any $b_2 \in A_2^{A_1}$, let $b_2(A_1)$ be the image of A_1 under b_2 and define

$$f^*(a_2) = \begin{cases} f'(a_2), & \text{if } a_2 \in b_2^*(A_1) \\ \max_{a_1}\{u_2(a_1, a_2) - u_2(a_1, b_2^*(a_1)) + f'(b_2^*(a_1))\}, & \text{otherwise} \end{cases}$$

for any $a_2 \in A_2$, and

$$b_{2,f}^*(a_1) = \begin{cases} b_2^*(a_1), & f = f^* \\ \in \operatorname{argmax}_{a_2} u_2(a_1, a_2) - f(a_2), & f \neq f^* \end{cases}$$

for any $f \in \mathscr{C}$ and $a_1 \in A_1$. Consider the assessment (β^*, μ^*) of $\Gamma(G)$, where $\beta_2^*[\emptyset] = f^*, \beta_1^*[\mathscr{C}] = b_1^*, \beta_3^*[f, a_1] = b_{2,f}^*(a_1)$ for all $f \in \mathscr{C}$ and $a_1 \in A_1$, and $\mu^*[\mathscr{C}](f^*) = 1$. Clearly, this assessment induces (b_1^*, b_2^*). Also,

$$f^*(b_{2,f^*}^*(b_1^*)) = f^*(b_2^*(b_1^*)) = f'(b_2^*(b_1^*)) = u_2(b_1^*, b_2^*(b_1^*)) = u_2(b_1^*, b_{2,f^*}^*(b_1^*))$$

and hence condition (20) of Definition 13 is satisfied. Since (b_1^*, b_2^*) is a Nash equilibrium and $b_2^* = b_{2,f^*}^*$, condition (21) is satisfied as well.

Condition (22) is satisfied by definition of $b_{2,f}^*$ for any $f \neq f^*$. To show that it is satisfied when $f = f^*$, fix $a_1 \in A_1$ and take any $a_2 \in A_2$. If $a_2 \in b_2^*(A_1)$, then

$$u_2(a_1, b_2^*(a_1)) - f'(b_2^*(a_1)) \geq u_2(a_1, a_2) - f'(a_2)$$

since, by condition 3 in Proposition 1,

$$u_2(a_1, b_2^*(a_1)) - f'(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f'(b_2^*(a_1')), \text{ for all } a_1' \in A_1.$$

Therefore,

$$\begin{aligned}
u_2(a_1, b^*_{2,f^*}(a_1)) - f^*(b^*_{2,f^*}(a_1)) &= u_2(a_1, b^*_{2,f^*}(a_1)) - f'(b^*_{2,f^*}(a_1)) \\
&= u_2(a_1, b^*_2(a_1)) - f'(b^*_2(a_1)) \\
&\geq u_2(a_1, a_2) - f'(a_2) \\
&= u_2(a_1, a_2) - f^*(a_2)
\end{aligned}$$

If, on the other hand, $a_2 \notin b^*_2(A_1)$, then

$$\begin{aligned}
u_2(a_1, b^*_{2,f^*}(a_1)) - f^*(b^*_{2,f^*}(a_1)) &= u_2(a_1, b^*_2(a_1)) - f'(b^*_2(a_1)) \\
&\geq u_2(a_1, a_2) - \max_{a'_1}\{u_2(a'_1, a_2) - u_2(a'_1, b^*_2(a'_1)) + f'(b^*_2(a'_1))\} \\
&= u_2(a_1, a_2) - f^*(a_2)
\end{aligned}$$

proving that

$$b^*_{2,f}(a_1) \in \operatorname*{argmax}_{a_2 \in A_2} u_2(a_1, a_2) - f(a_2), \text{ for all } f \in \mathscr{C} \text{ and } a_1 \in A_1$$

Since condition (23) is also satisfied by definition we need only to establish condition (19) to prove that the above assessment is a perfect Bayesian equilibrium of $\Gamma(G)$. So, take any $f \in \mathscr{C}$. If $f(b^*_{2,f}(b^*_1)) \leq u_2(b^*_1, b^*_{2,f}(b^*_1))$, then $f(b^*_{2,f}(b^*_1)) \leq u_2(b^*_1, b^*_2(b^*_1)) = f^*(b^*_{2,f^*}(b^*_1))$, and we are done. If, on the other hand, $f(b^*_{2,f}(b^*_1)) > u_2(b^*_1, b^*_{2,f}(b^*_1))$, then the agent rejects the contract and player 2 receives $-\infty$, and we are done again.

[Only if] Now, suppose that (b^*_1, b^*_2) can be supported. Then, there exists a perfect Bayesian equilibrium (β^*, μ^*) that induces (b^*_1, b^*_2), i.e., $\beta^*_2[\emptyset] = f^*, \beta^*_1[\mathscr{C}] = b^*_1, \beta^*_3[f^*, a_1] = b^*_2(a_1)$ for all $a_1 \in A_1$. Suppose, for contradiction, that (b^*_1, b^*_2) is not a Nash equilibrium of G. First assume that

$$u_1(a_1, b^*_2(a_1)) > u_1(b^*_1, b^*_2(b^*_1)), \text{ for some } a_1 \in A_1.$$

This contradicts (21). So, suppose

$$u_2(b^*_1, a_2) > u_2(b^*_1, b^*_2(b^*_1)), \text{ for some } a_2 \in A_2.$$

Let $a'_2 \in \arg\max_{a_2} u_2(b^*_1, a_2)$, $0 < \varepsilon < u_2(b^*_1, a'_2) - u_2(b^*_1, b^*_2(b^*_1))$, and consider the contract $f'(a_2) = u_2(b^*_1, a'_2) - \varepsilon$, for all $a_2 \in A_2$. Condition (22) implies that

$$\beta^*_3[f', a_1] \in \operatorname*{argmax}_{a_2 \in A_2} u_2(a_1, a_2), \text{ for all } a_1 \in A_1.$$

Also,

$$u_2(b^*_1, \beta^*_3[f', b^*_1]) - f'(\beta^*_3[f', b^*_1]) = \varepsilon > 0$$

and hence the agent accepts the contract offer f'. Therefore,

$$f'(\beta^*_3[f', b^*_1]) = u_2(b^*_1, a'_2) - \varepsilon > u_2(b^*_1, b^*_2(b^*_1)) \geq f^*(b^*_2(b^*_1)),$$

by (20), which contradicts (19). Therefore, (b^*_1, b^*_2) must be a Nash equilibrium.

We now show that f^* satisfies conditions 2 and 3 stated in Proposition 1. Suppose, for contradiction, that $f^*(b_2^*(b_1^*)) < u_2(b_1^*, b_2^*(b_1^*))$ and consider

$$f'(a_2) = \begin{cases} f^*(a_2) + \varepsilon, & a_2 = b_2^*(b_1^*) \\ f^*(a_2) + 2\varepsilon, & a_2 \neq b_2^*(b_1^*) \end{cases}$$

where $0 < \varepsilon < u_2(b_1^*, b_2^*(b_1^*)) - f^*(b_2^*(b_1^*))$. Note that $f'(b_2^*(b_1^*)) < u_2(b_1^*, b_2^*(b_1^*))$ and, therefore, is accepted by the agent. Also

$$\begin{aligned} u_2(b_1^*, b_2^*(b_1^*)) - f'(b_2^*(b_1^*)) &= u_2(b_1^*, b_2^*(b_1^*)) - f^*(b_2^*(b_1^*)) - \varepsilon \\ &> u_2(b_1^*, b_2^*(b_1^*)) - f^*(b_2^*(b_1^*)) - 2\varepsilon \\ &\geq u_2(b_1^*, a_2) - f^*(a_2) - 2\varepsilon, && \text{[by (22)]} \\ &= u_2(b_1^*, a_2) - f'(a_2), \end{aligned}$$

for all $a_2 \in A_2 \setminus \{b_2^*(b_1^*)\}$. Therefore, $\beta_3^*[f', b_1^*] = b_2^*(b_1^*)$ and

$$f'(\beta_3^*[f', b_1^*]) = f'(b_2^*(b_1^*)) = f^*(b_2^*(b_1^*)) + \varepsilon > f^*(b_2^*(b_1^*)),$$

contradicting (19). Therefore, $f^*(b_2^*(b_1^*)) = u_2(b_1^*, b_2^*(b_1^*))$.

Finally, (22) implies

$$u_2(b_1, b_2^*(a_1)) - f^*(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f^*(b_2^*(a_1)), \text{ for all } a_1, a_1' \in A_1,$$

completing the proof. □

Before we turn to the proof of Theorem 1 we introduce some notation and prove a supplementary lemma. Let the number of elements in A_1 be equal to n and order its elements so that $a_1^n \succsim_1 a_1^{n-1} \succsim_1 \cdots a_1^2 \succsim_1 a_1^1$. Let e_i be the i^{th} standard basis row vector for $\mathbb{R}^n$ and define the row vector $d_i = e_i - e_{i+1}, i = 1, 2, \ldots, n-1$. Let D be the $2(n-1) \times n$ matrix whose row $2i-1$ is d_i and row $2i$ is $-d_i$, $i = 1, \ldots, n-1$. Define $U(b)$ as a column vector with $2(n-1)$ component, where component $2i-1$ is given by $u_2(a_1^i, b(a_1^i)) - u_2(a_1^i, b(a_1^{i+1}))$ and component $2i$ is given by $u_2(a_1^{i+1}, b(a_1^{i+1})) - u_2(a_1^{i+1}, b(a_1^i))$, $i = 1, 2, \ldots, n-1$.

Notation 1. Given two vectors $x, y \in R^n$

1. $x \geq y$ if and only if $x_i \geq y_i$, for all $i = 1, 2, \ldots, n$;

2. $x > y$ if and only if $x_i \geq y_i$, for all $i = 1, 2, \ldots, n$ and $x \neq y$;

3. $x \gg y$ if and only if $x_i > y_i$, for all $i = 1, 2, \ldots, n$.

Similarly for $\leq$, $<$, and $\ll$.

For any $b_2 \in A_2^{A_1}$ and $f \in \mathscr{C}$ let $f(b_2)$ be the column vector with n components, where i^{th} component is given by $f(b_2(a_1^i)), i = 1, 2, \ldots, n$.

It is well-known that if the agent's strategy is increasing, then incentive compatibility reduces to local incentive compatibility under increasing differences. We state it as a lemma for future reference and prove it for completeness.

Lemma 1. *If u_2 has increasing differences and $b_2 \in A_2^{A_1}$ is increasing in $(\succsim_1, \succsim_2)$, then for any $f \in \mathscr{C}$*

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^j)) - f(b_2(a_1^j)), \text{ for all } i, j = 1, 2, \ldots, n \tag{24}$$

holds if and only if

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^{i-1})) - f(b_2(a_1^{i-1})), \text{ for all } i = 2, \ldots, n, \tag{25}$$

and

$$u_2(a_1^i, b_2(a_1^i)) - f(b_2(a_1^i)) \geq u_2(a_1^i, b_2(a_1^{i+1})) - f(b_2(a_1^{i+1})), \text{ for all } i = 1, 2, \ldots, n-1. \tag{26}$$

Proof of Lemma 1. For notational convenience define

$$V_2(j|i) = u_2(a_1^i, b_2(a_1^j)) - f(b_2(a_1^j)), \text{ for all } i, j = 1, 2, \ldots, n.$$

With this new notation, we need to show that

$$V_2(i|i) \geq V_2(j|i), \text{ for all } i, j = 1, 2, \ldots, n \tag{27}$$

holds if and only if

$$V_2(i|i) \geq V_2(i-1|i), \text{ for all } i = 2, \ldots, n, \tag{28}$$

and

$$V_2(i|i) \geq V_2(i+1|i), \text{ for all } i = 1, 2, \ldots, n-1. \tag{29}$$

Also note that u_2 has increasing differences if and only if

$$V_2(l|j) - V_2(k|j) \geq V_2(l|i) - V_2(k|i), \text{ for all } j \geq i \text{ and } l \geq k. \tag{30}$$

Clearly, (27) implies (28) and (29). Fix $i \in \{1, 2, \ldots, n\}$ and take any $j \in \{1, 2, \ldots, n\}$. If $j = i$, then (27) holds trivially. If $j < i$, then

$$V_2(i|i) - V_2(j|i) = \sum_{k=j+1}^{i} V_2(k|i) - V_2(k-1|i) \geq \sum_{k=j+1}^{i} V_2(k|k) - V_2(k-1|k) \geq 0$$

and hence (27) holds. If $j > i$, then

$$V_2(i|i) - V_2(j|i) = \sum_{k=i}^{j-1} V_2(k|i) - V_2(k+1|i) \geq \sum_{k=i}^{j-1} V_2(k|k) - V_2(k+1|k) \geq 0$$

and therefore (27) holds. □

Proof of Theorem 1. **[Only if]** Suppose that (b_1^*, b_2^*) can be supported with incomplete and non-renegotiable contracts. Then, there exists a perfect Bayesian equilibrium (β^*, μ^*) of $\Gamma(G)$ that induces (b_1^*, b_2^*), i.e., $\beta_2^*[\emptyset] = f^*, \beta_1^*[\mathscr{C}] = b_1^*, \beta_3^*[f^*, a_1] = b_2^*(a_1)$ for all $a_1 \in A_1$. Given Proposition 1 we only need to prove that b_2^* is increasing. Fix orders $(\succsim_1, \succsim_2)$ in which u_2 has strictly increasing differences. Take any $a_1, a_1' \in A_1$ and assume, without loss of generality, that $a_1 \succsim_1 a_1'$. Suppose, for contradiction, that

$b_2^*(a_1') \succ_2 b_2^*(a_1)$. Condition (22) implies that

$$u_2(a_1, b_2^*(a_1)) - f^*(b_2^*(a_1)) \geq u_2(a_1, b_2^*(a_1')) - f^*(b_2^*(a_1'))$$
$$u_2(a_1', b_2^*(a_1')) - f^*(b_2^*(a_1')) \geq u_2(a_1', b_2^*(a_1)) - f^*(b_2^*(a_1))$$

and hence

$$u_2(a_1, b_2^*(a_1')) - u_2(a_1, b_2^*(a_1)) \leq u_2(a_1', b_2^*(a_1')) - u_2(a_1', b_2^*(a_1)),$$

contradicting that u_2 has strictly increasing differences. Therefore, b_2^* must be increasing.

[If] Let (b_1^*, b_2^*) be a Nash equilibrium of G such that b_2^* is increasing and $b_1^* = a_1^k$, for some $k = 1, 2, \ldots, n$. Given Proposition 1, all we need to prove is the existence of a contract $f \in \mathscr{C}$ such that $f(b_2^*(a_1^k)) = u_2(a_1^k, b_2^*(a_1^k))$ and

$$u_2(a_1^i, b_2^*(a_1^i)) - f(b_2^*(a_1^i)) \geq u_2(a_1^i, b_2^*(a_1^j)) - f(b_2^*(a_1^j)), \text{ for all } i, j = 1, 2, ..., n. \tag{31}$$

By Lemma 1, (31) holds if and only if $Df(b_2^*) \leq U(b_2^*)$. Therefore, we need to show that there exists $f(b_2^*) \in \mathbb{R}^n$ such that $Ef(b_2^*) \leq V$ where

$$E = \begin{pmatrix} D \\ e_k \\ -e_k \end{pmatrix}$$

and

$$V = \begin{pmatrix} U(b_2^*) \\ u_2(a_1^k, b_2^*(a_1^k)) \\ -u_2(a_1^k, b_2^*(a_1^k)) \end{pmatrix}$$

By Gale's theorem for linear inequalities (Mangasarian (1994), p. 33), there exists such an $f(b_2^*) \in \mathbb{R}^n$ if and only if for any $y \in \mathbb{R}_+^{2n}$, $E'y = 0$ implies $y'V \geq 0$. It is easy to show that $E'y = 0$ if and only if $y_1 = y_2, y_3 = y_4, \cdots, y_{2n-1} = y_{2n}$. Let $U(b_2^*)_i$ denote the i^{th} row of $U(b_2^*)$ and note that since b_2^* is increasing and u_2 has strictly increasing differences, $U(b_2^*)_{2i-1} + U(b_2^*)_{2i} \geq 0$, for any $i = 1, 2, \ldots, n-1$. Therefore,

$$y'V = \sum_{i=1}^{n-1} (U(b_2^*)_{2i-1} + U(b_2^*)_{2i}) y_{2i-1} \geq 0$$

and the proof is completed. □

Proof of Theorem 2. By definition $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is not renegotiation proof if and only if there exist $i = 1, 2, \ldots, n$ and incentive compatible $(g, b_{2,g}) \in \mathscr{C} \times A_2^{A_1}$ such that $u_2(a_1^i, b_{2,g}(a_1^i)) - g(b_{2,g}(a_1^i)) > u_2(a_1^i, b_{2,f}(a_1^i)) - f(b_{2,f}(a_1^i))$ and $g(b_{2,g}(a_1^j)) > f(b_{2,f}(a_1^j))$ for all $j = 1, 2, \ldots, n$. For any $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$, let $f(b_{2,f}) \in \mathbb{R}^n$ be a vector whose row $j = 1, 2, \ldots, n$ is given by $f(b_{2,f}(a_1^j))$. Note that incentive compatibility of $(g, b_{2,g}) \in \mathscr{C} \times A_2^{A_1}$ is equivalent to $Dg(b_{2,g}) \leq U(b_{2,g})$. Therefore, $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is not renegotiation proof if and only if there exist $i = 1, 2, \ldots, n$ and $(g(b_{2,g}), b_{2,g}) \in \mathbb{R}^n \times A_2^{A_1}$ such that $Dg(b_{2,g}) \leq U(b_{2,g})$, $u_2(a_1^i, b_{2,g}(a_1^i)) - g(b_{2,g}(a_1^i)) > u_2(a_1^i, b_{2,f}(a_1^i)) - f(b_{2,f}(a_1^i))$, and $g(b_{2,g}) \gg f(b_{2,f})$. Also note that $g(b_{2,g}) \gg f(b_{2,f})$ if and only if there exists an $\varepsilon \gg 0$ such that $g(b_{2,g}) = f(b_{2,f}) + \varepsilon$. Therefore, we have the following

Lemma 2. $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ *is not renegotiation proof if and only if there exist* $i = 1, 2, \ldots, n$, $b_{2,g} \in A_2^{A_1}$, *and* $\varepsilon \in \mathbb{R}^n$ *such that* $D(f(b_{2,f}) + \varepsilon) \leq U(b_{2,g})$, $\varepsilon_i < u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i))$, *and* $\varepsilon \gg 0$.

We first state a theorem of the alternative, which we will use in the sequel.

Lemma 3 (Motzkin's Theorem)**.** *Let A and C be given matrices, with A being non-vacuous. Then either*

1. $Ax \gg 0$ *and* $Cx \geq 0$ *has a solution* x

   or

2. $A'y_1 + C'y_2 = 0$, $y_1 > 0$, $y_2 \geq 0$ *has a solution* y_1, y_2

   but not both.

Proof of Lemma 3. See Mangasarian (1994), p. 28. □

For any $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$, $b_{2,g} \in A_2^{A_1}$, and $i = 1, 2, \ldots, n$, define $V = U(b_{2,g}) - Df(b_{2,f})$, $C = \begin{pmatrix} V & -D \end{pmatrix}$, and

$$A = \begin{pmatrix} I_{n+1} \\ l_i \end{pmatrix}$$

where $l_i = (u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i)))e_1 - e_{i+1}$. Note that C and A depend on and are uniquely defined by $(f, b_{2,f})$ and $(i, b_{2,g})$ but we suppress this dependency for notational convenience. The following lemma uses Motzkin's Theorem to express renegotiation proofness as an alternative.

Lemma 4. $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ *is renegotiation proof if and only if for any* $i = 1, 2, \ldots, n$ *and* $b_{2,g} \in A_2^{A_1}$ *there exist* $y \in \mathbb{R}^{n+2}$ *and* $z \in \mathbb{R}^{2(n-1)}$ *such that* $A'y + C'z = 0$, $y > 0$, $z \geq 0$.

Proof of Lemma 4. By Lemma 2, $(f, b_{2,f})$ is not renegotiation proof if and only if there exist $i = 1, 2, \ldots, n$, $b_{2,g} \in A_2^{A_1}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(b_{2,f}) + \varepsilon) \leq U(b_{2,g})$, $\varepsilon_i < u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i))$, and $\varepsilon \gg 0$. This is true if and only if for some i and $b_{2,g}$ there exists an $x \in \mathbb{R}^{n+1}$ such that $Ax \gg 0$ and $Cx \geq 0$. To see this let $\xi > 0$ and define

$$x = \begin{pmatrix} \xi \\ \xi\varepsilon \end{pmatrix}$$

Then $D(f(b_{2,f}) + \varepsilon) \leq U(b_{2,g})$ if and only if $Cx \geq 0$. Also, $\varepsilon \gg 0$ and $\varepsilon_i < u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i))$ if and only if $Ax \gg 0$. The lemma then follows from Motzkin's Theorem. □

For any $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$, $b_{2,g} \in A_2^{A_1}$, and $i = 1, 2, \ldots, n$, let $U(b_{2,g})_j$ denote the j-th row of vector $U(b_{2,g})$ and define $\alpha_1 = 1$, $\alpha_{i+1} = u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i))$, and

$$\alpha_{k+1} = \sum_{j=k}^{i-1} U(b_{2,g})_{2j-1} + \alpha_{i+1} - f(b_{2,f}(a_1^k)) + f(b_{2,f}(a_1^i)), \qquad \text{for } k = 1, 2, \ldots, i-1,$$

$$\alpha_{l+1} = \sum_{j=i+1}^{l} U(b_{2,g})_{2(j-1)} + \alpha_{i+1} - f(b_{2,f}(a_1^l)) + f(b_{2,f}(a_1^i)), \qquad \text{for } l = i+1, i+2, \ldots, n,$$

$$\beta_j = U(b_{2,g})_{2j} + U(b_{2,g})_{2j-1}, \qquad \text{for } j = 1, 2, \ldots, n-1.$$

Again, note that α_j and β_j depend on and are uniquely defined by $(f, b_{2,f})$ and $(i, b_{2,g})$ but we suppress this dependency. We have the following lemma.

Lemma 5. *For any* $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$, $b_{2,g} \in A_2^{A_1}$, *and* $i = 1, 2, \ldots, n$, *there exist* $y \in \mathbb{R}^{n+2}$ *and* $z \in \mathbb{R}^{2(n-1)}$ *such that* $A'y + C'z = 0$, $y > 0$, *and* $z \geq 0$ *if and only if there exist* $\hat{y} \in \mathbb{R}^{n+1}$ *and* $\hat{z} \in \mathbb{R}^{(n-1)}$ *such that* $\hat{y} > 0$, $\hat{z} \geq 0$, *and*

$$\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0 \tag{32}$$

Proof of Lemma 5. Fix $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$, $b_{2,g} \in A_2^{A_1}$, and $i = 1, 2, \ldots, n$. First note that for any y and z, $A'y + C'z = 0$ if and only if

$$y_1 + (u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i)))y_{n+2} + V'z = 0 \tag{33}$$

$$D'z = \left[A'y\right]_{-1} \tag{34}$$

where $\left[A'y\right]_{-1}$ is the n-dimensional vector obtained from $A'y$ by eliminating the first row. Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce $\begin{pmatrix} D' & \left[A'y\right]_{-1} \end{pmatrix}$ to a row echelon form and show that (34) holds if and only if

$$z_{2j-1} = z_{2j} + \sum_{k=1}^{j} y_{k+1}, \quad j = 1, 2, \ldots, i-1 \tag{35}$$

$$z_{2j} = z_{2j-1} + \sum_{k=j+1}^{n} y_{k+1}, \quad j = i, i+1, \ldots, n-1 \tag{36}$$

$$y_{n+2} = \sum_{k=1}^{n} y_{k+1} \tag{37}$$

Substituting (34)-(37) into (33) we get

$$y_1 + \alpha_{i+1} \sum_{k=1}^{n} y_{k+1} + \sum_{j=1}^{i-1} U(b_{2,g})_{2j-1} \sum_{k=1}^{j} y_{k+1} + \sum_{j=i}^{n-1} U(b_{2,g})_{2j} \sum_{k=j+1}^{n} y_{k+1} + \sum_{j=1}^{i-1} (U(b_{2,g})_{2j-1} + U(b_{2,g})_{2j}) z_{2j}$$
$$+ \sum_{j=i}^{n-1} (U(b_{2,g})_{2j-1} + U(b_{2,g})_{2j}) z_{2j-1} - \sum_{k=1}^{n} (f(b_{2,f}(a_1^k)) - f(b_{2,f}(a_1^i))) y_{k+1} = 0 \tag{38}$$

Therefore, $A'y + C'z = 0$ if and only if equations (35) through (38) hold. Now suppose that there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $y > 0$, $z \geq 0$, and (35) through (38) hold. Define $\hat{y}_j = y_j$, for $j = 1, \ldots, n+1$ and

$$\hat{z}_j = \begin{cases} z_{2j}, & j = 1, \ldots, i-1 \\ z_{2j-1}, & j = i, \ldots, n-1 \end{cases}$$

It is easy to verify that $\hat{y} > 0$, $\hat{z} \geq 0$, and $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0$.

Conversely, suppose that there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and (32) holds. Define $y_j = \hat{y}_j$ for $j = 1, \ldots, n+1$ and $y_{n+2} = \sum_{i=1}^{n+1} \hat{y}_j$. For any $j = 1, \ldots, i-1$, let $z_{2j-1} = \hat{z}_j + \sum_{k=1}^{j} y_{k+1}$ and $z_{2j} = \hat{z}_j$, and for any $j = i, \ldots, n-1$, let $z_{2j-1} = \hat{z}_j$ and $z_{2j} = \hat{z}_j + \sum_{k=j+1}^{n} y_{k+1}$. It is straightforward to show that $y > 0$, $z \geq 0$, and (35) through (38) hold. This completes the proof of lemma 4. □

Lemmas 4 and 5 imply that $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is renegotiation proof if and only if for any $i \in \{1, 2, \ldots, n\}$ and $b_{2,g} \in A_2^{A_1}$, there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and equation (32) holds. We can now complete the proof of Theorem 2.

[Only if] Suppose,for contradiction, that there exist $i = 1,2,\dots,n$ and an increasing $b_{2,g} \in A_2^{A_1}$ such that $u_2(a_1^i, b_{2,g}(a_1^i)) > u_2(a_1^i, b_{2,f}(a_1^i)$, but there is no $k = 1,2,\dots,i-1$ such that (4) holds and no $l = i+1,\dots,n$ such that (5) holds. This implies that $\alpha_j > 0$ for all $j = 1,\dots,n+1$. Since u_2 has increasing differences, $\beta_j \geq 0$ for all $j = 1,\dots,n-1$. Therefore, $\hat{y} > 0$ and $\hat{z} \geq 0$ imply that $\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j > 0$, which, by Lemma 5, contradicts that $(f, b_{2,f})$ is renegotiation proof.

[If] Fix arbitrary $i = 1,2,\dots,n$ and increasing $b_{2,g} \in A_2^{A_1}$ such that $u_2(a_1^i, b_{2,g}(a_1^i)) > u_2(a_1^i, b_{2,f}(a_1^i))$. Suppose first that there exists a $k \in \{1,\dots,i-1\}$ such that (4) holds. This implies that $\alpha_{i+1} > 0$ and $\alpha_{k+1} \leq 0$. Let $\hat{y}_{k+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{k+1}}{\alpha_{i+1}} \geq 0$, and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This implies that equation (32) holds and, by Lemma 5, that $(f, b_{2,f})$ is renegotiation proof. Suppose now that there exists an $l \in \{i+1,\dots,n\}$ such that (5) holds. Then, $\alpha_{i+1} > 0$ and $\alpha_{l+1} \leq 0$. Let $\hat{y}_{l+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{l+1}}{\alpha_{i+1}} \geq 0$ and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This, again, implies that (32) holds and that $(f, b_{2,f})$ is renegotiation proof. □

Proof of Proposition 2. **[If]** Let (b_1^*, b_2^*) be a Nash equilibrium of G such that b_2^* is increasing and renegotiation proof. This implies that there exists $f' \in \mathscr{C}$ such that (f', b_2^*) is incentive compatible and renegotiation proof. Let $f^*(b_2^*(a_1)) = f'(b_2^*(a_1)) + u_2(b_1^*, b_2^*(b_1^*)) - f'(b_2^*(b_1^*))$ for all $a_1 \in A_1$ and note that $f^*(b_2^*(b_1^*)) = u_2(b_1^*, b_2^*(b_1^*))$. Furthermore, it can be easily checked that (f^*, b_2^*) is incentive compatible and renegotiation proof. For any $f \neq f^*$ and $a_1 \in A_1$, let $b_{2,f}(a_1) \in \operatorname{argmax}_{a_2} u_2(a_1, a_2) - f(a_2)$ and $g_{(f,a_1)} \in \operatorname{argmax}_g u_2(a_1, b_{2,g}(a_1)) - g(b_{2,g}(a_1))$ subject to $g(b_{2,g}(a_1')) \geq f(b_{2,f}(a_1'))$ for all a_1'.

Consider the following assessment (β^*, μ^*) of $\Gamma_R(G)$: $\beta_2^*[\emptyset] = f^*$; $\beta_1^*[\mathscr{C}] = b_1^*$; $\beta_3[f^*, a_1] = b_2^*(a_1)$ for all a_1;

$$\beta_3[f, a_1] = \begin{cases} g_{(f,a_1)}, & \text{if } u_2(a_1, b_{2,g_{(f,a_1)}}(a_1)) - g(b_{2,g_{(f,a_1)}}(a_1)) > u_2(a_1, b_{2,f}(a_1)) - f(b_{2,f}(a_1)) \\ b_{2,f}(a_1), & \text{otherwise} \end{cases}$$

for any $f \neq f^*$ and a_1; $\beta_3[f, a_1, g, y] = b_{2,g}(a_1)$ and $\beta_3[f, a_1, g, n] = b_{2,f}(a_1)$ for all (a_1, f, g);

$$\beta_2[I_2(f^*, g)] = \begin{cases} y, & \text{if } g(b_{2,g}(a_1)) > f^*(b_2^*(a_1)) \quad \forall a_1 \\ n, & \text{otherwise} \end{cases}$$

and

$$\beta_2[I_2(f, g)] = \begin{cases} y, & \text{if } g(b_{2,g}(a_1)) \geq f(b_{2,f}(a_1)) \quad \forall a_1 \\ n, & \text{otherwise} \end{cases}$$

for any g and $f \neq f^*$; $\mu^*[\mathscr{C}](f^*) = 1$; For any g, $\mu[I_2(f^*, g)](b_1^*) = 1$ if $g(b_{2,g}(a_1)) > f^*(b_2^*(a_1))$ for all a_1 and $\mu[I_2(f^*, g)](a_1') = 1$ if there exists a_1' such that $f^*(b_2^*(a_1')) \geq g(b_{2,g}(a_1'))$; For any $f \neq f^*$ and g $\mu[I_2(f, g)](b_1^*) = 1$ if $g(b_{2,g}(a_1)) \geq f(b_{2,f}(a_1))$ for all a_1 and $\mu[I_2(f, g)](a_1') = 1$ if there exists a_1' such that $f(b_{2,f}(a_1')) > g(b_{2,g}(a_1'))$. It is easy to check that this assessment induces (b_1^*, b_2^*) and is a renegotiation proof perfect Bayesian equilibrium.

[Only if] Necessity of (b_1^*, b_2^*) being a Nash equilibrium of G and b_2^* being increasing follows from Theorem 1. On the other hand, $\Gamma_R(G)$ has a renegotiation proof perfect Bayesian equilibrium that induces (b_1^*, b_2^*) only if b_2^* is renegotiation proof. Indeed, if b_2^* is not renegotiation proof, then for any contract f such that (f, b_2^*) is incentive compatible, there exists an a_1' and an incentive compatible $(g, b_{2,g})$ such that $u_2(a_1', b_{2,g}(a_1')) - g(b_{2,g}(a_1')) > u_2(a_1', b_2^*(a_1')) - f(b_2^*(a_1'))$ and $g(b_{2,g}(a_1)) > f(b_2^*(a_1))$

for all a_1. This implies that, in any perfect Bayesian equilibrium, after history (f, a_1') the agent strictly prefers to renegotiate and offer g and the principal accepts it. In other words, there exists no renegotiation proof perfect Bayesian equilibrium which induces (b_1^*, b_2^*). □

Proof of Proposition 3. Suppose that b_2^* is renegotiation proof and fix an $i = 1, \dots, n$ and a $b_2^i \in \mathfrak{B}(i, b_2^*)$. For any $j = 1, \dots, n$, let $c_j = e_i - e_j$, where e_j is the j^{th} standard basis row vector for $\mathbb{R}^n$, and define

$$E_j = \begin{pmatrix} D \\ c_j \end{pmatrix}$$

where D is as defined in the proof of Theorem 2. Also let

$$w_k = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=k}^{i-1} U(b_2^i)_{2j-1}$$
$$w_l = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \sum_{j=i+1}^{l} U(b_2^i)_{2(j-1)}$$

for any $k \in \{1, \dots, i-1\}$ and $l \in \{i+1, \dots, n\}$ and define

$$V_j = \begin{pmatrix} U(b_2^*) \\ -w_j \end{pmatrix}$$

Incentive compatibility of (f, b_2^*) implies that $Df(b_2^*) \leq 0$. Renegotiation proofness, by Theorem 2, implies that $c_k f(b_2^*) \leq -w_k$ for some $k \in \{1, \dots, i-1\}$ or $c_l f(b_2^*) \leq -w_l$ for some $l \in \{i+1, \dots, n\}$. Suppose first that there exists a $k \in \{1, \dots, i-1\}$ such that $c_k f(b_2^*) \leq -w_k$. Then we must have $E_k f(b_2^*) \leq V_k$. By Gale's theorem of linear inequalities, this implies that $x \geq 0$ and $E'x = 0$ implies $x'V_k \geq 0$. Denote the first $2(n-1)$ elements of x by y and the last element by z. It is easy to show that $E'x = 0$ implies that $y_{2j-1} = y_{2j} + z$ for $j \in \{k, k+1, \dots, i-1\}$ and $y_{2j-1} = y_{2j}$ for $j \notin \{k, k+1, \dots, i-1\}$. Therefore,

$$\begin{aligned} x'V_k &= \sum_{j=1}^{n-1} U(b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z w_k \\ &= \sum_{j=1}^{n-1} (U(b_2^*)_{2j} + U(b_2^*)_{2j-1}) y_{2j} + \sum_{k} z(-w_k + \sum_{j=k}^{i-1} U(b_2^*)_{2j-1}) \\ &\geq 0 \end{aligned}$$

Increasing differences imply that $-w_k + \sum_{j=k}^{i-1} U(b_2^*)_{2j-1} \geq 0$ and hence k is a blocking action.

Similarly, we can show that, if there exists an $l \in \{i+1, \dots, n\}$ such that $c_l f(b_2^*) \leq -w_l$, then l is a blocking action, and this completes the proof. □

Proof of Proposition 4. We will show that there exists an $f \in \mathscr{C}$ such that (f, b_2^*) is incentive compatible and renegotiation proof. For any $i = 1, \dots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ pick a blocking action $m(b_2^i)$ that satisfies the conditions of the proposition. Let $c_{b_2^i} = e_i - e_{m(b_2^i)}$ for each i and $b_2^i \in \mathfrak{B}(i, b_2^*)$, and let $\sum_i |\mathfrak{B}(i, b_2^*)| \times n$ matrix C have row $c_{b_2^i}$ corresponding to each b_2^i. Let E be given by

$$E = \begin{pmatrix} D \\ C \end{pmatrix}$$

where D is as defined in the proof of Theorem 2. Also let

$$w_{b_2^i} = u_2(a_1^i, b_2^i(a_1^i)) - u_2(a_1^i, b_2^*(a_1^i)) + \mathbf{1}_{\{m(b_2^i)\le i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(b_2^i)_{2j-1} + \mathbf{1}_{\{i\le m(b_2^i)-1\}} \sum_{j=i+1}^{m(b_2^i)} U(b_2^i)_{2(j-1)}$$

and $\sum_i |\mathfrak{B}(i, b_2^*)| \times 1$ vector W have row $w_{b_2^i}$ corresponding to each b_2^i. Define

$$V = \begin{pmatrix} U(b_2^*) \\ -W \end{pmatrix}$$

Observe that if $Ef(b_2^*) \le V$, then $Df(b_2^*) \le U(b_2^*)$, and hence (f, b_2^*) is incentive compatible. Furthermore, $Ef(b_2^*) \le V$ implies $W \le -Cf(b_2^*)$, and, by Theorem 2, that (f, b_2^*) is renegotiation proof. Therefore, if we can show that there exists $f(b_2^*) \in \mathbb{R}^n$ such that $Ef(b_2^*) \le V$, the proof would be completed. By Gale's theorem of linear inequalities this is equivalent to showing $x \ge 0$ and $E'x = 0$ implies $x'V \ge 0$. Decompose x into two vectors so that the first $2(n-1)$ elements constitute y and the remaining $\sum_i |\mathfrak{B}(i, b_2^*)|$ components constitute z. Notice that for any $i = 1, \ldots, n$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$ there is a corresponding element of z, which we will denote $z_{b_2^i}$.

Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce E' to a row echelon form and show that $E'x = 0$ if and only if

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} [\mathbf{1}_{\{m(b_2^i)\le j\le i-1\}} - \mathbf{1}_{\{i\le j\le m(b_2^i)-1\}}] \tag{39}$$

for $j = 1, \ldots, n-1$.

Let $J_- = \{j \in \{1, \ldots, n-1\} : \exists b_2^i$ such that $i \le j \le m(b_2^i)-1\}$ and $J_+ = \{j \in \{1, \ldots, n-1\} : \exists b_2^i$ such that $m(b_2^i) \le j \le i-1\}$ and note that $J_- \cap J_+ = \emptyset$. To see this, suppose, for contradiction, that there exists a $j \in J_- \cap J_+$. Therefore, there exists a b_2^i such that $i \le j \le m(b_2^i) - 1$ and $b_2^{i'}$ such that $m(b_2^{i'}) \le j \le i' - 1$. This implies that $i < i'$, $m(b_2^i) > i$, $m(b_2^{i'}) < i'$, but $m(b_2^i) > m(b_2^{i'})$, contradicting the conditions of the proposition. We can therefore write (39) as

$$y_{2j} = y_{2j-1} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{i\le j\le m(b_2^i)-1\}} \tag{40}$$

for $j \in J_-$ and

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{m(b_2^i)\le j\le i-1\}} \tag{41}$$

for $j \in J_+$.

Finally note that

$$x'V = \sum_{j=1}^{n-1} U(b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} U(b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z_{b_2^i} w_{b_2^i}$$

Substituting from (40) and (41) we obtain

$$x'V = \sum_{j\in J_-} \left[U(b_2^*)_{2j} + U(b_2^*)_{2j-1}\right] y_{2j-1} + \sum_{j\in J_+} \left[U(b_2^*)_{2j} + U(b_2^*)_{2j-1}\right] y_{2j}$$
$$+ \sum_{b_2^i} z_{b_2^i} \left[-w_{b_2^i} + \mathbf{1}_{\{m(b_2^i)\le i-1\}} \sum_{j=m(b_2^i)}^{i-1} U(b_2^*)_{2j-1} + \mathbf{1}_{\{i\le m(b_2^i)-1\}} \sum_{j=i}^{m(b_2^i)-1} U(b_2^*)_{2j} \right]$$

Increasing differences, the definition of $m(b_2^i)$, and $y, z \ge 0$ imply that $x'V \ge 0$, and the proof is completed. □

Proof of Proposition 5. Fix an $i \in \{1,\dots,n\}$ and $b_2^i \in \mathfrak{B}(i, b_2^*)$. Assume first that $b_2^i(a_1^i) \succsim_2 b_2^*(a_1^i)$ and note that $R(i) \neq \emptyset$ by assumption. Let $J = \{i+1 \le j \le \min R(i) - 1 : b_2^*(a_1^j) \succ_2 b_2^i(a_1^j)\}$. If $J = \emptyset$, let $m(b_2^i) = \min R(i)$ and if $J \neq \emptyset$, let $m(b_2^i) = \min J$ and note that we have

$$\sum_{j=i+1}^{m(b_2^i)} \Big(u_2(a_1^j, b_2^i(a_1^{j-1})) - u_2(a_1^j, b_2^*(a_1^{j-1})) - [u_2(a_1^{j-1}, b_2^i(a_1^{j-1})) - u_2(a_1^{j-1}, b_2^*(a_1^{j-1}))] \Big)$$
$$+ u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) \ge 0 \quad (42)$$

which implies that $m(b_2^i)$ is a blocking action.

Assume now that $b_2^*(a_1^i) \succsim_2 b_2^i(a_1^i)$ and note that $L(i) \neq \emptyset$. Let $J = \{\max L(i)+1 \le j \le i-1 : b_2^i(a_1^j) \succ_2 b_2^*(a_1^j)\}$. If $J = \emptyset$, let $m(b_2^i) = \max L(i)$ and if $J \neq \emptyset$, let $m(b_2^i) = \max J$ and note that

$$\sum_{j=m(b_2^i)}^{i-1} \Big(u_2(a_1^{j+1}, b_2^*(a_1^{j+1})) - u_2(a_1^{j+1}, b_2^i(a_1^{j+1})) - [u_2(a_1^j, b_2^*(a_1^{j+1})) - u_2(a_1^j, b_2^i(a_1^{j+1}))] \Big)$$
$$+ u_2(a_1^{m(b_2^i)}, b_2^*(a_1^{m(b_2^i)})) - u_2(a_1^{m(b_2^i)}, b_2^i(a_1^{m(b_2^i)})) \ge 0 \quad (43)$$

which, again, implies that $m(b_2^i)$ is a blocking action.

Finally, suppose that there exist $i_1 < i_2$ such that $m(b_2^{i_1}) > i_1$ and $m(b_2^{i_2}) < i_2$. This implies that i_1 has right deviation and i_2 has left deviation at b_2^*, and hence $R(i_1) \cap L(i_2) \neq \emptyset$. But this implies that $m(b_2^{i_1}) \le m(b_2^{i_2})$ and the proof is completed by applying Proposition 4. □

Proof of Proposition 6. We first prove a supplementary lemma.

Lemma 6. *For any selection (br_1, br_2), $a_1 \succsim_1 \overline{a}_1^{NE}$ implies $a_1 \succsim_1 br_1(br_2(a_1))$ and $a_1 \precsim_1 \underline{a}_1^{NE}$ implies $a_1 \precsim_1 br_1(br_2(a_1))$.*

Proof of Lemma 6. Fix $a_1^* \succsim_1 \overline{a}_1^{NE}$ and note that, by definition,

$$\overline{a}_1^{NE} = \max_{\succsim_1}\{a_1 \in A_1 : a_1 = br_1(br_2(a_1)) \text{ for some } (br_1, br_2)\}.$$

Therefore, $\overline{a}_1^{NE} \succsim_1 br_1(br_2(a_1))$ for all (br_1, br_2) and $a_1 \in A_1$. This implies that $a_1^* \succsim_1 br_1(br_2(a_1^*))$ for any selection (br_1, br_2). Similarly,

$$\underline{a}_1^{NE} = \min_{\succsim_1}\{a_1 \in A_1 : a_1 = br_1(br_2(a_1)) \text{ for some } (br_1, br_2)\}.$$

Therefore, $a_1^* \precsim_1 \underline{a}_1^{NE}$ implies that $a_1^* \precsim_1 br_1(br_2(a_1^*))$ for any selection (br_1, br_2). $\square$

[If] Fix an $a_1^* \succsim_1 \overline{a}_1^{NE}$ and a selection (br_1, br_2) such that (12) is satisfied. Define

$$b_2(a_1) = \begin{cases} \underline{a}_2, & a_1 \prec_1 a_1^* \\ br_2(a_1^*), & a_1^* \precsim_1 a_1 \prec_1 \overline{a}_1 \\ br_2(\overline{a}_1), & a_1 = \overline{a}_1 \end{cases}$$

Note that b_2 is increasing and satisfies the conditions of Proposition 5. Therefore, if we can show that (a_1^*, b_2) is a Nash equilibrium of the principals-only game G we will be done. By definition $b_2(a_1^*) \in BR_2(a_1^*)$. Condition (12) implies that $u_1(a_1^*, b_2(a_1^*)) \geq u_1(a_1, b_2(a_1))$ for all $a_1 \prec_1 a_1^*$ and $u_1(a_1^*, b_2(a_1^*)) \geq u_1(\overline{a}_1, b_2(\overline{a}_1))$. Therefore, take any a_1 such that $a_1^* \prec_1 a_1 \prec_1 \overline{a}_1$. By Lemma 6, $a_1 \succ_1 a_1^* \succsim_1 br_1(br_2(a_1^*))$, which, together with single-peakedness, implies that

$$u_1(a_1^*, b_2(a_1^*)) = u_1(a_1^*, br_2(a_1^*)) \geq u_1(a_1, br_2(a_1^*)) = u_1(a_1, b_2(a_1)).$$

Therefore, $a_1^* \in \operatorname{argmax}_{a_1} u_1(a_1, b_2(a_1))$ and hence (a_1^*, b_2) is a Nash equilibrium of G.
[Only if] Suppose that $(a_1^*, a_2^*) \in A_1 \times A_2$ can be supported with incomplete and renegotiable contracts. This, by Theorem 1, implies that there exists an increasing $b_2 \in A_2^{A_1}$ such that (a_1^*, b_2) is a Nash equilibrium of G and $b_2(a_1^*) = a_2^*$. This, in turn, implies that $a_2^* \in BR_2(a_1^*)$. Suppose, for contradiction, that $a_1^* \prec_1 \underline{a}_1^{NE}$. Lemma 6 implies that $a_1^* \precsim_1 br_1(a_2^*)$, for any br_1. Fix a br_1 and let $a_1' = br_1(a_2^*)$. Note that $a_1' \succsim_1 a_1^*$ and $u_1(a_1^*, a_2^*) < u_1(a_1', a_2^*)$, for otherwise the game $S(G)$ would have a Nash equilibrium smaller than $(\underline{a}_1^{NE}, \underline{a}_2^{NE})$. Therefore,

$$u_1(a_1^*, b_2(a_1^*)) = u_1(a_1^*, a_2^*) < u_1(a_1', a_2^*) = u_1(a_1', b_2(a_1^*)) \leq u_1(a_1', b_2(a_1')),$$

where the last inequality follows from positive externality. This contradicts that (a_1^*, b_2) is a Nash equilibrium of G.

Suppose now that (12) is not satisfied. If there exists a_1' such that $u_1(a_1', \underline{a}_2) > u_1(a_1^*, br_2(a_1^*))$ for all br_2, then

$$u_1(a_1', b_2(a_1') \geq u_1(a_1', \underline{a}_2)) > u_1(a_1^*, a_2^*) = u_1(a_1^*, b_2(a_1^*))$$

where the first inequality follows from positive externality. This contradicts that (a_1^*, b_2) is a Nash equilibrium.

To prove that $u_1(\overline{a}_1, br_2(\overline{a}_1)) \leq u_1(a_1^*, br_2(a_1^*))$ for some (br_1, br_2), we first prove the following lemma.

Lemma 7. *If $b_2 \in A_2^{A_1}$ is renegotiation proof, then $\overline{a}_1$ does not have right deviation.*

Proof of Lemma 7. Let $a_1^n = \overline{a}_1$ and suppose, for contradiction, that a_1^n has right deviation, i.e., there exists $a_2' \succ_2 b_2(a_1^n)$ such that $u_2(a_1^n, a_2') > u_2(a_1^n, b_2(a_1^n))$. Define

$$b_2'(a_1) = \begin{cases} a_2', & a_1 = a_1^n \\ b_2(a_1), & a_1 \prec_1 a_1^n \end{cases}$$

Note that b_2' is increasing and hence incentive compatible. Also,

$$\begin{aligned} &u_2(a_1^n, b_2'(a_1^n)) - u_2(a_1^n, b_2(a_1^n)) - [u_2(a_1^{n-1}, b_2'(a_1^n)) - u_2(a_1^{n-1}, b_2(a_1^n))] > 0 \\ &\qquad = \sum_{j=k}^{n-1} u_2(a_1^j, b_2'(a_1^j)) - u_2(a_1^j, b_2(a_1^j)) + \sum_{j=k}^{n-2} u_2(a_1^j, b_2(a_1^{j+1})) - u_2(a_1^j, b_2'(a_1^{j+1})) \end{aligned}$$

for all $k < n$, which, by Proposition 3, contradicts that b_2 is renegotiation proof. □

If $u_1(\overline{a}_1, br_2(\overline{a}_1)) > u_1(a_1^*, br_2(a_1^*))$ for all br_2, then

$$u_1(\overline{a}_1, b_2(\overline{a}_1)) \geq u_1(\overline{a}_1, br_2(\overline{a}_1)) > u_1(a_1^*, br_2(a_1^*))$$

for all br_2, where the first inequality follows from no right deviation at $\overline{a}_1$ (Lemma 7) and positive externality. Therefore, $u_1(\overline{a}_1, b_2(\overline{a}_1)) > u_1(a_1^*, b_2(a_1^*))$, which contradicts that (a_1^*, b_2) is a Nash equilibrium of G, and the proof is completed. □

Proof of Theorem 3. By definition $(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is not strongly renegotiation proof if and only if there exist $i = 1, 2, \ldots, n$ and incentive compatible $(g, b_{2,g}) \in \mathscr{C} \times A_2^{A_1}$ such that $u_2(a_1^i, b_{2,g}(a_1^i)) - g(b_{2,g}(a_1^i)) > u_2(a_1^i, b_{2,f}(a_1^i)) - f(b_{2,f}(a_1^i))$, $g(b_{2,g}(a_1^i)) > f(b_{2,f}(a_1^i))$, and $g(b_{2,g}(a_1^j)) - f(b_{2,f}(a_1^j)) > \min\{0, u_2(a_1^j, b_{2,g}(a_1^j)) - u_2(a_1^j, b_{2,f}(a_1^j))\}$ for all $j = 1, 2, \ldots, n$. The following lemma easily follows.

Lemma 8. *$(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is not renegotiation proof if and only if there exist $i = 1, 2, \ldots, n$, $b_{2,g} \in A_2^{A_1}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(b_{2,f}) + \varepsilon) \leq U(b_{2,g})$, $0 < \varepsilon_i < u_2(a_1^i, b_{2,g}(a_1^i)) - u_2(a_1^i, b_{2,f}(a_1^i))$, and $\varepsilon_j > \min\{0, u_2(a_1^j, b_{2,g}(a_1^j)) - u_2(a_1^j, b_{2,f}(a_1^j))\}$ for all $j = 1, 2, \ldots, n$.*

Define the matrices V and C as in the proof of Theorem 2, and define the matrix A as follows: its row 1 is e_1, row $n+2$ is l_i, and row $j+1$, for $j = 1, \ldots, n$, is given by $-\min\{0, u_2(a_1^k, b_{2,g}(a_1^k)) - u_2(a_1^k, b_{2,f}(a_1^k))\}e_1 + e_{j+1}$. We have the following lemma, whose proof is similar to that of Lemma 4.

Lemma 9. *$(f, b_{2,f}) \in \mathscr{C} \times A_2^{A_1}$ is strongly renegotiation proof if and only if for any $i = 1, 2, \ldots, n$ and $b_{2,g} \in A_2^{A_1}$ there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, $z \geq 0$.*

The rest of the proof is almost identical to that of Theorem 2, and therefore is omitted. □

References

[1] Beaudry, P. and M. Poitevin (1995), "Contract Renegotiation: A Simple Framework and Implications for Organization Theory," *Canadian Journal of Economics*, 28, 302-335.

[2] Bensaid, B. and R. J. Gary-Bobo (1993), "Commitment Value of Contracts under Renegotiation Constraints," *Econometrica*, 61, 1423-29.

[3] Caillaud, B., B. Jullien, and P. Picard (1995), "Competing Vertical Structures: Precommitment and Renegotiation", *Econometrica*, 63, 621-646.

[4] Cho, I. K. and D. Kreps (1987), "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 102, 179-221.

[5] Dewatripont, M. (1988), “Commitment through Renegotiation-Proof Contracts with Third Parties,” *Review of Economic Studies*, 55, 377-389.

[6] Fershtman, C. and K. Judd (1987), “Equilibrium Incentives in Oligopoly,” *American Economic Review*, 77, 927-940.

[7] Fershtman, C., K. Judd, and E. Kalai (1991), “Observable Contracts: Strategic Delegation and Cooperation,” *International Economic Review*, 32, 551-559.

[8] Fudenberg, D. and J. Tirole (1991), “Perfect Bayesian and Sequential Equilibrium,” *Journal of Economic Theory*, 53, 236-260.

[9] Gatsios, K. and L. Karp (1991), “Delegation Games in Custom Unions,” *Review of Economic Studies*, 58, 391-398.

[10] Katz, M. (1991), “Game-Playing Agents: Unobservable Contracts as Precommitments,” *Rand Journal of Economics*, 22, 307-328.

[11] Katz, M. (2006), “Observable Contracts as Commitments: Interdependent Contracts and Moral Hazard,” *Journal of Economics & Management Strategy*, 15, 685-706.

[12] Koçkesen, L. (2007), “Unobservable Contracts as Precommitments,” *Economic Theory*, 31, 539-552.

[13] Koçkesen, L. and E. A. Ok (2004), “Strategic Delegation by Unobservable Incentive Contracts,” *Review of Economic Studies*, 71, 397-424.

[14] Mangasarian, O. L. (1994) *Nonlinear Programming*, New York: McGraw-Hill.

[15] Persson, T. and G. Tabellini (1993), “Designing Institutions for Monetary Stability,” *Carnegie-Rochester Conference Series on Public Policy*, 39, 53-84.

[16] Polo, M. and P. Tedeschi (2000), “Delegation Games and Side-Contracting,” *Research in Economics*, 54, 101-116.

[17] Schelling, T. (1960), *The Strategy of Conflict*, Harvard University Press.

[18] Topkis, D. (1979), “Equilibrium Points in Nonzero-sum N-person Submodular Games,” *SIAM Journal on Control and Optimization*, 17, 773-787.