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Transition to Equilibrium in International Trades.*

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Abstract.– Building on Giraud & Tsomocos (2009), we develop a model of nonequilibrium trades with incomplete markets. Out-of-equilibrium trades occur in continuous time, both on international and domestic markets. Traders are assumed to exhibit locally rational expectations on future prices, interest rates and exchange rates. Although currencies turn out to be non-neutral, if their stock grows sufficiently rapidly and if agents can trade assets during a sufficiently long period, the world economy converges in probability towards some interim constrained efficient state. Moreover, a random localized version of the Quantity Theory of Money holds provided the economy is not trapped in a liquidity hole. The traditional theory of comparative advantages, however, turns out to be challenged by international capital mobility.

Keywords. Non-tâtonnement; Equilibrium transition; Incomplete Markets ; Non-arbitrage ; Price-quantity Dynamics; Liquidity; Comparative Advantages; Quantity theory of money.

JEL classification: E5, E6, F1, F2, F3.

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1 Introduction

This paper is part of a programme to verify whether it is possible to obtain a micro-economically founded dynamics of out-of-equilibrium monetary trades with heterogeneous boundedly rational agents that enables to explain the main phenomena observed in economic applications — and which yet cannot be explained within standard general equilibrium theory or only on ad hoc grounds.

This programme started with Giraud & Tsomocos (2009).[1] There, trade paths (i.e., orbits of our dynamics) were shown to converge to some locally Pareto-optimal point provided the quantity of inside money injected by the Central Bank grows sufficiently rapidly. Otherwise, the economy enters into a liquidity trap from which it can be saved only by a massive injection of money. Money was shown to be non-neutral (both in the short- and in the long-run), although a localized version of the quantity theory of money could be deduced. The consequence was that, whenever the quantity of injected money increases more rapidly than its optimal growth rate, this has no impact on the real economy, and produces only additional inflation. By contrast with Friedman's "golden rule", however, this optimal growth rate of money heavily depends upon the real characteristics of the economy. In particular, it depends on local gains-to-trade available at each point of time.

There were three major restrictions in the above mentioned paper: We restricted ourselves to complete markets and to national trades within a single country with a unique currency. In addition, no saving account was available where cash could be stored. The aim of the present paper is to extend our analysis to the case of international trades with incomplete markets, various currencies, national saving accounts, and heterogeneous households within each country. To simplify the presentation, we have introduced these new ingredients step by step. Sections 2 and 3 deal with incomplete barter markets. International trades, currencies and saving accounts are treated from section 4 on.

The dynamics is based on the continuous double auction, which is probably the most widely used price formation mechanism in modern financial markets. In a continuous-time setting, traders (buyers and sellers) are allowed to place or cancel trading orders whenever they like, at the prices of their choice. If an order to buy meets an order to sell, there is a transaction.[2] The specific micro-structure used in this paper is Mertens' (2003) limit-price mechanism.[3]

As appealing as it may be, however, empirical tests of the continuous

[1]Of course, there are many predecessors: the whole macro-economic literature of the 70s, devoted to disequilibrium theory (see, e.g., Herings (1996) for a synthesis), Smale (1976), Fisher (1983), Champsaur & Cornet (1990) for the dynamical aspects; Dubey & Geanakoplos (2003) for the treatment of money.

[2]See, e.g., Glosten & Milgrom (1985).

[3]see Giraud (2003) for the link between this mechanism, double auctions and strategic market games.

double auction show that it does not match reality very well.[4] This is partly the reason why double auctions on markets populated by traders with a "low level of intelligence" were developped, and turn out to fit data much more accurately. Such models assume that agents place orders more or less at random.[5] Here, following Giraud & Tsomocos (2009),[6] we assumed instead that traders are myopic in the following sense: 1) Instead of heroically solving the Hamilton-Jacobi-Bellman equation associated to his intertemporal optimization programme, each individual simply seeks to maximize the first-order linear approximation of his current utility. 2) Households entertain expectations about the distant future that may not coincide with perfect forecast. These expectations are reflected through the saving behavior of agents. One consequence of the postulated myopia of traders is that they continuously trade (adjusting their portfolio and commodity bundle according to their current utility gradient). This is in accordance with the well-known empirical observation that, on average, investors trade much more than they should according to standard equilibrium theory.[7] Global trading in financial markets is of an order of hundred times as large as global production.[8]

Our paradigm includes (possibly incomplete) markets of multiple financial assets (to allow for a study of relative price changes), heterogeneous households in each country, trade in continuous time in multiple currencies, riskless bonds, and national monetary policies conducted by budget-balanced Central Banks. Hahn's (1965) standard puzzle regarding the value of money is solved in a way similar to Dubey and Geanakoplos (2003): as long as there are effective trades along a solution path to our dynamics, money has a positive value (due to a cash-in-advance constraint a la Clower (1967)). However, as soon as the amount of inside money injected at some time t by the Central Bank in of each country is lower than a certain threshold (which depends upon the amount of outside money and the potential, local gains to trade available within the country[9]), then the world economy breaks down in a liquidity hole: Trades collapse, prices become indeterminate, and money has no more value. The surprising lessons to be drawn from this property are that 1) money is not neutral in our model ; 2) the collapse of the whole world economy can be circumvented as long as at least one country provides a sufficient liquidity for the rest of the world. This provides an optimal random growth rate of inside money for national monetary policies.

Although we maintain market-clearing all the time, our conclusions are a mix of Keynesian and monetarist wisdoms: The effect of national monetary

[4]See Sandas (2001).

[5]Becker (1962) is one of the pioneering approaches of non-rational individual behavior; see Gode & Sunder (1993), Bouchaud et al. (2002), Mike & Farmer (2008), for further developments on "zero intelligence" or "low intelligence" models.

[6]See also Champsaur & Cornet (1990), as well as Geanakoplos and Gray (1991) for a justification of why seeing further is not seeing better.

[7]Cf. Geanaloplos & Sebenius (1983).

[8]Cf. Shiller (1981, 1987) and Odean (1999).

[9]Calculated in the same way as in Geanakoplos & Tsomocos (2002).

policy depends upon whether the threshold just alluded to (which depends on the heterogeneity of households of the whole world economy) is surpassed or not. If a benchmark rate of growth leads, at some time T, to a liquidity hole, then an expansionary monetary policy enables to escape from the hole. However, if the instantaneous amount of inside money was already larger than the threshold mentioned above, then an expansionary open market operation will only induce domestic inflation, a fall in nominal interest rates, a currency depreciation, and no change in the real terms of trade.

Due to the myopia of traders, familiar properties such as uncovered interest parity or Fisher's effect need not be valid along the transition to equilibrium. This is roughly due to the possibility of intertemporal arbitrages that can be observed ex post and are not forecast by myopic households. As for the famous purchasing power parity, it may also fail in our setting due to the fact that each agent faces two (infinitesimal) budget constraints, one for each currency. Notice, nevertheless, that all these properties, despite being standard in static equilibrium theory, are hardly confirmed by empirical tests.

There is a number of papers to which this one is closely related. Geanakoplos & Tsomocos (2002) deal with international trades within a static general equilibrium set-up with complete markets. Dubey & Geanakoplos (2003) treat the case of incomplete markets but within a single country and still in static framework. Giraud & Tsomocos (2009), as already alluded to, introduce the monetary transition process to equilibrium but with complete markets in a single country. Champsaur & Cornet (1990), Bonnisseau & Nguénamadji (2009) consider a transitional process equivalent to the present one (in continuous and discrete time respectively) but in barter trades. Appart from Giraud & Tsomocos (2006), none of the already mentioned papers take advantage from the detailed micro-structure of Mertens (2003) double auction.

The paper is organized as follows: The next section details the dynamics of trades on incomplete markets. We prove that trajectories solving our dynamics converge in probability to some allocation of assets which is approximately interim second-best efficient (to be defined there). Section 3 provides a discussion on arbitrage and the consequences of the main result in terms of progression towards market efficiency. Section 4 extends the previous analysis to an economy with multiple currencies, exchange rates and international trades. An optimal random growth of national inside money is provided in section 5, together with a discussion of the classical theory of comparative advantages. Technical proofs are gathered in the Appendix.

2 The model

In this section, we lay out the assumptions that will be maintained throughout the paper and then construct the dynamics.[10]

2.1 The fundamentals

We are working in a stylized long-run international economy $\mathcal{E}$, with two countries, A and B, and a finite time horizon $T > 0$.[11]

Risk is captured through a metric dynamical system $\big(\Omega, \mathcal{F}, \mathbf{P}, (\phi_t)_t\big)$ where $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space of uncertain states of nature ω, and $(t, \omega) \mapsto \phi_t\omega$ is a measurable flow which leaves $\mathbf{P}$ invariant, i.e., $\phi_t\mathbf{P} = \mathbf{P}$ for all $t \in [0, T]$. Information is revealed over time according to $\phi_t\omega$. Indeed, given $(\phi_t)_t$, one can construct a filtration over the completion of $\mathcal{F}$ which can be interpreted as capturing the onfolding of information in the way it is usually done in economics (see Arnold (1998, p. 72). All uncertainty is resolved at time $T > 0$. (See Remark 1 below for examples of such dynamical systems.)

There are L non-perishable commodities $\ell = 1, ..., L$ in each country. We denote by ℓ^k the ℓ^{th} commodity produced in country k. (Nigeria and Venezuela may both sell crude oil, but it is Nigerian oil and Venezuelian oil.) There are also two real long-lived securities in each country, denoted X and Y, with the same maturity T.[12] The amount of security X held by agent 1 in country A is X_1^A and similarly for the other securities. Security X in country k is a European contingent claim to some nonnegative quantity of spot k-commodities, $R_X^k \in L^\infty(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{R}_+^L) \setminus \{0\}$, to be delivered at time T. Let $R := (R_X^A, R_Y^A, R_X^B, R_Y^B)$ represent the global return. As soon as $\text{Span}R(\Omega) \neq L^\infty(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{R}^L)$, markets are incomplete. For each agent i in country $k = A, B$, a portfolio $\theta_i = (X_i^A, Y_i^A, X_i^B, Y_i^B) \in \mathbb{R}^4$ is a holding of each asset, and may include short sales. We will often denote by $\theta_{i,j} \in \mathbb{R}$ (resp. R_j) the element of the portfolio θ_i (resp. of the return vector R) that corresponds to security j.

Remark 1. Since myopic traders do not perfectly forecast future (random) prices, the space of uncertain states Ω can be understood, if one wishes so, as being the space of continuous asset price functions $q : \mathbb{R} \to \mathbb{R}_+^{4+2L}$ which satisfy some initial condition. $\mathcal{F}$ can then be taken as the Borel σ-field induced by the compact-open topology on Ω and $\mathbf{P}$, e.g., the Wiener measure on $(\Omega, \mathcal{F})$, i.e., the distribution on $\mathcal{F}$ of a standard Wiener process, as is traditional in finance. One would then have: $W_t(\omega) = \omega(t)$ and $\phi_s\omega(t) = \omega(s + t)$.[13] It should be clear in the sequel, however, that the randomness of

[10] By contrast with Giraud & Tsomocos (2009), in this section, markets are incomplete but there is no money.

[11] Our whole analysis goes through with an arbitrary, finite number of countries.

[12] This is for convenience only: We could work with an arbitrary finite number of securities.

[13] See, e.g., Arnold (1998, p. 535 sq) for details.

the stochastic process $q(\cdot)$ arising from the local interactions of boundedly rational investors can be much wilder and more chaotic than a standard white noise — which is consistent with empirical observation.

2.1.1 Households

Each country is populated by two types of agents $i = 1, 2$. Each (type of) agent belongs to a single country. In order to wipe out any phenomenon of imperfect competition, we assume that each type i is represented by a continuum of identical clones $[0, 1]$, equipped with the restriction, λ, of the Lebesgue measure. (The space of individuals is therefore $\big([0, 1]^4, \lambda^{\otimes 4}\big)$.)

To each agent i in country k is associated:

(i) a measurable initial portfolio $\omega \mapsto \theta_i(0, \omega) \in \mathbb{R}^4_+$ of real securities at time zero (no agent starts with short positions[14]); a measurable initial endowment $\omega \mapsto x_i(0, \omega) \in \big(\mathbb{R}^L_+\big)^2$ of commodities arising from both countries. (Citizens from the US may drive with Japanese cars and vice-versa.)

(ii) a random endowment $e_i(\cdot) \in L^\infty_+ := L^\infty\big(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{R}^{2L}_+\big)$ of spot consumption commodities at time T;

(iii) von Neumann preferences over time T-consumption bundles. These preferences are assumed to be represented by a (Bernoulli) utility function $u_i : L^\infty_+ \to \mathbb{R}$.

A pair of financial allocation of portfolios, $\theta = (\theta_i)_i \in \big(\mathbb{R}^4\big)^4$ and commodity allocations, $x = (x_i) \in \big(\mathbb{R}^{2L}_+\big)^4$ is feasible if:[15][16]

$$\sum_i \theta_i = \overline{\theta}(0) \text{ and } \sum_i x_i = \overline{x}(0) \text{ and for every } i, \quad e_i + \theta_i \Box R \geq 0 \quad \mathbf{P}-\text{a.s.} \tag{1}$$

The last inequality requires that no investor goes bankrupt at time T.[17] Feasibility therefore imposes that all promises be honored at T: Agents are allowed to go arbitrarily short $\theta_{i,j} < 0$ in any asset j provided they ultimately keep their promises by reducing their initial holding at time T by $\theta_{i,j} R_j$. Let $\tau(\omega) \subset \mathbb{R}^{12+6L}$ denote the subset of feasible allocations in state ω.

For all i, $V_i : \mathbb{R}^{4+2L}_+ \times \Omega \times [0, T] \to \mathbb{R}$ denotes the expected indirect utility derived by agent i at time T from a bundle $z_i(t) = (x_i(t), \theta_i(t)) \in \mathbb{R}^{4+2L}$ conditional on the information available at time $t \in [0, T]$:

$$V_i\big(z_i(t), \omega, t\big) := E^{\mathbf{P}}\Big[u_i[e_i + x_i(t) + \theta_i(t)\Box R] \mid \mathcal{F}_t\Big](\omega). \tag{2}$$

[14] Securities are to be thought of as real contracts, not as potential ones as it is often the case in static GEI.

[15] Throughout this paper, $\overline{z}$ stands for the sum $\sum_i z_i$ for every variable z.

[16] For all $\theta_i = (X^A_i, Y^A_i, X^B_i, Y^B_i) \in \mathbb{R}^4$, $\theta_i \Box R := (X^A_i \cdot R^A_X + Y^A_i \cdot R^A_Y, X^B_i \cdot R^B_X + Y^B_i \cdot R^B_Y) \in \mathbb{R}^{2L}$, a country-by-country scalar product.

[17] For a static general equilibrium analysis of international trades with default, see Peiris & Tsomocos (2009).

The following assumption will hold throughout the paper.

Assumption (C).

(i) $\forall i, u_i$ is $\mathcal{C}^1$, $\nabla u_i(\cdot) >> 0$, u_i is strictly quasi-concave and verifies the boundary condition: $u_i^{-1}(\lambda)$ is closed in $\mathbb{R}^{2L}_{++}$.

(ii) $(\Omega, \mathcal{F}) \subset (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\mathbf{P}$ is Borel.

(iii) For every $z = (z_i)_i \in \tau$ and every i, the mapping $\omega \mapsto u_i\big[e_i(\omega) + x_i + \theta_i \square R(\omega)\big]$ belongs to $L^1(\Omega)$.

(iv) $\forall K \subset \mathbb{R}^{2L}_+$ compact (nonvoid), $\exists g \in L^1(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{R}_+)$ s.t.

$$\left|\frac{\partial}{\partial z} u_i(z, \omega, t)\right| \leq g(\omega), \quad \forall z \in \tau(\omega) \cap K, \ \mathbf{P}- \text{ a.e. } \ \omega \in \Omega.$$

Assumptions (ii-iv) are technical, and are fulfilled, e.g., as soon as $|\Omega| < +\infty$.[18]

2.2 The barter dynamics

In this section, only barter trades take place. Our exposition follows Giraud & Tsomocos (2009), and we sometimes refer to this paper for supplementary information.

Suppose that, along a path of trades, investors have no expectation about the future, hence are entirely myopic. (Expectations will be introduced in the next subsection.) When restricted to this class of economies, and under the myopia assumption, our dynamics becomes similar to the ones first analyzed by Champsaur & Cornet (1990), Bottazzi (1994) and Giraud (2004)[19]. Actually, it can be viewed as an extension to incomplete markets of Giraud (2004). That is, the configuration set of the continuous-time dynamics is given by the feasible set τ ; at each instant t, when the asset allocation is $z(t)$, agents exchange infinitesimal trades in a tangent market $T_{z(t)}\mathcal{E} : \langle -z_i(t), \nabla V_i(z_i(t))\rangle_i$. The latter is defined as a linear auxiliary economy with the same set of individuals, the same set of commodities and securities except that, now, each individual's set of trades is the shifted cone: $-z_i(t) + \mathbb{R}^{4+2L}_{++}$, her initial endowment is 0 (both in commodities and securities), and i values the infinitesimal trade $\dot{z}_i(t)$ according to:[20]

$$\nabla V_i(z_i(t)) \cdot \dot{z}_i(t). \tag{3}$$

In other words, in the tangent market $T_{z(t)}\mathcal{E}$, trades are net, and i's short-sale upper-bound is given by her current stock $z_i(t)$.[21] Traders meet every time on the tangent market, myopically trading in the direction of

[18] One could presumably weaken some of these assumptions. For the sake of not overburdening the presentation, we shall not strive for the utmost generality.

[19] See also Smale (1976b) for a seminal contribution along the same line.

[20] Assumptions (C)(iii) and (iv) guarantee that one can differentiate with respect to z_i under the integral of (2), so that the gradient, $\nabla V_i(z_i(T), \omega, t)$, is well-defined.

[21] One can think of $z_i(t)$ as implicitly playing the role of i's collateral.

the steepest increase in their own, current expected utility. The budget constraint of individual i is:

$$q(t) \cdot \dot{z}_i(t) \leq 0, \text{ and } \dot{z}_i(t) \geq -z_i(t), \tag{4}$$

where $q(t) \in \mathbb{R}_+^{4+2L}$ is a vector of prices. The set of infinitesimal moves, $\dot{z}_i(t)$, that will actually take place is then given by the Walrasian allocations of the linear economy $T_{z(t)}\mathcal{E}$, taking place at the corresponding competitive price $q(t)$. The dynamics is therefore given by the following random differential inclusion:

$$(\dot{z}(t), q(t)) \in \mathrm{WE}\big[T_{z(t)}\mathcal{E}\big](\phi_t\omega), \quad (z(0), q(0)) = (z_0, q_0)(\omega), \tag{5}$$

where (z_0, q_0) is some (measurable) random variable, and $\mathrm{WE}\big[T_{z(t)}\mathcal{E}\big]$ is the subset of Walras equilibria in net trades of $T_{z(t)}\mathcal{E}$. That is, for each i, $\dot{z}_i(t)$ solves the linear optimization programme of (3) subject to (4) while $\sum_i \dot{z}_i(t) = 0$. A random trade path can also be viewed as a measurable map $\varphi : [0,T] \times \Omega \times \mathbb{R}_+^{16+8L}$ with $(t,z) \mapsto \varphi(t,\omega,z)$ continuous for every ω,[22] for which there exists a measurable price curve $q : [0,T] \times \Omega \to \mathbb{R}_+^{4+2L}$, also continuous in t, and such that:

$$(\dot{\varphi}(t,\omega,z_0), q(t,\phi_t\omega)) \in \mathrm{WE}\big[T_{\varphi(t,\omega,z_0)}\mathcal{E}\big](\phi_t\omega), \quad (\varphi(0,\omega,z_0), q(0)) = (z_0, q_0)(\omega).$$

When uncertainty is absent, a random trade path reduces to the usual flow of a (deterministic) differential inclusion.[23]

> Proposition 1. Under assumptions (C), every world economy $\mathcal{E}$ admits a random trade path and price curve, i.e., for every starting random variable $(z(0,\cdot), q(0,\cdot))$, (5) admits a solution.

Proof. (1) According to Lemma 3.3 (vii) in Arnold & Schmalfuss (2001), the sup over an uncountable set can be replaced by the one over an exhaustible countable set. Hence, $v_i(z_i,\omega,t) := \max\{\nabla V_i(z_i,\omega,t) \cdot \dot{z}_i(t) \mid q(t) \cdot \dot{z}_i(t) \leq 0 \text{ and } \dot{z}_i(t) \geq -z_i\}$ is a measurable function of (z_i,ω,t). It folllows that $(z,\omega,t) \mapsto \mathrm{WE}[T_z\mathcal{E}]$ is measurable.

(2) For all $(\omega,t) \in \Omega \times [0,T]$, $z \mapsto \mathrm{WE}[T_z\mathcal{E}]$ is u.s.c. with nonempty, compact, convex values.

(3) Since, for all ω and a.e. t, $\dot{z}(t)$ verifies $\sum_i \dot{z}_i(t) = 0$, then, $z(t) \in \tau$ a.e. Hence, for all ω, the short-sale constraint $\dot{z}_i(t) \geq z_i(t)$ implies that $\forall\omega$ $\|\dot{z}(t)\| \leq \sum_i \|z_i(t)\|(\omega) = \sum_i \|z_i(0)\|(\omega)$ a.e. in t. As for prices, they can be normalized into, say, the unit sphere, so that $\|\mathrm{WE}[T_{z(t)}\mathcal{E}]\|(\omega) \leq a(\omega,t)$ with $a(\omega,\cdot) \in L^1_+([0,T])$.

[22] φ need not be continuous with respect to ω.

[23] See Giraud & Tsomocos (2009), subsection 2.1, in order to grasp intuition on this dynamics in the complete markets case.

The conclusion then follows, e.g., from Theorem 4.1. in Papageorgiou (1988).

□

Remark 2. This existence result should be contrasted with Hart's (1975) example of non-existence of static GEI equilibria, and compared with the (generic both in endowments and asset structure) existence result of static equilibria for GEI economies with real assets (Duffie & Shafer (1986)). Clearly, this is due to the (random) constraint $\dot{z}_i(t) \geq -z_i(t)$ which provides a (random) upper-bound on short sales, unlike in standard GEI where no such upper-bound is available (at least when there are several spot commodities).

2.3 Convergence toward Interim constrained optimality

The next issue is whether solution trajectories of our stochastic dynamical process will converge towards anything akin to the Pareto set. First, we need to define efficiency in a way well-suited to our environment. The following definition is inspired from the literature devoted to efficiency in game-theoretic environments with incomplete information[24] as well, of course, as from the notion of constrained optimality in static general equilibrium theory with incomplete markets.[25]

> Definition. (i) An admissible curve at $z \in \tau$, is a continuous random path $\varphi : [0, \eta) \times \Omega \to \tau$ such that $\varphi(0, \omega) = z$, φ is differentiable at 0 and $\frac{d}{dt} V_i(\varphi(t, \omega)) \geq 0$ for all i with one strict inequality.
>
> (ii) An allocation of assets z is constrained locally optimal *interim* at time $t \in [0, T)$ and in state ω if there does not exist any admissible curve at z.

Such allocations are called "locally" optimal because welfare improvements are tested only for infinitesimal reallocations. "*Interim*" stresses that such allocations are optimal at some intermediary time where all the information is not yet known by the households. "Constrained" underlines the fact that optimality is tested only with respect to the available (possibly incomplete) asset structure. Let Θ denote the random subset of *interim* constrained locally optimal allocations.

Remark 3. This definition of optimality departs from the one used, say, in Geanakoplos & Polemarchakis (1986, and in most of the literature devoted to GEI) in as much it does not rely upon some second-period equilibrium (which requires rational expectations in some way or another). As a consequence, even if an interim constrained optimal allocation in assets is reached in finite time, there may still be a need for reopening spot markets after time T.

[24] Cf., e.g, Myerson (1991, p. 487 sq.).

[25] See Geanakoplos (1990) for an introduction.

Since Θ is evaluated with the help of conditional expected utilities, the set of allocations and portfolios that are constrained interim optimal *at time* t depend upon t; This dependence is captured through the randomness of $\omega \mapsto \Theta(\omega)$: As time evolves, so does ω via $\phi_s \omega_t$, hence also Θ. Intuition borrowed from Bayesian decision theory suggests, however, that *interim* constrained efficiency at time $t < T$ should imply *interim* constrained efficiency at any posterior time $t' \in [t, T)$. Intuitively, Pareto efficiency with respect to the information known at time t does not only require that the allocation be Pareto efficient for each state ω (*ex post* optimality) but also that the allocation "optimally" insures all risk-averse agents over the different states of the world — at least, given $\phi_t \omega$. The finer the information, the easier it should be to insure oneself optimally, so that sub-optimality at time t' should imply sub-optimality at time $t < t'$. A simple explanation is that agents in the standard Bayesian model are dynamically consistent. For instance, the well-known no-trade theorem rests on the statement that any *ex ante* efficient allocation is *interim* efficient[26] so that purely speculative trade is impossible. This property, however, need not hold in our set-up for a general metric dynamical system $(\Omega, \mathcal{F}, \mathbf{P}, (\phi_t)_t)$. This calls for the following restriction on the onfolding of information: $\Theta(\omega) \subset \Theta(\phi_t \omega)$ for any ω and $t \in [0, T]$. We shall impose a slightly stronger restriction:

Assumption (B).

The random set Θ verifies: $\Theta(\omega) = \Theta(\phi_t \omega) \quad \forall \omega, \forall t \in [0, T]$.

Assumption (B) says that, for every state ω, if at some time t, an allocation is optimal, then it was already so earlier, and it will remain so later on, whatever being the revealed information. In other words, if no-trade occurs at some stage t because, at the given allocation of assets, all gains to trade seem to have been exhausted, no-trade will still prevail in the future, and the same allocation would also have induced no-trade in the past. Notice that this assumption does not imply that the onfolding of information has no influence on the current gains to trade: They may well vary across time, depending upon the information shared by the investors. Therefore, (B) is a limit condition on the dynamic consistency of Bayesian agents: Whether gains to trade are *exactly* zero or not does not depend upon information.

Several concepts of convergence are available in our stochastic setting. Here, we use one of the weakest possible criteria. A sequence $(X_n)_n$ of random variables converges to a random variable X in probability if:

$$\lim_n \mathbf{P}\Big[\omega \ : \ |X_n(\omega) - X(\omega)| > \varepsilon\Big] = 0 \quad \forall \varepsilon > 0.$$

Given some random trade path $\varphi(\cdot)$, a compact random set A[27] is stable under φ if

[26] see Milgrom & stokey (1982), Holmström & Myerson (1983).

[27] Since $\mathcal{F}$ is complete, A is a compact random set if it takes closed/compact values and its graph is measurable.

(i) it is invariant under φ, i.e., $\varphi(t,\omega)\Theta(\omega) = \Theta(\phi_t\omega)\forall t \in [0,T]$.
(ii) $\forall \varepsilon > 0$, there exists a random neighborhood, C, of A, such that:[28]

$\triangleright$ $\mathbf{P}\big[d(C|A) \geq \varepsilon\big] < \varepsilon$,

$\triangleright$ $\varphi(t,\omega,C(\omega)) \subset C(\phi_t\omega)\ \forall t \in [0,T]$, i.e., C is forward invariant under φ.

A is a global attractor of φ if, for any random variable z_0,

$$\mathbf{P}\lim_{t\to\infty} d\big(\varphi(t,\cdot,z_0(\cdot)), A(\phi_t\cdot)\big) = 0.$$

Finally, A is globally asymptotically stable if it is stable and a global attractor. The following result states that, if we wait long enough, continuous trades will utlimately enable households to converge (in probability) toward an interim constrained efficient allocation of assets. It can be interpreted as a version of "Coase theorem" within our specific set-up. The proof is given in the Appendix.

THEOREM 1.— Under (C) and (B), and for $T = \infty$, $\Theta(t)$ is globally asymptotically stable for any trade path, $\varphi(\cdot)$, solving (5).

The next Corollary provides a uniform attraction property easier to interpret economically.

COROLLARY.— Under (C) and (B), for any random compact neighborhood C of Θ, then there exists a ϕ-invariant set $\overline{\Omega} \subset \Omega$ of full measure such that for every $\omega \in \overline{\Omega}$ and any random compact set D of $\mathbb{R}^{4L+16}$, if $z(0,\omega) \in D(\omega)$, there exists a $\tau(\omega) > 0$ with

$$\varphi(t,\omega) \in \text{int}C(\omega) \quad \forall t \geq \tau(\omega).$$

In words: If households trade sufficiently enough, the economy will be ε-close to the set of interim constrained efficient allocations in the metric of convergence in probability. Another interpretation is in terms of speed of trades. Notice, indeed, that there is no absolute speed in this barter dynamics. That is, the units in which infinitesimal trades are computed is partially arbitrary: So far, in our theory, nothing prevents trades from occuring at, say, the celerity of light.[29] Therefore, for a fixed T, if the speed of trades is sufficiently high, the same convergence result holds.

Remark 4. The stability definition actually allows the random neighborhood $C(\omega)$ to lie possibly far away from $\Theta(\omega)$ on a ω-set of small probability. Therefore, the notion of probabilistic convergence used here is very weak.

[28]As usual, the distance between a point x and a set A is defined as $d(x,A) = \inf_{y\in A} d(x,y)$, while $d(C|A) := \sup_{x\in C} d(x,A)$.

[29]Provided it is compatible with the short-sale constraint which nevertheless provides an upper-bound on the scale (hence, the speed) of infinitesimal trades.

The possible convergence ω-almost surely (or in L^p) is left for further research, and is likely to fail in general.

Remark 5. Duffie & Huang (1985, following many others) consider a problem that is close, but not identical to, the one studied here. They prove that continuous-time retrading enables to mimic every (complete markets) Arrow-Debreu equilibrium by means of an (incomplete markets) Radner equilibrium. The main difference with our approach is that, here, agents are myopic, and we get (only) a probabilistic convergence towards approximate second-best efficiency. There, agents are far-sighted in the sense that they perfectly forecast the law of the whole stream of future random prices and are able to solve the corresponding intertemporal optimization programme. As a consequence, Duffie & Huang (1985) get exact first-best efficiency. Otherwise stated, the whole literature devoted to spanning properties of retrading in incomplete markets still consider equilibrium properties (comparing various competing notions of equilibria), while we are dealng with a transitional process *to* equilibrium.

2.4 Locally rational expectations

Let us now introduce agents' expectations. Agent i's expectations are captured through a saving function, $s_i : \mathbb{R}^{4+2L}_{++} \to \mathbb{R}^{4+2L}_{+}$ which associates to current asset allocation, $z_i(t)$, a bundle of saved commodities and securities: $0 \leq s_i[z_i(t)] \leq z_i(t)$. Let us denote by $\delta_i(t) := z_i(t) - s_i[z_i(t)]$ the bundle of objects of exchange that, given his expectations, agent i is ready to put on the market at time t. Of course, $0 \leq \delta_i(t) \leq z_i(t)$. A tangent market $T_{z(t),\delta(t)}\mathcal{E}$ is defined in the same way as in the barter case, except that the budget constraint (4) is replaced by:

$$q(t) \cdot \dot{z}_i(t) \leq 0, \quad \text{and} \quad \dot{z}_i(t) \geq -\delta_i(t). \tag{6}$$

When agents' expectations become so pessimistic that $\delta_i(t) = 0$, for every i, then no-trade will occur in the tangent market (even though different expectations might reveal beneficial gains-to-trade). The myopic economy of the previous subsection corresponds to the particular no-saving case: $\delta_i(t) = z_i(t)$, every t and i.[30] The dynamics with expectations is now given by:

$$(\dot{z}(t), q(t)) \in \mathrm{WE}\big[T_{z(t),\delta(t)}\mathcal{E}\big], \quad (z(0), q(0), \delta(0)) = (z_0, q_0, \delta_0). \tag{7}$$

The next Proposition is a straightforward corollary of Theorem 1:

PROPOSITION 2.— Provided that, for every household i, expectations verify: $\delta_i(t) > 0$ a.e. t, the conclusions of Proposition 1 and Theorem 1 hold after having replaced (5) with (7).

[30] This case —where everybody is forced to put for sale all of her current endowments— corresponds to Lucas' initial approach.

3 On market efficiency once again

The theory of market arbitrage efficiency is usually justified by the assertion that if there were profit-making opportunities in flexible markets, they would be quickly found and exploited, and the resulting trading activity would change prices in ways that would remove them. This informational efficiency of prices translates into the non-arbitrage hypothesis common to financial pricing and static general equilibrium (with incomplete markets) or, equivalently, into the martingale property of discounted (fair or equilibrium) prices. It should not be confused, in general, with Pareto-optimality. The absence of arbitrage attributed to informationally efficient markets, however, raises an embarrassing puzzle: To remove market inefficiencies we must have traders who are motivated to exploit them. But if the market is already perfectly efficient, there is no possibility to make excess profits, so that the very notion of a fully efficient market is inherently contradictory. As convincingly argued by Geanakoplos & Farmer (2009), "the need for a nonequilibrium theory is apparent. Equilibrium theory predicts that markets are perfectly efficient, and thus violations of efficiency cannot be addressed without going outside it. The time scale for the degradation of a profitable strategy is inherently a disequilibrium phenomenon".

In our set-up, a trivial answer goes as follows, at least when $|\Omega| < \infty$. As long as $t < T$, it may well be the case that the asset price system $q(t) \in \mathbb{R}_+^{4+2L}$ turns out, ex post, to exhibit some arbitrage opportunity in the sense of static GEI theory, i.e., is such that there exists some portfolio $\theta(t) \in \mathbb{R}^4$ with

$$\begin{pmatrix} -q(t) \\ P(T) \circ R \end{pmatrix} \theta(t) > 0,$$

where, here, $\circ$ is the usual state-by-state product of GEI, and $P(T)$ is the, say square-integrable, state-dependent spot price of commodities at maturity.[31] Because of the myopia of households, nothing prevents the price system $(q(t), P(T))$ from exhibiting ex-post arbitrage — and, in fact, nothing proves that $P(T)$, as defined here, will have a finite variance... This stands in sharp contrast with the static GEI approach where equilibrium prices with perfect forecast preclude arbitrage. As is well-know, however, absence of arbitrage is probably one of the most popular properties of economic and financial models which fares most badly with empirical tests (more on this in the next subsection).

In this section, we explore an alternate interpretation of arbitrage in order to examine whether our transition to equilibrium narrative permits to give some sense to the highly disputed no-arbitrage property. For this purpose, let us recall the microstructure underlying infinitesimal trades in the tangent market, $T_{z,\delta}\mathcal{E}$.

[31] The integrability or regularity of $P(T)$ is immaterial for our discussion.

3.1 Limit-orders as fictitious agents

As we have just seen, our dynamics results in a set of curves $z(\cdot)$ (possibly degenerate at some point whenever the economy reaches some interim second-best efficient point).[32] Imagine, now, an electron microscope aimed at a point $z \in \tau$. Under enlargement, the neighborhood of z and the environment above it become linear: We get the tangent market $T_z\mathcal{E}$. So far, we simply postulated that infinitesimal flows are induced by the Walrasian equilibria of $T_z\mathcal{E}$. For reasons already spelt out in Giraud & Tsomocos (2009), this is just a proxy of the actual mechanism at work in our continuous double auction. With even greater magnification, we see, instead of the reduced-form concept of Walras equilibrium, a strategic market-game $G[T_z\mathcal{E}]$, where investors send limit-price orders in continuous time to a central clearing house which instantaneously execute some of them according to the rules of Mertens (2003) limit-price mechanism. From the standpoint of the central clearing house, investors are "invisible": all they can see is the order book populated by a myriad of anonymous orders.

For simplicity, let's begin with a two-good barter economy. A strategy of player i in the local game $G[T_z\mathcal{E}]$ associated to the tangent market $T_z\mathcal{E}$ is to send a limit-price order to the market. Only *selling* orders are allowed — but this implies no loss of generality: if a player wants to buy a commodity or a security, he just has to sell another item in exchange. A limit-order to sell item commodity ℓ in exchange for item c gives a quantity Q_ℓ to be sold, and a relative price q_ℓ^+/q_c^+. The order is to sell up to $Q_{\ell c}$ units of item ℓ in exchange for item c if the actual relative price verifies $q_\ell/q_c \geq q_\ell^+/q_c^+$. The amount Q_ℓ put up for sale will stay untouched at any price $q_\ell < q_c q_\ell^+/q_c^+$, and is intended to be fully sold at any price $q_\ell \geq q_c q_\ell^+/q_c^+$. When $\frac{q_\ell^+}{q_c^+} = 0$, one gets a familiar market order. A limit-order to "sell" commodity ℓ against c, at relative prices $q_c^+ = 0, q_\ell^+ > 0$ is, in fact, an order *not* to buy c, and to sell as much of ℓ as possible. The key in understanding the relationship between the reduced-form model and $G[T_z\mathcal{E}]$ lies in Mertens' trick, which we now recall.

Mertens' trick. Suppose that the central clearing house fixes q as a current price vector. Checking whether a sell-order $(Q_\ell, q_\ell^+/q_c^+)$ must be (totally or, at least, partially) executed at q, is equivalent to solving the following programme (see Fig. 1):

$$\max\Big\{\frac{q_\ell^+}{q_c^+}\dot{z} \quad | \quad q \cdot \dot{z} \leq 0 \text{ and } \dot{z} \geq (0, ..., -Q_\ell, ...0)\Big\}. \tag{8}$$

[32] Actually, Giraud & Tsomocos (2009) provide sufficient conditions for the (generic) uniqueness of the solution curve $z(\cdot)$ (locally in time). One could provide an analogous analysis here, provided $|\Omega| < \infty$.

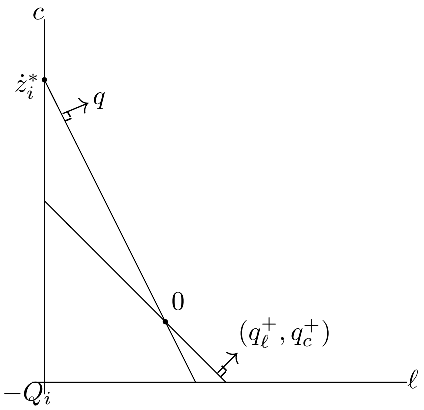

Fig. 1. Limit-price orders are fictitious linear agents.

Here, indeed, the relative price q_c/q_ℓ (set by the market) is lower than the limit-price q_c^+/q_ℓ^+, so that the order to sell ℓ against c is entirely executed: The agent i who sent this order will sell the quantity Q_i^ℓ of good ℓ against $(q_\ell/q_c)Q_i^\ell$ units of commodity c, and will end up at $\dot z_i^*$. But $\dot z_i^*$ coincides with the Walrasian demand of the fictitious agent with linear utility $v_i(\dot z) := (q_c^+/q_\ell^+)\dot z$ and short-sale bound $\delta_i := Q_i$. In other words, a limit-price order can be viewed as a fictitious linear "agent", whose (non-normalized) "utility" is given by the relative price (or "exchange rate", or "personal price" at which the sender of the order agrees to exchange one good for another) q_ℓ^+/q_c^+, and whose "short-sale bound" is the offer $(0, ..., Q_\ell, ..., 0)$.

More generally, in a L-commodity barter tangent market, an order is a vector of relative prices $b^i := (q_1^+, ..., q_L^+)$ and a vector of offers $e^i := (Q_1, ..., Q_L)$, to be understood as follows: If the actual relative price of good c against good ℓ verifies:

$$\frac{q_c}{q_\ell} \geq \frac{q_c^+}{q_\ell^+},$$

then player i is ready to sell up to quantity Q_ℓ of commodity ℓ against c. There is no loss of generality in restricting ourselves to *sell*-orders since a buying order, say of commodity ℓ, can be replicated as an order to sell *any* other commodity against ℓ. For simplicity, we assume that, within a single period t, a player i can send a single order (b^i, e^i) to the market. This also involves no loss, since this player's short-run utility is linear in $T_z\mathcal{E}$, so that his demand and supply correspondence can be mimicked by means of a single limit-order.[33]

[33] By contrast, in a non-linear economy, to mimick the demand and supply correspondences of a player would require a continuum of limit-orders.

Next, the order book obtained by pooling together the whole set of limit-orders sent by "true" agents to the clearing house at time t can be viewed as a fictitious linear (possibly nonatomic) economy $\mathcal{L}_t$. In the economy $\mathcal{L}_t$, the clearing house selects a trade price and executes orders accordingly. Recall, however, that each agent i in $\mathcal{E}$ actually stands for a continuum, $[0, 1]$, of identical clones, having the same characteristics. As a consequence, the unique strategy-proof strategy-profile of the game $G[T_z\mathcal{E}]$ consists, for a.e. clone $h \in \otimes_{i=1}^4 [0, 1]$ in reporting truthfully his current characteristics at time t, i.e., in quoting $\nabla V_h(z(t), \omega, t)$ as limit-price, and sending $\delta_h(t)$ as supply.

3.2 Progression toward market efficiency ?

In this perspective, Theorem 1 provides a first step toward a out-of-equilibrium narrative of how arbitrage opportunities disappear in a perfectly flexible market. Indeed, a standard characterization of Pareto-optimality is that in $\Theta_0(t) \cap \mathrm{int}(\tau)$, all the traders should have collinear gradients.[34] Thus, what Theorem 1 says is that, up to a normalization, agents' gradients, $\nabla V_i(z_i(t), \omega, t)$ will converge (in probability) towards approximately the same vector, say ∇^*. But the micro-structure of trades given by our continuous double auction is such that, at every time t, trader's i current gradient, $\nabla V_i(z_i(t), \omega, t)$ will exactly coincide with his quoted limit-price. That is, at time t, i will send a signal to the clearing house saying that, if, say,

$$q_j(t) \geq \frac{\partial V_i(z_i(t), \omega, t)}{\partial \theta_{i,j}(t)},$$

then, i is willing to sell a certain amount of security j. If, the last inequality is reversed, then i wishes to buy a certain amount of j. Suppose that two traders, i and h, have different normalized gradients at time t, so that

$$\frac{\partial V_i(z_i(t), \omega, t)}{\partial \theta_{i,j}(t)} > \frac{\partial V_h(z_h(t), \omega, t)}{\partial \theta_{h,j}(t)}.$$

This means that there is an arbitrage opportunity: An arbitrageur (whose own marginal utility for j lies somewhere in between those of i and h) will buy the asset j cheap from h and sell it dear to i. This arbitrage opportunity is not infinite, because of the constraint $\dot{\theta}_j(t) > -\theta_j(t)$, so that no trader can go arbitrarily short. However, the arbitrageur will obviously try to benefit as much as possible from the situation given his own current constraint. Of course, prices at time t will change as the arbitrageur carries out her trade: The double auction will determine the amount of the surpluses that can be realized, as well as the fraction of this total surplus captured by the arbitrageur. To put it differently, positive local gains-to-trade (as defined in this paper) can be readily interpreted as a measure of the arbitrage opportunities surviving on the market.

[34] $\mathrm{int}(X)$ denotes the interior of the set X.

More precisely, let us formally define the measure $\chi(z)$ of local gains-to-trade, following Dubey & Geanakoplos (2003). Let $\dot{z}_i \in \mathbb{R}^{4+2L}$ be an infinitesimal trade vector of i in some tangent market, with positive component representing purchases and negative ones representing sales. For any scalar $\chi \geq 0$, define:

$$\dot{z}_{i,j}(\chi) := \min\Big\{\dot{z}_{i,j}, \frac{\dot{z}_{i,j}}{1+\chi}\Big\}. \tag{9}$$

There are local gains to χ-diminished trades in the barter tangent market $T_{\gamma(t)}\mathcal{E}$ if there exist feasible infinitesimal trades $(\dot{z}_i(t))_i$ such that $\dot{z}_i(t) \geq -\delta_i(t)$ for all i, and $V_i(\dot{z}_i(\chi), \omega, t) \geq 0$ for all i with at least one strict inequality. In words, it should be possible for households to Pareto-improve on no-trade in spite of the χ-handicap on infinitesimal trades. For every $z \in \tau$, the measure $\chi(z)$ is the supremum of all handicaps that permit Pareto-improvement. Clearly, z is Pareto-optimal if, and only if, $\chi(z) = 0$. According to the previous discussion, $\chi(z)$ can be equivalently interpreted as a measure of the lack of constrained Pareto-optimality (interim) of z or as a measure of arbitrage opportunities.

Since traders are myopic and trades take time, they cannot exhaust instantaneously $\chi(z) > 0$. But Theorem 1 tells us that, if consumers can trade during a sufficiently long time, then eventually, roughly all the arbitrage opportunities should disappear (at least in probability). An estimate of the time scale for the progression toward market efficiency is suggested in Geanakoplos & Farmer (2009) as being at least eight years. Of course, as investors start to recognize the opportunity and exploit it, it will make it shrink (i.e., it will reduce $\chi(z)$) and thus make it harder for others to see it, extending the time until its superior profitability is completely extinguished.[35] In any case, we should not be surprised that inefficiencies disappear slowly. The speed of convergence in our model provides a first attempt, within a general-equilibrium framework, for quantifying the time-scale for the progression toward market efficiency.

3.3 Is elimination of arbitrage Pareto-improving ?

An important caveat must be added, however. Farmer & Geanakoplos (2009) provide, indeed, anecdotal empirical evidence that not only financial markets might not always remove every arbitrage opportunity (even slowly) but, in certain circumstances, they may even amplify them. Can this be reconciled with our theory ?

We answer to this question by means of the simplest possible example of market-making "arbitrage", which we henceforth refer to as arbitrage, though it may also include financial innovation or other forms of market-

[35]see Farmer (2002).

completion.[36] Two markets for assets are segmented (physically or institutionally) and unable to trade with one another without the help of an arbitrageur. In the classic case, when arbitrage is beneficial, the initial distribution of the assets between the two markets is Pareto-inefficient ; the entrance of an arbitrageur helps smooth risk, making all parties better off. This is classic gains from trade: it is the same as saying that the explorers who traded British luxuries for Indian spices benefited to both countries. This is essentially the content of Theorem 1 above. However, it holds under the implicit assumption that all the traders are "rational" (albeit myopic) in the sense that they correctly forecast the "true" probability measure $\mathbf{P}$. The following example shows that elimination of arbitrage may not be Pareto-improving when traders do not perfectly forecast $\mathbf{P}$.

Example. There are two traders, two spot commodities, (x, y), one riskless asset promising one unit of commodity x for sure, and a risky asset which yields as real dividend at time T, the random variable Y of commodity y, whose objective law is $N(\mu, 1)$, with $\mu > 0$. Consumer 1 has utility $u_1(x_1, y_1) := x_1 - e^{-cy_1}$, where $(x_1, y_1) \in \mathbb{R}^2_+$ is 1's time T-consumption of spot commodities (with $c > 0$ the absolute risk aversion). Consumer 2 has utility: $u_2(x_2, y_2) = x_2 + by_2$. At time 0, initial endowments in commodities are zero, and in securities: $\theta_1(0) = (1, 2)$, $\theta_2(0) = (2, 1)$. To begin with, suppose that both individuals correctly forecast Y's Gaussian distribution. No additional news are delivered between time 0 and T. The exogenous parameters, c, μ and b are chosen so that:

$$ce^{-2\mu c + 2c^2} < 1 < b \quad \text{and} \quad b\mu > 1. \tag{10}$$

At each time $t \geq 0$, given her current portfolio $\theta_1(t) = (\theta_{1,x}(t), \theta_{1,y}(t))$, the marginal utility of trader 1 (which is also the limit-price he sends to the clearing house at t) for securities is $\left(1, E^{\mathbf{P}}\left[ce^{-c\theta_{i,y}(t)Y}\right]\right)$ with:

$$E^{\mathbf{P}}\left[ce^{-c\theta_{i,y}(t)Y}\right] = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-c\theta_{i,y}(t)y - \frac{(y-\mu)^2}{2}} dy = e^{-\theta_{i,y}(t)c\mu + \frac{1}{2}(c\theta_{i,y}(t))^2}.$$

The gradient of trader 2, at the same time t, is: $(1, b)$. Suppose that the markets of both consumers are segmented (physically, or institutionally) and unable to trade with one another without the help of an arbitrageur. In particular, at time 0, their quoted limit-prices, $q_1(0) = \left(1, ce^{-2\mu c + 2c^2}\right)$ and $q_2(0) = (1, b)$, will stand as prices prevailing on each individual's own market. Condition (10) then implies that an arbitrageur who could buy or sell the risky asset at (internal, normalized) market prices in both consumers' markets, would observe an arbitrage opportunity by buying cheap 1's risky asset and selling it dear to trader 2 while doing the opposite transaction

[36] Glen Weyl (2007) proposes an analogous example but without making explicit the underlying microstructure of trades.

for the riskless asset. According to Theorem 1, actually, connecting both markets suffices to partially eliminate the arbitrage opportunity: if T is sufficiently large and agents trade according to our continuous double auction, then, with probability 1, almost all social surplus available from trades will be realized. Actually, trades conducted by the double auction will follow a straight line toward the final allocation $\theta_1(T) = (3, 0), \theta_2(T) = (0, 3)$. The welfare of consumer 1 has increased from $1 - e^{-2\mu c + 2c^2}$ (at time 0) to 2 (at time T^-); that of consumer 2, from $2 + b\mu$ to $3b\mu$. Along the trade path, (normalized) prices resulting from trades will be constantly equal to $(1, 1)$. As $t \to T^-$, the arbitrage induced by the disagreement of traders about the value of their securities does not disappear because the traders' gradient do not coincide (even asymptotically) while the resulting allocation of portfolio converges toward the boundary of the Edgeworth box. Nevertheless, in the same way as local gains-to-trade decrease across time, trades diminish the size of arbitrage so that the traditional wisdom is confirmed.[37]

What happens, now, if agents do not perfectly forecast $\mathbf{P}$? Suppose that Y actually follows $N(0, 1)$, while both consumers still believe that $Y \sim N(\mu, 1)$, with $\mu > 0$. Keep all the other characteristics of the previous example unchanged, and suppose, in addition, that $ce^{2c^2} = b$. Then, the initial situation at time 0 is (objectively) already Pareto-optimal since the two consumers have collinear gradients. Subjectively, however, they both believe, as in the previous example, that there are beneficial local gains to trade. They will therefore trade in the same way as earlier, ending up with $\big((3, 0), (0, 3)\big)$ as final allocation. The asymptotic social welfare will be 2, whereas it initially was $3 - e^{c^2}$ inducing a global welfare loss of $e^{c^2} - 1 > 0$. Reducing arbitrage now causes a suboptimal allocation of risk, harming the investors.

4 Money and Central Banks

So far, the issue of international trading and exchange rates could not arise since all trades were barter. Let us now introduce *fiat* money. Any purchase is subject to a cash-in-advance constraint. The money of country k is called k-money. Money is present in the private endowments of households. Following Dubey and Geanakoplos (2003), we call it "outside money". At time t, agent i has an endowment $m_i^k(t)$ of outside k-money, $k = A, B$. Also, in each country k, a government acts on markets, through a Central Bank by injecting k-money into the world economy, termed inside money. Outside money is owned by the agents free and clear of debt; inside money is always accompanied by debt when it comes into the hands of the households. The quantities $m^k = (m_i^k(0))_i$ of outside money at time 0 are exogenously fixed, and $\overline{m}^k(0) = \sum_i m_i^k(0)$ represents the aggregate stock of outside k-money

[37]Notice, however, that the reallocation of securities induced by trades does not increase individual portfolio's diversification, on the contrary.

held by the agents throughout the world at the beginning of time.

There are two kinds of bonds. Short-term (intraperiod or overnight) k-bonds at time t promise one unit of k-money at the "end" of period t. Long-term (interperiod) k-bonds at time t promise one unit of k-money at the "beginning" of period $t+dt$.[38] By rolling over a long deposit by means of interperiod loans, agents can save money against any future time period. Let $r^k(t)$ (resp. $\hat{r}^k(t)$) be the interest rates on the short- (resp. long-) term loans in country k. There is no need for considering interperiod loans of longer duration since with rollover, any zero-coupon bond, $B^k(t,T) := e^{\int_t^T \hat{r}^k(s)ds}$, can be mimicked by means of instantaneous interperiod bonds. In order to keep a clear distinction between the roles of the two types of interest rates, households are allowed only to *sell* intraperiod bonds and to *buy* interperiod bonds in all countries. Equivalently, they can *borrow* on the intraperiod loan markets, while they can *deposit* money on interperiod markets.[39] An agent who borrows ε in the intraperiod (resp. deposits ε in the interperiod) loan market at the beginning (resp. end) of period t will have a debt of $e^{r^k(t)dt}\varepsilon$ (resp. will owe $e^{\hat{r}^k(t)dt}\varepsilon$) at the end of period t (resp. beginning of period $t+dt$). Short-term k-interest rates, $r^k(\cdot)$, are exogenously fixed by the k-Central Bank and publicly known in advance by every household of the world economy. By contrast, long-term rates, $\hat{r}^k(t)$, are endogenously determined on the loan market.

4.1 Monetary policy

The Central Bank of country k has two monetary policy instruments: It sets the short-term interest rates $r^k(t)$ and clears its budget at each period t.[40] For this purpose, it auctions off in continuous time the quantities of k-inside money, $M^k(t)$ and $\hat{M}^k(t)$.[41] Borrowing and depositing occur in local currencies. We assume that the amount of k-money inventoried by each household at the end of period t is automatically deposited or, equivalently, invested in the zero-coupon bond bearing the interperiod (long-run) interest rate, $\hat{r}^k(t)$. Hence, money can serve as a profitable store of value.

An asset issued in country k can be bought only with k-money. Hence, the unique way for a household of country A to buy a security of country B

[38] Giraud and Tsomocos (2009) only dealt with intraperiod bonds.

[39] The difference, $r^k(t) - \hat{r}^k(t)$, can be interpreted either as a short-run bid-ask spread or as the difference between short-term and long-term rates. In order to authorize long term borrowing, one would need to introduce private banks. We here focus on the role of the Central Bank which only lends over short periods. On the other hand, in this stylized world with infinite horizon, permitting households to borrow money on the interperiod market would open the door for Ponzi schemes (obtained by financing an intraperiod debt by means of further interperiod borrowing.

[40] Permanent public deficit is not allowed in this paper. We leave this issue for further research.

[41] For simplicity, the Bank of country k does not have currency reserves in k'-currency. This could be added to our model without impairing the essence of our results.

is to pay in B-money for this asset. But if this agent does not already hold sufficiently outside B-money in her pocket, the unique way for her to get B-money consists in selling A-money on the corresponding foreign exchange (FX) market or else in borrowing B-money to the B-Central Bank. Even in this second case, this agent will have to pay back $e^{r^k(t)}$ times her loan in B-money. Unless she transacted only B-assets, she may need to go through the foreign exchange market in order to change part of her A-cash (obtained from trading A-assets) into B-money. Thus, there is room for opening FX markets.

Three types of demands for money therefore emerge in this setup. Money serves as a store of value. In addition, since there is uncertainty (captured by the filtration $\mathbf{F}$), agents hold money for speculative and precautionary reasons. When inside money is borrowed on the intra-day market at rate $r^k(t)$, we speak of the transactions demand for money, as discussed in Giraud and Tsomocos (2009): The higher the interest rate, the fewer infinitesimal transactions in securities will be sought, and —*ceteris paribus*— the lower will be the demand for inside money. The higher the nominal value of infinitesimal purchases desired (and the higher the money value of sales desired), the greater will be the demand for money, given the same intra-day rate. Finally, since countries A and B may set different interest rates, $r^A(t) \neq r^B(t)$, there will be an arbitrage among traders between borrowing money from their own domestic Central Bank or else from the foreign Central Bank. This arbitrage will result in the exchange rate $\pi^{AB}(t)$ between A and B-money.

We denote by $m_i^k(t)$ the quantity of k-money owned by household i at the beginning of time t. Depending upon its locally rational expectations, i decides to spend the amount $0 \leq \mu_i^k(t) \leq m_i^k(t)$ on various markets at time t, and to save $m_i^k(t) - \mu_i^k(t)$.

4.2 The time structure of markets

In a way similar to Giraud & Tsomocos (2009) and Dubey and Geanakoplos (2003)[42], and for the sake of clarity, in each period t, six steps meet in the following order:[43]

α) The cash gained from deposits comes due, i.e., i receives $e^{\hat{r}^k(t^-)dt}(m_i^k(t^-) - \mu_i^k(t^-)) \geq 0$, $k = A, B$ (where $\hat{r}^k(t^-)$ is the left-side limit $\lim_{\tau \to t, \tau < t} r^k(\tau)$).

β) the FX market where both currencies can be exchanged against each other;

γ) the inside monetary market, where agents and the Central Bank meet within each country k. There, agent i borrows the quantity $\tilde{m}_i^k(t) \geq 0$ to each k-Central Bank ($k = A, B$).

[42] See also Geanakoplos & Tsomocos (2002).

[43] This time structure is crucial for our results. In particular, a different analysis would emerge if FX markets were to open *after* the inside money market.

δ) Then, the domestic financial markets open, where securities issued in country k can be traded in k-money. The flow, $\dot{z}_i(t)$, captures the impact of infinitesimal trades on i's portfolio, $z_i(t)$.

ε) Intraperiod loans come due: Household i having borrowed $\tilde{m}_i^k(t)$ has to repay back $e^{r^k(t)dt}\tilde{m}_i^k(t)$ to the k-Central Bank, and this for each country k.

η) Inventoried money, $m_i^k(t) - \mu_i^k(t)$, is invested on the interperiod loan market.

At time t_δ, the profits of the k-Central Bank will be equal to $r^k(t)M^k(t)dt$. They are distributed at time $(t + dt)_\alpha$ by each Central Bank as interperiod-returns so as to balance its budget. Since the excess return of i's deposit (received at time $(t+dt)_\alpha$) in k-currency will be $\hat{r}^k(t)(m_i^k(t) - \mu_i^k(t))dt$, budget balancedness of the k-Central Bank means that $\hat{M}^k(t)$ is chosen so that the resulting interperiod interest rate, $\hat{r}^k(t)$, solves the following equation:

$$r^k(t)M^k(t) = \hat{r}^k(t)\Big(\overline{m}^k(t) - \overline{\mu}^k(t)\Big), \tag{11}$$

where the right-hand side is the total amount of interest earned by investors on the k-interperiod time t-loan market.

4.3 The foreign exchange market

An endogenous exchange rate, $\pi^{AB}(t)$, is determined at each time t_β as a foreign market clearing price between A and B currencies. The rate $\pi^{AB}(t)$ is the amount of A-currency that one unit of B-currency will buy at time t (of course, $\pi^{BA}(t) = 1/\pi^{AB}(t)$). Each household can participate in both sides of the FX market with her domestic currency or the foreign currency she has accumulated in previous periods. Let us denote by $0 \leq \kappa_i^k(t) \leq \mu_i^k(t)$ the part of $\mu_i^k(t)$ that household i is ready to change into k'-currency at time t_β. One more notation, $\rho^k(t) := \overline{\kappa}_i^k(t) - \pi^{kk'}(t)\overline{\kappa}_i^{k'}(t)$, and we are able to define the world quantity of outside k-money that will be available in order to borrow inside k-money at time t_β as: $\overline{\mu}^k(t) - \rho^k(t)$. The exchange rate, $\pi^{AB}(t)$, is then defined as the clearing market price on the FX market:[44]

$$\pi^{AB}(t) := \frac{\overline{\kappa}^A(t)}{\overline{\kappa}^B(t)}. \tag{12}$$

Now, the short-term intraperiod rate, $r^k(t)$, in country k is determined by the ratio between outside and inside money:[45]

$$r^k(t) = \frac{\overline{\mu}^k(t) - \rho^k(t)}{M^k(t)} \geq 0. \tag{13}$$

[44] $\frac{x}{0} := 0$.

[45] We refer to Giraud & Tsomocos (2009) for the mechanics of interest rates behind (13) (to be understood with the convention $x/0 := 0$).

For simplicity, (13) assumes that no individual exchanges k-money for k'-money at time t with no intention to spend it on commodity or security k'-markets at the very time t. We shall drop this restriction in subsection 4.8 infra. Thanks to (12), (13) simplifies to:

$$r^k(t) = \frac{\overline{\mu}^k(t)}{M^k(t)}. \tag{14}$$

The interperiod rate, $\hat{r}^k(t)$, is given by:

$$\hat{r}^k(t) = \frac{\hat{M}^k(t)}{\overline{m}^k(t) - \overline{\mu}^k(t)} \geq 0. \tag{15}$$

It follows from (11) and (15) that $r^k(t)M^k(t) = \hat{M}^k(t)$. Therefore, the quantity of k-currency, $r^k(t)M^k(t)$, that is taken away from the economy by the k-Central Bank will return to citizens in the form of interperiod excess returns. As a consequence, the global stock, $\overline{m}^k(t)$, remains constant and variations in the world amount of outside k-money available for trades, $\overline{\mu}^k(t)$, are solely due to the investors' expectations. An important point is that the k-cash received from the k-Bank as interperiod return will be received by k-citizens only at time $(t+dt)_\alpha$. Hence, although we do not explicitly impose a budget constraint on each Central-Bank, each government budget is almost always balanced. Indeed, at the end of period t, the k-Central Bank has a non-negative profit $r^k(t)M^k(t)$. At the beginning of period $t+dt$, this profit is entirely redistributed to his shareholders, and the government budget is balanced. "Between" t_α and t_δ, however, the government k "temporarily" incurs a deficit of $M^k(t)$ that will be immediately replenished once all the agents repay their borrowed inside k-money at t_δ.

4.4 Budget constraints

We denote by $q_k(t) \in \mathbb{R}_+^{2+L}$ the vector of prices at time t of assets issued in k, expressed in k-currency. The stock of outside k-money hold by agent i (belonging to k) must satisfy the following differential equation (where the variation $\dot{m}_i^k(t)$ is computed at time $(t+dt)_\alpha$:

$$\dot{m}_i^k(t) = e^{\hat{r}^k(t)}\Big[m_i^k(t) - \mu_i^k(t)\Big] + \pi^{kk'}(t)\kappa_i^{k'}(t) - \mu_i^k(t) - q_k(t)\cdot \dot{z}_i^k(t) \tag{16}$$

Finally, let us denote by $\tilde{m}_i^k(t)$ the amount of inside k-money borrowed by household i to Central Bank k. One gets:[46]

$$\tilde{m}_i^k(t) := \frac{\mu_i^k(t) + \pi^{kk'}(t)\kappa_i^{k'}(t) - \kappa_i^k(t)}{\overline{\mu}^k(t)} M^k(t), \tag{17}$$

[46] Notice that, for (17) to make sense, we need FX markets to open at t_α "before" households borrow inside money from Central Banks at t_β.

Of course, $\sum_{i\in k,k'} \tilde{m}_i^k(t) = M^k(t)$. Now, the budget set of household i is defined by:

(i) a short-sale constraint: $\dot{z}_i^k(t) \geq -\delta_i^k(t)$,

(ii) a cash-in-advance constraint in each currency:[47]

$$q_k(t) \cdot \dot{z}_i^{k+}(t) \leq \mu_i^k(t) + \pi^{kk'}(t)\kappa_i^{k'}(t) - \kappa_i^k(t) + \tilde{m}_i^k(t), \qquad (18)$$

(iii) a budget constraint given by the duty of fully delivering on one's loan in each currency k:[48]

$$(1 + r^k(t))\tilde{m}_i^k(t) \leq q_k(t) \cdot \dot{z}_i^{k-}(t) + \Delta(18),$$

where $\Delta(18)$ denotes the difference between the right- and the left-hand sides of inequality (18). Fortunately, equation (17) enables us to reduce the last budget constraint to the following more familiar one (in net trades):

$$q_k(t) \cdot \dot{z}_i^k(t) \leq 0 \quad \text{ for } k = A, B. \qquad (19)$$

4.5 A local random Quantity Theory of Money

Summing (18) over i, and using (12) yields the following (localized) version of Fisher's celebrated quantity theory of money, for every k:

$$\overline{\mu}^k(t) + M^k(t) \geq q_k(t) \cdot \Big(\sum_i \dot{z}_i^{k+}(t)\Big), \qquad (20)$$

Provided trades are effective —that is: $\sum_i \dot{z}_i^{k+}(t) \neq 0$—, and if a.e. consumer i verifies (18) as an equality, then (20) holds as an equality.[49] Notice that, in (20), "income" corresponds to the current value (at market prices) of infinitesimal trades, and not to initial endowments. Moreover, at variance with the textbook analysis of Fisher's equation, and apart from the quantity $M^k(\cdot)$ of inside money, (20) only involves endogeneous random variables: infinitesimal trades $\dot{z}^k(t)$, as well as prices $q(t)$ and available cash $\overline{\mu}^k(\cdot)$ in k-currency are all determined endogenously along a trade path. Being stated in the tangent bundle of τ, our quantity theory of money involves only flows (and no stock). Finally, the "velocity of money" is variable, and always greater than, or equal to, 1 in our theory. It equals 1 whenever (20) is

[47] $z^+ := z \vee 0$. Again, (18) presupposes that the FX market as well as the loan market for each Central Bank open "before" the security markets.

[48] $z^- := (-z) \wedge 0$. Here, $1 + r^k(t)$ stands for the first-order, linear approximation (in the tangent market $T_{\gamma(t)}\mathcal{E}$) of the cost $e^{r^k(t)}\tilde{m}_i^k(t)$.

[49] When no-trade prevails within country k, prices are indeterminate and the QTM breaks down.

binding, and can be interpreted asthe ratio between the rate of growth of circulating money and the rate of growth of the real economy. (Indeed, the speed of infinitesimal trades on the right hand-side of (20) also characterize the speed of growth of stocks $z_i^k(t) = z_i^k(0) + \int_0^t \dot{z}_i^k(t)dt$. Although they are oviously linked together, the velocity of money should therefore not be confused with the speed of trades, $\|\sum_i \dot{z}_i^{k+}(t)\|$, which is also endogenously determined.

4.6 The local interaction of investors

At each time t, the *state* of the monetary economy $\mathcal{E}$ contains both a "real" and a monetary part: $\gamma(t) := (z^k(t), \delta^k(t), m^k(t), \mu^k(t), M^k(t))_k$. The configuration space, $\mathcal{M} := \tau \times \left(\mathbb{R}_+^{2L+4}\right)^4 \times \mathbb{R}_+^{10}$, is the set of *feasible states* of our dynamics, i.e., of feasible allocations in assets, savings, and stocks of money $\gamma = (x, \delta, m, \mu, M)$ with $\sum_i m_i^k = \overline{m}^k$, $0 \leq \delta_i^k \leq z_i^k$ and $0 \leq \mu_i^k \leq m_i^k$, for every i, k.

4.6.1 Pseudo-flows

We begin with a generalization of the standard Walrasian equilibrium concept.

DEFINITION.[50] A monetary pseudo-flow of $T_{\gamma(t)}\mathcal{E}$ in (uncertain) state ω is a price, $q(t) = (q^A(t), q^B(t)) \in \left(\mathbb{R}_+^{4+2L}\right)^2 \setminus \{0\}$, a feasible 4-tuple of borrowed money $(\tilde{m}_i(t))_i \in \mathbb{R}_+^4$, and a feasible infinitesimal net trade in assets, $\dot{\mathbf{z}}(t) \in \left(\mathbb{R}_+^{4+2L}\right)^4$, such that:

(i) For every i, $q(t) \cdot \nabla V_i(z_i, \omega, t) = 0$ implies $\dot{\mathbf{z}}_i(t) = 0$. Moreover, if, for some k, $\overline{\mu}^k(t) > 0$, then $\tilde{m}_i^k(t)$ verifies for every i: $\tilde{m}_i^k(t) = (\mu_i^k(t)/\overline{\mu}^k(t))M^k(t)$.

(ii) For every i, $\dot{\mathbf{z}}_i(t)$ maximizes $\dot{z}_i(t) \cdot V_i(\dot{x})$ subject to the short-sale, cash-in-advance and budget constraints, for every k:

$$\dot{z}^k(t) \geq -\delta_i^k(t), \quad q^k(t) \cdot \dot{z}^k(t) \leq 0, \quad q^k(t) \cdot \dot{z}^{k+}(t) \leq \tilde{m}_i^k(t) + \mu_i^k(t) \qquad (21)$$

and for every asset c, $q_c^k(t) = 0 \Rightarrow \dot{z}_{i,c}(t) = 0 \quad$ every i.

(iii) For every asset c, $q_c^k(t) = 0$ implies that, for every i, $\frac{\partial V_i(z_i(t), \omega, t)}{\partial z_{i,c}(t)} = 0$.

If $\mu_i^k(t) = 0$ while $\overline{\mu}^k(t) > 0$, then i has no endowed money and can no more borrow k-money (because of (17)). Hence, the cash-in-advance constraint (18) implies that i is excluded from trades in currency k. If, in

[50] In the parlance of Giraud & Tsomocos (2009) the present definition corresponds to first-order pseudo-flow. Since, in the setting of this paper, every flow is of first-order, we skip this distinction.

addition, $\overline{\mu}^k(t) = 0$ but $M^k(t) > 0$ for both $k = A, B$, then $r^A(t) = r^B(t) = 0$, the cash-in-advance constraint vanishes and $(q, \dot{z})$ reduces to a pair of Walras allocations (in net trades) and price ratios of the linear economy $T_\gamma\mathcal{E}$ (cf. section 2 above). The same outcome obtains dually whenever $M^k(t) \to \infty$ while $\overline{\mu}^k > 0$, for both k. Then, indeed, $r^k \to 0^+$ and, at the limit, the final infinitesimal trades induced by a pseudo-flow are not different from the Walrasian net trades obtained in $T_\gamma\mathcal{E}$ in an idealized world without money at all, where prices only have the meaning of exchange rates between pairs of commodities. Finally, if $\overline{\mu}^k = M^k = 0$, no-trade is the unique outcome in country k.

LEMMA 1.— Under (C), (a) Every monetary tangent market $T_{\gamma(t)}\mathcal{E}$ admits a monetary pseudo-flow, and (b) every such pseudo-flow verifies: $q_k(t) \cdot \dot{z}_i^k(t) = 0, \quad$ a.e. i.

Proof. In order to simplify notations, we drop the time and ω indices.

(a) For every country k, if $\overline{\mu}^k = 0$ and $M^k = 0$, no-trade is the unique pseudo-flow. If $\overline{\mu}^k = 0$ and $M^k > 0$, a monetary pseudo-flow boils down to a "pseudo-equilibrium" (in the sense of Mertens (2003)) of $T_\gamma\mathcal{E}$, expressed in net trades. Existence of such pseudo-equilibria follows from Mertens (2003, Lemma 3). If, now, $\overline{\mu}^k > 0$, then all the traders i for whom $\mu_i^k = 0$ can be ignored since $\dot{z}_i^k = 0$ (by definition of a pseudo-flow). Consider the restriction of the linear economy $T_\gamma\mathcal{E}$ to those individuals i with $\overline{\mu}_i^k > 0$. If, in Definition 1(i), the cash-in-advance constraint $q^k \cdot \dot{z}^{k+} \leq \tilde{m}_i^k + \mu_i^k$ is temporarily omitted, then the part $(q^k, \dot{z}^k)$ of a monetary pseudo-flow reduces, once again, to a "pseudo-equilibrium" in net trades. Such a pseudo-equilibrium is defined up to a normalization constant $\lambda^k > 0$ of k-prices. It therefore only remains to check that we can choose λ so that $\lambda^k q^k \cdot \dot{z}^{k+} \leq \tilde{m}_i^k + \mu_i^k$ is fulfilled for every i. This is easy since, by construction, $\tilde{m}_i^k + \mu_i^k > 0$ for every i (in the restricted economy).

(b) According to part (ii) of the definition of a pseudo-flow, $q^c = 0 \Rightarrow \dot{z}_{i,c} = 0$. Thus, we can ignore commodities with zero price, i.e., we assume $q \gg 0$. It then follows from the short-run utility maximization of Def1.(ii) that $q \cdot \dot{z}_i = 0$. □

The equality $q_k(t) \cdot \dot{z}_i^k(t) = 0$ means that, along a trade path, an investor i is never forced to spend more money $\mu_i^k(t)$ than she initially decided to (for a.e. time t). Therefore, the ODE satisfied by each i's stock of k-currency, $m_i^k(\cdot)$, simplifies to (cf. (16)):

$$\dot{m}_i^k(t) = e^{\hat{r}^k(t)} \left[m_i^k(t) - \mu_i^k(t) \right] + \pi^{kk'}(t)\kappa_i^{k'}(t) - \mu_i^k(t). \tag{22}$$

Another noticeable consequence concerns international balances of trades. Here, indeed, the balance of trade surplus of A is given by:

$$\mathbf{B}(A, B, t) := \sum_{i \in B} q_A(t) \cdot \dot{z}_i^A(t) - \pi^{AB}(t) \sum_{i \in A} q_B(t) \cdot \dot{z}_i^B(t). \tag{23}$$

Lemma 1 therefore implies that $\mathcal{B}(A, B, t) = 0$. In our stylized setting, a country can never run balance of payment deficits during some time interval.[51]

4.6.2 Monetary flows

DEFINITION. A monetary flow $(\overline{q}, \dot{z})$ of $T_{\gamma(t)}\mathcal{E}$ is defined as follows:

> (a) As for the asset flow, $\dot{z}$, one applies "Mertens' algorithm":[52] Select any pseudo-flow $(q, \dot{z})$, next start again with the truncated economy restricted to zero-price assets $\{c \ : \ q_c = 0\}$, as long as this set is non-empty. Since there are finitely many assets, the algorithm must end.
>
> (b) As for prices, for each country k, fix them in the unit sphere, and choose the largest gauge parameter, $0 < \lambda^k \leq 1$, such that each individual i still verifies her cash-in-advance constraint with respect to $\lambda^k q^k$: $\lambda^k q^k(t) \cdot \dot{z}^{k+}(t) \leq \tilde{m}_i^k(t) + \mu_i^k(t)$.

Part (b) of the Definition is consistent with everyday practice of clearing houses consisting in maximizing the value of trades.

> LEMMA 2.— *Regardless of the partition of commodities chosen at every step, Mertens' algorithm produces the same final asset flow $\dot{z}$. Unless $\dot{z} = 0$ or $\overline{\mu} = 0$, the corresponding price $\overline{q} > 0$ is unique.*

It is worth noticing that, when $\dot{z} = 0$, Mertens' algorithm may end up with various price ratios, while, when $\overline{\mu}^k = 0$ and $M^k > 0$, the players' amounts of borrowed money, $\tilde{m}_i^k \geq 0$ and the price normalization factor $\lambda^k > 0$ may take various values compatible with $\sum_i \tilde{m}_i^k = M^k$ and $\lambda^k q^k \cdot \dot{z}^{k+} \leq \tilde{m}_i^k$.

Let $Z(T_{\gamma(t)}\mathcal{E})$ denote the unique asset flow of the monetary tangent market $T_{\gamma(t)}\mathcal{E}$, and $Q(T_{\gamma(t)}\mathcal{E})$ the set of associated prices. Given some initial conditions at time 0, our dynamics is defined by the pair of equations:

$$\dot{z}(t) = Z(T_{\gamma(t)}\mathcal{E}) \text{ and } q(t) \in Q(T_{\gamma(t)}\mathcal{E}). \tag{24}$$

> PROPOSITION 3.— Under (C), (24) admits a random trade path and price curve.

Proof. The proof mimics that of Proposition 1 above. The normalization of prices in the unit ball provides the needed upper-bound on prices. □

[51] See nevertheless section 4.2 below.
[52] See section VIII.A.Def. 5 in Mertens (2003).

5 An optimal random growth of national currencies

We now prove that, for the conclusion of Theorem 1 to hold in our monetary set-up, this requires national currencies to grow at a minimal speed. Since, above this speed a further acceleration would have no real impact but produces inflation alone (because of the local Quantity Theory of Money), this provides an optimal (random) growth of currencies.

5.1 Monetary local gains-to-trade

The monetary multi-national version of $\chi(\cdot)$ (cf. section 3.2 supra) goes as follows: (i) If $\overline{\mu}^k > 0$ for some k, then consider the linear economy obtained by ignoring, in $T_\gamma\mathcal{E}$, those traders i with $\mu_i^k = 0$. Denote by $\chi^{*k}(\delta)$ the measure applied on the short-sale constraints of those traders (of both countries) who are positively endowed with k-money. k-monetary local gains-to-trade are then defined as: $\chi^k(z) := \chi^{*k}(\delta)$.[53] (ii) If $\overline{\mu}^k = 0$, then $\chi^k(z) := \chi^k(\delta)$.

The next Lemma is the key result for Theorem 2 to follow, but is also interesting in its own right. Its proof is in the Appendix.

LEMMA 3.—Under (C), (i) for every trade path $\varphi(\cdot)$, if it verifies $\dot{\varphi}^k(t) = 0$, then one of the following must be true:

(a) Either $\chi^k(z(t)) = 0$,
(b) or $r^k(t) > \chi^k(z(t))$ for both $k = 1, 2$,
(c) else, $\delta^k(t) = 0$.

(ii) If $r^k(t) > \chi^k(z(t))$ for both $k = 1, 2$, the unique monetary flow is no-trade.

The previous Lemma tells us under which conditions no trade will occur on a tangent market. Three situations may be identified: Either (a) investors do not believe that there are gains to trade –that is, the current allocation of assets put on the market as a result of agents' expectations is already Pareto-optimal. Or (b) the cost of borrowing inside money is too heavy in comparison with current gains-to-trade as envisaged by agents –i.e., the cash-in-advance constraint prevents from trading agents who, otherwise, would be willing to do so. Else (c), investors are so pessimistic that they refuse to trade and save all their assets. Lemma 1 (ii) provides a partial converse: If the current k-interest rate is above the threshold provided by current gains-to-trade (given households' savings), then no-trade in k-currency must occur.

[53]They are said to be local because they only depend upon the local geometry of households'preferences. Notice, however, that $\chi^k(z)$ embraces the world virtual trade opportunities and is not country-specific but currency-specific. (Remember that a citizen of country A may possess B-currency as well as assets issued in country B.

5.1.1 Convergence with money

We are now ready to state the central result of this paper. Its proof is given in the Appendix.

THEOREM 2.—Under (C) and (B), for every feasible initial state $z(0) \gg 0$,

(i) Suppose that the two following conditions are satisfied:

(a) Expectations are such that, for every i, k, $z^k_{i,j}(t) > 0 \Rightarrow \delta^k_{i,c}(t) > 0$ for every asset c, and almost every t;
(b) For every $k = 1, 2$, $t \mapsto M^k(t)$ grows sufficiently rapidly, so that:

$$M^k(\phi_t\omega) \geq \frac{\overline{\mu}^k(t)}{\chi(z(t))}(\phi_t\omega), \quad \text{a.e. } t, \forall\omega, \tag{25}$$

then the conclusion of Theorem 1 obtains, i.e., every trade curve converges in probability to some approximately constrained interim efficient allocation for T large enough.

(ii) On the contrary, if the length of time where markets in country k are not liquid enough verifies:[54]

$$\lambda\Big[\Big\{t \;:\; \chi^k(z(t)) < \frac{\overline{\mu}^k(t)}{M^k(t)} = r^k(t)\Big\} \cap \big\{t \;:\; \overline{\mu}^k(t) > 0\big\}\Big] > 0, \tag{26}$$

then, at some time t^*, the flow of $T_{\gamma(z(t^*))}\mathcal{E}$ coincides with no-trade, and the state will rest on $z(t^*)$ as long as the ratio of inside to outside money does not increase sufficiently so as to verify (25) for a subset of time $t \geq t^*$ of positive measure.

Remark 6. The condition $\chi^k(z(t)) < r^k(t)$ will open a liquidity trap in country k only whenever, at the same time, $\overline{\mu}^k(t) > 0$. Otherwise, we know indeed that the monetary flow degenerates to some Walras equilibrium which becomes independent from the monetary sector. Equation (25) provides an optimal random growth rate of money that need be verified by each of the two countries if we want markets to function properly. Notice that if country A is experiencing a "black-out", the markets of B may still remain alive.

This result provides us with an optimal random rate of growth for inside money. Unlike Friedman's famous "Golden rule", this rate depends on households' expectations but also on the real fundamentals of the economy. Notice, in particular, that each country's optimal rate depends on the real

[54]Here, $\lambda(\cdot)$ is the Lebesgue measure.

parameters of the whole world economy. Theorem 2 also shows that currencies are not neutral, neither in the long-, nor in the short-run. In the short-run, because when an country is trapped in a liquidity hole, a supplementary injection of inside money can induce new (infinitesimal) trades (provided it suffices to reduce the intraperiod k-interest rate, $r^k(t)$, below the local gains-to-trade). In the long-run, because an increase of inside k-money from $M^k(t)$ to $M^k(t) + \varepsilon$ at some time t will have an impact, at least later on (say, at time $\tau > t$), whenever local gains-to-trade will have sufficiently diminished in order to fall below $\overline{\mu}^k(t)/M^k(t)$. At this time, without the additional increment ε, the economy would have been stuck in some liquidity trap. Thanks to ε, this event is postponed. Since $M^k(t)$ must grow to infinity for Θ to be eventually reached, this time τ will come sooner or later (provided T is sufficiently large) — which proves that the increment ε has an impact in the long-run. This non-neutrality property, however, should not be confused withy money-illusion: One readily sees from the definition of a monetary pseudo-flow that a proportional increase of $m_i^k(t)$, for all i, and $M^k(t)$ does not affect trade paths. And it has no effect on prices apart from the obvious rescaling. That is, if EU switches from Euros to cents while England sticks to pounds and the U.S.A. to dollars, then only European prices and Euro exchange rates will change.

5.2 Comparative advantages

As a Classical economist, Ricardo adhered to the labor theory of value. As a result, he used labor hours contained in one unit of a good as a measure of cost, and calculated comparative cost accordingly. Hecksher, Ohlin and Samuelson[55] then provided a rewriting of this theory in terms of factor proportions. In our pure-exchange setting, these standard ideas can be reconsidered by replacing ratios of factor proportions or labour hours with marginal rates of utility substitution. Consider the following textbook example:

	Cheese (1 pound)	Wine (1 gallon)
Country A	2 gallon wine	0.5 pound cheese
Country B	1.25 gallon wine	0.8 pound cheese

It can be translated into our set-up by considering a linear world economy with two commodities, no financial security, where country A is populated by a single representative household $u_A(x, y) := 2x + y$, while citizens in country B are also identical: $u_B(x, y) := 1.25x + y$ (x stands for cheese and y for wine). Traditional wisdom asserts that A should sell some gallons of wine to country B for extra cheese whereas country B sells cheese to country A for extra wine. In the barter version of this paper (section 2), this is obviously confirmed (although it may be that, due to the linearity of preferences, B

[55]Cf. Ohlin (1933) and Stolper & Samuelson (1941).

does not gain — nor does it loose— anything from opening its frontiers to A). Does this story hold water within our monetary transition to equilibrium with heterogenous agents ?

Suppose that agent i exchanges $\kappa^A(t)+\varepsilon$ A-money for B-money in order to deposit $\varepsilon > 0$ on the B-interperiod loan market instead of using it on B-commodity markets. Then, according to (12), the floating exchange rate will adjust to:

$$\tilde{\pi}^{BA}(t) = \frac{\overline{\kappa}^B(t)}{\overline{\kappa}^A(t)+\varepsilon} := \pi^{BA}(t) - \eta.$$

Each individual's h budget constraint in B-money (19) will stay unchanged, but combining the last equation with (14) and (17), her cash-in-advance constraint in B-money (18) becomes:

$$q_B(t) \cdot \dot{z}_h^{B+}(t) \leq \mu_h^B(t) + (\pi^{BA}(t) - \eta)\kappa_h^A(t) - \kappa_h^B(t) + \tilde{m}_h^B(t) - \eta\frac{\kappa_h^A(t)}{r^B(t)},$$

provided $r^B(t) > 0$. For η sufficiently large (or $r^B(t)$ sufficiently small), this implies that the cash-in-advance contraint will prevent from trading in B-currency any agent h for whom $\kappa_h^A(t) > 0$. In particular, for A-citizens, $\kappa_h^A(t)$ is very likely to be positive, so that these consumers won't be able to trade in B-money. Therefore, international movements of capital may challenge the traditional conclusion of "comparative advantages".

Prima facie, this is not related to any trade imbalance since, as already seen, countries always have a zero balance of (net) trade surplus. A little reflection, however, suggests that this is due to the failure of our definition (23) to capture what is going on. One might suspect, indeed, that whenever speculative capital movements occur, the balance of trade surplus of country A with respect to country B at time t should be nonzero.[56] This conclusion would obtain by defining A's balance of trade surplus slighlty differently:

$$\mathcal{B}(A, B, t) := \sum_{i \in B} q_A(t) \cdot \dot{z}_i^{+A}(t) - \pi^{AB}(t) \sum_{i \in A} q_B(t) \cdot \dot{z}_i^{+B}(t). \qquad (27)$$

With such a definition in hand, a country may run a balance of trade deficit (or a surplus). If an agent exchanges ε units of her domestic currency A for B-money, and does spend this additional B-money at time t, this will raise the A-balance of trade surplus as defined by (27). Contrary to the finite-horizon setting of Geanakoplos & Tsomocos (2006), here there is no terminal date from which a backward induction argument would enable to deduce that a balance of trade deficit today must result into a balance of trade surplus in the future.

The analysis conducted in this paper could be runned without assuming that no consumer exchanges money on the FX-market for speculative

[56] As it is the case in Geanakoplos & Tsomocos (2006).

purposes, as we did so far through equation (13). As already alluded to, this would lead to a more complicated cash-in-advance constraint. Hence, additional restrictions on the exchange rate, $\pi^{AB}(\cdot)$, would be needed for Theorem 2, in order to ensure that no currency is artificially overvalued at the price of making trades in this currency too costly for foreigners.

6 Appendix

Proof of Theorem 1.

Not surprisingly, the proof consists in extending Lyapounov's second method to our set-valued random dynamical system set-up. We follow Arnold & Schmalfuss (2001). Let A be a φ-invariant random set. $\mathcal{V} : \Omega \times \mathbb{R}^{16+8L} \to \mathbb{R}_+$ is a Lyapounov function for A under φ if:

(i) $\mathcal{V}(\cdot, z)$ is measurable $\forall z \in \mathbb{R}^{16+8L}$, and $\mathcal{V}(\omega, \cdot)$ is continuous $\forall \omega \in \Omega$.

(ii) $\mathcal{V}$ is uniformly unbounded, i.e., $\lim_{\|z\| \to \infty} \mathcal{V}(\omega, z) = +\infty \; \forall \omega$.

(iii) $\mathcal{V}(\omega, z) = 0$ for $z \in A(\omega)$ and $\mathcal{V}(\omega, z) > 0$ for $z \notin A(\omega)$.

(iv) $\mathcal{V}$ is strictly decreasing along orbits of φ not in A:

$$\mathcal{V}(\phi_t \omega, \varphi(t, \omega, z)) < \mathcal{V}(\omega, z), \forall t \in (0, T], z \notin A(\omega).$$

Lemma 0. Under (C), $\mathcal{V}(\omega, \varphi(t, \omega, z)) := -\sum_i \nabla_z V_i(\omega, \varphi(t, \omega, z)) \cdot \dot{z}_i(t)$, is a Lyapounov function for Θ under φ.

Proof. Properties (i) and (ii) are obvious. (iii) and (iv) follow from an ω-adaptation of Lemma 2 in Champsaur & Cornet (1990), which states that, under (C):

$$\sum_i \nabla_z V_i(\omega, \varphi(t, \omega, z)) \cdot \dot{z}_i(t) \begin{cases} = 0 & \text{if } z \in \Theta, \\ > 0 & \text{otherwise.} \end{cases}$$

Indeed, let $(\dot{z}, q)$ be a Walras equilibrium of $T_z\mathcal{E}$. From the duality theorem, one gets, for every i (with obvious notational simplifications):

$$\begin{aligned} 0 &= \nabla^k V_i(z_i) \cdot \dot{z}_i \\ &= -\nabla^k V_i(z_i) \cdot z_i + q \cdot z_i \max \left\{ \frac{\nabla^k V_i(z_i)}{q_k} \mid \text{ for any asset } k \right\}. \end{aligned}$$

Consequently,

$$q_k \geq \left[\frac{q \cdot z_i}{\nabla^k V_i(z_i) \cdot z_i} \right] \nabla_i^k V_i(z_i)$$

for every i and any asset k such that $\nabla^k V_i(z_i)\cdot z_i > 0$. But, as preferences are strictly monotone, the boundary condition on utilities and the fact that asset returns are positive ensure that this latter condition is verified. It remains to check that the above inequalities are in fact equalities for each asset k such that $z_i^k > 0$. Suppose the contrary for some pair (i,k), multiply each inequality by z_i^k, and sum over i in order to get a contradiction.
Assumption (B) then ensures that Θ is φ-invariant since $\varphi_\Theta = \mathrm{Id}_\Theta$.

□

Theorem 1 then follows from Theorem 6.5 in Arnold & Schmalfuss (2001)). Its Corollary follows from Proposition 4.4 (loc. cit.).

Incidentally, Lemma 0 implies the following Proposition, which shows that when shifting from exact optimality to approximate optimality, (B) implies the intuitive Bayesian dynamical consistency.[57]

Proposition 0.— Under (C) and (B), for any $\varepsilon > 0$, the set

$$C_\varepsilon := \mathrm{cl}\big\{z \in \tau \mid \mathcal{V}(\omega, z) < \varepsilon\big\}$$

is a forward invariant compact random set.

Proof. It follows from Lemma 6.3. (i) in Arnold & Schmalfuss (2001) that the auxiliary set

$$D_\varepsilon := \mathrm{cl}\big\{z \mid \mathcal{V}(\omega, z) < \varepsilon\big\}$$

is forward invariant with respect to ϕ. Since τ is also random compact, so is $C_\varepsilon = \tau \cap D_\varepsilon$. For $z \in C_\varepsilon(\omega)$, we have $0 \leq \mathcal{V}(\omega, z) < \varepsilon$, hence $0 \leq \mathcal{V}\big(\phi_t\omega, \varphi(t,\omega,z)\big) < \varepsilon$ for any $t \in (0,T]$ by property (iv) of a Lyapounov function. This says that $\mathcal{V}^{-1}\big(\phi_t\omega, [0,\varepsilon)\big) \subset \mathrm{int} C_\varepsilon(\phi_t\omega)$ for any $t \in (0,T]$.

□

In words: When gains to trade are small *interim*, the onfolding of information *per se* does not increase them.

Proof of Lemma 2.

(i) Suppose that, in country k, $\gamma^k(x,\omega,t) \geq r^k(\omega,t) \geq 0$, and that nevertheless the monetary flow of $T_{\gamma(t)}\mathcal{E}$ involves no-trade. Then, for every household i, the cash-in-advance constraint in k-currency $q^k(t)\cdot \dot{z}_i^{k+}(t) \leq \tilde{m}_i^k(t) + mu_i^k(t)$ is trivially satisfied, whatever being people's initial endowment in money as well as the factor $\lambda^k(t) > 0$ chosen by the clearing house in order to fix the price level in country k. Imagine therefore that the k-clearing house commits to set $\lambda^k(t) > 0$ and let $\lambda^k(t) \to +\infty$. Since, at a monetary flow, $q^k(t) \gg 0$ by definition, this means that $q^{k\lambda}(t) := \frac{\lambda^k(t) q^k(t)}{\|q^k(t)\|} \to +\infty$ as well. As a consequence, the purchasing power of the endowed k-money $m_i^k(t)$ as well as that of the k'-money $\kappa_i^k(t) \leq m_i^{k'}(t)$ that i may wish to change into k, both go to zero and may be ignored. At the limit, the trading opportunity

[57] $\mathrm{cl}X$ designates the topological closure of X.

on k-assets for any household is to purchase assets solely out of the borrowed money and to pay back the loan, $\tilde{m}_i^k(t)$, at the intraperiod interest rate $r^k(t)$, out of her sales revenues – conducting all infinitesimal trades in k-assets at the limiting price ratios obtained as $\lambda^k(t) \to \infty$. To be more precise, consider $q^{k*}(t)$ given by:[58]

$$q^{k*}(t)_c := \lim_{\lambda^k(t) \to \infty} \frac{q_c^{k\lambda}(t)}{\sum_{d \in k} q_d^{k\lambda}}, \quad \text{every asset } c,$$

and let us denote by $B_i(q^{k\lambda}(t), \mu^{k\lambda}(t), r^k(t))$, the budget set of agent i defined by those infinitesimal k-trades $\dot{z}_i^k(t) \geq -\delta_i^k(t)$ such that the cash-in-advance and the no-default constraint (21) are satisfied. As shown by Lemma 3 in Giraud & Tsomocos (2009), this is equivalent to the non-linear budget constraint:

$$q^{k\lambda}(t) \cdot \dot{z}_i^{k+}(t) + \frac{1}{1 + r^k(t)} q^{k\lambda}(t) \cdot \dot{z}_i^{k-}(t) \leq \mu_i^k(t).$$

The budget set being therefore homogeneous with respect to $(q^{k\lambda}, \mu^k)(t)$, one has:

$$B_i(q^{k\lambda}(t), \mu^{k\lambda}(t), r^k(t)) = B_i\Big(\frac{q^{k\lambda}(t)}{\|q^{k\lambda}(t)\|_{\ell_1}}, \frac{\mu^k(t)}{\|q^{k\lambda}(t)\|_{\ell_1}}, r^k(t)\Big)$$

where $\|q^{k\lambda}(t)\|_{\ell_1} := \sum_{c \in k} q_c^{k\lambda}(t)$. For any ω, as $\lambda^k(t) \to \infty$, one has the set convergence (e.g., for the Hausdorff metric) of this budget set towards $B_i(q^{k*}(t), 0, r^k(t))$. If, now, given, $\dot{z}_i^{k'}(t)$, $\dot{z}_i^k$ is $\nabla^k V_i(z_i(t))$-optimal in $B_i\Big(\frac{q^{k\lambda}(t)}{\|q^{k\lambda}(t)\|_{\ell_1}}, \frac{\mu^k(t)}{\|q^{k\lambda}(t)\|_{\ell_1}}, r^k(t)\Big)$, its limit, $\dot{z}_i^k \to 0$, must be $\nabla^k V_i(z_i)$-optimal in $B_i(q^{k*}(t), 0, r^k(t))$. (This argument is possible because each agent faces two separate budget- and cash-in-advance constraints for each currency.) In the same way as in Theorem 2 of Dubey & Geanakoplos (2003a), this is tantamount to performing standard Walrasian trades in k-assets at $q^{k*}(t)$ but consuming only the fraction $1/(1 + r^k(t))$ of purchases. In turn, a change of variable shows that this may be viewed as performing the whole Walrasian net trades via modified utilities $v_i^{r^k(t)}(\cdot)$ defined as follows:

$$v_i^{r^k(t)}(\dot{z}_i^k(t)) := \nabla^k V_i(z_i) \cdot \big(\dot{z}_i^{k'}(t), \dot{z}_i^k(r^k(t))\big),$$

where $\dot{z}_i^k(r^k(t))$ is defined as in (9). Thus, no-trade is a Walras allocation in k-assets for the whole world population $(v_i^{r^k(t)})_i$ at k-prices $q^{k*}(t)$, and must be Pareto-optimal (in the short-run, i.e., with respect to infinitesimal reallocations within the tangent market $T_{\gamma(t)}\mathcal{E}$) wrt $(v_i^{r^k(t)})_i$. Since $r^k(t) \leq \chi^k(z(t))$, 0 is also Pareto optimal in the short-run with respect to $(v_i^{\gamma^k(z(t))})_i$. But we know from Lemma 2 in Dubey & Geanakoplos (2003a) that there are no local gains to $\chi^k(z(t))$-diminished k-trades in $T_{\gamma(t)}\mathcal{E}$ if, and only if, the (concave

[58] The sum in the denominator is taken over all the k-assets.

but non-linear) economy $(v_i^{\chi^k(z(t))})_i$ has a no-trade Walras equilibrium. This contradicts the gains-to-trade hypothesis $\chi^k(x) > 0$. So, no-trade cannot be a monetary flow of $T_{\gamma(t)}\mathcal{E}$.

(ii) is a direct consequence of Theorem 6 in Dubey & Geanakoplos (2003a). There, it is proven that, under the stated condition, no individual can have effective trades since she could then be able to improve her short-run welfare by slightly perturbing her trades. The same argument shows, here, that the unique monetary flow must be no-trade. Details are left to the reader. □

Proof of Theorem 2. Lemma 2 enables to prove, again, that $\mathcal{V}(\cdot)$, as defined in Lemma 0 supra, is still a random Lyapounov function. Theorem 2 then follows from Theorem 1. □

References

[1] Alvarez, F., A. Atkeson, & P. Kehoe (2007) "Time-Varying Risk, Interest Rates, and Exchange Rates in General Equilibrium", Federal Reserve Bank of Minneapolis Research Department Staff Report 371.

[2] Arnold, L. (1998) *Random Dynamical Systems*, Springer-Verlag, Berlin, Heidelberg, New-York.

[3] Arnold L. & B. Schmalfuss (2001) "Lyapunov Second Method for Random Dynamical Systems", *Journal of Differential Equations* 177, 235-265.

[4] Becker, G. (1962) "Investment in Human Capital: A Theoretical Analysis," *Journal of Political Economy* LXX, 9 - 49.

[5] Bhatia, N. P. & G.P. Szego (1970). *Stability Theory of Dynamical Systems*, Collection: Grundlehren der mathematischen wissenschaften in einzeldarstellungen (161) Springer.

[6] Bouchaud, J.-P., M. Mezard & M. Potters (2002) "Statistical Properties of the Stock Order Books: Empirical Results and Models", *Quantitative Finance* 2, 251-256.

[7] Bottazzi, J.-M. (1994) "Accessibility of Pareto Optima by Walrasian Exchange Processes", *Journ. of Math. Economics*, 23, 585-603.

[8] Champsaur, P., and B. Cornet (1990) "Walrasian Exchange Processes", in: Gabszewicz, J.-J., Richard, J.-F., Wolsey, L.A. (eds.) *Economic Decision Making: Games, Econometrics and Optimizaiton.* Amsterdam: Elsevier.

[9] Clower, R. (1967) "A Reconsideration for the Microeconomic Foundations of Monetary Theory", *Western Economic Journal*, 6, 1-8.

[10] Drèze, J. & H. Polemarchakis (1998) "Money and Monetary Policy in General Equilibrium", in L.-A. Gérard-Varet, A. P. Kirman & M. Ruggiero (eds.), *Economics, the Next Ten Years*, Oxford, Oxford University Press.

[11] Drèze, J. & H. Polemarchakis (1998) "Intertemporal General Equilibrium and Monetary Theory", in A. Leijonhufvud (ed.), *Monetary Theory as a Basis for Monetary Policy*, Macmillan.

[12] Drèze, J. & H. Polemarchakis (2000) "Monetary Equilibrium", in G. Debreu, W. Neuefeind & W. Trockel (eds.) *Economics Essays — A Festschrift for Werner Hildenbrand*, Springer.

[13] Dubey, P. & J. Geanakoplos (2003) "Inside and Outside Money, Gains to Trade and IS-LM", *Economic Theory* 21, 347-397.

[14] Duffie, D, & C. Huang (1985) "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities", *Econometrica* Vol. 53, 1337-1356.

[15] Duffie, D, & W. Shaffer (1986): Equilibrium in Incomplete Financial Markets: II Generic Existence in Stochastic Economies, *Journal of Mathematical Economics* 15, 199-216.

[16] Farmer, J. D. (2002) "Market force, ecology and evolution,"in Industrial and Corporate Change, Oxford University Press, vol. 11(5), pages 895-953, November.

[17] Filippov, A.I. (1988) *Differential Equations with a Discontinuous Right-hand Side*, Kluwer Academic Publisher.

[18] Fisher, F. (1983) *Disequilibrium Foundations of Equilibrium Economics*, Cambridge, UK & New York: Cambridge University Press.

[19] Geanakoplos, J. (1990) "Introduction to General Equilibrium with Incomplete Asset Markets," *Journal of Mathematical Economics* 19 (1-2), 1-38.

[20] Geanakoplos, J. & J. D. Farmer (2009) "The Virtues and Vices of Equilibrium and the Future of Financial Economics," *Complexity* 14(3), 11-38.

[21] Geanakoplos, J. & J. D. Farmer (2009) Hyperbolic Discounting is Rational: Valuing the Far Future with Uncertain Discount Rates , Cowles Foundation Discussion Paper.

[22] Geanakoplos, J. & L. Gray (1991) "When Seeing Further is not Seeing Better", Bulletin of the Santa Fe Institute, 6(2).

[23] Geanakoplos, J. & D. Tsomocos (2002) "International Finance in General Equilibrium", *Research in Economics*, 56, 85-142.

[24] Geanakoplos, J. & J. Sebenius (1983) Dont Bet On It: A Note on Contingent Agreements with Asymmetric Information, *Journal of American Statistical Association*, 78 (382), 224-226.

[25] Giraud, G. (2003) "Strategic Market Games: an Introduction", *Journ. of Math. Econ.*, 39, 355-375.

[26] Giraud, G. (2008) "The Limit-Price Exchange Process", PSE WP, Université Paris-1.

[27] Giraud, G. & D. Tsomocos (2009) "Nominal Uniqueness and Money Non-neutrality in the Limit-Price Exchange Process ", forthcoming in *Economic Theory*.

[28] Glosten, L. R. & P. Milgrom (1985) "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 14(1), pages 71-100.

[29] Gode, D. K. & Sunder, S. (1993). "Allocative Efficiency of Markets with Zero Intelligence Traders: Market as a Partial Substitute for Individual Rationality." *Journal of Political Economy* 101-1, 119-137.

[30] Hart, O. (1975) "On the Optimality of Equilibrium When the Market Structure Is Incomplete," *Journal of Economic Theory* XI, 416-43.

[31] Ohlin, B. (1933) *Interregional and International Trade*, Harvard University Press.

[32] Herings, P.J.J. (1996), "Static and Dynamic Aspects of General Disequilibrium Theory", in *Theory and Decision Library Series C: Game Theory, Mathematical Programming and Operations Research*, Kluwer Academic Publishers, Dordrecht.

[33] Holmström, B. & Myerson, R. (1983) "Efficient and Durable Decision Rules with Incomplete Information", *Econometrica*, 51, 1799-1819.

[34] Karatzas, I. (1996) *Lectures on the Mathematics of Finance*, CRM monograph Series 8, AMS.

[35] Lucas, R. (1972) "Expectations and the Neutrality of Money", *Journ. of Economic Theory* 4, 103-124.

[36] Lucas, R.E. Jr. (1982). "Interest Rates and Currency Prices in a Two-Country World." *Journal of Monetary Economics,* 10, 335359.

[37] Lucas, R.E. Jr. (1990). "Liquidity and Interest Rates." *Journal of Economic Theory,* 50, 237264.

[38] Mertens, J.-F. (2003) “The Limit-Price Mechanism”, *Journ. of Math. Economics*, 39, 433-528.

[39] Mike, S. & J.D. Farmer (2008), “An Empirical Behavioral Model of Liquidity and Volatility”, *Journal of Economic Dynamics and Control* 32, 200-234.

[40] Milgrom P. & Stokey, N. (1982) “Information, Trade and Common Knowledge”, *Journ. of Econ. Theory*, 26, 17-27.

[41] Odean, T. (1999) “Do Investors Trade Too Much?,” *American Economic Review* 89, 1279-1298.

[42] Papageorgiou, N. (1988) “Random Fixed Points and Random Differential Inclusions”, *Intern. Journ. of Math. & Math. Sci.*, 11(3), 551-560.

[43] Pavlova, A., & R. Rigobon (2003) “Asset Prices and Exchange Rates”, mimeo.

[44] Peiris, U. & D. Tsomocos (2009) “International Monetary Equilibrium with Default”, Saïd Business School WP.

[45] Sandas, P. (2001) Adverse Selection and Competitive Market Making: Empirical Evidence from a Limit Order Market, *Review of Financial Studies* 14(3), 705734.

[46] Shiller, R. (1981) “Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?” *American Economic Review* 71, 421-436.

[47] Shiller, R. (1987) “Ultimate Sources of Aggregate Variability,” *American Economic Review* 77.

[48] Smale, S. (1976) “Global Analysis of Pareto Optima and Price Equilibria under Classical Hypotheses”, *Journ. of Math. Economics*, 3, 1-14.

[49] Smale, S. (1976a) “Dynamics in General Equilibrium Theory”, *American Economic Review*, 66, 288–294.

[50] Smale, S.(1976b) “Exchange Processes with Price Adjustment”, *Journ. of Math. Econ.*, 3, 211–226.

[51] Stolper, W. & Samuelson, P., ‘Protection and Real Wages”, *Review of Economic Studies*, 9, p. 58-67.

[52] Weyl, G. (2007) “Is Arbitrage Socially Beneficial?” Princeton University, mimeo.