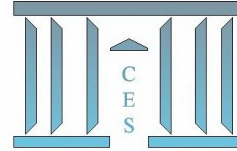




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## Financial Crashes versus Liquidity Trap : the Dilemma of Monetary Policy

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# Financial Crashes versus Liquidity Trap: the Dilemma of Monetary Policy

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**ABSTRACT.**— This paper considers a two-period monetary double auction with incomplete markets of securities and derivatives. Players may share heterogenous beliefs. Short positions in derivatives are constrained by collateral requirements. A central Bank stands ready to lend money or engage in unconventional monetary policy such as quantitative easing. In sharp contrast with the usual picture of equilibrium properties, I show that only three scenarios are compatible with the Nash equilibrium condition: 1) either the economy enters a liquidity trap in the first period ; 2) or the money injected by the Central Bank fuels a financial inflation driven by “rational exuberance”, whose burst leads to a global crash in the next period, 3) else a significant inflation of commodity prices accompanies the functioning of markets. In particular, neither Friedman’s golden rule, nor the Taylor rule turn out to be compatible with the third scenario: Both inevitably lead to a liquidity trap. An example shows that quantitative easing does not provide, in general, any escape from the monetary dilemma.

**Keywords:** Central Bank, Gains to trade, Liquidity trap, Collateral, Default, Crash, Taylor rule, Deflation, Bubble, Rational exuberance, Heterogenous belief.

**JEL Classification Numbers :** D50, E40, E44, E50, E52, E58, G38, H50.

## 1 introduction

The Great Moderation lulled macroeconomists and policymakers alike in the belief that we knew how to conduct macroeconomic policy. The current crisis, started in 2007, forces us to question the assessment. In this paper, I provide a general equilibrium monetary set-up where some elements of the pre-crisis consensus can be

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reviewed, in order to identify what tenets of the pre-crisis framework still hold, and to take a tentative first pass at the contours of a diagnosis of the current crisis.

For simplicity, let us consider a two-period economy with finitely many states of Nature in the second period, finitely many types of households, a Central Bank and no private banking sector. There are financial markets where investors can raise money and trade derivatives in order to hedge themselves against risk. Investors may share heterogeneous beliefs about the uncertain states of Nature.<sup>1</sup> Short positions in financial assets must be secured by durable collaterals. Loans are non-recourse and there is no utility penalty for defaulting. Whenever the face value of the security is higher than the value of the collateral, the seller of the security can choose to default. In this case, however, I assume that, instead of receiving the face value of the security, the security buyer seizes the collateral *plus* part of the cash hoard by the seller. It is only when the face value of the collateral together with the cash holding of the seller are still lower than the face value of the security that a default is actually recorded. To the best of my knowledge, all the previous literature devoted to default dealt with cashless economies, and therefore assumed that, absent of utility penalty, a seller can default without further consequences.<sup>2</sup>

#### *The monetary dilemma*

The main argument driving the result can be informally stated as follows: The need to improve the efficiency of trades calls for an increase on the quantity of money injected in the economy by the Central Bank. The impact of such an increase of money essentially depends upon the agents' expectations. If investors trust that there won't be enough money in the next period (relatively to the current one), then the economy enters a global liquidity trap: the short-term interest rate shrinks to zero while real cash balances held by households increase with no effect on the real economy. This is, roughly speaking, the *deflationary scenario*.

The alternative goes as follows: If, on the contrary, they anticipate that the Central Bank will pump in enough money in the second period, agents go on trading in commodities and assets. The impact of this increase of monetary liquidity, however, is ambiguous. If the leverage ratio on financial markets is small<sup>3</sup>), then a sufficiently large additional quantity of money will significantly raise commodity prices.<sup>4</sup> This is the *inflationary scenario*. By contrast, if the leverage ratio is high enough, the additional quantity of money injected may fuel inflation on assets and collaterals, eventually resulting in a financial crash. This is the *crash scenario*.

<sup>1</sup>However, they are assumed to coordinate on a Nash equilibrium, hence, to have perfect forecasts about future equilibrium prices and interest rates.

<sup>2</sup>See, e.g., Zame (1993), Kubler & Schmedders (2001), Geanakoplos & Zame (2002), Fostel and Geanakoplos (2008), Araujo, Kubler & Schommer (2009), to name but a few. Barrett (2000) considers collateral requirements in an economy where investors may have incorrect forecasts about future prices (which is not allowed here).

<sup>3</sup>Equivalently, if margin requirements are high.

<sup>4</sup>In Dubey & Geanakoplos (2003a), it is shown that adding a small (relatively to the money already available) amount of cash in the economy may reduce the equilibrium level of prices, while a large monetary increment has the standard inflationary impact. I shall not consider this issue in the present paper, and will focus on large monetary increments.

Let us briefly explain why a crash may occur at equilibrium when most of the liquidity injected by the Central bank migrates towards financial markets. Recall that investors have heterogenous beliefs and that the prices of assets and collaterals are given by the marginal utility of marginal buyers. When the most optimistic agents are wealthy, they are most likely to be the marginal buyers, so that the price of securities and collaterals will be higher. Conversely, when the most optimistic are poor, the marginal purchaser of securities and collaterals is more likely to be someone less optimistic, so that prices on security and collateral markets will be lower. The injection of further quantity of money by the Central Bank reduces the cost of money, hence leverages purchasing power, and thus drives up the security and collateral prices. When the state of nature is revealed (in the second period), bad news may be announced, and margin purchasers may end up poorer than they would otherwise have been. Indeed, the inflation on the security market leaves them with a larger debt and a larger fraction of their wealth in securities and collaterals. Because the most optimistic are poorer, the most pessimistic are more likely to become the marginal buyers of assets in the second period. The price of collaterals may then be driven down by the (low) marginal utility of the pessimists with respect to collaterals. There is a shift of ownership of collaterals, from optimists to pessimists, with its attendant shift of prices which provokes a margin crash.<sup>5</sup>

#### *Rational exuberance*

The higher was inflation on financial markets, the deeper will be the crash in the bad state of the second period. What makes this phenomenon compatible with our standard rational expectations framework is the assumption that investors share heterogenous beliefs. The 2007-2009 crisis highlighted the role of belief heterogeneity and how financial markets allow investors with different beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds took advantage from the securities by short-selling them. One reason for why there has been little attention paid to belief heterogeneity is Friedman's (1953) celebrated market selection hypothesis: On the long-run, there should be little differences in beliefs because agents with wrong beliefs should be driven out of the market by those who share correct beliefs.<sup>6</sup> Following Cao (2010), I show, on the contrary, that collateral requirements prevent the market forces from driving out investors with wrong beliefs. As a consequence, belief heterogeneity survives at equilibrium and can fuel what is called, in this paper, "rational exuberance": some investors, being convinced that an asset is undervalued by the market, will keep buying it, hence further driving its market price. Combined with a high leverage ratio and a lax monetary policy, I show by means of an example (section 2), that such a rational exuberance may lead to a global crash in some uncertain state of the

<sup>5</sup>For a first statement of this impact of collateral constraints on price volatility, see Geanakoplos & Zame (2002). For further work, see Kubler & Schmedders (2001), Kiyotaki & Moore (1997), Geanakoplos (2001).

<sup>6</sup>See also Blume & Easley (2006) and Sandroni (2000).

second period. The main departure with Cao (2010) is the introduction of money within a pure-exchange setting, while Cao (2010) considers a cashless production economy with heterogenous beliefs.

One by-product of this analysis is that, in the absence of a liquidity trap, what enables inflation over commodities to remain bounded despite the increase of liquidity, is the fact that most of the injected liquidity migrates towards financial markets, thus fueling some rational exuberance phenomenon. The example of section 2 below exhibits an economy where this happens for a high enough leverage ratio. This provides a theoretical narrative for the Great Moderation of inflation experienced by Western countries for the last two decades that contrasts sharply with the conventional wisdom. Furthermore, this example shows that a constant level of domestic prices is compatible with an increase of the quantity of money injected by the Bank, together with a huge inflation both on the financial markets and the market for collaterals. This happens, e.g., under the condition that the leverage ratio increases at the same speed as the quantity of money injected in period 0. The reason why collaterals are not immune against inflation is that they play a dual role: they are used both for the intrinsic value and as collaterals. The larger is the financial rational exuberance, the more attractive are collaterals.

Before entering into the details of the argument, let us immediately mention which remedy can be thought of for this unfortunate dilemma which sheds some light about current debates on most Central Banks' non-conventional policies. We do not offer any magic alternate solution. However, our model points in the following direction. In order to avoid the threat of the liquidity trap, the Bank should convince economic actors that it will not contract its monetary policy in the future. Long-term interest rates emerge as a good instrument for this purpose (see the discussion at the end of section 2). However, quantitative easing in the sense of outright asset purchasing can turn out to be ineffective in order to fight against the liquidity trap (see subsection 2.4 *infra*). A non-conventional policy designed to affect the yield curve at longer-than-usual horizons therefore seems unavoidable. On the other hand, in order to get rid of financial rational exuberance, regulatory authorities should reduce the leverage power of financial derivatives. What will be the upshot of such a policy mix ? Since the reduction of leverage on financial markets will make them less attractive, the quantity theory of money implies that it will but induce domestic inflation on consumption commodities. Therefore, a supposedly fostered inflation eventually emerges as a *pis-aller* in order to avoid both deflation and financial crashes.

The paper is organized as follows. The next section exhibits an example where all the conclusions of this paper are present. Section 3 develops the general model. In section 4, I provide some general properties of monetary equilibria such as the survival of traders with wrong beliefs, an endogenously determined quantity theory of money, and the non-arbitrage relations within the yield curve. Section 5 is devoted to proving the alternative faced by monetary authorities : either they refrain from injecting money in the economy, at the cost of leaving an inefficient monetary equilibrium take place; or, they pump in a virtually infinite quantity of money but

then, in order to escape from a liquidity trap, they must convince the economic actors that they will still inject a lot of money in the future. If they succeed in this non-conventional task, then they face two alternative risks: either a huge domestic inflation (when the leverage ratio on financial markets is low) or a financial rational exuberance (when the leverage ratio is high) whose burst may induce a global collapse of the economy in at least one second-period state of nature. A concluding section discusses the results of the paper in light of the 2007-09 crisis. The proof of the existence theorem is relegated to the Appendix. There, the double auction underlying our model is made explicit.

## 2 An example

The next example can be seen as a monetary version of example 8 in Geanakoplos & Zame (2002). It exhibits a situation where the three scenarios alluded to in the Introduction can occur. In particular, a global crash occurs with positive probability provided the leverage ratio is sufficiently large on financial markets.

### 2.1 The monetary economy

Consider an exchange economy with two dates, 0, 1, two states  $g$  (good) and  $b$  (bad) in the second period, two goods,  $F$  (food) and  $S$  (stock), and two agents 1 (the wealthy pessimist) and 2 (the poor optimist). Each type of investor is represented by a continuum of clones having identical characteristics. Thus, each agent takes the macrovariables (prices and interest rates) as given.<sup>7</sup>

Only the stock can serve as collateral. Expected utility functions are:

$$u^1(F_0, S_0; (F_g, S_g), (F_b, S_b)) := F_0 + \frac{1}{2}(F_g + 7.33S_g) + \frac{1}{2}(F_b + 2S_b),$$

$$u^2(F_0, S_0; (F_g, S_g), (F_b, S_b)) := F_0 + .9(F_g + 7.33S_g) + .1(F_b + 6S_b).$$

Obviously, agents have heterogenous beliefs. Endowments are:

$$\mathbf{e}^1 = ((40, 4); (40, 0), (40, 0)); m^1 = (2; 0, 5.49).$$

$$\mathbf{e}^2 = ((24, 0); (7, 0), (6.41, 0)); m^2 = (2; 3.39, 1.22).$$

The quantity of outside money owned by  $h$  in state  $s$ , free and clear of debt, is  $m_s^h \geq 0$ . Agent 1 has no monetary endowment in state  $g$ . A Central Bank stands ready to lend or borrow money by buying or selling IOUs from the agents. For simplicity, only two kinds of IOUs are considered throughout this paper. A short-term IOU of state  $s$  is traded at the beginning of that state and promises €1 at its end. The long-term IOU is traded at the beginning of state 0 and promises €1 just before commodities trade in every future state  $s \in S$  in period 1.

In period 0, the Central Bank injects  $M_0$  on the short-term loan market and  $M_{\bar{0}}$  on the long-run one. In the second-period state  $s$ , it injects  $M_s$  on the short-term market. When the Bank borrows money by selling IOUs, it temporarily reduces the

<sup>7</sup>Throughout the paper, I focus on type-symmetric equilibria.

stock of money. But when the IOU comes due, it returns the borrowed money, and in addition creates more outside money to pay the interest.<sup>8</sup> One may suppose the Bank has quantity targets and pre-commits to the size of its borrowing or lending, letting interest rates be determined endogenously at equilibrium. This would be faithful to the quantitative “pillar” of the ECB, inspired by Friedman’s celebrated “Golden rule”. Alternatively, one may as well suppose that the Bank has interest rate targets, and precommits to supplying whatever money or IOUs are demanded at those rates. This would reflect the ECB’s second pillar, as well as most contemporary Central Banks’ mandate, in the line of Taylor’s rule. I shall consider both policies in the sequel.

Consumption goods can be stored. For simplicity, there is no capital depreciation.<sup>9</sup> Food is perishable and stock is durable: When consumed in period 0, food cannot be inventoried into period 1 while stock can be consumed in period 0, stored into period 1 and consumed in this last period.

I assume that there is no default on the short-term loans market.<sup>10</sup>

Suppose that there is an asset,  $A$ , which promises delivery of  $\beta$  times the price of one unit of food in each state, collateralized by a unit of stock. In a barter framework (where money is only implicit), the actual delivery of one unit of asset in state  $s \in S$  would be:<sup>11</sup>

$$A_{\beta}^s := \min\{\beta p_s^F; p_s^S\}.$$

Here, I assume instead that, whenever agent  $h$  defaults, not only is her collateral forfeit but also (part of) her cash. It is only when the value of the collateral plus the cash available to the borrower do not suffice to fulfill her promise that default is registered.

In period 0, first agents trade IOUs with the Central Bank in order to borrow and lend money both on the short- and long-run monetary markets. Second, they trade commodities and assets. Third, they repay back their debt on the short-term loan market. In state  $s = g, b$ , at time 1, first agents deposit or borrow money on the state- $s$ -short-term loan market. Second, the long-term IOUs from the previous period deliver. Third, agents trade commodities. Fourth, they repay back their debt on the short-term loan market and the financial asset delivers. Fifth, the agents consume. In the fourth step, if necessary, the cash obtained through the sale of commodities in the previous step of the same period can be used in order to finance the assets’ return.

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<sup>8</sup>This may be viewed as one of the origins of outside money in this model, inherited from some unmodelled past.

<sup>9</sup>It would be only a notational matter —here and in the sequel of the paper— to allow for capital depreciation and to let this depreciation depend upon the second-period state of nature.

<sup>10</sup>This is in conformity with what we currently observe. On the Repo market, for instance, where there is day-to-day lending, there is virtually no default and even in crisis periods, like 2007-2010 or 1998 or 1994, the rate of default remained hardly significant.

<sup>11</sup>See, e.g., Geanakoplos & Zame (2002).



## 2.2 Monetary equilibria

The utility maximization programme of agent  $h$  can be conveniently written as:

$$\begin{aligned} & \max_{\theta^h, x^h \geq 0} u^h(x^h) \\ \text{s.t. } & p_0 \cdot (x_0^h - \mathbf{e}_0^h) + q_\beta \theta^h + \frac{r_0}{1+r_0} \mu_0^h + \tilde{\mu}_0^h \leq m_0^h \end{aligned} \quad (1)$$

$$p_s \cdot (x_s^h - \mathbf{e}_s^h) + \frac{r_s}{1+r_s} \mu_s^h \leq m_s^h + \mathbf{A}_\beta^s \theta^h + (1+r_s) \tilde{\mu}_0^h + \Delta(1) \quad \forall s = g, b \quad (2)$$

where  $\theta^h \in \mathbb{R}$  is  $h$ 's portfolio,  $q_\beta$  is the asset's price, and  $\mathbf{A}_\beta^s$  stands for the actual monetary delivery of one unit of asset in state  $s \in S$ , given by:

$$\mathbf{A}_\beta^s := \min \left\{ \beta p_s^F; p_s^S - p_s \cdot (x_s^h - \mathbf{e}_s^h) - \frac{r_s}{1+r_s} \mu_s^h + m_s^h + (1+r_s) \tilde{\mu}_0^h + \Delta(1) \right\}.$$

Notice that (2) does not imply that a borrower is forced to sell commodities in state  $s$  whenever the value of the collateral,  $p_s^S$ , falls below the value of the promise,  $\beta p_s^F$ . It simply says that, if the latter happens and if the borrower has sold some commodities in state  $s$ , then the cash she received against these sales will be forfeited in order to pay her debt. Of course, at equilibrium, no agent will sell any commodity of which she derives a positive utility in a state where she defaults.

In (1), the bid-ask spread<sup>12</sup> is given by

$$\sigma_0 := r_0 / (1 + r_0). \quad (3)$$

A *monetary equilibrium* is defined as a collection  $(p_s, r_s, (\theta^h, x_s^h)_h)_{s \in S^*}$  of prices, interest rates, portfolios and commodity allocations such that each  $(\theta^h, (x_s^h)_s)$  solves  $h$ 's utility maximization problem subject to (1) and (2), and all markets clear:

$$\sum_h \theta^h = 0, \text{ (asset market clearing)}$$

$$\sum_h (x_s^h - \mathbf{e}_s^h) \leq 0, \quad \forall s \in S, \text{ (commodity market clearing)}$$

$$\sum_h (\mu_s^h + \tilde{\mu}_0^h) \leq (1 + r_s) M_s \quad \forall s, b, \text{ (money market clearing at } t = 1)$$

$$\sum_h (\mu_0^h - \tilde{\mu}_0^h - \tilde{\mu}_0^h) \leq (1 + r_0) M_0. \text{ (money market clearing in state 0).}$$

The parameter  $\beta$  is exogenous. In practice, margin requirements (which, in the US, are set by the Federal Reserve) are usually expressed in terms of a cash down

<sup>12</sup>Obviously, instead of imposing a cash-in-advance constraint—which is sometimes viewed as being an artificial way to introduce money—we could as well start with a bid-ask spread,  $\sigma_s$ , for each state  $s$ , and link it with the Bank's policy through (3). Cf. Duffie (1990) for an approach in terms of bid-ask spread. In the present model, the two viewpoints are entirely equivalent provided the spread can be linked with the nominal interest rate via some equation akin to (3) (and not, say, with the volume of trades).



payment as a fraction of the sale price. The margin requirement on the asset  $A_\beta$  is then  $\frac{(1+r_0)p_{0s}-q_\beta}{(1+r_0)p_{0s}}$ . We shall also see that  $\beta$  drives the leverage ratio in the financial market.

Some useful remarks before embarking in the analysis of this example:

1) In either the interest or quantity target policies,  $r_s \geq 0 \forall s \in \{0, b, g\}$ , and  $r_{\bar{0}} \geq r_0$ . Indeed, if  $r_s < 0$  in state  $s$ , the households could infinitely arbitrage the Central Bank. If  $r_{\bar{0}} < r_0$ , then we must be in the quantity target model with  $M_0 > 0$ . But then, nobody would borrow on the short-term loan market, which would therefore not clear.

2) When  $r_s > 0 \forall s$ , then agents spend all the money at hand on purchases: Indeed, they can deposit money they do not intend to spend (or else borrow less), receiving the money back with interest, before they face the next buying opportunity.<sup>13</sup>

3) As a consequence, by summing over  $h$  the (binding) cash-in-advance constraints in every state, we get a quantity theory of money:<sup>14</sup>

$$p_0 \cdot \sum_h (x_0^h - e_s^h)^+ + q_\beta \sum_h \theta^{h+} \leq \frac{1}{1+r_0} M_0 + \frac{1}{1+r_{\bar{0}}} M_{\bar{0}} + \sum_h m_0^h. \quad (4)$$

The difference between the right- and the left-hand sides of (4) is precisely the amount of cash inventoried by households from time 0 to time 1. In (4), the implicit velocity of money can be lower than 1, depending upon the saving behavior of agents on the long-run deposit market. When there is no inventoried cash, (4) is satisfied as an equality (and money velocity = 1).

4) There is no worthless cash at end:  $\Delta(2) = 0$ . Otherwise, agent  $h$  could borrow (a little) more on  $r_s$ , use the additional cash to buy some commodity, and would still have enough money in order to pay back her debt.

5) If  $r_{\bar{0}} > r_0$ , for any monetary equilibrium, there is another equilibrium leaving prices, interest rates and consumptions untouched, but where no agent deposits on short-loans, and no one deposits and borrows simultaneously on the long-loan market.<sup>15</sup> From now on, I shall always consider such an equivalent monetary equilibrium.

6) If  $r_{\bar{0}} > r_0$ , any monetary equilibrium admits an equivalent one where each inventorying agent deposits exactly  $m_0^h$  and borrows a positive amount of money on the short-loan market, i.e.,  $\mu_0^h > 0$ .

7) If agent  $h$  borrows some money on the long-loan market (or, else, inventories money from period 0 into period 1), then the first-order condition on the long-loan market requires that

<sup>13</sup>This is Lemma 2 in Dubey & Geanakoplos (2006).

<sup>14</sup> $x^+ := \max\{x, 0\}$ .

<sup>15</sup>This is Lemma 4 in Dubey & Geanakoplos (2006a); the argument is not repeated here.

$$\frac{\nabla_{0x}^h}{p_{0x}} = (1 + r_{\bar{0}}) E^h \left[ \frac{\nabla_{sx}^h}{p_{sx}} \right],$$

provided she buys commodity  $x$  in every state  $s \in S^*$  —where  $E^h[\cdot]$  is the expectation operator over period 1-uncertain states (computed according to  $h$ 's subjective beliefs) and  $\nabla_{sx}^h$  stands for the partial derivative of  $u^h$  with respect to state- $s$  commodity  $x = F_s, S_s$ .

8) The optimist will always find it advantageous to buy stock entirely on margin (because the cost of buying the asset in order to purchase the stock will never exceed the cost of the stock itself at time 1). Hence, the number of units of security sold by agent 2 is equal to the number of units of stocks she buys.

9) The relative price of the stock with respect to food in the good state will always be  $p_{gS}/p_{gF} = 10/(1 + r_g)$ . Indeed, suppose that agent  $h$  buys some shares and sells some food in state  $g$ . Then, denoting  $\tilde{\nabla}^h$  (resp.  $\bar{\nabla}^h$ ) the left-hand (resp. right-hand) derivative of  $u^h(\cdot)$ , one must have:

$$\frac{\tilde{\nabla}_S^h}{p_{gS}} \geq (1 + r_g) \frac{\bar{\nabla}_F^h}{p_{gF}},$$

otherwise  $h$  would do better by reducing (by a little) both her purchase of Stock and her sale of Food. But since there are only two households, the reverse inequality must hold for  $h$ 's partner.

10) The relative price of the stock in the bad state will be at least  $p_{bS}/p_{bF} \geq 2/(1 + r_b)$ .

11) The pessimist (i.e., the lender or, equivalently, the security-buyer) is risk-neutral and indifferent about the timing of food consumption, so that the price  $q_\beta$  of the asset  $A_\beta$  will be exactly the pessimist's expectation of its delivery. On the other hand, the relative value of collateral (stock) with respect to food in the good state being always  $10/(1 + r_g)$ , the optimist (i.e., the borrower or asset-seller) will never default in the good state as long as  $r_g \leq 9$ . Thus, whenever  $r_g \leq 9$ ,

$$q_\beta = \begin{cases} \beta p_{bF} & \text{if } \beta \leq \frac{p_{bS}}{p_{bF}} \\ \frac{p_{0F}}{2} (\beta + \frac{p_{bS}}{p_{bF}}) & \text{if } \beta > \frac{p_{bS}}{p_{bF}}. \end{cases}$$

12) The pessimist will not buy any stock in the bad state if  $p_{bS} > 2p_{bF}$ .

13) If the optimist does not inventory any money from period 0 into period 1, then, relative to  $p_{0F}$ , her implicit marginal utility of stock at date 0 is at least  $(1 + r_0)$  times her expectation of the stock price at date 1. Hence, her marginal utility of stock is at least  $9.2p_{0F}(1 + r_0)$ . If she does inventory money (or deposit cash on the long-run loan market), then, it is at least  $9.2p_{0F}(1 + r_{\bar{0}})$ .

14) If the pessimist does not inventory money, her implicit marginal utility of stock is  $(1 + r_0)$  times her expectation of the stock price at time 1. Hence, it is at

most  $8p_{0F}(1 + r_0)$ . Similarly, if she does transfer money from period 0 into period 1, then, it is at most  $8p_{0F}(1 + r_{\bar{0}})$ .

Now, we shall see that, whenever the leverage ratio,  $\beta$ , two main scenarios emerge: either *all* the households do believe, at equilibrium, that the Central Bank will inject a lot of money in period 2 (so that prices tomorrow will be high relatively to today's prices, and tomorrow's interest rate relatively low), in which case a rather "normal" functioning of markets is to be expected.<sup>16</sup> Else, there is at least one type of households which does not trust the Central Bank when it claims that it will lower the short-term interest rate in every second period-state. Then, the economy must fall in a liquidity trap whenever  $M_0$  exceeds a certain threshold.

If, on the contrary,  $\beta$  is large enough, then the unique escape road from the previous alternative obtains whenever the money injected by the Central Bank into the economy at time 0 fuels a bubble on the asset market, whose burst in the bad state induces a global collapse of the whole economy in that very state.

### 2.3 The monetary dilemma

Suppose, to begin with, that  $\beta = 1$ , and that the Central Bank targets  $r_0 = 9\%$ ,  $r_{\bar{0}} = 20\%$ , and  $r_b = 10\%$ . The unique equilibrium is:

$$\begin{aligned} p &= (1, 8; 1, 7.33; 1, 6), \quad M_0 \sim 44, \quad M_{\bar{0}} \sim 0.82, \quad \text{and} \quad M_b = 67.1, \\ x^1 &= (64, 0.61; 40, 0; 46.41, 0), \\ x^2 &= (0, 3.39; 7, 0; 0, 4). \end{aligned}$$

The asset price is  $q_\beta = \beta = 1$  and there is no default in period 1. At date 0, the optimist sells all her food and borrows to buy stock on the margin. But she cannot afford to buy all the stock; the pessimist keeps the remaining stock. Suppose that the optimist does not borrow money on the long-loan market. Her non-linear budget identity becomes:

$$8\theta^2 - \frac{1}{1 + r_0}(\theta^2 + 24) = 2.$$

Thus, the number of shares bought by the optimist in period 0 is  $\theta^2 = \frac{26+2r_0}{7+8r_0} \sim 3.39$ .

In the bad state, the optimist spends  $\beta\theta^2 p_{bF} = \theta^2$  to repay her risky loan on the asset market, gets  $\theta^2$  shares of stock that had been held as collateral, spends  $p_{bS}(4 - \theta^2)$  to purchase the remaining  $4 - \theta^2$  shares of stock, and sells part of her initial endowment in food in order to finance her purchases. But her budget constraint is:

$$\beta\theta^2 + 6(4 - \theta^2) - \frac{1}{1 + r_b}x_{bF}^2 = 1.22$$

which yields  $x_{bF}^2 \sim 6.41 = e_{bF}^2$ . Thus, having sold her entire endowment in food, the optimist has no additional income. Moreover, it is rational for her not to bor-

<sup>16</sup>As we focus on rational expectations equilibria, then what people expect to occur must happen, so that there will indeed be a high inflation in every second-period state.

row money on the long-run loan market, because she would incur an additional transaction cost that she cannot afford.

At the date 0 price vector of (1, 8), the pessimist is exactly indifferent between stock and food if she borrows money on the long-loan market. Indeed, her marginal utility for stock is 7.33 in the good state, and the stock will sell for 6 in the bad state, an event to which she assigns probability .5. According to remark 7 in the previous subsection, given  $r_{\bar{0}} = 1.2$ , this makes the discounted relative price,  $8/(1 + r_{\bar{0}})$ , of stock in period 0, equal to its expected value,  $1/2 \times 7.33 + 1/2 \times 6$ . The pessimist faces the following budget constraint in period 0:

$$\theta^2 + 24 - \frac{1}{1 + r_0} 8\theta^2 = 2 + \frac{1}{1 + r_{\bar{0}}} \mu_{\bar{0}}^1,$$

whence, she borrows  $\mu_{\bar{0}}^1/(1 + r_{\bar{0}}) \sim 0.82$  on the long-loan market. Her budget constraint in the bad state is therefore binding:

$$\beta\theta^2 + e_{bF}^2 - \frac{6}{1 + r_b} (4 - \theta^2) - \mu_{\bar{0}}^1 \sim 5.49 = m_b^1$$

Now, let us allow  $\beta$  to vary. For  $\beta$  large enough, the optimist must default entirely in the bad state whenever she buys a fixed quantity,  $\theta^2 > 0$ , of shares in period 0. Indeed, the total quantity of money available in the whole economy in the bad state being bounded, there is a threshold,  $\bar{\beta}$ , above which the promise,  $\theta^2\beta p_{0F}$ , will exceed this upper-bound.

If the optimist defaults in the bad state, she will deliver her whole portfolio,  $\theta^2$ , instead. Does she have an incentive to sell some food? For  $\beta$  large enough, the answer is no since the money she will acquire by selling her initial endowment in food will be used to repay her financial debt. Thus, the optimist should rather keep her endowment, forcing the pessimist to experience an even larger loss.

Is such a scenario compatible with the equilibrium condition? Let us show the answer to be yes by computing a concrete example:

Take  $\beta > 2$ ,  $p = (p_{0F}, p_{0S}, q_{\beta}, p_{gF}, p_{gS}, p_{bF} = 1, p_{bS} = 2)$ ,  $r_0 = 9\%$ ,  $r_{\bar{0}} = 0\%$ ,  $r_b = 10\%$ ,  $M_0 \sim 44.4$ ,  $M_{\bar{0}} = 0$  and  $M_b = 67.1$ .

$$x^1 = (64, 0; 40, 0; 40, 4)$$

$$x^2 = (0, 4; 7, 0; 6.41, 0).$$

In period 0, the budget constraint of the optimist is

$$4p_{0S} - \frac{1}{1 + r_0} (24p_{0F} + 4q_{\beta}) = 2. \quad (5)$$

The budget constraint of the pessimist is

$$4q_{\beta} + 24p_{0F} - \frac{4p_{0S}}{1 + r_0} = 2, \quad (6)$$

and the quantity theory of money yields:

$$24p_{0F} + 4q_\beta + 4p_{0S} = 4 + M_0,$$

provided that the long-term market remains inactive. Of course,  $q_\beta = \frac{p_{0F}}{2}(\beta + 2)$  and  $r_0 = \frac{4}{M_0}$ .

A simple computation yields:

$$p_{0S} = \frac{2(1 + r_0)}{4r_0}. \quad (7)$$

$$p_{0F} = \frac{2(1 + r_0)}{r_0(28 + 2\beta)}. \quad (8)$$

$$\frac{p_{0S}}{p_{0F}} = 7 + \frac{\beta}{2}. \quad (9)$$

Recall that an investor who borrows  $D$  and invests  $A = K + D$  in an asset faces a leverage ratio  $\ell := D/K$ . In our set-up, the leverage ratio is given by

$$\ell = \frac{q_\beta}{(1 + r_0)p_{0S} - q_\beta} = \frac{\beta + 2}{12 + r_0(14 + \beta)}.$$

For a fixed  $\beta$ , the impact of increasing the quantity of money,  $M_0$ , is transparent: as  $M_0$  grows to infinity,  $r_0$  shrinks to 0,  $\ell \rightarrow (\beta + 2)/12$ , and both  $p_{0F}, p_{0S}$  and  $q_\beta \rightarrow +\infty$ . However, for this to be compatible with the equilibrium conditions, the quantity of money injected in period 1 must increase as well, at least in the good state. Indeed, suppose that  $M_0 \rightarrow +\infty$  but  $M_g$  remains fixed. This means that prices in the good state will remain constant, say, equal to  $p_g = (1, 10)$ . But then, for  $M_0$  high enough, the sale of a quantity,  $\varepsilon > 0$ , of food in period 0 will enable each agent to save enough money into period 1 to be able to buy the whole aggregate endowment of commodities in the good state. This contradicts the equilibrium condition. Thus, *either*  $M_g$  increases proportionately to  $M_0$ , so that prices in the good state also increase to infinity, *or* the economy falls into a liquidity trap in period 0. In the latter case, there is a threshold,  $\overline{M}_0$ , such that, for every  $M_0 \geq \overline{M}_0$ , the short-term interest rate hits its floor,  $r_0 = 0$ , and the additional money,  $M_0 - \overline{M}_0$ , is hoarded by the agents at time 0, but remains unused (and flows back to the Central Bank at the end of period 0 at no cost).

Next, for a fixed  $M_0$  (or, equivalently, a fixed  $r_0$ ), if  $\beta \rightarrow +\infty$ , then the price of the asset,  $q_\beta \rightarrow 1/2r_0$ ,  $\ell \rightarrow 1/r_0$ , and  $p_{0S}$  remains constant while  $p_{0F} \rightarrow 0^+$ . Thus, increasing the leverage ratio while keeping the quantity of circulating money constant induces a deflation on the domestic sector. This is the phenomenon of migration of liquidity towards the financial market, due to its increasing attractiveness. At the limit, *all* the (constant quantity of) money is captured by the financial market. Now, the injection of further quantity of money,  $M_0$ , by the Central Bank reduces the cost,  $r_0$ , of money, hence leverages purchasing power, increases the leverage ratio,  $\ell$ , and drives up the security and collateral prices.

In which sense can the price,  $q_\beta$ , be interpreted as a bubble ? Since the two agents have heterogenous beliefs, the very definition of a bubble becomes ambiguous. According to the pessimist's viewpoint, the asset price remains correct whatever being the parameters  $\beta$  and  $r_0$ . According to the optimist, the asset's subjective value is  $p_{0F}(0.9\beta + 0.2)$ , so that the asset is viewed as being undervalued at time 0. The gap between the optimist's assessment and the market value being  $p_{0f}(0.4\beta - 0.3)$ , it will increase up to  $0.4/r_0$  as  $\beta$  grows to infinity, making the asset all the more attractive. This phenomenon cannot be interpreted as irrational exuberance, as both investors have rational expectations, but is clearly due to the heterogeneity of beliefs. Let us call this "rational exuberance". We shall see in section 4 below, that investors having heterogenous beliefs cannot be driven out of the market, so that "rational exuberance" is something we must live with. It is worth noticing that, in this example, an increase of  $M_0$  induces an increase of the maximal rational exuberance phenomenon (measured by  $0.4/r_0$ ).

Finally, if, say,  $\beta = 1/r_0$ , then:  $q_\beta$  and  $p_{0S}$  still explode, but  $p_{0F} = 1$ . This suggests that, whenever the leverage ratio increases at a speed similar to that of the quantity of circulating money, then, this additional money fuels rational exuberance on the financial market ( $q_\beta$  grows to infinity), but leaves domestic prices untouched (the price of food in period 0 is constant), while only the price of the collateral explodes (as did the housing market prices between 2001 and 2006). This means that the deflationary effect due to the migration of liquidity towards financial markets can be compensated by a lax monetary policy. As for the leverage ratio,

$$\ell = \frac{\beta^2 + 2\beta}{13\beta + 14},$$

it is increasing in  $\beta = M_0/4$ .

## 2.4 Quantitative easing

In order to escape from the crux highlighted by the previous example (inflation/liquidity trap/crash), the Central Bank may engage in quantitative easing (as the Banks of England and Japan, and the Federal Reserve did after 2009).<sup>17</sup> Recast in our set-up, such an unconventional policy consists in: either targeting the long-term interest rate,  $r_{\bar{0}}$ , or lending extra money by *buying* the asset  $A_\beta$  in period 0.

Let us begin with the first interpretation of quantitative easing. Manipulating  $r_{\bar{0}}$  clearly has an effect in our model, as soon as the long-term markets are active at equilibrium. This means that the usual explanation for the restriction of conventional policies to the short end of the yield curve —namely, that the determination of longer-term interest rates can be left to market mechanisms through no-arbitrage arguments— does not hold water in our setting: equilibrium conditions do not enable, in general, to deduce  $r_{\bar{0}}$  from  $r_0, (r_s)_s$ . Thus, there is room for a policy that

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<sup>17</sup>Cf. Meier (2009).

affects the yield curve at longer-than-usual horizons. No-arbitrage, however, does impose the following relationship within the yield curve:<sup>18</sup>

$$(1 + r_{\bar{0}}) \geq (1 + r_0) \min_{s \in S} (1 + r_s).$$

So that an increase of  $M_{\bar{0}}$  must imply, in general, an increase of  $M_s$  in at least one state.<sup>19</sup> Thus, this first version of quantitative easing may succeed in circumventing the liquidity trap but at the cost of forcing the Central Bank to commit to a high inflation in at least one future state. Consequently, playing with longer-than-usual interest rates amounts to shifting to our scenario 1, where inflation prevails in one state or another.

Let us turn to the second interpretation of quantitative easing. To keep the analysis simple, suppose that the Central Bank no more offers money on the long-term market but rather offers to buy the asset  $A_\beta$  against fresh money.

Clearly, when  $\beta > 2$ , this would have no effect on the equilibrium: the optimist already borrows to the pessimist the needed amount of money in order to purchase the 4 units of stock available in period 0. Hence, the optimist holds already the maximal amount of collateral and there is no additional collateral to secure any additional loan.

When  $\beta \leq 2$ , the picture is more interesting. Absent of such a quantitative easing policy, the optimist cannot borrow enough from the sole pessimist to buy all the stock at time 0, so that the price  $p_{0S}$  is the pessimist's expectation of the date 1 price of stock, i.e.,

$$p_{0S} = \frac{1}{2}(10 + p_{bS}).$$

The quantity theory of money at time 0 is:

$$24p_{0F} + \theta^2 \beta p_{bF} + p_{0S} \theta^2 = 4 + M_0.$$

If the monetary policy in period 1 is kept fixed,  $p_{bF}$  and  $p_{bS}$  are fixed. Hence,  $p_{0S}$  remains constant as well, so that, when  $M_0 \rightarrow +\infty$ ,  $p_{0F}$  must increase until the liquidity trap is reached.

Suppose, therefore, that the Central Bank buys  $A_\beta$  in place of the pessimist (who saves her money for a better use). If the quantity of fresh money thus injected is large enough, the optimist will now be able to buy *all* 4 shares of stock on margin at date 0. The budget identity of the optimist in period 0 is:

$$4p_{0S} - \frac{1}{1 + r_0}(24p_{0F} + 4b_F) = m_0^2,$$

while the pessimist's budget constraint now is:

$$24p_{0F} - \frac{1}{1 + r_0}4p_{0S} = m_0^1.$$

<sup>18</sup>See Proposition (4.2) infra.

<sup>19</sup>In the previous example, this would mean in the bad state, as the good one is irrelevant.



Solving these two equations yields:

$$p_{0F} = \frac{(1+r_0)^2}{24r_0(2+r_0)} \left[ \frac{4p_{bF}}{(1+r_0)^2} + \frac{m_0^2}{1+r_0} - m_0^1 \right]. \quad (10)$$

Thus, quantitative easing (in its second version) does have a real effect on the economy. Its weakness, of course, is that such a non-conventional policy is limited by the quantity of collateral already available in the economy. The Central Bank might therefore wish to supplement it with a more conventional policy consisting in reducing  $r_0$  at the same time. Equation (10) shows that  $p_{0F}$  will then explode to infinity again. For a fixed monetary policy at time 1, this is incompatible with the equilibrium condition, unless a liquidity trap is hit at some finite level,  $\bar{M}_0$ . As a conclusion, the second understanding of quantitative easing does not succeed, in general, in escaping from the liquidity trap.<sup>20</sup>

### 3 The general model

The preceding example shows that the three scenarios described in the introduction—inflation, deflation or crash—can be encountered. It raises the question as to whether there are other alternatives and how general the monetary dilemma is. I therefore move to the general case of a two-period monetary economy with capital markets and collateral constraints. Still I try to keep the model tractable so as to be able to exhibit the three stylized regimes that emerged in the preceding section.

#### 3.1 The physical economy

The set of states of nature is  $\mathbf{S}^* := \{0, 1, \dots, S\}$ . State 0 occurs in period 0, then Nature moves and selects one of the states in  $\mathbf{S} := \{1, \dots, S\}$ , which occurs in period 1. There are  $L \geq 1$  commodities—all of them are storable. Therefore, the commodity space is  $\mathbb{R}_+^{\mathbf{S}^* \times L}$ , where the pair  $sl$  denotes commodity  $l$  in state  $s$ .

The set of consumers is  $\mathbf{H} := \{1, \dots, H\}$ . Each household  $h$  is endowed with  $\mathbf{e}^h \in \mathbb{R}_+^{\mathbf{S}^* \times L}$ , and has a utility function:  $u^h : \mathbb{R}_+^{\mathbf{S}^* \times L} \rightarrow \mathbb{R}$ . There is little loss of generality in assuming that, in each state, no agent has the null endowment and that every marketed good is actually present in the economy, i.e.,  $\mathbf{e}_s^h := (\mathbf{e}_{s1}^h, \dots, \mathbf{e}_{sL}^h) > 0 \quad \forall i \in \mathbf{H}, s \in \mathbf{S}^*$ , and  $\sum_{h \in \mathbf{H}} \mathbf{e}_s^h \gg 0 \quad \forall s \in \mathbf{S}^*$ . Each  $u^h(\cdot)$  is assumed to be continuous, quasi-concave and verifies the local non-satiation property: for each  $x^h \in \mathbb{R}^{\mathbf{S}^* \times L}$  and each  $\varepsilon > 0$ , there exists some  $y^h$  in the open ball,  $B(x^h, \varepsilon)$ , of radius  $\varepsilon$  and centered at  $x^h$ , such that  $u^h(y^h) > u^h(x^h)$ .

<sup>20</sup>See McMahon & Polemarchakis (2011) for another work on quantitative easing within a GEI model.

### 3.2 Money

Money is fiat but is the sole medium of exchange. Hence, all purchases are out of cash (this is the so-called Clower cash-in-advance constraint, Clower (1965)).<sup>21</sup> Money enters the economy in two ways. Each agent  $h$  has endowments of money free and clear of debt,  $m_s^h \geq 0$ , in state  $s \in S^*$ , with  $\sum_{h \in H} m_0^h > 0$ . Following Woodford (2003) and Dubey & Geanakoplos (2003a,b), this is called outside money.<sup>22</sup>

The Central Bank that stands ready to make short loans totaling  $M_s > 0$  euros for one period in each state  $s \in S^*$  and also to make the long loans totaling  $M_0 > 0$  for two period starting at date 0. Money is perfectly durable. If the interest rate on loan  $n \in N := \{\bar{0}, 0, 1, \dots, S\}$  is  $r_n$ , then anyone can borrow  $\mu_n/(1+r_n)$  euros by promising to repay  $\mu_n$  euros at the time the loan comes due.

The macrovariables are:  $\eta := (r, \rho, p, \pi)$ , where:<sup>23</sup>

$r \in \mathbb{R}_+^N :=$  interest rates on bank loans,  $n \in N$ .

$p \in \mathbb{R}_+^{S^* \times L} :=$  commodity prices.

$\pi := (\pi_1, \dots, \pi_K) \in \mathbb{R}^K :=$  the price of derivatives (to be defined infra).

Sometimes, I write  $\eta = (\eta(0), (\eta_s)_s)$ , breaking  $\eta$  into its state components.

### 3.3 Collateralized assets

All asset deliveries are supposed to be non-negative, and must be made in money. When the asset promise includes commodities,  $A_{sk} = (a_{sk}^1, \dots, a_{sk}^L)$ , the seller is asked to deliver the money equivalent to  $p_s \cdot a_{sk}^j$  — where  $p_s \in \mathbb{R}_+^L$  is the spot commodity price in state  $s$ . Derivatives have payoffs that depend upon the fundamental macrovariables (see supra). For example, a *call option* on firm  $j$ , with strike  $\lambda_j$ , pays off  $(V_{sj} - \lambda_j)^+$  in each state  $s \in S$  (and usually, the strike is a function of some macrovariables). Another example is an inflation-indexed promise, which delivers  $p_s \cdot \Lambda_s$  in state  $s \in S$ , where  $\Lambda_s \in \mathbb{R}_+^L$  is a fixed basket of goods. More generally, derivative  $k \in \mathbf{K} := \{1, \dots, K\}$  promises payoffs  $A_{sk}(\eta_0, \eta_s)$  euros in each state  $s \in S$ , where  $A_{sk}(\cdot, \cdot)$  is a continuous function of  $\eta_0$  and  $\eta_s$ . For simplicity, we keep each firm out of the derivative asset markets and the capital markets of other firms. Only household  $h$  can buy or sell each derivative  $k$  at price  $\pi_k$ , and possibly the Central Bank whenever it engages into non-conventional monetary policies. Because there are no a priori endowments of derivatives, such sales are “short sales”. Notice that they are not a priori bounded.<sup>24</sup>

<sup>21</sup>As already mentioned above, this way to introduce money can be made equivalent, to a certain extent, with alternative modeling choices in terms of bid-ask spreads.

<sup>22</sup>This endowment can be interpreted as transfer payments that are independent of equilibrium prices. Thus, in the parlance of Woodford (1994), I consider non-Ricardian monetary policies.

<sup>23</sup>Buying and selling nominal prices are identical. The bid-ask spread will be implicitly determined, at equilibrium, by the cost of borrowing money in order to purchase.

<sup>24</sup>In GEI, this unboundedness destroys the existence of financial equilibria. However, the addition of money suffices to restore existence (Dubey & Geanakoplos (2003a)).

Agents can only sell the asset  $k$  if they hold shares of some collateral. Asset  $k$  is therefore associated with a vector,  $\kappa_k \in \mathbb{R}_+^L$ , of collateral requirement. If an agent sells one unit of security  $k$ , she is required to hold  $\kappa_k^\ell$  units of commodity  $\ell$  as collateral.<sup>25</sup> Since the same commodity can be used as collateral for different financial assets, the agent is required to invest  $\kappa_k^\ell$  in  $\ell$  for each  $k \in K$ .<sup>26</sup> Since there are no penalties for default, a seller of the financial asset  $k$  defaults in state  $s$  whenever the total value of collateral falls below the promise at that state. The cash she owns may then be forfeited in order to compensate for the default. Default will only be registered when the borrower's cash itself is not sufficient to pay the debt. The actual yield of asset  $k$  in state  $s$  can no more be defined independently of the entire portfolio,  $(\alpha_k^h)_k$ , of its owner,  $h$ . It is namely given by:

$$\sum_k \mathbf{A}_{sk}(\eta_0, \eta_s) \alpha_k^h := \min \left\{ \sum_k A_{sk}(\eta_0, \eta_1), \sum_k p_s^L \cdot \kappa_k + \text{cash available} \right\}, \quad (11)$$

where the (endogenous) quantity of cash available at the time of asset delivery will be defined infra.

Because of the scarcity of collaterals, collateral requirements introduce an endogenous bound on short sales. When  $\kappa_k^\ell = 0$  for each  $k, \ell$ , there is no collateral requirement, hence short sales are not limited. Whenever the span of the return matrix,  $A$ , is always equal to  $S$ , and absent of collateral constraints, we call this the “complete benchmark”. By contrast, the “incomplete case” divides itself into two subcases: a) either there are no collateral constraints but  $\text{span}A < S$  for some commodity prices — this is the standard GEI case ; or  $\kappa_{sk} > 0$  for each  $k$  and each  $s$ .<sup>27</sup>

### 3.4 Liquidity constraints

The sequence of events is as follows. In period zero, households borrow money either from the stock of outside money put on the loan markets by households or from the Bank. There are two loan markets: one for the short term — where the Bank injects the stock,  $M_0$ , of inside money — and one for the long-term — where the Bank injects  $M_{\bar{0}}$ . On each market, an interest rate emerges (resp.  $r_0$  and  $r_{\bar{0}}$ ) so as to clear the market.<sup>28</sup> Next, the capital markets meet for the trade of assets and derivatives, followed by the commodity markets. After this, there is a move of chance and the economy enters one of the state  $s \in S$  in period 1. In any state  $s \in S$ , there is a fresh disposal of Bank money  $M_s$  and of outside money put for lending by

<sup>25</sup>For the sake of simplicity, I do not allow the collateral to be held by the lender or to be warehoused (see Zame & Geanakoplos (2002)).

<sup>26</sup>This means that tranching is not allowed. For simplicity also, only commodities are eligible as collaterals. In particular, I do not allow assets to be used as collaterals (pyramiding).

<sup>27</sup>For simplicity, we shall focus on these three polar cases. The intermediary situation — where  $\kappa_{sk} > 0$  for some pair  $(s, k)$  but not all — is left for further study.

<sup>28</sup>Alternatively, one can imagine that the Central Bank targets the interest rates and adjust the (endogenous) quantities  $M_0$  and  $M_{\bar{0}}$  so as to clear markets.

those households which saved money from the previous period, at a common interest rate  $r_s$ . Money markets in state  $s$  are followed by another round of trade in spot commodities. Then, all the deliveries take place simultaneously: households deliver on their derivatives. Finally agents settle their debts with the Bank and with the households having lend money.

For any fixed choice of macrovariables  $\eta$ , I now describe the set  $\Sigma_\eta^h$  of feasible choices of  $h \in H$ , and the outcome that accrues to  $h$  as a function of  $\eta$  and of her strategy,  $\sigma^h \in \Sigma_\eta^h$ .

We denote:

$\mu_n^h :=$  IOUs (or Bank bonds) sold by  $h$  ( $h$  borrows  $\mu^h/(1+r_n)$  on the loan market  $n$ )

$\alpha_k^h :=$  asset  $k \in K$  sold by  $h$

$q_{s\ell}^h :=$  commodity  $\ell$  sold by  $h$  in state  $s \in S^*$ .

A tilde on any variable will denote the money spent on it, i.e.,

$\tilde{\mu}_n^h :=$  money deposited (money spent on Bank bonds of type  $n$ ) by  $h$

$\tilde{\alpha}_k^h :=$  money spent by  $h$  in asset  $k \in K$

$\tilde{q}_{s\ell}^h :=$  bid of  $h$  on  $\ell$  in state  $s \in S^*$ .

The choice

$$\sigma^h := ((\mu_n^h, \tilde{\mu}_n^h)_{n \in N}, (\alpha_k^h, \tilde{\alpha}_k^h)_{k \in K}, (q_{s\ell}^h, \tilde{q}_{s\ell}^h)_{s \in S^*, \ell \in L}) \geq 0$$

must satisfy the following *physical constraints*:<sup>29</sup>

$$q_{0\ell}^j + \sum_k \kappa_k^\ell \frac{\alpha_k^h}{\pi_k} \leq e_{0\ell}^h \quad \forall \ell \in L, \quad (12)$$

that is, the total amount of commodities sent to the clearing house + collateral held cannot exceed the quantity of commodities) at hand.<sup>30</sup>

$$q_{s\ell}^j \leq e_{s\ell}^h + \sum_k \kappa_k^\ell \frac{\alpha_k^h}{\pi_k} \quad \forall s, \ell \in S \times L, \quad (13)$$

that is, the total amount of commodities supplied in state  $s$  in the second period cannot exceed the initial endowment + the collateral stored from period 0.

The choice  $\sigma^h$  must as well satisfy the following *liquidity constraints*:

(i) Bank deposits in period 0  $\leq$  money endowed with:

$$\tilde{\mu}_0^h + \tilde{\mu}_0^h \leq m_0^h \quad (14)$$

<sup>29</sup>Such constraints are standard in strategic market games, cf. Giraud (2003). They have to be understood with the following convention  $x/0 := 0$ .

<sup>30</sup>The ratio  $\kappa_k^\ell (\tilde{\alpha}_k^h / \pi_k)$  is the quantity of goods stored by  $h$  as collateral for her sale of  $(\tilde{\alpha}_k^h / \pi_k)$  units of asset  $k$ .

(ii) Expenditures on assets and derivatives  $\leq$  money left in (14) + money borrowed:

$$\sum_k \tilde{\alpha}_k^h \leq \Delta(14) + \frac{\mu_0^h}{1+r_0} + \frac{\mu_0^h}{1+r_0}. \quad (15)$$

(iii) Expenditures on commodities  $\leq$  money left in (15) + money obtained from sales of assets and derivatives:

$$\sum_\ell \tilde{q}_{0\ell}^h \leq \Delta(15) + \sum_k \pi_k \alpha_k^h. \quad (16)$$

(iv) Money owed and repaid on loan 0  $\leq$  money left in (16) + money received from commodity sales

$$\mu_0^h \leq \Delta(16) + \sum_\ell p_{0\ell} q_{0\ell}^h. \quad (17)$$

In each state  $s \in S$  of period 1, we must have:

(v)<sub>s</sub> Money deposited on loan  $s \leq$  money inventoried from period 0 + fresh endowment of outside money:

$$\tilde{\mu}_s^h \leq \Delta(17) + m_s^h \quad (18)$$

(vi)<sub>s</sub> Expenditures on commodities  $\leq$  money left in (18)<sub>s</sub> + money borrowed on loan  $s$ :

$$\sum_\ell \tilde{q}_{s\ell}^h \leq \Delta(18) + \frac{\mu_s^h}{1+r_s}. \quad (19)$$

(vii)<sub>s</sub> Money delivered on derivatives (or on collaterals in case of default)  $\leq$  money left in (19) + money obtained from commodity sales:

$$\sum_k \mathbf{A}_{sk}(\eta_0, \eta_s) \alpha_k^h \leq \Delta(19) + \sum_\ell p_{s\ell} q_{s\ell}^h. \quad (20)$$

We are now ready to define the “cash available” in the definition of the asset return,  $\mathbf{A}_{sk}(\eta_0, \eta_s)$  in (11) by:

$$\text{cash available} = \Delta(19) + \sum_\ell p_{s\ell} q_{s\ell}^h.$$

With this definition, (20) is always satisfied (and, from now, will be omitted). In case  $h$  defaults on several assets simultaneously, an arbitrary rule divides the cash among the various assets and their owners.<sup>31</sup>

<sup>31</sup>The quantity of cash available may be rationed proportionately to the value of a promise relatively to the whole debt of its owner.

(viii)<sub>s</sub> Money owed and repaid on loan  $s$  and (long-term) loan  $\bar{0} \leq$  money left in (20) + money obtained from asset deliveries:

$$\mu_s^h + \mu_{\bar{0}}^h \leq \Delta(20) + \sum_k \mathbf{A}_{sk}(\eta_0, \eta_s) \frac{\tilde{\alpha}_k^h}{\pi_k}, \quad (21)$$

These constraints define the convex feasible set  $\Sigma_\eta^h$ . The consumption that accrues to  $h \in H$  on account of  $\eta$  and  $\sigma^h \in \Sigma_\eta^h$  is  $x^h \in \mathbb{R}_+^{L \times S^*}$ , with (for all  $sl \in S^* \times L$ ):

$$x_{0\ell}^h := \mathbf{e}_{0\ell}^h - q_{0\ell}^h + \frac{\tilde{q}_{0\ell}^h}{p_{0\ell}} - \sum_k \kappa_k^\ell \frac{\alpha_k^h}{\pi_k},$$

and,  $\forall s \in S$ ,

$$x_{s\ell}^h := \mathbf{e}_{s\ell}^h - q_{s\ell}^h + \frac{\tilde{q}_{s\ell}^h}{p_{s\ell}} + \sum_k \kappa_k^\ell \frac{\alpha_k^h}{\pi_k},$$

and yields utility  $u^h(x^h)$  to player  $h$ .

### 3.5 Monetary equilibrium

Our definition is identical to that introduced by Dubey & Geanakoplos (2003b) except that our convention ( $x/0 := 0$ ) allows for no-trade at equilibrium (in which case, prices are zero, and money has no value). This is consistent with the well-known resilience of autarkic Nash equilibria in strategic market games (Giraud (2003)). We shall see, however, that such self-fulfilling pessimistic prophecies (where everybody is confirmed in the opinion that nobody will trade on markets) do not occur when there are positive gains-to-trade.

We therefore say that  $\langle \eta, (\sigma^h)_{h \in H} \rangle$  is a Monetary Equilibrium (ME) for the economy  $\mathcal{E} := \langle (u^h, \mathbf{e}^h, m^h)_{h \in H}, \mathbf{A}, (M_0, M_{\bar{0}}, (M_s)_{s \in S}) \rangle$  if:

(i) All agents maximize:

$$\sigma^h \in \arg \max_{\tilde{\sigma}^h \in \Sigma_\eta^h} u^h(x^h(\eta, \tilde{\sigma}^h)) \quad \forall h$$

(ii) All markets clear:

(a) Loans,  $n \in N$ :

$$\frac{1}{1+r_n} \sum_h \mu_n^h \leq M_n + \sum_h \tilde{\mu}_n^h \quad (22)$$

(b) Assets,  $k \in K$

$$\pi_k \sum_{h \in H} \alpha_k^h \leq \sum_{h \in H} \tilde{\alpha}_k^h. \quad (23)$$

(c) Commodities,  $sl \in S^* \times L$

$$p_{sl} \cdot \sum_h q_{sl}^h \leq \sum_h \tilde{q}_{sl}^h. \quad (24)$$

Remark that, when cast as a Nash equilibrium of the underlying strategic market game, this definition of a monetary equilibria rests on the implicit assumption that players cannot condition their actions in period 1 on the actions observed from period 0. This is consistent with the anonymity property of large markets.<sup>32</sup> Prices are the unique signal on which players coordinate.

Let us briefly comment on some specific aspects of the model that are responsible for its upshot.

a) As in most strategic market games, every transaction that an agent undertakes requires the physical transfer of money out of what he has on hand at that time. This amounts to various liquidity constraints. The upshot is that we have a well-defined physical process in which effect follows cause in a time sequence. By contrast, general equilibrium analysis steers clear of liquidity constraints because all transactions are imagined to occur simultaneously. The point of this paper is to go beyond this and to analyze the effects of liquidity constraints when default is permitted to occur on markets with collateralized derivatives. As we assume that each type of investor is represented by a continuum of negligible clones, they all take prices as given, which considerably simplifies the analysis. The existence proof, however, provides the full-blown double auction underlying our model (see the Appendix).

b) We assume that agents may default on certain promises and not on others, and that the only consequence of default is forfeiture of collateral and cash. For pawn shop loans, overnight repurchase agreements, margin loans and home mortgages, these assumptions are relatively close to reality. For other types of collateral, these are rather strong assumptions in the sense that, usually, bankruptcy involves default on all collaterals, penalties and a broad spectrum of consequences in addition to forfeiture of collateral. I depart from the standard general equilibrium model with collateral constraints by assuming that cash can also be forfeited in case of default. To the best of my knowledge, however, all the papers devoted to bankruptcy and collateral constraints within a general equilibrium framework, consider an idealized, cashless economy, and assume either that only the collateral is forfeited, or that default is accompanied by some utility penalty. Here, I take advantage from the introduction of cash by assuming that, even when the value of one's collateral falls below that of the promise, a borrower may still circumvent default by thanks to her monetary endowment (or thanks to the money she inventoried from the previous period, if any). This reduces the opportunity of default when compared to the conventional, cashless modeling option of bankruptcy.

<sup>32</sup>See Giraud & Stahn (2003) for the impact of allowing for non-trivial monitoring in strategic market games with incomplete security markets.



c) Money plays here all its different roles: it can be held for transactional purposes (because of the liquidity constraints detailed *supra*) and as a store of value between periods 0 and 1. But it can also be used as an asset that permits transferring wealth from one state to another in period 1, hence as an insurance tool: if short-term interest rates are expected to be very high in some second-period state  $s$ , then the economic agents will try to acquire money in advance in period 0. Furthermore, there may be also a speculative demand for money: inventorying money from period 0 to period 1 is equivalent to holding an implicit (riskless, nominal) asset. If the return of this asset becomes more attractive, a speculative demand for it will appear. And finally, if commodity prices are expected to increase in the second period, there will be a demand for money on the long-term loan market driven by the fear for inflation.

It should be clear, however, that there is no money illusion: multiplying both outside and inside money by some constant  $\lambda$  solely amounts to computing prices, say, in cents rather than in euros). Since expectations are rational, the Central Bank's policy is also perfectly anticipated, so that the results to follow are not due to some irrational anticipations. And nevertheless, we shall see that the "stylized facts" evoked in the previous subsection can be recovered within the present setting.

## 4 General properties of monetary equilibria

Introducing collateral constraints in a model of incomplete markets has two well-known consequences. The standard non-arbitrage argument that lies at the core of pricing theory in the complete markets benchmark does no more hold, even *at equilibrium*, in our set-up where markets are endogenously incomplete due to the scarcity of collaterals. "Efficient financial markets" are usually said to be characterized by price processes that follow random walks. As is well-known, this martingale property is satisfied in GEI models (independently of the Pareto-inefficiency of its equilibria, see, e.g., Geanakoplos (1990)), but need no more be satisfied in our set-up with collateral requirements: when the collateral constraint is binding, its actual price is the sum of two shadow prices, the marginal value attributed to it by its marginal purchaser *plus* its value as a collateral (see, e.g., Cao (2010)). Hence, the market incompleteness induced by the collateralization of assets is of specific nature when compared to more classical models of market incompleteness. The second consequence is that equilibrium pricing is no more linear. Hence, the celebrated Modigliani-Miller theorem also fails in our setting as in any environment with non-linear pricing rules (which has long been recognized, see Stiglitz (1974), Hellwig (1981) or Geanakoplos (1990)).

### 4.1 The yield curve

It is easy to show that money, in our model, is non-neutral (see Dubey & Geanakoplos (2003a,b)). Nevertheless, we get the analogue of a quantity theory of money:

**Proposition 4.1** *At equilibrium, as soon as  $r_0, r_{\bar{0}} > 0$ , one has:*

$$\pi_k \cdot \sum_{k,h} \tilde{\alpha}_k^h + p_{0\ell} \cdot \sum_{\ell,h} \tilde{q}_{0\ell}^h = \sum_h m_0^h + M_0 + M_{\bar{0}}. \quad (25)$$

Similarly, if  $r_s > 0$ , then

$$p_{s\ell} \cdot \sum_{\ell,h} \tilde{q}_{s\ell}^h = \sum_h m_s^h + M_s. \quad (26)$$

**Proof.** The local non-satiation of each trader's preferences implies that, at equilibrium, the two liquidity constraint (16) must be binding. Suppose the contrary: Each individual  $h$  would gain by reducing (by a little) the right-hand side of each inequality in order to increase her final allocation in consumption commodities in state 0. A contradiction. Summing over  $h$  yields:

$$\sum_h \left[ \tilde{\mu}_0^h + \tilde{\mu}_{\bar{0}}^h + \sum_k \tilde{\alpha}_k^h + \sum_{\ell} \tilde{q}_{0\ell}^h \right] = \sum_h m_0^h + \frac{1}{1+r_0} \sum_h \mu_0^h + \frac{1}{1+r_{\bar{0}}} \sum_h \mu_{\bar{0}}^h.$$

The conclusion follows by (22), (23) and (24)—which must be binding at equilibrium as well.  $\square$

Notice that, at variance with Fisher's seminal version of the quantity theory of money (“ $Mv = pT$ ”), here, the velocity of money is constant, and equal to 1, while prices and the volume of transactions,  $T$ , are endogenous. This is why no “monetarist” conclusion can be drawn from Prop. 4.1, while this version of the quantity theory of money is compatible with the non-neutrality of money.

The next Proposition describes the term structure of interest rate, showing that the full interplay of all the demands for money can be captured in our model (transaction, precaution, speculation, storage, insurance against inflation). Its proof easily follows from Theorem 2 in Dubey & Geanakoplos (2003b). Details are left to the reader.

**Proposition 4.2** *At any ME,*

- (i)  $r_s \geq 0 \forall s \in S^*$ ;
- (ii)  $1 + r_{\bar{0}} \geq \min_{s \in S} (1 + r_0)(1 + r_s)$  with strict inequality unless all  $r_s$  are identical  $\forall s \in S$ ;
- (iii)  $r_0 \leq \sum_h m_0^h / M_0$  and  $r_s \leq \mu_s(m, M) \forall s \in S$ ;
- (iv)  $r_0 M_0 + r_{\bar{0}} M_{\bar{0}} + r_s M_s \leq \sum_h (m_0^h + m_s^h) \forall s \in S$  with an equality if, and only if, there is no default in state  $s$ .

Let us briefly comment property (iv). On the left hand of the inequality, there is the interest revenue of the Bank, and on the right, its expenditures (by way of, say, gifts, lump sum transfers of the payment of past interests on the government's public debt,  $m_s^h$ , to households). This equation thus says that the government is

balancing its budget on the long-run as long as there is no default, although  $M_s$  and  $m_s^h$  may be quite arbitrary. On the other hand, whenever there are defaults in some state  $s$ , the public deficit is given by:

$$\text{Public deficit} = \sum_h (m_0^h + m_s^h) - r_0 M_0 - r_{\bar{0}} M_{\bar{0}} - r_s M_s.$$

This should make clear that a sovereign default is compatible with our market equilibrium conditions as soon as investors may default on their collateralized derivatives. This is true, in particular, in the third scenario to be described in section 5 below, where *all* the private investors default in at least one second-period state.

## 4.2 Belief heterogeneity

The fourth property of monetary equilibria to be recorded in this section deals with the issue of belief heterogeneity.<sup>33</sup> We saw in the example of section 2 that it may play a significant role in the emergence of rational exuberance at equilibrium. In the complete markets case (where non collateral requirement is imposed and the available assets span the whole space,  $\mathbb{R}^S$ , of returns in period 1), agents having beliefs that are too far away from the “truth” are driven out of the market, as shown by the next Proposition. Let us denote by  $P^h \in \Delta(S)$  the subjective probability attributed by  $h$  on second-period states of nature. For the sake of simplifying the discussion, we assume that each state,  $s \in S$ , belongs to the support of  $P^h$ , for every  $h$ . Let  $\mathcal{H}$  be a measure of the belief heterogeneity among households, defined by:

$$\mathcal{H} := \ln \max_{h,i \in H, s \in S} \frac{P^h(s)}{P^i(s)}.$$

Of course, if beliefs are perfectly homogeneous,  $\mathcal{H} = 0$ . In the sequel, utility  $u^h$  is said to be *separable* whenever it is of the form:

$$u^h(x^h) = U_0^h(x_0^h) + \lambda^h E^{P^h} [U_1^h(x_1^h)],$$

with a discount factor,  $\lambda^h \in (0, 1)$ . Finally, the classical Inada condition reads:  $U_1^h(x_1^h) \rightarrow +\infty$  as  $x_1^h \rightarrow 0^+$ .

**Proposition 4.3** *When markets are complete, if utilities are  $C^1$  and separable, with  $U_1^h$  verifying Inada condition, then the consumption of at least one household gets arbitrarily close to zero as the degree,  $\mathcal{H}$ , of heterogeneity gets arbitrarily large. Formally,*

$$\lim_{\mathcal{H} \rightarrow +\infty} x_1^h = 0.$$

<sup>33</sup>For an early discussion of the common prior assumption, see Morris (1995).

**Proof.** Suppose, to begin with, that all interest rates are 0. From the first-order condition, at equilibrium, for any pair,  $h, i$ , of households, and any state  $s \in S$ ,

$$\frac{U_1^h(x_s^h)}{U_1^i(x_s^i)} = \frac{P^i(s)U_0^i(x_0^i)}{P^h(s)U_0^h(x_0^h)}.$$

Since  $x_s^i$  is bounded above by  $\bar{\mathbf{e}}_s := \sum_t \mathbf{e}_s^h$  and utilities are differentially concave, there exists some  $B > 0$  such that:  $U_1^i(x_s^i) \leq B$ . Thus,

$$U_1^h(x_s^h) \geq \frac{P^i(s)U_0^i(x_0^i)}{P^h(s)U_0^h(x_0^h)}B.$$

As the degree of heterogeneity,  $\mathcal{H}$ , increases to infinity, it must be the case that  $x_s^h$  goes to zero.

Suppose, now, that interest rates are positive.

A COMPLETER. □

By contrast, when markets are incomplete (because of collateral requirements), then every consumer survives at equilibrium, even those who share arbitrarily wrong beliefs. The proposition below shows, indeed, that, even with arbitrarily large difference in beliefs, no household's consumption will come arbitrarily close to 0 at some state. Therefore, incomplete markets differ from complete markets when consumers differ in their beliefs. The intuition for this result is the same as in Cao (2010) : if an agent believes that the likelihood of a state is much smaller than what other agents believe, the consumer will want to exchange her consumption in that state for consumption in other states. Complete markets allow her to do so but, in incomplete markets, collateral constraint limits the amount of consumption that she can sell in each state.

**Proposition 4.4** *Suppose that, for every  $h$ ,  $u^h$  is separable and  $\lim_{x_0^h \rightarrow 0^+} U_0^h(x_0^h) = \lim_{x_s^h \rightarrow 0^+} U_s^h(x_s^h) = -\infty$ ,<sup>34</sup> then, at a monetary equilibrium with collateral, every trader's consumption is bounded below in each state  $s \in S^*$  by some constant,  $\underline{c} > 0$ .*

**Proof.** In period 0, refusing to trade and consuming only one's initial endowment,  $\mathbf{e}_0^h$  is always a feasible strategy. In period 1, refusing to honor all one's debt, providing one's creditor with the stored collateral, and consuming only one's initial endowment is also a feasible strategy (because of the postulated limited liability of consumers), even when a state of nature is chosen to which she attributed an arbitrarily small occurrence probability. Hence, for every  $h$ , one has, at equilibrium:

$$u^h(x^h) \geq u^h(\mathbf{e}^h).$$

<sup>34</sup>Log-linear utility or CRRA utility with CRRA constant exceeding 1 clearly verify this assumption.

Since consumption in every state  $s \in S$  is bounded from above by the total endowment of commodities, there exists some  $B > 0$  such that  $U_s^h(x_s^h) \leq B$ . Hence,

$$U_0^h(x_0^h) \geq U_0^h(\mathbf{e}_0^h) + \lambda(U_s^h(\mathbf{e}_s^h) - B) > U_0^h(\underline{c}) \quad \forall s \in S,$$

for some  $\underline{c} > 0$  small enough. The same argument can be repeated for  $h$ 's consumption in every state  $s \in S$  of the second period. Hence the result.  $\square$

**Corollary 4.1** *Under the hypotheses of Propositions (4.3) and (4.4), there is a degree of belief heterogeneity,  $\overline{\mathcal{H}}$ , above which no monetary equilibrium with collateral is Pareto-optimal.*

**Proof.** In an incomplete markets set-up with collateral, each consumer's consumption is bounded away from 0 (Proposition (4.4)). In the complete market setting, the consumption of at least one consumer goes to 0 when  $\mathcal{H} \rightarrow +\infty$  (Proposition (4.3)). This holds, whatever being initial endowments. Thus, for any choice of initial endowments, the two sets of equilibria no more intersect for  $\mathcal{H}$  sufficiently high. The second welfare theorem then implies that no monetary equilibrium with collateral requirements can be Pareto-optimal.  $\square$

Notice that the preceding corollary holds for any monetary policy. Thus, *even if interest rates were 0*, the resulting equilibrium would be Pareto-suboptimal. The suboptimality, here, is not primarily due to the friction induced by a positive interest rate but to the survival of players with heterogenous beliefs, which itself is clearly due to the collateral requirements.

### 4.3 Gains to trade

In this section, we introduce the intratemporal gains-to-trade assumption borrowed from Dubey & Geanakoplos (2003b).

Let  $x^h \in \mathbb{R}_+^{S^* \times L}$  be any feasible allocation for household  $h$ . For any  $\gamma \geq 0$ , we say that  $x = (x^h)_h \in (\mathbb{R}_+^{S^* \times L})^H$  is *not  $\gamma$ -Pareto optimal in state  $s$*  if there exist some trades  $(\tau_s^h)_h \in (\mathbb{R}^L)^H$  in state  $s \in S^*$ , such that

$$\sum_h \tau_s^h = 0 \quad (\text{feasibility}) \quad (27)$$

$$x_s^h + \tau_s^h \in \mathbb{R}_+^L \quad \text{for all } h \in H \quad (\text{consumability}) \quad (28)$$

$$u^h(\bar{x}^h[\gamma, \tau_s^h]) \geq u^h(x^h) \quad \forall h \in H, \text{ with at least one strict inequality} \quad (\text{improvement}) \quad (29)$$

where, for every  $\ell \in L$ ,

$$\bar{x}[\gamma, \tau^h]_{t\ell} := \begin{cases} x_{t\ell}^h & \text{if } t \in S^* \setminus \{s\} \\ x_{s\ell}^h + \min\{\tau_{s\ell}^h, \frac{\tau_{s\ell}^h}{1+\gamma}\} & \text{for } t = s. \end{cases}$$

In words, the trades,  $\tau^h$ , considered as candidates to  $\gamma$ -Pareto-improve  $x^h$  involve a tax of  $\gamma/(1 + \gamma)$  on trade. Of course, 0-Pareto-optimality coincides with the standard notion of Pareto-optimality. The *gains to trade*,  $\gamma_s(x)$ , in state  $s$  at a point  $x \in (\mathbb{R}_+^{S^* \times L})^H$  is defined as the supremum of all  $\gamma$  for which  $x$  is not  $\gamma$ -Pareto-optimal in state  $s$ .

When entering state  $s \in S$  in period 1, the stock of outside money (owned by the households free and clear of any offsetting obligations) is equal to the money inventoried from period 0 less what is already owed on the long loan to the Bank. As in Dubey & Geanakoplos (2003b), it is easy to show that this stock is never more than

$$\hat{m}_s^h := \sum_h m_0^h - \min_{t \in S} \frac{\sum_h m_0^h}{M_0 + M_t} M_0,$$

where the stock of inside money injected in state  $s$  is  $M_s$ . The maximal ratio of outside to inside money in state  $s \in S$  is therefore given by:  $\hat{m}_s/M_s$ .

The gains-to-trade hypothesis can now be formulated as follows. For every state  $s \in S$ , let us denote by  $X_s$  the subset of feasible and consumable trades  $(\tau^h)_h$  that involve no trade in state  $s$ .

**Gains-to-trade hypothesis.** For all  $s \in S$  and every  $x \in X_s$ ,  $\gamma_s(x) > \frac{\hat{m}_s}{M_s}$ .

This assumption requires that there be gains to trade in *every* state  $s \in S$  in period 1, but not necessarily in period 0. It also rules out the case of only one commodity per state, because then, any feasible and consumable allocation would be automatically 0-Pareto optimal. Similarly, it rules out the representative agent case where  $H = 1$ , because, again, this would lead to Pareto-optimality for free.

If initial endowments in the economy  $\mathcal{E}$  are not Pareto-optimal, then as  $M_s \rightarrow +\infty$  leaving the economy otherwise fixed, the Gains-to-trade hypothesis will sooner or later be satisfied.

## 4.4 Existence

The next result (whose proof is in the Appendix) extends the existence theorem of Dubey & Geanakoplos (2003b) to the case where households can default on the delivery of collateralized derivatives. The main upshot is that, here, equilibrium asset prices may be zero. By contrast, as soon as households are endowed with a positive amount of costless outside money, no commodity price can reach 0. Similarly, even when individual monetary endowments vanish, if only firms are allowed to default, their positive endowment in commodities ensures that equilibrium prices will always be positive.

**Theorem 4.1** *Any monetary economy  $\mathcal{E}$  verifying our standing assumptions together with the gains-to-trade hypothesis has a ME.*

For the reader familiar with the incomplete markets literature with collateral or with money, this existence result should not be too surprising. Indeed, I allow for real assets, options, derivatives, and even more complicated non-linear assets. In the standard framework with no money and no collateral constraints, the presence of such assets implies that the space of feasible income transfers does not depend continuously on commodity prices, so that equilibrium may not exist.<sup>35</sup> In the present framework however, there are two forces which help restoring existence: both the collateral requirements and the cash-in-advance constraints place an endogenous bound on short-sales. The first one because of the scarcity of collateralized assets (see, e.g., Geanakoplos & Zame (2002)), the second, because of the scarcity of money (cf. e.g., Dubey & Geanakoplos 2003b)). As in the standard incomplete markets setup (see Radner (1972) for instance), a lower-bound on short-sales eliminates the discontinuity and guarantees existence.

It should be stressed that no Gains-to-trade hypothesis is needed in period 0 in order to guarantee the existence of an active monetary equilibrium. Nevertheless, it should be clear that the Pareto-optimality of any period 0 equilibrium allocation depends upon  $r_0$  and  $r_{\bar{0}}$ . The smaller are these interest rates, the closer will be the allocation to optimality. As a consequence, monetary authorities may be willing to increase  $M_0$  and  $M_{\bar{0}}$  in order to improve the optimality of trades. The next section is devoted to the implications of such a monetary policy.

## 5 Robust liquidity trap versus financial crash

We now show the central result of this paper, namely that monetary authorities face a universal dilemma. Quite robustly, the monetary authority will be able to improve the efficiency of trade, and thus total real output, by increasing supplies of Bank money or, equivalently, by lowering interest rates.<sup>36</sup> This will have the additional side cost of increasing price levels because of the quantity theory of money, as shown by the next lemma:

**Lemma 5.1** *If  $M_0 + M_{\bar{0}} \rightarrow +\infty$ , then  $\|\pi, p_0\| \rightarrow +\infty$ .*

### **Proof.**

Recall, indeed, that transactions are uniformly bounded, either because of the physical constraints on commodities or because of the collateral requirements on derivatives (12).

<sup>35</sup>See Duffie & Shafer (1985) for a generic existence proof, Ku & Polemarchakis (1996) for a robust example of non-existence with options.

<sup>36</sup>It should be clear, indeed, that in this model, the Bank's monetary policy can be equivalently understood as consisting in choosing  $M_0, M_{\bar{0}}$  and letting  $r_0, r_{\bar{0}}$  clear the loans markets or in choosing  $r_0, r_{\bar{0}}$  and adjusting  $M_0, M_{\bar{0}}$  in order to balance the demand for money. Thus, our approach is compatible both with the traditional inflationary target à la Taylor and with the quantitative targets to which some Central Banks seem to have gone back after the 2008 crisis (and which was never entirely abandoned by the ECB).



□

Thus an expansionary monetary policy will reduce the real wealth of those who begin with high initial stocks of money,  $m_s^h > 0$  (for  $s \in S^*$ ). It turns out, however, that the monetary authority will be stymied in its efforts to increase output and the price levels by increasing  $M_0$ : Either this leads the economy to a liquidity trap (in the Keynesian meaning of this word) or to a large bubble on the financial markets, whose burst induces the general collapse of the whole economy in at least one second-period state.

**Theorem 5.1** *Under the assumptions of Theorem 4.1,*

(i) *Fix  $\mathcal{E} \setminus M_0$ : There is a level of money  $M_0^*$  and commodity price levels  $p_0^*$  such that, for every monetary economy  $\mathcal{E} = \langle M_0, \mathcal{E} \setminus M_0 \rangle$  with  $M_0 \geq M_0^*$  and every monetary equilibrium,  $(\eta, (\sigma^h)_t)$  for  $\mathcal{E}$ , one of the three following situations arises:*

(ii a) *Either  $r_0 = 0$ ,  $p_0 \leq p_0^*$  and households hoard at least  $M_0 - M_0^*$  as unspent money balances in period 0 —this is the “liquidity trap”.*

(ii) *Else, for every state  $s$  in period 1, one of the two cases occurs:*

(ii b) *Either  $\forall \varepsilon > 0$ , there exists  $M_0^* > 0$  such that, for every  $M_0 \geq M_0^*$ ,  $r_s \leq \varepsilon$  in every state  $s$ .*

(ii c) *Or some agents default in state  $s$  and no-trade prevails.*

**Proof.** In equilibrium, the following holds:

$$r_0 M_0 + r_{\bar{0}} M_{\bar{0}} + r_s M_s \geq \sum_h (m_0^h + m_s^h) \quad \text{for all } s \in S,$$

with equality if there is no bankruptcy. The reason is that no agent will hold worthless money at the end of any state in period 1. Thus, all the money,  $M_0 + M_{\bar{0}} + M_s + \sum_h (m_0^h + m_s^h)$  is used to repay either the principal,  $M_0 + M_{\bar{0}} + M_s$  or the interest payments,  $r_0 M_0 + r_{\bar{0}} M_{\bar{0}} + r_s M_s$ .

We begin with  $(M, m)$  and a monetary equilibrium. Assume that  $r_0 > 0$ . Suppose the monetary authority increases  $M_0$  or, equivalently, lower the current short-term interest rate,  $r_0$ .

We know from Theorem 4.1 that a monetary equilibrium exists whatever being the Bank’s monetary policy. Since we are in a REE world, prices and monetary policy in period 1, in each state  $s \in S$ , are perfectly anticipated ex ante. Hence, there is a finite amount of money,  $M_s$ , injected at time 1, in state  $s$ , and agents know it. Thus, the total stock of money available to be spent in state  $s$  is no more than  $M_{\bar{0}} + M_s + m_0 + m_s$ .

Suppose, to begin with, that the volume,  $\sum_h z_s^{h-}$ , of aggregate equilibrium supply of spot commodities in each state  $s$  is bounded from below, and that this lower-bound is independent from  $M_0$ . This means that spot prices,  $p_s$ , at time 1, must have an upper-bound independent from  $M_0$ , as follows from QTM:

$$p_s \cdot \sum_h z_s^{h-} = p_s \cdot \sum_h z_s^{h+} \leq M_0 + M_s + m_0 + m_s.$$

Let denote  $\bar{p} = \max_{s \in S, \ell=1, \dots, L} p_{s\ell}$ . This implies that commodity and asset prices at time 0 must be bounded by some constant, say,  $K$ , independent of  $M_0$ . Otherwise, indeed, every one could sell  $\varepsilon > 0$  of any item (be it a commodity or a security) she is positively endowed (either directly or as a collateral) with at time 0, and buy the whole economy in some state in period 1 for a price at most equal to  $\max(\bar{p}, \dots, \bar{p}) \cdot z$ , where  $z$  is any allocation belonging to the (compact) feasible set. If  $r_0 > 0$  and the monetary equilibrium is active, then  $M_0$  must be borrowed and spent in time 0 : no agent would borrow money at positive interest unless she is going to spend it. But payments on assets and derivatives are all bounded above by the stock of money available in state  $s$ , the payments of any one of these securities is bounded above, independently of  $M_0$ . Hence, there is an upper-bound, independent of  $M_0$ , on how much money is spent on securities in period 0. The rest must be spent on consumption goods. Because no more than  $\sum_h \mathbf{e}_0^h$  can be put up for sale, prices  $p_0$  must become arbitrarily large as  $M_0$  increases because of QTM. This contradicts the boundedness of period 0 prices. Hence, as  $M_0$  rises, eventually,  $r_0 = 0$ , and the hoarding of real money balances thereafter increases proportionately with  $M_0 - M_0^*$ . The economy has therefore reached a liquidity trap when  $M_0 \geq M_0^*$ : even if the Bank pumps in more and more money, this has no real effect any more. This is case (iia) of the Theorem. It occurs essentially because people have rational expectations, and expect money at time 1 to be bounded across states, independently of  $M_0$ .

Next, if the economy does not fall into a global trap at time 0, this means that, at time 1, either the quantity of available money increases with  $M_0$  or that the size of aggregate supply,  $\sum_h z_s^{h-}$ , shrinks to zero. The first case means that second period interest rates,  $r_s \rightarrow 0^+$  as  $M_0 \rightarrow +\infty$ . This is case (ii b) in the Theorem.

Suppose that, in state  $s$ ,  $r_s$  is bounded below independently of  $M_0$ . How can

$$\lim_{M_0 \rightarrow +\infty} \sum_h z_s^h = 0$$

happen at equilibrium ? This is where default really enters the picture. We have supposed, indeed, that, in each state  $s \in S$ , there are positive gains to trade in spot commodities, and that there is enough inside money, so that the gains-to-trade hypothesis is fulfilled. Therefore, the unique reason why households would not trade in state  $s$  is because some assets defaulted so that sufficiently many agents have already lost their wealth when entering in state  $s$ . Prima facie, this seems impossible because the collateral ensures that nobody can lose its whole wealth due to default: everyone gets at least the collateral of the assets she bought in period 0. This is where the possible forfeiture of cash plays its role, as shown in the Example of section 2: default may prevent from trading because a significant number of agents have no cash to finance their purchases or to borrow money on the short-loan market.  $\square$

The strength of Theorem (5.1) is to show that there is actually no escape road from what we have called the monetary dilemma : Either the Central Bank commits to fostering inflation, or it takes the risk of either a liquidity trap or a financial crash. The weakness of this general theorem is that it does not prove that *each* of these three regimes may actually take place. This is why it needs to be supplemented by the Example of section 2.

## 6 Some comments in light of the 2007-09 crisis

This paper does not attempt to provide a unified analysis of the global crisis that started in 2007. Nevertheless, the model presented above, and its main results, shed some light about what we may have learned from the crisis and the policy issues raised by the response of the authorities to it. The monetary dilemma highlighted by Theorem (5.1) says that there are 3, and only 3, scenarios compatible with the Nash equilibrium conditions:

- scenario 1: No liquidity trap and no default in the second period, but the size of injected money,  $(M_0, (M_s)_s)$  allows to improve the efficiency of trades at the cost of a possibly unbounded inflation of commodity and asset prices in both periods.

- scenario 2: Inflation is prevented in the second period but at the cost of a liquidity trap in the first one.

- scenario 3: rational exuberance on financial markets leads to a global crash in the second period.

The introduction of collateral requirements into monetary general equilibrium analysis enables to emphasize the role of leveraging as one of the microeconomic roots of financial crashes. Indeed, as shown by section 2 above, the larger the leverage ratio, the larger are the debts of optimistic investors in case of default. It has been argued, e.g., by Adrian & Shin (2008) that, even in the absence of a true bankruptcy, the very fact that a bank's assets have lost value implies a sudden rise in the leverage ratio, which is likely to lead the bank to sell off assets or restrict credit in order to deleverage. This, however, is a partial equilibrium argument. Here, we can recast the argument within a general equilibrium framework: it is the shift of wealth between optimistic investors and pessimistic ones that can create a dramatic fall in prices and, eventually, a crash (see Geanakoplos (2001)) for the seminal statement of this phenomenon).<sup>37</sup>

This is not to say that our story depicts financial crashes as “black swans”, i.e., as large-impact, low-probability events against which any protection would be exceedingly costly.<sup>38</sup> According to our model, there are two ways to circumvent the risk of a big crash. The first one consists in turning to a contractionary monetary policy —at the cost of running the risk of falling into the liquidity trap (i.e., of shifting from scenario 3 to scenario 2). The second way consists in regulating

<sup>37</sup>Large European banks in 2007 had leverage ratios between 20 in the UK and 35 in Switzerland (see Panetta & Angelini (2009)).

<sup>38</sup>For a defense of the “Black Swan” viewpoint, see Blanckfein (2009).

financial markets so as to reduce their leverage ratio (driven by  $\beta$  in the Example of section 2) —at the cost of having to accept a high inflation rate on consumer prices whenever the monetary policy remains expansionary.

On the other hand, the interplay between money and collaterals enables to show how the macroeconomic environment may contribute to the excessive leveraging and risk-taking. US and global monetary instances have been criticized as excessively expansionary (Taylor (2008)). According to this view, monetary policy in the aftermath of the 2001 recession remained too lax for too long and this triggered asset-price inflation, primarily but not exclusively on the US housing market, and a generalized leverage boom. Had it followed the Taylor rule, so goes the argument, the Fed would have tightened rates faster, instead of lowering interest rates further to counter perceived deflation risks. Accordingly, short-term rates would have been higher between 2001 and 2005, making the subsequent burst less pronounced. This paper recasts this debate within a general equilibrium set-up with rational expectations: it is indeed the quantity  $M_0$  of money injected on the short-term loans market that may fuel an inflation of asset prices (“rational exuberance”) when leverage on the financial markets is sufficiently large. This means that one explanation of the Great Moderation (and of the fact that consumer price inflation remained rather subdued throughout the 2000-06 period) might rest on the sharp increase of leverage ratios on financial markets. Despite the vivid growth of the world monetary base (15% each year since 1997, 30% since 2007) we did not observe the domestic inflation we should have experienced according to the Quantity Theory of Money (equation (??)) because this huge amount of fresh liquidity migrated from the real sector to the financial sphere.

On the other hand, however, Theorem 2 shows that the deflation risk is perfectly compatible with rational expectations and market clearing. Thus, when then-board member Ben Bernanke famously outlined a contingency plan to avoid the repetition of the Japanese experience (Bernanke (2002)), he was not referring to some improbable curiosity: the liquidity trap is part of an equilibrium story with rational expectations. Moreover, our dilemma shows that, whenever a central bank efficiently accomplishes its mission dedicated to consumer-price stability (i.e., avoids scenario 1), then it faces only two alternative scenarios: Either inflation on financial markets driven by some “rational exuberance” whose burst may induce a global collapse (scenario 3) or a liquidity trap (scenario 2). In scenario 3, if the central Bank sticks to consumer-price stability it will have little reason to raise interest rates aggressively, and will therefore be unable to fight against rational exuberance, hence, to prevent a crash. Thus, our approach provides a theoretical ground for a plea in favor of Central Banks standing ready to depart from their price stability goal in the name of financial stability.

In scenario 2, application of the Taylor benchmark encounters the zero-bound problem: While the Taylor rule would recommend a negative interest rate, this is impossible to achieve because rational depositors are not prepared to pay for keeping deposits. Thus, our approach also makes the case for unconventional monetary policies in order to avoid liquidity traps. The recommended policy, however, cannot

be the zero-interest rate policy (ZIRP), at least in our model. Imagine, indeed, that the central Bank prints vast amounts of banknotes and drops them from helicopters. Individuals receiving banknotes from heaven could feel suddenly richer and could spend at least of this money, especially if they have heard about monetarism and fear that relying on the printing press will in the end induce inflation. Demand should pick up and inflation would indeed follow later on. This conventional reasoning, however, does not hold here: the quantity theory of money, in the present model, is not monetarist in essence, nor are rational economic agents. Thus, they know that the Central Bank's power to create money does not automatically result in inflation. If, on the contrary, people are convinced that, in the second period, the Central Bank won't pursue its easing policy, the scenario 2 of my narrative tells us that they will hoard the helicopter money unspent. For this additional money to help the economy escape from the liquidity trap, the Bank must convince the economic agents that it will go further in its zero-interest rate policy, hence should commit to put zero interest rates in the second period as well: we are then back to our scenario 1.

The issue at hand therefore becomes to find channels by which the central Bank can commit, implicitly or explicitly, to higher inflation in the future (i.e., to even lower rates and more liquidity in the second period). As was said by Krugman, it amounts to "committing to be irresponsible". Such a commitment allows to shift from scenario 3 to scenario 1. How can the central Bank proceed to such a commitment? For instance, by monitoring long-term rates,  $r_{\bar{0}}$ . Central Banks normally only target the short end of the yield curve, leaving the determination of longer-term interest rates to market mechanisms. In a situation of near-deflation, however, the central Bank can commit to keep policy rates low for an extended period and inter into refinancing operations with extended maturity, thereby imposing a ceiling on interest rates at the corresponding horizon. Here, there is room for a monitoring of long-term rates: if  $r_{\bar{0}}$  decreases (say, by the increase of  $M_{\bar{0}}$ ), then the no-arbitrage relation between first-period long-term rates and second-period short-term rates implies that  $r_s$  must decrease for each  $s$ .

Thus, our approach sustains the viewpoint vividly expressed by Krugman (1998a, 2000) (and later by Orphanides (2004)) in the context of the Japanese crisis: the Central Bank of Japan "needs a credible commitment to expand not only the current but also *future* money supplies, which therefore raises future expected prices —or, equivalently, a credible commitment to future inflation" (Krugman (2000)). Theorem 2 shows that, *there is no alternative* to such an "irresponsible" commitment, as otherwise the central Bank faces two major failures —either the deflationary liquidity trap or the possibility of a financial crash. This absence of fourth scenario (where the central Bank could avoid any disaster and still commit to be "responsible" with respect to consumer prices) is what I have called the "dilemma of monetary policy".

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## 7 Appendix

This Appendix provides the proof of the existence theorem (4.1). We prepare the proof with a lemma.

**Lemma 7.1** *Let  $p$  be an eME price vector of  $(\mathcal{E}(\mathbf{e}), (\mu_i), M)$ . There exists  $\eta > 0$  such that  $p_\ell > \eta$  for every commodity  $\ell$ .*

### Proof of Lemma 7.1

Let  $K > \sum_h \sum_\ell \mathbf{e}_\ell^i$ . Define  $u_i^* := \max_{\tau \in [0,1]} u_i^\tau(K, \dots, K)$ , fix some  $\tau \in [0, 1]$ , and let  $H^\tau > 0$  be chosen large enough so that

$$u_i^\tau(0, \dots, 0, H^\tau, 0, \dots, 0) > u_i^*,$$

for  $H^\tau$  in any component. The following argument (adapted from footnote 19 in Dubey & Geanakoplos (2003a)) proves that such an  $H^\tau$  exists. Let  $\square$  be the cube with sides of length  $K$  in  $\mathbb{R}_+^L$ . Define  $\tilde{u}_i^\tau : \mathbb{R}_+^L \rightarrow \mathbb{R}$  by  $\tilde{u}_i^\tau(y) := \inf\{L_x(y), x \in u, L_x$  is an affine function representing the supporting hyperplane to the graph of  $u_i^\tau$  at

the point  $(x, u_i^\tau(x))$ .  $\tilde{u}_i^\tau$  coincides with  $u_i^\tau$  on  $\square$ , and there exists some  $H^\tau$  such that  $\tilde{u}_i^\tau(0, \dots, 0, H^\tau, 0, \dots, 0) > u_i^*$  for  $H^\tau$  in any component.

Now, if  $\mu^* := \max_i \mu_i$  and  $H := \max_{\tau \in [0,1]} H^\tau$ , we claim that

$$p_\ell^\tau \geq \frac{\mu^*}{H}.$$

Otherwise, any agent  $i$  with  $\mu_i = \mu^*$  could spend her money in order to buy  $H$  units of commodity  $\ell$ , thus obtaining a final utility  $\tilde{u}_i^\tau(0, \dots, 0, H, 0, \dots, 0)$  higher than  $u_i^*$ . A contradiction.  $\square$

**Proof of Theorem 4.1** The main difficulty lies in the fact that *financial asset* prices may be zero.<sup>39</sup> As in Geanakoplos & Zame (2003), I first construct, for each  $\pi > 0$ , an auxiliary economy,  $\mathcal{E}^\pi$ , which differs from  $\mathcal{E}$  only in that asset promises are given by:

$$\mathbf{A}_{sk}^\pi(\eta_0, \eta_1) := \mathbf{A}_{sk}(\eta_0, \eta_1) + \pi.$$

In these economies, I prove existence of an active monetary equilibrium with derivative prices that are bounded away from 0. I then construct an equilibrium for  $\mathcal{E}$  by taking the limit as  $\pi \rightarrow 0^+$ .

*Step 1.* Existence in  $\mathcal{E}^\pi$  with a dummy player.

For any  $\varepsilon > 0$ , we define a truncated generalized game  $\Gamma_\pi^\varepsilon$  on a continuum player-set with types  $H$ . Each time  $h$  corresponds to, say, the unit interval  $[0, 1]$  of identical players, equipped with the restriction of the Lebesgue measure. Following Dubey & Geanakoplos (2003a, 2006a,b), we add a dummy player who puts up for sale  $\varepsilon$  units for sale of each instrument (commodities, assets, loans) except for derivatives, for which s/he puts only  $\varepsilon^2$ . Furthermore, s/he puts up  $\varepsilon$  units of money for purchase on every market. This external player fully delivers on her/his promises. However, on her/his  $\varepsilon^2$  sale of derivatives, s/he delivers only up to a cap of  $\varepsilon$  dollars, actually delivering the minimum between  $\varepsilon$  and what s/he owes.

The other players act strategically, and prices form so as to clear every market (taking the dummy player into account). A type-symmetric Nash equilibrium (NE) of  $\Gamma^\varepsilon$  will be called an  $\varepsilon$ -Monetary Equilibrium ( $\varepsilon$ -ME).<sup>40</sup>

I first construct truncated strategy sets in the auxiliary game  $\Gamma_\pi^\varepsilon$ . By assumption, collateral requirements for each asset are non zero. Choose a constant  $\mathbf{M}$  so large that, for each  $s, k$ ,  $\mathbf{M}\kappa_{s,k} \geq \bar{\mathbf{e}} := \sum_h \mathbf{e}_h$ . Thus, to sell  $\mathbf{M}$  units of asset  $\mathbf{A}_{sk}^\pi$  would require more collateral than is available in the entire economy.

<sup>39</sup>Given our rational expectations set-up, the price of any asset yielding a 0 return will necessarily be 0 at equilibrium. For instance, a call to purchase an ounce of gold at € 800 will be priced 0 if, at equilibrium, the price of gold is always strictly less than € 800.

<sup>40</sup>Throughout the proof, we confine ourselves to type-symmetric action profiles. By a slight abuse of notations, the action,  $\sigma^h$ , of type  $h$  will denote either the aggregate action,  $\int_{[0,1]} \sigma^\tau d\lambda(d)$ , or the action of a single, negligible, individual  $\tau \in [0, 1]$ . The interpretation should be clear from the context.

For any  $\varepsilon > 0$ , define  $\Sigma_\varepsilon^h := \{\sigma^h : 0 \leq \sigma^h \leq 1/\varepsilon\}$ , the ambient strategy space of type  $h$  where derivatives purchases and sales are bounded:

$$\forall h, 0 \leq \alpha_k^h \leq NM \text{ and } \tilde{\alpha}_k^h \leq NM.$$

Given an action profile  $\sigma = (\sigma^h)_t \in \prod_{t \in T} \Sigma_\varepsilon^h$ , define the macrovariables  $\eta(\sigma) := (r, \pi, p)$ , recursively as follows:

$$\begin{aligned} \frac{1}{1+r_n} &:= \frac{\varepsilon + M_n + \sum_{h \in H} \tilde{\mu}_n^h}{\varepsilon + \sum_h \mu_n^h} && (\text{n}^{\text{th}} \text{ loan market}) \\ \pi_k &:= \frac{\varepsilon + \sum_{h \in H} \tilde{\alpha}_k^h}{\varepsilon + \sum_{h \in H} \alpha_k^h} && (\text{k}^{\text{th}} \text{ derivative}) \\ p_{s\ell} &:= \frac{\varepsilon + \sum_h \tilde{q}_{s\ell}^h}{\varepsilon + \sum_h q_{s\ell}^h} && (\text{commodity } s\ell) \end{aligned}$$

Furthermore, define:

$$\mathbf{A}_{sk}^{\varepsilon, \sigma}(\tilde{\eta}_0, \tilde{\eta}_s) := \left( \frac{\varepsilon^2}{\varepsilon^2 + \sum_h \alpha_k^h} \right) \min \left\{ \mathbf{A}_{sk}(\tilde{\eta}_0; \tilde{\eta}_s), \frac{1}{\varepsilon} \right\} + \left( \frac{\sum_h \alpha_k^h}{\varepsilon^2 + \sum_h \alpha_k^h} \right) \mathbf{A}_{sk}(\tilde{\eta}_0, \tilde{\eta}_s),$$

which is the effective delivery of a derivative  $k$  in state  $s$ . The payoff to any player  $h$  is:

$$\Pi^h(\sigma, \sigma^h) := \begin{cases} u^h \left( (\mathbf{e}_{s\ell}^h - q_{s\ell}^h + \frac{\tilde{q}_{s\ell}^h}{p_{s\ell}})_{s\ell \in S^* \times L} \right) & \text{if } t := h \in H \\ u^j(\eta_\varepsilon(\sigma), \sigma^j) & \text{if } t := j \in J \end{cases}$$

Finally, the subset of  $\Sigma_\varepsilon^h$  that is feasible for player  $h$ , given  $\sigma$ , is  $\Sigma_\varepsilon^h \cap \hat{\Sigma}_{\eta_\varepsilon(\sigma)}^h$ , where  $\hat{\Sigma}_{\eta_\varepsilon(\sigma)}^h$  is defined in the same way as  $\Sigma_{\eta_\varepsilon(\sigma)}^h$ , but replacing  $A_{sk}(\tilde{\eta}_0, \tilde{\eta}_s)$  with  $A_{sk}^{\varepsilon, \sigma}(\tilde{\eta}_0, \tilde{\eta}_s)$  because of the specific behavior of the dummy player.

This completes the construction of the generalized market game  $\Gamma^\varepsilon$ . Since players can bid and supply on each side of each market, it can be interpreted as a double auction where only market orders are allowed (and not limit-price orders). Because, thanks to the introduction of the dummy player, all the standard convexity and continuity assumptions are satisfied, best-reply correspondences have a closed graph and convex values. They also have non-empty values at every price because asset purchases and sales are bounded (by  $M$ ). This bounds the potential arbitrages that may occur when the sale of an asset enables to finance an additional loan of money and the purchase of its collateral requirement.

Thus, the standard Kakutani-fixed-point argument ensures that there exists a type-symmetric pure NE in the truncated generalized game  $\Gamma_{M, \varepsilon}^\pi$ .

*Step 2.*  $\underline{\pi} \rightarrow 0^+$ .

Let  $(p..)$  be a monetary equilibrium of  $\mathcal{E}^\pi$  and let  $\pi \rightarrow 0^+$ . By construction, prices, consumption plans lie in a bounded set, so that we can choose a convergence subsequence. Let the limit prices be  $(p^*, \pi^*)$ . I claim that the player's best-reply correspondence verifies the standard boundary condition shared in GET by the aggregate excess demand set-valued map:

$$\|\text{BR}(p)\| \rightarrow \infty \text{ if } p \rightarrow \partial\mathbb{R}_+^L.$$

The proof of the claim follows closely the proof of Theorem 1 in Zame & Geanakoplos (2002). The unique difference lies in the cash-in-advance constraints which provide an upper-bound on the excess demand due to the scarcity of money. But, as prices shrink to zero, the purchasing power of money becomes infinite and the cash-in-advance constraints loose any bite. Details are omitted.

Commodity prices,  $p^*$  do not lie on the boundary  $\partial\mathbb{R}_+^L$ , for otherwise the best-reply map at prices  $(p^*, \pi^*)$  would be unbounded rather than zero. Therefore, the consumption plans are utility optimal in consumers' (truncated) budget set at prices  $(p^*, \pi^*)$ . Being the limit of feasible plans, they are feasible. Hence, the artificial bounds on asset purchases and sales are not binding at the prices  $(p^*, \pi^*)$ . Hence the limit associated with  $(p^*, \pi^*)$  constitutes a full-blown monetary equilibrium of  $\mathcal{E}^\varepsilon$ .

*Step 3.* Dropping the dummy player.

It remains to show that a limit of  $\varepsilon$ -ME, as  $\varepsilon \rightarrow 0^+$  is a *bona fide* ME of  $\mathcal{E}$ . The main steps of the proof are identical to that of Dubey & Geanakoplos (2003a, 2006b) and are not repeated here. This time, the variance with the papers just cited is that, here, we must take into account the durability of collaterals, the storability of commodities and the specific definition of an asset's return. But this does not impair the argument, which can be roughly summarized as follows. As  $\varepsilon \rightarrow 0^+$ , if prices remain bounded, then, up to a subsequence, the limit must be a ME of  $\mathcal{E}$ . Else, some prices diverge to infinity as  $\varepsilon \rightarrow 0^+$ , but then, the purchasing power of outside money goes to zero. This implies that the limit must be a no-trade equilibrium, which contradicts the gains-to-trade hypothesis. Details are left to the reader.

□