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Abstract This study analyses the dynamics of a two-dimensional overlapping generations economy with endogenous labour supply à la Reichlin (1986) and aspirations, i.e. effective consumption by individuals of the current generation depends on the standard of living (based on consumption experience) of those that belong to the previous generation. We show that the relative importance of aspirations in utility is responsible for the existence of either one (normalised) steady state or two steady states. In particular, when the relative degree of aspiration is fairly high, the supply of labour becomes higher than those corresponding to the normalised steady state because individuals want to increase the amount of time spent at work when they are young in order to increase consumption possibilities when they are old, since the relative importance of past consumption is high in such a case. As regards local stability, the normalised steady state can be determinate or indeterminate and can undergo either a transcritical bifurcation or supercritical flip bifurcation depending on the intensity of the taste externality. Moreover, some interesting global dynamic properties emerge: indeed, when the relative importance of aspirations is strong enough, cyclical or quasi-cyclical behaviour and/or coexistence of attractors may occur. In particular, this last phenomena may cause global indeterminacy even if the stationary equilibria are locally determinate.

Keywords Aspirations; Indeterminacy; Labour supply; OLG model; Nonlinear dynamics

JEL Classification C61; C62; C68; J22; O41

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1. Introduction

The treatment of economic models with habit and aspiration formation (namely, past actions) of individuals has received in depth attention in both the theoretical and empirical literature in recent years. In particular, habits are defined as the case under which individual preferences depend on both current and past consumption experiences of the individual under scrutiny. In contrast, aspirations are assumed to represent the case under which preferences of an individual are affected by own consumption as well as by the consumption experience of ancestors. In a context with overlapping generations (OLG), which seems to be a natural basis where studying habit and aspiration formation, the existence of habits implies that preferences over old-age consumption by the current generation depends also on young-age consumption, while the existence of aspirations implies that preferences over consumption by the current generation are affected by the standard of living based on consumption experience by past generations (parents), which represents a reference to compare the level of current consumption.

On theoretical grounds, studies especially regard: (*i*) the role that habits and aspirations can play in affecting the way intergenerational transfers operate on the mechanics of capital accumulation and wealth inequality (see, e.g., de la Croix and Michel, 2001; Alonso-Carrera et al., 2007), and (*ii*) the effects of habits and aspirations on economic growth and stability of long-run equilibria (see, e.g., de la Croix, 1996; de la Croix and Michel, 1999; de la Croix, 2001; Artige et al., 2004; Alonso-Carrera et al., 2005).[1] On empirical grounds, evidence exists about the way habit and aspiration formation affect consumers' tastes over time (see, e.g., Ferson and Constantinides, 1991; de la Croix and Urbain, 1998; Carrasco et al., 2005).

The present study aims at analysing the role played by inherited tastes (aspirations) on the dynamics of a two-dimensional OLG economy where individuals work when they are young and consume only when they are old (see Reichlin, 1986). The interest in studying growth models that generate endogenous deterministic fluctuations that resemble random ones dates back at least to, e.g., Grandmont (1985), Farmer (1986), Reichlin (1986) and Azariadis (1993), and subsequently several other authors have dealt with this topic in OLG models either with exogenous (e.g., Yokoo, 2000) or endogenous (e.g., Nourry, 2001; Nourry and Venditti, 2006) labour supply.

The papers that are more closely related to ours in the theoretical literature are de la Croix (1996) and de la Croix and Michel (1999). However, both papers deal with an OLG economy where the labour supply is inelastic and individuals consume in both the first period and second period of their life (as in Diamond, 1965). Indeed, in such a case the existence of inherited tasted makes the dynamics of the economy be characterised by a two-dimensional system because of the accumulation of the stock of physical capital and the accumulation of the stock of aspirations. In particular, de la Croix (1996) shows that when the intensity of aspirations in utility is fairly high, adult individuals want to increase consumption because the standard of living of their parents is high and then savings becomes low. This can indeed generate a Neimark-Sacker bifurcation and endogenous cycles when savings experience too high a contraction because of the importance of past consumption levels. De la Croix and Michel (1999), instead, concentrates on the optimality issue in a growth model with

[1] Another interesting study that deals with the effects of habits on savings and equilibrium dynamics in a pure exchange economy is Lahiri and Puhakka (1998).

aspirations that generate a negative consumption externality (when individual are selfish) that can be corrected by an adequate use of investment subsidies and lump-sum transfers. Moreover, the authors also show that (*i*) when the taste externality is strong enough, the competitive equilibrium can be destabilised through a Neimark-Sacker bifurcation and endogenous fluctuations can occur, and (*ii*) the planner solution can actually experience damped oscillations even under the case of no discounting of utilities of future generations.

The novelty of the present paper is represented by the introduction of aspirations in an OLG growth model with endogenous labour supply à la Reichlin (1986), i.e. individuals work when they are young and consume only when they are old. First, we show that the relative importance of aspirations in utility is responsible for the existence of either one (normalised) steady state (which can be determinate or indeterminate) or two steady states. Second, some interesting local and global dynamic properties of the two-dimensional decentralised economy emerge: indeed, when the relative importance of aspirations in utility is strong enough cyclical or quasi-cyclical behaviour and/or coexistence of attractors may occur. In particular, this last phenomenon as well as the existence of some global bifurcations may cause global indeterminacy to the model while the stationary equilibria are locally determinate.

The rest of the paper is organised as follows. Section 2 builds on the overlapping generation model with aspirations and labour supply. Section 3 studies the conditions for the existence of steady states, and analyses both the local and global properties of the two-dimensional dynamic system. Section 4 concludes.

2. The model

2.1. Individuals

We consider an OLG closed economy populated by a continuum of perfectly rational and identical two-period lived (selfish) individuals of measure one per generation (see Diamond, 1965), and a new generation is born in every period. In the first period of life (youth), the individual of generation t is endowed with two units of labour and supplies the share $\ell_t \in (0,2)$ to firms, while receiving the wage w_t per unit of labour. Indeed, the remaining share $2-\ell_t$ is used for leisure activities. Individuals consume only in the second period of life (see, e.g., Reichlin, 1986; Galor and Weil, 1996; Grandmont et al., 1998; Antoci and Sodini, 2009; Gori and Sodini, 2011). Therefore, the budget constraint of an individual born at time t simply reads as $s_t = w_t \ell_t$, that is the (labour) income earned in the first period of life is entirely saved (s_t) for consumption purposes in the second period of life (C_{t+1}). In such a period (old age), individuals retire and consumption is therefore constrained by the amount of resources saved when young plus the expected interest accrued from time t to time $t+1$ at rate $r^e{}_{t+1}$, that is $C_{t+1} = R^e{}_{t+1} s_t$, where $R^e{}_{t+1} := 1 + r^e{}_{t+1}$ is the (expected) interest factor.[2]

[2] The existence of inter-generational transfers, e.g., intentional bequests (see Chakraborty and Das, 2005; Alonso-Carrera et al., 2007) is avoided in the present study. The reasons why people make intergenerational transfers can be different. However, the empirical economic literature has found evidence about intergenerational altruism as being one of the most important reasons for the existence of these transfers (see, e.g., Hurd, 1987; Gale and Scholz, 1994).

Therefore, the lifetime budget constraint of an individual born at time t is expressed as follows:

$$C_{t+1} = R^e{}_{t+1} w_t \ell_t \,. \tag{1}$$

Individuals have preferences towards leisure time when young and material consumption when old. Moreover, individual preferences by the current generation are negatively affected by the level of consumption of the old-aged (parents) that belong to the previous generation (C_t).[3] The existence of inherited tastes represents an inter-generational (negative) consumption externality (see de la Croix, 1996; de la Croix and Michel, 1999), because they give rise to a reference against which consumption when old within a certain generation is compared to, and an increase in it increases the need for an individual of the current generation to raise her own consumption bundles to keep utility unaffected. Effective consumption of the old-aged, therefore, depends on consumption experience of their predecessors, and this gives rise to a form of aspirations in our model.[4]

We assume that the lifetime utility index of generation t is described by a twice continuously differentiable utility function $U_t\left(\hat{C}_{t+1}, \ell_t\right)$, where $\hat{C}_{t+1}$ is a variable that represents effective consumption when old. We now assume that the individual that belongs to the current generation "inherits standard-of-living aspirations a_t from… parents" (de la Croix and Michel, 1999, p. 521). Since in the present study individuals consume only when they are old, the consumption experience of the old-aged that belong to the past generation (i.e., those born at time $t-1$), C_t, affects desired consumption by the current generation (i.e., those born at time t) in such a way that $a_t = C_t$ for every t. This means *de facto* that aspirations are not forgotten at the end of every period, that is the rate of depreciation of aspirations is zero and thus they affect consumption in the second period of life (old-age). By assuming that aspirations take the multiplicative form $\hat{C}_{t+1} = \frac{C_{t+1}}{a_t^{\rho}}$, introduced by Abel (1990) and Galí (1994) for the case of habits, and used, amongst others, by Carroll (2000), Bunzel (2006) and Hiraguchi (2011), we specify the lifetime utility index by using the following Constant Inter-temporal Elasticity of Substitution (CIES) formulation:

$$U_t\left(\hat{C}_{t+1}, \ell_t\right) = \frac{(2-\ell_t)^{1-\gamma}}{1-\gamma} + \frac{B}{1+\theta} \frac{\left(\dfrac{C_{t+1}}{a_t^{\rho}}\right)^{1-\sigma}}{1-\sigma}, \tag{2}$$

where $\theta \geq 0$ is the psychological subjective discount rate, B is a scale parameter in the utility function that allows us to define the (normalised) fixed point $(1,1)$ when the basic parameters of the problem are continuously changed, $\sigma > 0$ ($\sigma \neq 1$) and $\gamma > 0$ ($\gamma \neq 1$) are two parameters attached to effective consumption and leisure, respectively, and represent a measure of the reciprocal of the elasticity of marginal utility with respect to the corresponding arguments in the utility function. Moreover, $\rho > 0$ indexes the relative importance of aspirations in utility (i.e., the aspiration intensity):

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[3] Note that every agent of previous generations is the parent of agents that belong to subsequent generations, as is usual in the overlapping generations literature.

[4] Alternatively, this means that the standard of living of an individual born at time t is determined by the consumption experience of an individual that belongs to the previous generation (time $t-1$).

if $\rho = 0$ the model collapses to the standard model in which consumers only care about the level of current consumption (i.e., aspirations are irrelevant) and leisure; if $\rho = 1$ current and past consumptions are equally weighted; if $\rho > 1$ aspirations strongly matter.

By taking factor prices and the consumption reference C_t of parents as given, the individual of generation t chooses ℓ_t to maximise Eq. (2) subject to Eq. (1) and $\ell_t \in (0,2)$. Therefore, the first order conditions for an interior solution are given by:

$$-(2-\ell_t)^{-\gamma} + \frac{B}{1+\theta} \cdot \frac{\left(\frac{R^e{}_{t+1} w_t \ell_t}{a_t{}^\rho}\right)^{1-\sigma}}{\ell_t} = 0 . \tag{3}$$

2.2. Firms

Since in the present study we concentrate on the dynamic effects of the existence of a negative externality on the consumers side, then in contrast to, e.g., amongst others, Grandmont et al. (1998) and Cazzavillan (2001), which adopt a Constant Elasticity of Substitution (CES) technology or consider externality in the production sector, we assume that at time t identical and competitive firms produce a homogeneous good, Y_t, by combining capital and labour, K_t and L_t, respectively, through the *constant returns to scale Cobb-Douglas technology* $Y_t = A \cdot F(K_t, L_t) = AK_t{}^\alpha L_t^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$ are a scale parameter and the capital share in production, respectively. The equilibrium supply of labour at time t is given by the equality $L_t = \ell_t$. Then, by assuming that capital fully depreciates at the end of every period and output is sold at unit price, profit maximisation implies that factor inputs are paid their marginal products, that is:

$$R_t = \alpha A k_t{}^{\alpha-1} , \tag{4}$$

$$w_t = (1-\alpha) A k_t{}^\alpha , \tag{5}$$

where $k_t := K_t / \ell_t$ is capital per efficient worker.

2.3. Equilibrium

The market-clearing condition in the capital market can be expressed as:

$$k_{t+1} \ell_{t+1} = s_t = w_t \ell_t . \tag{6}$$

Then, by exploiting Eqs. (3), (4), (5) and (6), and knowing that: (*i*) individuals have perfect foresight, that is the expected interest factor $R^e{}_{t+1}$ is a function of the capital stock at time $t+1$ (see the one-period forward Eq. 4), and (*ii*) the consumption reference can be expressed as $a_t = C_t = \alpha A k_t{}^\alpha \ell_t$, the equilibrium condition for an economy with young labour supply and aspirations is described by the following equations:

$$-(2-\ell_t)^{-\gamma} + \frac{B}{1+\theta} \cdot \frac{\left[\frac{\alpha A^2 (1-\alpha) k_t{}^\alpha k_{t+1}{}^{\alpha-1} \ell_t}{\left(\alpha A k_t{}^\alpha \ell_t\right)^\rho}\right]^{1-\sigma}}{\ell_t} = 0 , \tag{7}$$

$$k_{t+1}\ell_{t+1} = A(1-\alpha)k_t^{\alpha}\ell_t. \tag{8}$$

3. Dynamic analysis

3.1. Existence of steady states

The dynamic system characterised by Eqs. (7) and (8) defines the variables k_{t+1} and ℓ_{t+1} as functions of k_t and ℓ_t. In this section, we study the stability of steady states of such a discrete dynamical system. To simplify the analysis, we apply the geometrical-graphical method developed by Grandmont et al. (1998) that allows us to characterise the stability properties of a specific steady state of a two-dimensional dynamic system. We now impose some conditions on parameters under which a steady state (k_{ss}, ℓ_{ss}) with $k_{ss} = \ell_{ss} = 1$ does exist. This permits to analyse the effects on stability due to changes in some parameter values while being sure that the steady state does not vanish. Therefore, by setting $k_{ss} = \ell_{ss} = 1$ and using Eqs. (7) and (8) we find that the following restrictions on the scaling parameters A and B must hold, that is:

$$A = A^* := \frac{1}{1-\alpha}, \tag{9}$$

$$B = B^* := \frac{1+\theta}{\left(\frac{\alpha}{1-\alpha}\right)^{(1-\sigma)(1-\rho)}}. \tag{10}$$

Now, by using Eqs. (7), (8), (9) and (10), the system that characterises the dynamics of the economy, which is defined on the set $D := \{(k_t, \ell_t) \in R^2 : k_t > 0,\ 0 < \ell_t < 2\}$, can explicitly be written in the following way:

$$k_{t+1} = V(k_t, \ell_t) := \left[k_t^{\alpha(\rho-1)(\sigma-1)}\ell_t^{-\sigma-\rho+\sigma\rho}(2-\ell_t)^{\gamma}\right]^{\frac{1}{(1-\sigma)(1-\alpha)}}, \tag{11}$$

$$\ell_{t+1} = Z(k_t, \ell_t) := \ell_t k_t^{\alpha}\left[\frac{1}{k_t^{\alpha(\rho-1)(\sigma-1)}\ell_t^{-\sigma-\rho+\sigma\rho}(2-\ell_t)^{\gamma}}\right]^{\frac{1}{(1-\sigma)(1-\alpha)}}. \tag{12}$$

Notice that $k = 1$ always holds as a steady state of the system defined by Eqs. (11) and (12), while the steady state values of ℓ are determined as solutions of the following equation:

$$g(\ell) := \ell^{\rho(\sigma-1)-\sigma}(2-\ell)^{\gamma} = 1. \tag{13}$$

Of course, $\ell = 1$ is a solution of Eq. (13), that is the normalised steady state exists for every constellation of parameters.

We are now in a position to state the following proposition as regards the existence of steady states of the discrete dynamic system given by Eqs. (11) and (12).

Proposition 1. *[Existence of steady states]. If either $\sigma < 1$ or $\sigma > 1$ and $\rho < \frac{\sigma}{\sigma-1}$, then $(1,1)$ is the unique steady state. If $\sigma > 1$ and $\frac{\sigma}{\sigma-1} < \rho < \frac{\sigma+\gamma}{\sigma-1}$ another steady state exists, with $\ell < 1$. If $\sigma > 1$ and $\rho > \frac{\sigma+\gamma}{\sigma-1}$ another steady state exists, with $\ell > 1$.*

Proof. Notice that $\lim_{\ell \to 2} g(\ell) = 0$ for every constellation of parameters, while $\lim_{\ell \to 0} g(\ell) = +\infty$ (resp. $\lim_{\ell \to 0} g(\ell)$) if and only if (*a*) $\sigma < 1$ or (*b*) $\sigma > 1$ and $\rho < \frac{\sigma}{\sigma - 1}$ (resp. $\sigma > 1$ and $\rho > \frac{\sigma}{\sigma - 1}$). Moreover, with direct computation we have that $\mathrm{sgn}\{g'(\ell)\} = \mathrm{sgn}\{(\rho(\sigma - 1) - \sigma + \gamma)\ell + 2(\sigma + \rho - \rho\sigma)\}$ and $\mathrm{sgn}\{g'(1)\} = \mathrm{sgn}\{\rho(\sigma - 1) - \sigma - \gamma\}$. It follows that $g(\ell)$ is either monotone or unimodal. In the former case, no solution other than $\ell = 1$ of Eq. (13) does exist. In the latter case, a further solution of Eq. (13) exists on the left (resp. on the right) of $\ell = 1$ if $\rho < \frac{\sigma + \gamma}{\sigma - 1}$ (resp. $\rho > \frac{\sigma + \gamma}{\sigma - 1}$). **Q.E.D.**

Proposition 1 shows the crucial role played by both the reciprocal of the elasticity of marginal utility of effective consumption and intensity of aspirations in utility (i.e., the relative importance of the taste externality), in determining the existence of either one steady state or two steady states. In particular, when σ is low (i.e., the elasticity of marginal utility of effective consumption is high) and/or the importance of the taste externality (the relative degree of aspiration, ρ) is low, a unique (normalised) steady state does exist. When σ raises together with the relative degree of aspirations, a second steady state appears with leisure being either lower or higher than the level corresponding to the normalised steady state. In particular, when the relative degree of aspirations is fairly high, the supply of labour becomes higher than 1 because individuals want to increase the amount of time spent at work when they are young in order to increase consumption possibilities when they are old, since the relative importance of past consumption is high in such a case.

3.2. Local bifurcation and stability

This section starts by analysing the local dynamics around the normalised steady state. In the present model, the stock of capital K_t is a state variable, so its initial value K_0 is given, while the supply of labour ℓ_t is a control variable. It follows that individuals of the first generation ($t = 0$) choose the initial value ℓ_0 (and then the initial value of capital per efficient worker $k_t = K_t / \ell_t$, namely $k_0 = K_0 / \ell_0$). If the normalised steady state is a saddle and the initial condition of K is close enough to 1, then, given the expectations on the interest rate, there exists a unique initial value of ℓ_t (ℓ_0) such that the orbit that passes through (k_0, ℓ_0) approaches the steady state. In contrast, when the steady state is a sink, given the initial value K_0 and the expectations on the interest rate, there exists a continuum of initial values ℓ_0 such that the orbit that passes through (k_0, ℓ_0) approaches the steady state. As a consequence, the orbit that the economy will follow is "locally indeterminate" because it depends on the choice of ℓ_0.

The Jacobian matrix of the dynamic system defined by Eqs. (11) and (12), evaluated at the normalised steady state $(1,1)$ is:

$$J=\begin{pmatrix} \frac{\alpha(1-\rho)}{1-\alpha} & -\frac{\gamma+\sigma+\rho(1-\sigma)}{(1-\sigma)(1-\alpha)} \\ \frac{\alpha(\rho-\alpha)}{1-\alpha} & \frac{(\rho-\alpha)(1-\sigma)+\gamma+1}{(1-\sigma)(1-\alpha)} \end{pmatrix}. \tag{14}$$

The trace and determinant of the Jacobian matrix Eq. (14) are the following:

$$Tr(J)=\frac{1+\gamma}{(1-\alpha)(1-\sigma)}+\rho, \tag{15}$$

$$Det(J)=\frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}. \tag{16}$$

Given the values of the other parameters of the model, when the index that measures the relative importance of aspirations in utility, ρ, varies, the point (see Eqs. 15 and 16)

$$(P_1,P_2):=\left(\frac{1+\gamma}{(1-\alpha)(1-\sigma)}+\rho,\frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}\right), \tag{17}$$

drawn in the $(Tr(J),Det(J))$ plane, describes a horizontal half-line T_1 that starts from the point

$$(\overline{P}_1,\overline{P}_2):=\left(\frac{1+\gamma}{(1-\alpha)(1-\sigma)},\frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}\right), \tag{18}$$

when $\rho=0$.

In turn, when the capital share in production, α, varies, the point $(\overline{P}_1,\overline{P}_2)$ in Eq. (18), drawn in $(Tr(J),Det(J))$ plane, describes a half-line T_2 (with slope equal to 1) that starts from the point $\left(\frac{1+\gamma}{1-\sigma},0\right)$ when $\alpha=0$. If $\sigma\in(0,1)$ (resp. $\sigma>1$), then $(\overline{P}_1,\overline{P}_2)\to(+\infty,+\infty)$ (resp. $(\overline{P}_1,\overline{P}_2)\to(-\infty,-\infty)$) for $\alpha\to 1$. Moreover, regardless of the value of σ, $(P_1,P_2)\to\left(+\infty,\frac{\alpha(1+\gamma)}{(1-\alpha)(1-\sigma)}\right)$ for $\rho\to+\infty$.

From the above geometrical findings and Proposition 1, we can state the following proposition as regards local bifurcations.

Proposition 2. *[Local bifurcation]. Let* $\rho_{fl}:=\frac{2+\sigma(\alpha-1)+\gamma(1+\alpha)}{(1-\alpha)(\sigma-1)}$ *and* $\rho_{tc}:=\frac{\sigma+\gamma}{\sigma-1}$ *hold. Then, (1) if* $\sigma\in(0,1)$*, the normalised (unique) steady state is determinate (see the first quadrant of Figure 1); (2) if* $\sigma>2+\gamma$ *and* $\alpha<\frac{\sigma-2-\gamma}{\sigma+\gamma}$ *the normalised steady state is indeterminate for* $\rho\in(0,\rho_{tc})$*, it undergoes a transcritical bifurcation for* $\rho=\rho_{tc}$*, and it is a saddle for* $\rho>\rho_{tc}$ *(see the second and third quadrants of Figure 1); (3) if* $\sigma>2+\gamma$ *and* $\alpha>\frac{\sigma-2-\gamma}{\sigma+\gamma}$ *or* $1<\sigma<2+\gamma$*, the normalised steady state is determinate for* $\rho<\rho_{fl}$*, it undergoes a supercritical flip bifurcation for* $\rho=\rho_{fl}$*, it is indeterminate for* $\rho\in(\rho_{fl},\rho_{tc})$*, it undergoes a transcritical bifurcation for* $\rho=\rho_{tc}$*, and it is a saddle for* $\rho>\rho_{tc}$ *(see the second and third quadrants of Figure 1).*

Proof. In order to find the bifurcation values of ρ, we impose the condition that (P_1, P_2) belongs to: (*i*) the straight line $1 - Tr(J) + Det(J) = 0$, to obtain the transcritical bifurcation value ρ_{tc}, and (*ii*) the straight line $1 + T(J) + Det(J) = 0$, to obtain the flip bifurcation value ρ_{fl}. Then, we identify Cases 1-3 by considering the position of the starting points $(\overline{P}_1, \overline{P}_2)$ and (P_1, P_2) with respect to the stability triangle delimited by $1 \pm Tr(J) + Det(J) = 0$ and $Det(J) = 1$ (see Grandmont and al 1998 for details). **Q.E.D.**

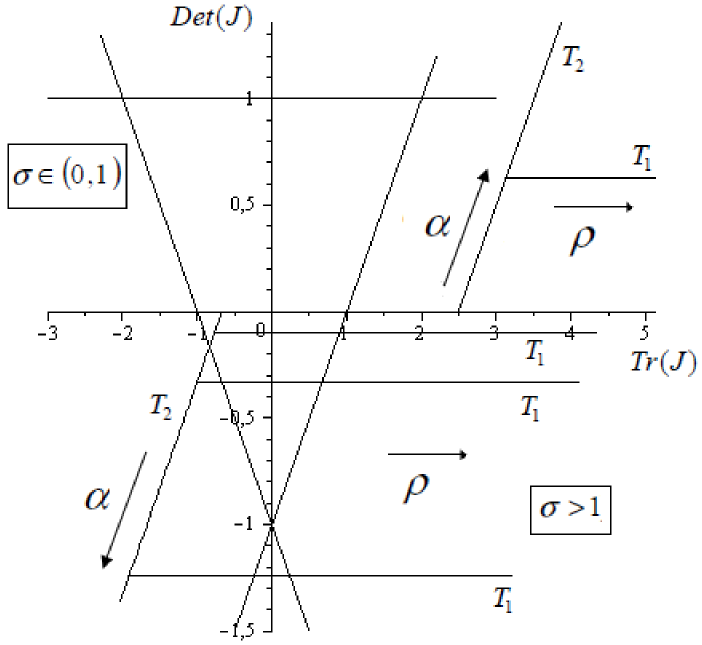


Figure 1. Stability triangle and local indeterminacy. If $\sigma \in (0,1)$ a unique steady state (saddle) exists; if $\sigma > 1$ multiple steady states and local indeterminacy may occur.

3.3. Global analysis

In the previous section we have characterised the local properties of the dynamic system in the neighbourhood of the normalised steady state $(1,1)$. In particular, we have shown that quasi periodic trajectories obtained through a Neimark-Sacker bifurcation cannot be observed in our model with young labour supply and aspirations.

In this section we show how the study of (*i*) the dynamics around the non-normalised steady state, and (*ii*) the global structure of the dynamic system defined by Eqs. (11) and (12), permit us to explain interesting phenomena that cannot indeed be observed with the local analysis (as regards this point, see Pintus et al., 2000).

We start the global analysis by showing that the map M described by Eqs. (11) and (12) is invertible. The invertibility of a map is an important result when the global properties of a dynamic system are studied. For instance, it implies that the basins of attraction of any attracting set of a map are connected sets. Furthermore, by making use of the inverse map, we can obtain the boundary of the attracting sets and, more generally, the stable manifolds of saddle points.

As regards the map defined by Eqs. (11) and (12), the following lemma holds.

Lemma 1. *The map M defined by Eqs. (11) and (12) is invertible on the set D.*

Proof. First, we note that M can be expressed as follows:

$$k_{t+1} = V(k_t, \ell_t), \tag{19}$$

$$\ell_{t+1} = \frac{\ell_t k_t^{\alpha}}{V(k_t, \ell_t)}, \tag{20}$$

where $V(k_t, \ell_t)$ is defined by Eq. (11). As a consequence, the Jacobian matrix is given by:

$$Q = \begin{pmatrix} V'_{k_t} & V'_{\ell_t} \\ \ell_t k_t^{\alpha} \dfrac{\alpha \dfrac{V(k_t,\ell_t)}{k_t} - V'_{k_t}}{V^2(k_t,\ell_t)} & \ell_t k_t^{\alpha} \dfrac{\dfrac{V(k_t,\ell_t)}{\ell_t} - V'_{\ell_t}}{V^2(k_t,\ell_t)} \end{pmatrix}. \tag{21}$$

The determinant of the Jacobian matrix Eq. (21) is computed as follows:

$$Det(Q) = \frac{\ell_t k_t^{\alpha}}{V^2(k_t,\ell_t)} \left[V'_{k_t} \left(\frac{V(k_t,\ell_t)}{\ell_t} - V'_{\ell_t} \right) - V'_{\ell_t} \left(\alpha \frac{V(k_t,\ell_t)}{k_t} - V'_{k_t} \right) \right]. \tag{22}$$

It follows that $Det(Q) = 0$ on the set D if and only if the expression in brackets of Eq. (22) vanishes. Rearranging terms from Eq. (22), therefore, we get:

$$Det(Q) = 0 \Leftrightarrow \frac{V'_{k_t} k_t}{V'_{\ell_t} \ell_t} - \alpha = 0 \Leftrightarrow \ell_t = \frac{2}{1-\gamma}. \tag{23}$$

Since $\gamma > 0$, then we have the result. **Q.E.D.**

Notice that it is impossible to have a closed-form expression of the inverse map of M, i.e. M^{-1}, but after some algebraic manipulations it is possible to find that it is solution of the following system:

$$M^{-1} : \begin{cases} \ell_t^{\frac{1-\alpha}{1-\rho}} (2 - \ell_t)^{\frac{\gamma(1-\alpha)}{\rho-1}} = \dfrac{\ell_{t+1}^{(\sigma-1)(\alpha-1)}}{k_{t+1}^{\frac{(\sigma-1)(\alpha-1)(\alpha-\rho)}{\rho-1}}} \\ k_t = \left(\dfrac{k_{t+1} \ell_{t+1}}{\ell_t} \right)^{\frac{1}{\alpha}} \end{cases}. \tag{24}$$

Before performing the global analysis of the dynamic system defined by the map M, we recall the definitions of both stable

$$W^s(p) = \{ x : M^{zn}(x) \to p \ as \ n \to +\infty \}, \tag{25}$$

and unstable

$$W^u(p) = \{ x : M^{zn}(x) \to p \ as \ n \to -\infty \}, \tag{26}$$

manifolds of a periodic point p of period z. If the periodic point $p \in R^2$ is a saddle, then the stable (resp. unstable) manifold is a smooth curve through p, tangent at p to the eigenvector of the Jacobian matrix evaluated at p corresponding to the eigenvalue λ with $|\lambda| < 1$ (resp. $|\lambda| > 1$), see, e.g., Guckenheimer and Holmes (1983). Outside the neighbourhood of p, the stable and unstable manifolds may even intersect each other with dramatic consequences on the global dynamics of the model (see, e.g., Guckenheimer and Holmes, 1983, p. 22).

Non-trivial intersection points of stable and unstable manifolds of a unique saddle cycle are known as homoclinic points. However, when multiple saddle cycles exist, heteroclinic bifurcations may also occur. We remember that given two saddle cycles h_1 and h_2, a heteroclinic bifurcation is defined as the birth of a non trivial point E of intersection between the stable manifold of one cycle and the unstable manifold of the other cycle. Starting from this new configuration, it is possible to find a path on the two manifolds that connects the cycles. This phenomenon is interesting from an economic point of view because is related to global indeterminacy. We recall that global indeterminacy occurs when, starting from the same initial condition K_0 of the state variable K_t, different fixed points or other ω-limit sets can be reached according to the initial value ℓ_0 of the jumping variable ℓ_t chosen by individuals of the first generation (see, e.g., Agliari and Vachadze, 2011, Gori and Sodini, 2011 and the related literature cited therein).

Furthermore, when saddle cycles exist, heteroclinic connections as well as homoclinic tangles (see Agliari et al., 2005) may be the engine of the birth either of repelling or attractive closed invariant curves. Let us remind that the dynamics of the restriction of a map to a closed invariant curve is either quasi-periodic or periodic but of very high period, so that it is numerically indistinguishable from a quasi-periodic one.

In the analysis we are going to perform below, we fix these parameter values (chosen only for illustrative purposes): $\alpha = 0.35$ (see Gollin, 2002, for estimates of the capital share in production), $\gamma = 1.2$, $\sigma = 3$ and let the parameter ρ (that is, the intensity of aspirations in the utility function) vary. We start our analysis from the value $\rho = 2.1$, corresponding to which the normalised steady state $(1,1)$ is saddle stable, that is it is locally determinate (see Section 3.2), while the other fixed point $(1,\hat{\ell})$ is indeterminate, and its basin of attraction is defined by the stable manifold of the normalised steady state. Even if the normalised steady state is locally determinate, by considering a small neighbourhood of $(1,1)$, there exist trajectories that converge to the interior equilibrium, and a heteroclinic connection between $(1,1)$ and $(1,\hat{\ell})$ does exist, which is made up by the branch of the unstable manifold of the saddle that converges to $(1,\hat{\ell})$, as shown in Figure 2. While this result is not surprising and generically occurs when the saddle point belongs to the border of the basin of attraction of the attracting steady state, the economic literature on local (in)determinacy does not stress the importance of the existence of a continuum of equilibria around the determined one (an exception is represented by Agliari and Vachadze, 2011), which can indeed be of interest especially from a policy perspective.

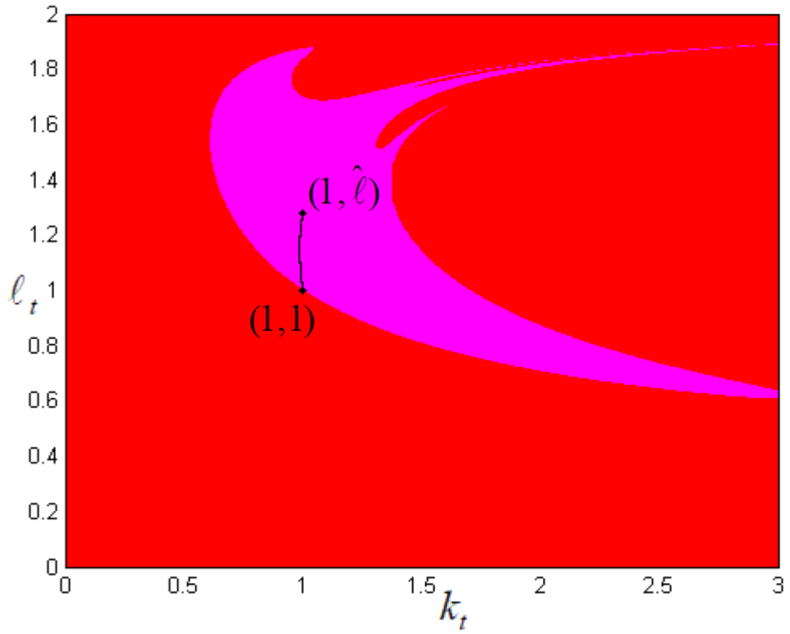


Figure 2. ($\rho = 2.1$). The fixed point $(1,\hat{\ell})$ (note that $\hat{\ell} \cong 1.2818$) is the unique attractor of the system (indeterminate equilibrium), while the normalised fixed point $(1,1)$ belongs to the boundary of the attractor. Legend: the basin of the attraction of $(1,\hat{\ell})$ is pink-coloured while the region of unfeasible trajectories is red-coloured. The curve that connects $(1,\hat{\ell})$ and $(1,1)$ is the simulated unstable manifold of $(1,1)$ (i.e., the heteroclinic connection), obtained by iterating a small segment in the direction of the unstable eigenvector. If we consider an economy that starts exactly on the stable manifold that converges to $(1,1)$, a small change (shock) in the expectations about the future interest rate r^e_{t+1} (see Section 2) may cause the convergence to $(1,\hat{\ell})$, where the capital stock is the same and the labour supply higher than the values corresponding to the normalised steady state.

If we let ρ increase, a flip bifurcation occurs for the equilibrium $(1,\hat{\ell})$ at $\rho \cong 2.3078$, and a two-period cycle captures almost all non diverging trajectories. At this stage, the stable manifold of $(1,1)$ persists to define the basin of attraction of the attractor, while the point $(1,1)$ is globally indeterminate in a more general form: in fact, even if we restrict the model to a small neighbourhood, there exist (*i*) an infinite number of trajectories that converge to the attractor of the system, and (*ii*) a unique trajectory, i.e. the unstable manifold of $(1,1)$ that converges to the interior saddle $(1,\hat{\ell})$.

In order to better illustrate the dynamic properties of the map at this stage, we study (through numerical simulations) the second (forward) iterate of the map M, namely M^2. We now recall that when the second iterate of a map is studied, the fixed points of the original map hold as fixed points of the second iterate, while the two-period cycles of the original map become fixed points. In other words, the flip bifurcation previously with the study of the map M becomes a pitchfork bifurcation of M^2 and two attracting fixed points are separated by the stable manifold of the intermediate saddle.

Trajectories, as those drawn in Figure 3, starting just out the left or the right of the stable manifold, follow the curve almost until the saddle $(1,\hat{\ell})$ but converge to V_1 and

V_2. As regards the economic implication of this framework, by considering an historically given value of the capital stock K_0, individuals may either coordinate themselves on the corresponding stable manifold(s) of the system or choose another feasible value of labour supply ℓ_t. This implies that, after the transient, the economy oscillates between the two points.

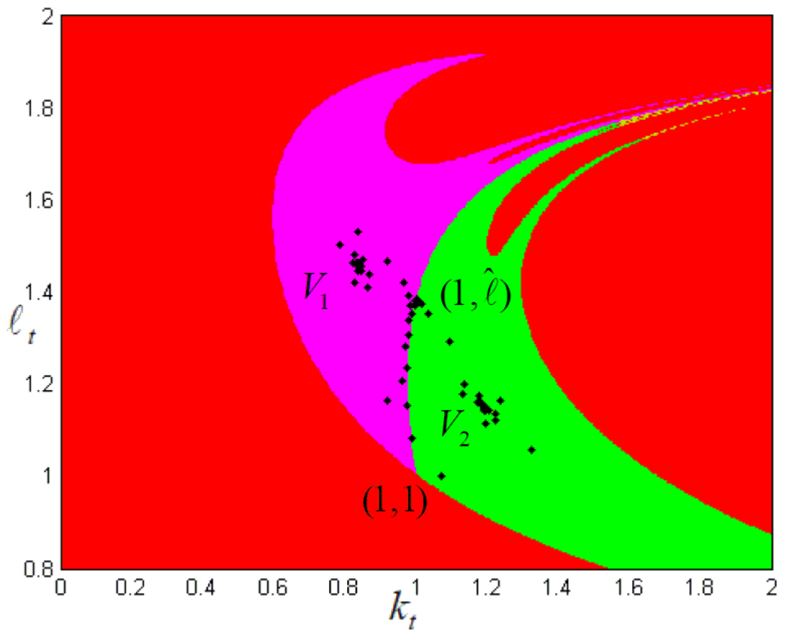
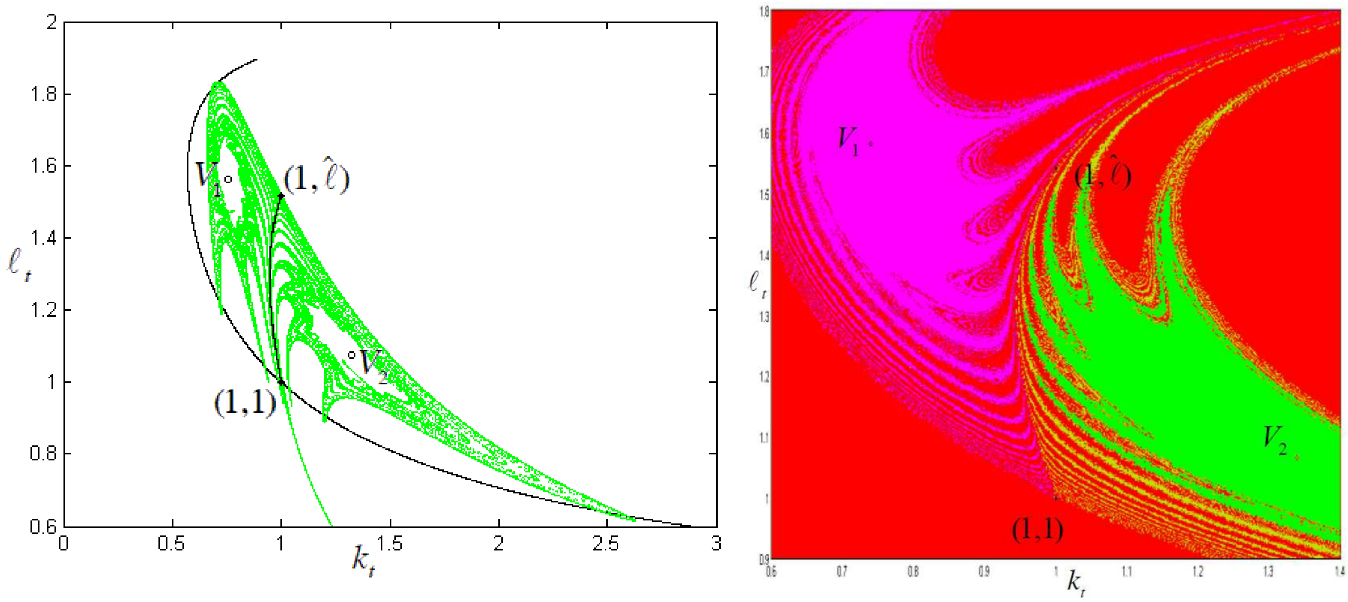
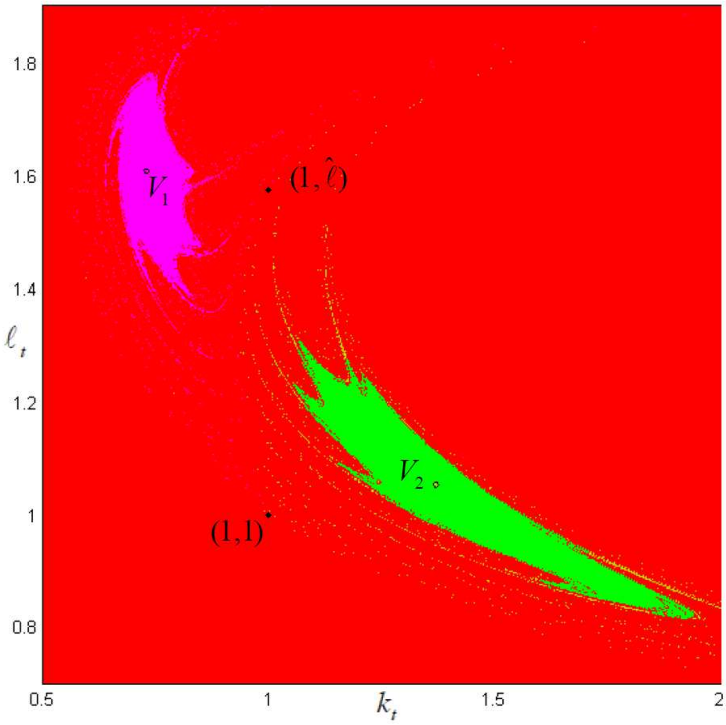


Figure 3. ($\rho = 2.4$). The map M^2 has two attracting fixed points $V_1 = (0.8642, 1.428)$ and $V_2 = (1.1592, 1.1719)$. The stable manifold of $(1,\hat{\ell})$ (note that $\hat{\ell} \cong 1.3575$) separates the basins of attraction of V_1 (pink-coloured) and V_2 (green-coloured), while the stable manifold of $(1,1)$ defines the boundary of the feasible trajectories (the region of unfeasible trajectories is red-coloured).



(a) (b)



(c)

Figure 4. (a) The upper branch of the unstable manifold $(1,1)$ converging to $(1,\hat{\ell})$ that coincides with the lower branch of the saddle $(1,\hat{\ell})$ and the unstable manifold of $(1,\hat{\ell})$ that forms, together with the intersection with the stable manifold of $(1,1)$, the heteroclinic orbits from $(1,\hat{\ell})$ to $(1,1)$ ($\rho = 2.58$). (b) Basin of attraction of M^2 when $\rho = 2.58$. (c) Evolution of the basin of attraction of M^2 when $\rho = 2.628$: at this stage, the figure shows that around $k = 1$ only a small subset of values of ℓ converges to the equilibria of M^2.

When we let ρ increase, we can see more and more convolutions of the boundary of the basins of the two foci around the normalised steady state. In particular, as shown in Figures 4.a and 4.b for $\rho = 2.58$, a heteroclinic bifurcation has involved the stable manifold of $(1,1)$ and the unstable manifold of $(1,\hat{\ell})$, and a heteroclinic path that starts from $(1,\hat{\ell})$ and converges to $(1,1)$ does exists. This implies that even if the economy lies exactly on the path that converges to $(1,\hat{\ell})$, a small change on the expectations on the future interest rate $r^e{}_{t+1}$ may lead the economy on a path that converges to the normalised steady state $(1,1)$. From the numerical study, it seems that the heteroclinic orbits tend to survive at least until the attractors exist. However, if we let ρ increase further on (see Figure 4.c plotted for $\rho = 2.628$), the convolutions evolve in an ever more important way and, given an initial value of K_t (K_0), only a smaller subset of initial values of ℓ_t (ℓ_0) may generate feasible trajectories. This phenomenon can have interesting economic consequences because even if the 2-period cycle is stable for M, i.e., there exists a continuum of initial conditions such that the trajectories converge towards the 2-period cycle, starting from values of the capital stock around $k = 1$, only a few values of ℓ can now generate feasible trajectories. From an economic point of view this result suggests the importance of coordination of individuals on the choice of ℓ.

We note that for this range of values of the parameter ρ, the stable and unstable manifolds of $(1,\hat{\ell})$ seem to become tangent (this can be ascertained by looking at Figure 4.a), and the structure of the basin of attraction resembles the phenomenon described by Brock and Hommes (1997) starting from the "homoclinic kissing lemma".

The last part of the section is devoted to the study of periodic and or quasi-periodic orbits generated by the system. To make the subsequent analysis more precise, we now continue with the study of the second iterate of the map M, namely M^2, and concentrate on the evolution of the dynamics on the upper part of the domain D (the dynamics on the lower part being symmetric).

The bifurcation diagram depicted in Figure 5, which is obtained by using $(k_0,\ell_0)=(0.7,1.6)$ as the initial condition, shows some apparent discontinuities starting from $\rho \cong 2.6295$. They are caused by a 5-period cycle born through a saddle node bifurcation that captures the given initial condition, for some ranges of the parameter ρ.

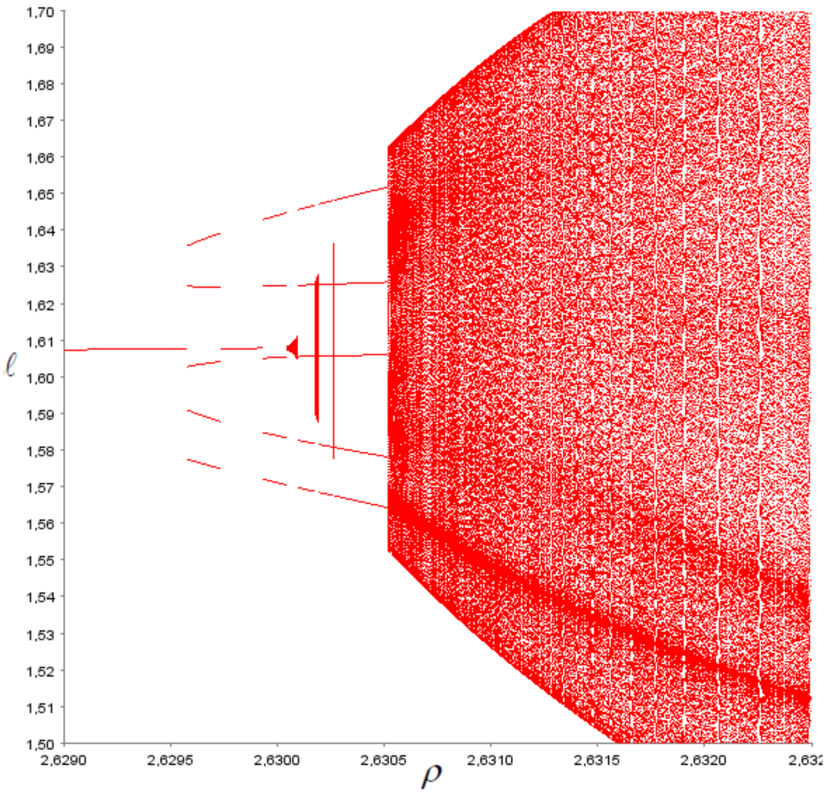
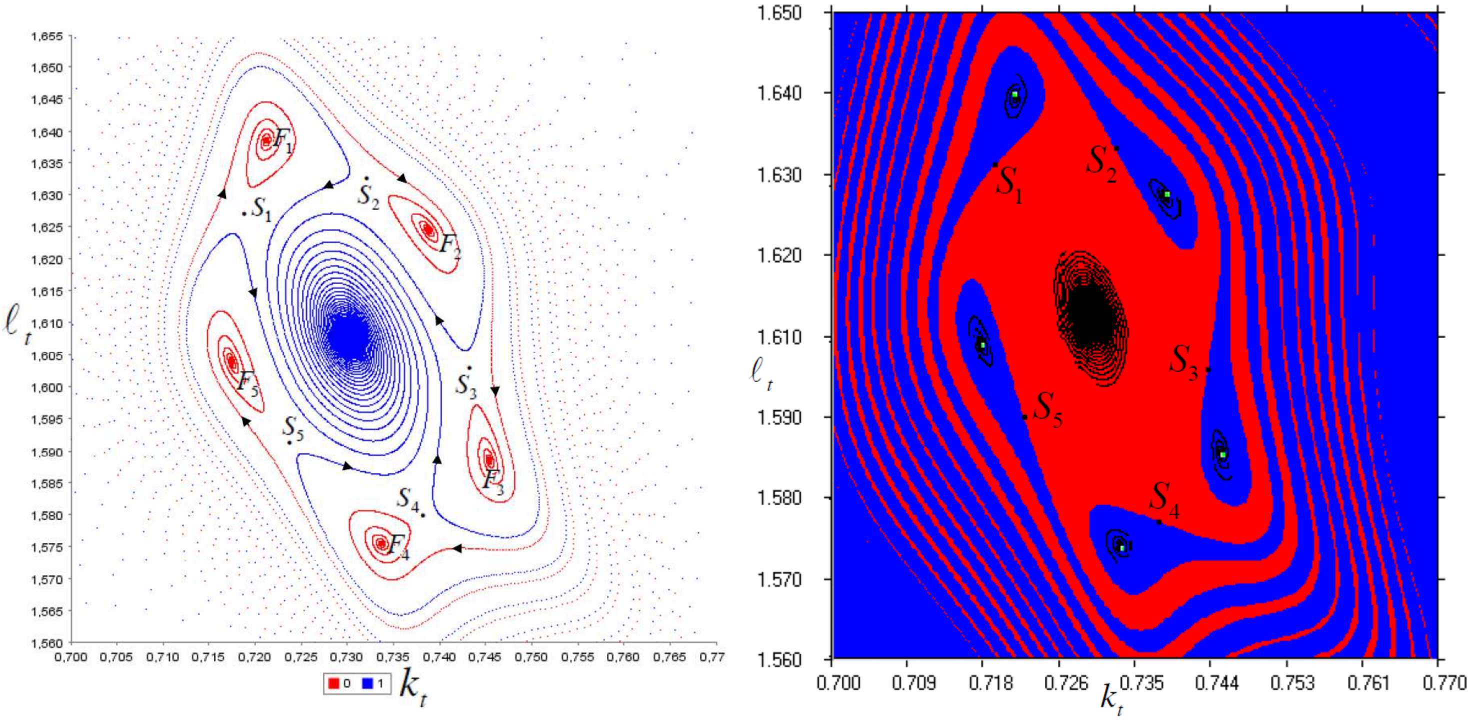


Figure 5. Bifurcation diagram for ρ. We follow the long-run evolution of the starting point $(k_0,\ell_0)=(0.7,1.6)$ when ρ increases. The discontinuities in the picture are due to the birth of another attractor of the system, while the apparent superposition of the curves just beyond $\rho=2.63$ is due to the projection of the dynamics on the ℓ axis.

At this stage the dynamics may converge to the interior fixed point or to the 5-period cycle. The basin of attraction of the 5-period cycle is defined by the stable manifold of the saddles (see Figures 6.a and 6.b).



(a) (b)

Figure 6. ($\rho = 2.6297$). (a) A period 5-cycle coexist with an attracting fixed point. Two converging trajectories are drawn (red- and blue-coloured). F_i (S_i) indicates the ith point of the attracting (saddle) 5-period cycle. (b) A portion of the basins of attraction of M^2: the basins of attraction of the interior fixed point (V_1 not reported in the figure) and of the 5-period cycle are red-coloured and blue-coloured, respectively. Notice the role of the stable manifold of the saddle 5-period cycle in defining the boundary of the attracting 5-period cycle.

Following now the evolution of the fixed point V_1 of M^2, we can see that it undergoes a supercritical Neimark-Sacker bifurcation at $\rho \cong 2.629732$ (see Figure 5) and an attracting invariant curve (Γ) may be observed around the interior equilibrium of M^2 (see Figure 7, where the coexistence of the attractors is illustrated when $\rho = 2.6302$ and two trajectories are drawn), corresponding to which the dynamics may be cyclical or quasi-cyclical.

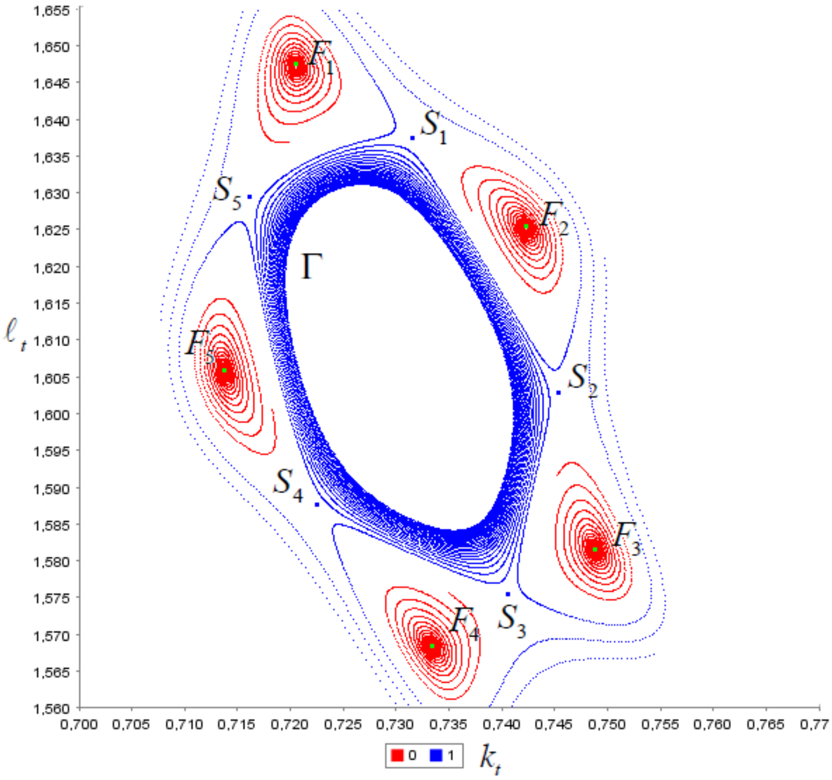


Figure 7. ($\rho = 2.6302$). Coexistence of both an attracting 5-period cycle and closed invariant curve (Γ). Notice that we do not report the basin of attraction when $\rho = 2.6302$ because its structure is similar to those presented in the case depicted in Figure 6.b.

When the dynamics is characterised by two coexisting attractors, as in the cases depicted in Figures 6 or 7, some exogenous changes in the parameters as well as in the expectations of individuals about the future interest rate, may cause the switch to another attractor of the system: if, for instance, a trajectory is converging to the interior equilibrium (as in Figure 6), the coordination on a fairly high value of ℓ causes the switch towards an attractor with larger oscillations.

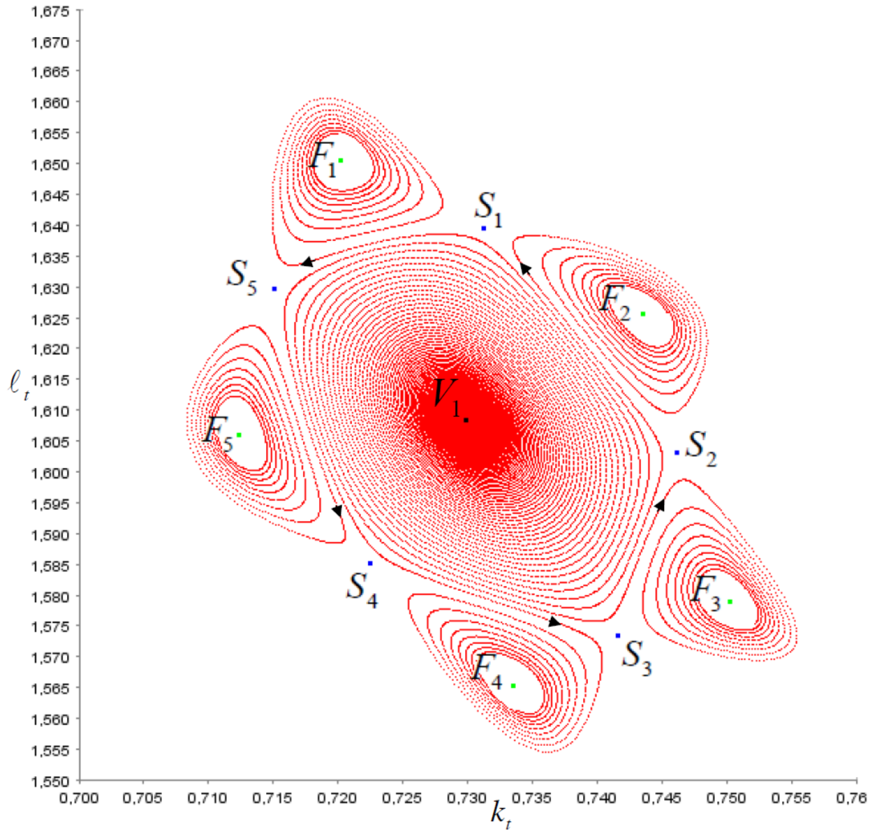
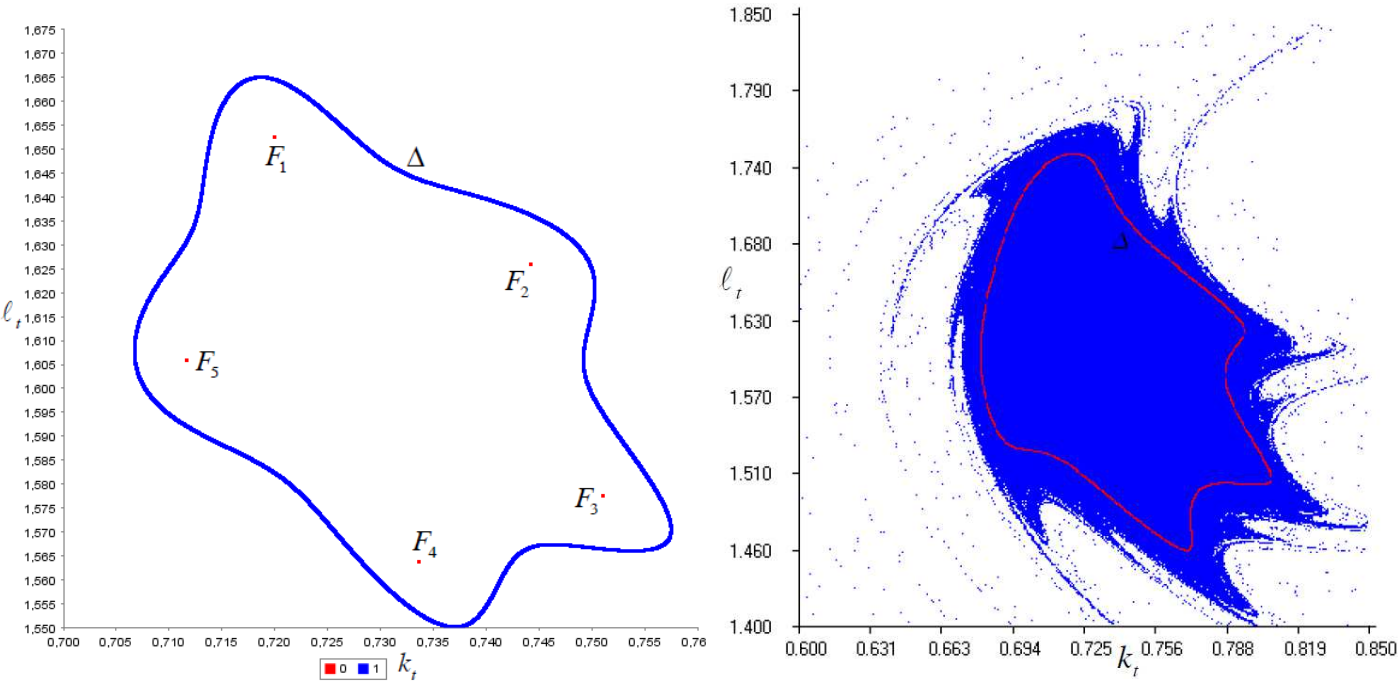


Figure 8. ($\rho = 2.63044$). A basin boundary bifurcation has destroyed the closed invariant curve and a unique attracting 5-period cycle survives.

The enlargement of the invariant curve causes a collision between the stable manifold of the 5-period cycles and the curve itself at $\rho \cong 2.6303$ (see Figure 8). This fact indeed causes the death of the invariant curve and a unique attracting 5-period cycle of the system does exist.

Now, to understand the sudden explosion in the bifurcation diagram at $\rho \cong 2.63052$ (see Figure 5), we refer to the theoretical results on invariant curves proposed by Agliari et al. (2005): essentially, a global bifurcation involving the stable and unstable manifolds of the 5-period cycle (see Figure 9.a) has occurred at $\rho = 2.63056$: a larger invariant curve (Δ) surrounds an attracting 5-period cycle. For higher values of ρ, another basin boundary bifurcation causes the death of the 5-period cycle of M^2 and the invariant curve remains the unique attractor of the system (see Figure 9.b plotted for $\rho = 2.6312$).



(a) (b)

Figure 9. (a) A large invariant curve (Δ) surrounds a 5-period cycle ($\rho = 2.63056$). (b) When ρ becomes larger ($\rho = 2.633$), the invariant closed curve (Δ) becomes larger and remains the unique attractor of the system. Note that: (i) in Figure 9.a the basin of attraction of the attractors is not reported because of the long transient, and (ii) in Figure 9.b both the attractor and its basin of attraction are depicted.

4. Conclusions

We studied the dynamic properties of a two-dimensional overlapping generations growth model with endogenous labour supply (see Reichlin, 1986) and inherited tastes. In particular, following de la Croix (1996) and de la Croix and Michel (1999), we assumed that preferences over consumption of individuals that belong to the current generation are affected by consumption experiences of the past generations (aspirations), i.e., there exists a consumption reference used by individuals to judge their standard of living period by period. Different from de la Croix (1996) and de la Croix and Michel (1999), which studied the dynamic effects of the existence of inherited tasted in an overlapping generations model à la Diamond (1965), i.e. with both young-age and old-age consumptions and exogenous labour supply, and where the dynamics of the economy is characterised by a two-dimensional system because of the accumulation of the stocks of both physical capital and aspirations, the dynamics in our economy is described by a two-dimensional system because of the accumulation of capital and the time evolution of the individual supply of labour. This implicitly derives by the fact that aspirations are assumed to be not forgotten at the end of every period, that is the rate of depreciation of aspirations is zero and thus they affect consumption in the second period of life (old-age).

We showed that the intensity or degree of aspirations in utility matters for the existence either of one (normalised) steady state (when the intensity of aspirations is fairly low) or two steady states (when the intensity of aspirations is fairly high).

Moreover, some interesting local and global stability properties arise when the taste externality gradually increases. In particular, we find that the coexistence of attractors may cause global indeterminacy even if the stationary equilibria are locally determinate.

Interesting extensions of the present study can be the analysis of (*i*) labour/leisure decisions of individuals in an OLG model à la Diamond (1965), i.e. in a framework where individuals choose the inter-temporal allocation of consumption over the lifecycle, and (*ii*) labour/leisure decisions when young and when old.

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