

Introducing Asymmetric Separability in the FAST Multistage Demand System

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Abstract

This paper determines the set of parametric restrictions required to maintain flexibility under asymmetric weak separability for the flexible and separable translog (FAST) multistage demand system. Because there is not a unique set of parametric restrictions that ensures separability and the values of the unconditional price and expenditure elasticities depend on the parametric restrictions imposed, the appropriateness of a chosen set of parametric restrictions should be tested empirically. An empirical example that illustrates how the choice of parametric restrictions affects the estimation results and the functional form of the price and expenditure elasticities is provided.

Keywords: Asymmetric weak separability, FAST multistage demand system, demand elasticities

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1. Introduction

For the purposes of conducting policy analyses, demand elasticities should be unconditional, that is price and expenditure elasticities should depend on total expenditures and not expenditures within a specified group of goods. Moschini (2001, p. 24) states that “conditional demand functions cannot provide the parameters (i.e. elasticities) that are typically of interest for policy questions. This is because the optimal allocation of expenditures to the goods in any one partition depends on all prices and total expenditure.” However, conditional demand models are frequently used because they only depend on a small set of prices for all goods within the partition of a group and group expenditures (for example see Brown et al, 1994) Thus, this smaller amount of required data allows for the conditional demand model to be more readily implemented and estimated.

To obtain unconditional demand elasticities for a specified group of goods, one must consider expenditures on all other goods and services. However, it is often the case that a researcher is only interested in a relatively small subset of goods, such as meats or beverages (for example see Helen and Willett, 1986) and invokes the assumption of weak separability. In these cases, the easiest way to create an unconditional demand system is to create an asymmetric partition that contains a single “all other goods” aggregate.¹ Even if one were to use several other composite goods (e.g., non-food, services, etc.) rather than a single composite good due to aggregation concerns, it may still may be reasonable to create an asymmetric partition for each of these composite goods (see Edgerton, 1997).²

Given the potential use of asymmetric separability to construct unconditional empirical demand models, the purpose of this paper is to extend the flexibility propositions developed by

Moschini (2001) for the case of asymmetric separability and identify the parametric restrictions required to maintain flexibility for the flexible and separable translog (FAST) multistage demand system developed by Moschini.³ We have chosen to focus on the FAST model because it provides a theoretically consistent parametric specification of both the conditional and unconditional demand functions of a weakly separable preference structure. As we will show there is no unique set of parametric restrictions that will ensure flexibility for the case of asymmetric separability. Because the unconditional elasticities depend on the choice of parametric restrictions, it will be important to empirically test the appropriateness of each set of restrictions. We provide an empirical example to illustrate how the choice of parametric restrictions affects the estimation results and the functional form of the price and expenditure elasticities.

2. The FAST Multistage Demand System

Following Moschini (2001), the FAST multistage demand system is based on the assumption of indirect weak separability. Preferences are said to be weakly separable in the partition $\hat{I} = \{I^1, \dots, I^N\}$ if the indirect utility function $V(p/y)$ can be represented as:

$$V(p/y) = V^0[V^1(p^1/y), \dots, V^N(p^N/y)], \quad (1)$$

where p^r is the vector of prices for the r th group of goods ($r = 1, \dots, N$) and $V^r(p^r/y)$ are indices dependent only on p^r and total expenditure (y). The function $V^0(\cdot)$ is assumed to be continuous, non-increasing and quasiconvex, and $V^r(\cdot)$ is assumed to be continuous, non-decreasing and quasiconcave. These assumptions ensure that $V(p/y)$ retains the usual properties of an indirect utility function.

Using Roy's identity along with equation (1), conditional and unconditional demand functions can be obtained.⁴ Adopting the translog specification of Christensen et al (1975) for $V^0(\cdot)$ and $V^r(\cdot)$ gives:

$$V^0(\cdot) = - \left[\gamma_0 + \sum_{r=1}^N \gamma_r \log V^r(\cdot) + \frac{1}{2} \sum_{r=1}^N \sum_{s=1}^N \gamma_{rs} \log V^r(\cdot) \log V^s(\cdot) \right], \text{ and} \quad (2)$$

$$\log V^r(p^r/y) = \beta_0^r + \sum_{i \in I^r} \beta_i \log(p_i/y) + \frac{1}{2} \sum_{i \in I^r} \sum_{j \in I^r} \beta_{ij} \log(p_i/y) \log(p_j/y). \quad (3)$$

The following specification satisfies the assumption of homogeneity by construction and symmetry by setting $\beta_{ij} = \beta_{ji} \forall i, j$ and $\gamma_{rs} = \gamma_{sr} \forall r, s$. To ensure that the indirect utility function given by equations (2) and (3) is flexible and satisfies the properties of indirect weak separability, Moschini (2001) suggests imposing the following restrictions:

$$\beta_0^r = 0 \text{ for } r = 1, \dots, N, \quad (4)$$

$$\gamma_0 = 0, \quad (5)$$

$$\sum_{i \in I^r} \beta_i = 1 \text{ for } r = 1, \dots, N, \quad (6)$$

$$\sum_{r=1}^N \gamma_r = 1, \text{ and} \quad (7)$$

$$\sum_{s=1}^N \gamma_{r,s} = 0 \text{ for } r = 1, \dots, N. \quad (8)$$

In the case of asymmetric partitions or groups, Moschini suggests replacing condition (8) with:

$$\sum_{i \in I^r} \sum_{j \in I^r} \beta_{ij} = 0 \text{ for } r = 1, \dots, N_1, \quad (9)$$

where N_1 denotes the number of symmetric partitions. This alternative set of restrictions allows for the case of asymmetric separability, where the asymmetric groups have only one price.

Moschini (2001) derives conditional share equations and group share equations allowing the FAST multistage demand system to be estimated using a two-step process. The conditional share equations:

$$w_i^r = \frac{\beta_i + \sum_{j \in I^r} \beta_{ij} \log(p_j/y)}{1 + \sum_{k \in I^r} \sum_{j \in I^r} \beta_{kj} \log(p_j/y)} \quad \forall i \in I^r, \quad (10)$$

where $w_i^r = (p_i q_i) / y_r$ and y_r is the within-group expenditure, are estimated in the first stage.

In the second stage, the modeler estimates the following group share equations:

$$w^r = \frac{B^r(p^r/y) \left(\gamma_r + \sum_{s=1}^N \gamma_{rs} \log V^s(p^s/y) \right)}{\sum_{g=1}^N B^g(p^g/y) \left(\gamma_g + \sum_{s=1}^N \gamma_{gs} \log V^s(p^s/y) \right)} \quad \text{for } r = 1, \dots, N, \quad (11)$$

where $w^r = y^r / y$ and $B^g(p^g/y) = 1 + \sum_{j \in I^g} \sum_{i \in I^g} \beta_{ij} \log(p_i/y)$ for $g = 1, \dots, N$; and the indices

$\log V^r$ and B^g are computed using the estimated parameters of the conditional share equations in the first step.

3. Flexibility and Asymmetric Separability

Moschini (2001) follows Diewert (1974) in defining $F(x)$, where x is a $(n \times 1)$ vector, as “a flexible functional form (FFF) for $V(x)$ if $F(x)$ can provide a second-order approximation to $V(x)$ at a point \bar{x} (p. 27).” Thus, $F(x)$ must satisfy the following conditions:

$$F(\bar{x}) = V(\bar{x}) \quad (\text{FLEX0})$$

$$F_i(\bar{x}) = V_i(\bar{x}) \quad \forall i = 1, \dots, n \quad (\text{FLEX1})$$

$$F_{ij}(\bar{x}) = V_{ij}(\bar{x}) \quad \forall i, j = 1, \dots, n. \quad (\text{FLEX2})$$

Note that due to the ordinality property of indirect utility functions, the function value is not meaningful. Thus, (FLEX0) need not be satisfied in this case.

Using the above definition of flexibility, we next turn to determining the number of independent effects necessary to maintain flexibility for a weakly separable indirect utility function with asymmetric partitions. Consider the indirect utility function $V(x)$ that is separable in the partition $I = \{I^1, \dots, I^{N_1}, I^{N_1+1}, \dots, I^N\}$. The partitions $r = 1, \dots, N_1$ designate the N_1 partitions with two or more goods and the partitions $a = N_1 + 1, \dots, N$ designate the asymmetric partitions (with one good). Next define the function $F(x)$ to be separable in the same partitions:

$$F(x) = F^0 \left\{ F^1(x^1), \dots, F^{N_1}(x^{N_1}), F^{N_1+1}(x^{N_1+1}), \dots, F^N(x^N) \right\}$$

Following the treatment of asymmetric partitions in Driscoll and McGuirk (1992), let F^a be the identity function such that $F^a(x^a) = x_a$. Note that this is equivalent to setting $\beta_i = 1$ and $\beta_{ii} = 0$ in equation (3) for all asymmetric partitions.

Results from Driscoll, McGuirk, and Alwang (1992) and Moschini, Moro, and Green (1994) show that a symmetric weakly separable utility function has

$$\Omega_S = n + \sum_{r=1}^N k_r(k_r + 1)/2 + N(N-1)/2 \text{ independent effects, where } n \text{ is the total number of}$$

goods, k_r is the number of goods in the r^{th} partition, and N is the number of partitions. When there are asymmetric partitions present, the number of independent effects is larger because fewer restrictions are placed on the marginal rates of substitutions between goods in different partitions. The number of independent effects when asymmetric partitions are presents is

$\Omega_A = n + \sum_{r=1}^{N_1} k_r(k_r + 1)/2 + (N - N_1) + N(N - 1)/2$. Thus, a weakly separable FFF with

asymmetric partitions must have at least Ω_A independent parameters.

With only the symmetry conditions imposed on the second-order terms in equations (2)

and (3), the FAST model has $\Omega_T = n + N_1 + \sum_{r=1}^{N_1} k_r(k_r + 1)/2 + N(N + 1)/2$ independent

parameters for the case of asymmetric partitions. In order for the FAST model to be

parsimonious, there must be $\Omega_T - \Omega_A = 2N_1$ additional parametric restrictions.⁶ This leads to the

following proposition.

PROPOSITION: When $V(x)$ is an indirect utility function, the function

$F(x) = F^0 [F^1(x^1), F^2(x^2), \dots, F^{N_1}(x^{N_1}), F^{N_1+1}(x^{N_1+1}), \dots, F^N(x^N)]$ is a FFF for

$V(x) = V^0 [V^1(x^1), V^2(x^2), \dots, V^{N_1}(x^{N_1}), V^{N_1+1}(x^{N_1+1}), \dots, V^N(x^N)]$ where partitions $r = 1, \dots, N_1$

are symmetric and partitions $a = N_1 + 1, \dots, N$ are asymmetric if:

- a. each F^r for $r = 1, \dots, N_1$ satisfies (FLEX1) for all but one first derivative, F^0 satisfies (FLEX1) for all first derivatives and satisfies (FLEX2) for all second derivatives of partitions $N_1 + 1, \dots, N$, and one of the following conditions is satisfied,
- b. if F^r for $r = 1, \dots, N_1$ satisfies (FLEX2) for all but one second derivative then F^0 satisfies (FLEX2) for all second derivatives of that partition, or
- c. if F^r for $r = 1, \dots, N_1$ satisfies condition (FLEX2) for all second derivatives then F^0 satisfies (FLEX2) for all but one second derivative of that partition.

The proof of this proposition is given in the Appendix. Note that condition (a) provides N_1 parametric restrictions on the first derivatives of F^r for the symmetric partitions, which may be represented by the parametric restrictions given in equation (6). Conditions (b) and (c) provide N_1 parametric restrictions on the second derivatives of F^r and F^0 . If condition (b) holds for all F^r , then the set of parametric restrictions given in equation (9) can be applied. If condition (c) holds for all F^r , then:

$$\sum_{s=1}^N \gamma_{rs} = 0, \forall r = 1, \dots, N_1 \quad (12)$$

can be applied. Of course it is also possible that condition (b) holds for a subset of the symmetric partitions and condition (c) holds for the remaining symmetric partitions.

The implication of the above proposition is that there no unique set of parametric restrictions that will maintain flexibility for the FAST model when asymmetric partitions are present. The restrictions given by equation (9) have an intuitive appeal because they suggest that the sub-indirect utility functions have PIGLOG preferences (see Deaton and Muelbauer, 1980). However, the entire FAST demand system would not reflect PIGLOG preferences. The restrictions in equations (9) or (12) have the appeal of a uniform set of parametric restrictions. But it also may be the case that the results of hypothesis tests indicate that a mix of the restrictions in equations (9) and (12) fit the data better than the uniform set of restrictions in either equation (9) or (12). The choice of parametric restrictions imposed is important because it will affect the parameter estimates and the values of the price and expenditure elasticity estimates.

3. Unconditional Price and Expenditure Elasticities

The primary benefit of using the FAST multistage demand system is the derivation of a complete matrix of unconditional Marshallian expenditure and price elasticities. Following Moschini (2001), the demand elasticities are derived by normalizing the data so that $p_i = y = 1$ for all i goods and imposing the parametric restrictions given by equations (4) through (7) and equations (9) or (12). Bergtold et al, (2004, p. 285-6) have derived the price and expenditure elasticity formulas for the case of asymmetric partitions when equations (4) through (7) and (9) are imposed on the model.⁷ This leaves the case where a mix of the restrictions given by equations (9) and (12) is used. To derive this case, let R_1 represent the set of partitions (with two or more goods) where the restrictions given by equation (12) are imposed and R_2 the set of partitions where the restrictions given by equation (9) are imposed. Note that the set of asymmetric partitions is a subset of R_2 . Then normalizing such that $p_i = y = 1$ for all i goods the unconditional price (ε) and expenditure (η) elasticities are:

$$\varepsilon_{ij} = \begin{cases} \frac{\beta_{ij}}{\beta_i} + \frac{\gamma_{rr}\beta_j}{\gamma_r} - \gamma_r \left(\sum_{q \in I^r} \beta_{jq} \right) - \delta_{ij}, \text{ for } (i, j) \in I^r \in R_1, \\ \frac{\beta_{ij}}{\beta_i} + \frac{\gamma_{rr}\beta_j}{\gamma_r} - \gamma_r \left(\sum_{q \in I^r} \beta_{jq} \right) - \beta_j \left(\sum_{s=1}^N \gamma_{rs} \right) - \delta_{ij}, \text{ for } (i, j) \in I^r \in R_2 \end{cases} \quad (13)$$

$$\varepsilon_{ik} = \begin{cases} \frac{\gamma_{rs}\beta_k}{\gamma_r} - \gamma_s \left(\sum_{q \in I^r} \beta_{qk} \right), \text{ for } i \in I^r \text{ and } k \in I^s \in R_1 \\ \frac{\gamma_{rs}\beta_k}{\gamma_r} - \gamma_s \left(\sum_{q \in I^r} \beta_{qk} \right) - \beta_k \left(\sum_{g=1}^N \gamma_{gs} \right), \text{ for } i \in I^r \text{ and } j \in I^s \in R_2 \end{cases}, \text{ and} \quad (14)$$

$$\eta_i = \begin{cases} 1 - \frac{\sum_{j \in I^r} \beta_{ij}}{\beta_i} + \sum_{g=1}^N \sum_{s \in R_2} \gamma_{gs} + \sum_{s \in R_1} \gamma_s \left(\sum_{q \in I^s} \sum_{p \in I^s} \beta_{qp} \right), & \text{for } i \in I^r \in R_1 \\ 1 - \frac{\sum_{j \in I^r} \beta_{ij}}{\beta_i} - \frac{\sum_{s=1}^N \gamma_{rs}}{\gamma_r} + \sum_{g=1}^N \sum_{s \in R_2} \gamma_{gs} + \sum_{s \in R_1} \gamma_s \left(\sum_{q \in I^s} \sum_{p \in I^s} \beta_{qp} \right), & \text{for } i \in I^r \in R_2 \end{cases}, \quad (15)$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, and 0 otherwise).⁸

These elasticity formulas emphasize the dependent nature of these formulas on the choice of parametric restrictions. Furthermore, the choice of parametric restrictions may affect estimation results as well, by altering the underlying probabilistic properties of the statistical model being estimated. These two factors together make the a priori choice of parametric restrictions even more difficult.

4. Empirical Examination

To empirically examine the implications of the a priori imposition of different sets of restrictions on the FAST multistage demand system, the estimation results from a complete demand system with ten categories of products are presented. The ten categories represent nine composite categories of processed foods and one “all other goods” category. The nine processed food categories are: (1) coffee, tea and creamer, (2) soft drinks and bottled water, (3) juices, (4) milk products, (5) condiments, sauces and dressings, (6) baking products, bread and pasta, (7) deserts and candy, (8) fruits and vegetables, and (9) cheese products. These categories are then partitioned into three weakly separable partitions, giving rise to the following indirect utility function:

$$V(p/y) = V^0(V^1(p_1/y, \dots, p_4/y), V^2(p_5/y, \dots, p_9/y), p^3/y), \quad (15)$$

where the superscripts 1 and 2 refer to the beverage and non-beverage product groupings respectively. Group 3 is an asymmetric partition with one good, the “all other goods” composite good.

Data for prices and total sales are obtained from the Information Resources, Inc. (IRI) InfoScan[®] retail database.⁹ A detailed description of the data are provided in Bergtold et al (2004). The original data set contained 140 processed food products for 42 U.S. metropolitan areas from 1988 to 1992 on a quarterly basis. The 140 categories were aggregated into the 9 processed food categories presented above.¹⁰ Price indices for each processed food category were derived by dividing total sales by total sales in the first quarter of 1988 for each metropolitan area. In addition, the data were supplemented with information on median household income for each metropolitan area, which was reallocated across quarters to match the price and quantity data using the methods provided in Bergtold et al (2004). Price indices for the “all other goods” category were computed using regional consumer price indices (U.S. Department of Labor) for “All Urban Consumers,” due to the large share of total expenditures represented by this category. Nonparametric WARP and GARP tests indicate that the data are consistent with the maintained hypothesis of utility maximization for 39 of the 42 metropolitan areas. Thus, the three metropolitan areas not satisfying these tests were excluded from the data set, providing 780 observations.

To examine the different sets of a priori parametric restrictions that could be imposed on the FAST multistage demand system, four separate models are examined. The models are represented by the different combinations of parametric restrictions given by equations (9) and (12) imposed on the beverage (I^1) and non-beverage (I^2) product categories. The different

models are provided in Table 1. In addition, all the models are subject to the parametric restrictions given by conditions (4) through (7), as well as symmetry.

Models 1 through 4 are estimated by imposing the appropriate parametric restrictions on equations (10) and (11) using the two-stage process presented by Moschini (2001). In the first stage of estimation for each model, the conditional share equations for the beverage and non-beverage groups are estimated.¹¹ Using the estimation results in the first stage, a system of group share equations is estimated in the second stage. To avoid singularity of the variance/covariance matrix, one equation from each system was dropped. Each model had a total of 38 parameters to be estimated after all restrictions were imposed. The “full information maximum likelihood” (FIML) procedure in SAS was used to estimate each system of equations for both stages of estimation. This estimation procedure presumes that the errors for each system of equations are distributed multivariate normal.

The unconditional own-price and expenditure elasticities for each model are presented in Tables 2.¹² Comparisons across the different models show substantial differences in the estimated elasticities. All of the own-price and expenditure elasticities for model 2 are substantially larger (in absolute terms for the own-price elasticities) than for the other models. The magnitude of these estimates is certainly much higher than those typically found in the literature for food products. The estimated elasticities for models 1 and 4 are fairly similar with several product categories (coffee, tea, and creamer and processed fruits and vegetables) being inferior products. The estimated expenditure elasticities for model 3 are all positive, and with the exception of coffee, tea, and creamers and processed fruits and vegetables, significantly different from zero. The own-price elasticities for goods in the non-beverage partition are all own-price elastic.

Given the variation in the elasticity estimates between the different models, it would be useful to be able to determine empirically, which of the different parametric restrictions underlying each model is compatible with the data. Due to the use of a two-stage estimation process, nested tests that simultaneously tested for the restrictions in each model can not be performed. A nested test requires that all the systems of equations be estimated simultaneously, which would have been equivalent to estimating the system of unconditional demand equations for each model (see Moschini, 2001). Given the degrees of freedom required to estimate multistage demand systems this approach is not practical. Thus, an alternative procedure is used that examined the restrictions imposed on the systems of conditional and group share equations independently.

To test the parametric restrictions given by equation (9), asymptotic likelihood ratio tests are used to determine the appropriateness of those restrictions on conditional share equations. Likewise, the parametric restrictions imposed by equation (12) are tested using asymptotic likelihood ratio tests on the group share equations. However, because of the two-stage estimation process, to perform the likelihood ratio tests on the group share equations requires that the estimates from the beverage and non-beverage conditional systems of share equations are used to provide consistent estimates of $\log V^r$ and B^g . For models 3 and 4 where a mix of parametric restrictions on the conditional and group share equations are imposed, the restrictions on the conditional share equations are tested first. If parametric restrictions on the conditional share equations are rejected, then there is no need to test the parametric restrictions on the group share equations. Only if the parametric restrictions on the conditional share equations are not rejected, are the parametric restrictions on the group share equations tested.

The asymptotic likelihood ratio test used takes the form:

$$-(2\bar{T}/T)\left(\ln L(\tilde{\theta}; x) - \ln L(\hat{\theta}; x)\right) \underset{\infty}{\overset{H_0}{\sim}} \chi^2(m \cdot p),$$

where $\bar{T} = (T - k - 0.5(m - p + 1))$, T is the total number of observations, k is the number of parameters estimated in the system, m is the number of equations, and p is the number of restrictions (Spanos, 1986 and Schatzoff, 1966). When restrictions on the systems of group share equations were tested, k included the number of parameters estimated in the corresponding systems of conditional share equations.

The results of the tests for the parametric restrictions given by equations (9) and (12) are presented in Table 3. Given that the restrictions on the beverage conditional share equations are the same for models 1 and 3, as well as the restrictions on the non-beverage conditional share equations in models 1 and 4, the test results for these sets of restrictions are identical. The results in Table 3 indicate the parametric restrictions in the non-beverage conditional share equations for models 1 and 4 and the group share equations for model 2 are not supported by the data. This also implies that the test results for the restrictions in the group share equations in model 4 may be misleading; given the restrictions on the non-beverage conditional share equations are not appropriate. The only model where the data provide some support that the restrictions may be compatible with the data is model 3.

5. Summary and Conclusions

The use of the FAST multistage demand system provides an internally consistent and parsimonious method for obtaining unconditional price and expenditure elasticities. Use of the FAST model can be problematic when the modeler is confronted with using weakly separable asymmetric partitions or groups, given the a priori parametric restrictions needed to obtain flexibility and meet the assumptions of weak separability are not unique. In fact, a number of

combinations of restrictions can be considered, and this number increases dramatically as the number of partitions with two or more goods increases. This predicament is further complicated by the fact that both estimation and elasticity estimates are affected by the choice of restrictions, which was evident in the empirical example presented above. After testing the restrictions imposed on the systems of conditional and group share equations, it was determined that a mixture of the parametric restrictions in equations (9) and (12) should be used.

Appendix: Proof of Proposition

PROOF: Given the assumed partition and separable structures indicated for $V(\bullet)$ and $F(\bullet)$, conditions (FLEX0) – (FLEX2) imply the following:

$$V^0 = F^0$$

$$V_r^0 V_i^r = F_r^0 F_i^r \quad \forall i \in I^r, \quad r = 1, \dots, N_1$$

$$V_a^0 = F_a^0 \quad \text{for } a = N_1 + 1, \dots, N$$

$$V_{rr}^0 V_i^r V_j^r + V_r^0 V_{ij}^r = F_{rr}^0 F_i^r F_j^r + F_r^0 F_{ij}^r \quad \forall i, j \in I^r, \quad r = 1, \dots, N_1$$

$$V_{rs}^0 V_i^r V_k^s = F_{rs}^0 F_i^r F_k^s \quad \forall i \in I^r, k \in I^s, r \neq s, \quad r, s = 1, \dots, N_1$$

$$V_{ab}^0 = F_{ab}^0 \quad \text{for } a, b = N_1 + 1, \dots, N$$

$$V_{ar}^0 V_i^r = F_{ar}^0 F_i^r \quad \forall i \in I^r, r = 1, \dots, N_1, a = N_1 + 1, \dots, N,$$

where all of the functions are evaluated at the point \bar{x} . Rewriting these conditions in terms of the first and second derivatives of $F(\bullet)$ gives:

$$V^0 = F^0 \tag{A1}$$

$$\frac{V_r^0 V_i^r}{F_r^0} = F_i^r \quad \forall i \in I^r, \quad r = 1, \dots, N_1 \tag{A2}$$

$$V_a^0 = F_a^0 \quad \text{for } a = N_1 + 1, \dots, N \tag{A3}$$

$$\frac{V_{rr}^0 V_i^r V_j^r + V_r^0 V_{ij}^r - F_{rr}^0 F_i^r F_j^r}{F_r^0} = F_{ij}^r \quad \forall i, j \in I^r, \quad r = 1, \dots, N_1 \tag{A4}$$

$$\frac{V_{rs}^0 F_r^0 F_s^0}{V_r^0 V_s^0} = F_{rs}^0 \quad \forall i \in I^r, k \in I^s, r \neq s, \quad r, s = 1, \dots, N_1 \tag{A5}$$

$$V_{ab}^0 = F_{ab}^0 \quad \text{for } a, b = N_1 + 1, \dots, N \tag{A6}$$

$$\frac{V_{ar}^0 F_r^0}{V_r^0} = F_{ar}^0 \quad \forall i \in I^r, r = 1, \dots, N_1, a = N_1 + 1, \dots, N, (A7)$$

where conditions (A5) and (A7) use condition (A2).

Because of ordinality, the exact function value of an indirect utility functions is not meaningful. Thus, condition (A1) does not need to be satisfied. For all symmetric partitions, there are k_r first derivatives of F^r and one first derivative of F^0 in condition (A2). This indicates that one may either identify all of the first derivatives of F^r or $(k_r - 1)$ first derivatives of F^r and F_r^0 to satisfying condition (A2). Note that condition (A3) will be satisfied if F^0 satisfies condition (FLEX1) for all $a = N_1 + 1, \dots, N$. By combining the conditions for (A2) and (A3), one obtains that each F^r satisfies condition (FLEX1) for all but one first derivative and F^0 satisfies condition (FLEX1) for all first derivatives.

One can obtain two different sets of conditions to satisfy conditions (A4) through (A7) depending on whether F_{rr}^0 needs to obtain an arbitrary value or not. Note that condition (A4) can be written as:

$$\frac{V_{rr}^0 V_i^r V_j^r + V_r^0 V_{ij}^r - F_0^r F_{ij}^r}{F_i^r F_j^r} = F_{rr}^0$$

for all $(i, j) \in I^r$. Thus:

$$\frac{V_{rr}^0 V_i^r V_i^r + V_r^0 V_{ii}^r - F_0^r F_{ii}^r}{F_i^r F_i^r} = \frac{V_{rr}^0 V_j^r V_j^r + V_r^0 V_{jj}^r - F_0^r F_{jj}^r}{F_j^r F_j^r} = \frac{V_{rr}^0 V_i^r V_j^r + V_r^0 V_{ij}^r - F_0^r F_{ij}^r}{F_i^r F_j^r} = F_{rr}^0,$$

which may be rewritten as:

$$\left(\frac{V_{rr}^0 V_i^r V_i^r + V_r^0 V_{ii}^r}{F_i^r F_i^r} - \frac{V_{rr}^0 V_j^r V_j^r + V_r^0 V_{jj}^r}{F_j^r F_j^r} - 2 \frac{V_{rr}^0 V_i^r V_j^r + V_r^0 V_{ij}^r}{F_i^r F_j^r} \right) - \frac{F_0^r}{F_i^r F_i^r} F_{ii}^r + \frac{F_0^r}{F_j^r F_j^r} F_{jj}^r + \frac{2F_0^r}{F_i^r F_j^r} F_{ij}^r = 0.$$

This condition implies that one of the second derivatives F_{ij}^r is not independent if F_{rr}^0 is required to obtain an arbitrary value. If F_{rr}^0 does not need to obtain an arbitrary value, then condition (A5) shows that for symmetric partition r , only $(N_1 - 1)$ of the values of $F_{rs}^0 \forall s = 1, \dots, N_1$ are independent. This leads to two cases:

- a. if F^r for $r = 1, \dots, N_1$ satisfies condition (FLEX2) for all but one second derivative then F^0 satisfies (FLEX2) for all second derivatives of that partition, or
- b. if F^r for $r = 1, \dots, N_1$ satisfies condition (FLEX2) for all second derivatives then F^0 satisfies (FLEX2) for all but one second derivative for that symmetric partition.

Note that F^0 will satisfy (FLEX2), and in turn conditions (A6) and (A7), for all second derivatives of the asymmetric partitions in either case.

Endnotes

- ¹ Asymmetric weak separability refers to the case where at least one of the partitions of the (indirect) utility function contains a single good
- ² Utilizing asymmetric partitions for the composite goods also reduces the number of unknown parameters that must be estimated. This may be a consideration for analyses with a small number of observations.
- ³ Moschini briefly mentions the case of asymmetric separability in one paragraph of his paper.
- ⁴ The general formulas for the conditional and unconditional demand functions are provided by Moschini (2001).
- ⁵ A single subscript denotes the first derivative with respect to category i and a double subscript denotes the second order derivative with respect to category i and category j .
- ⁶ For the case of symmetric partitions, the number of additional parametric restrictions is $2N$.
- ⁷ The formulas given by Moschini (2001) for this case do not take account of the change in restrictions and were re-derived by Bergtold et. al. (2004).
- ⁸ For asymmetric partitions, $\beta_{ii} = 0$ and $\beta_i = 1$.
- ⁹ These data were made available via an arrangement with Professor Ron Cotterill at the Food Marketing Policy Center at the University of Connecticut.
- ¹⁰ These product categories represent a further aggregation of the processed food categories presented in Bergtold et. al. (2004).
- ¹¹ The “all other goods” category is a trivial estimation given the parameters in the conditional equation are restricted to take particular values.
- ¹² The parameter estimates are available from the authors upon request.

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Table 1: Models Estimated and Varying Sets of Parametric Restrictions Used

Model	Parametric Restrictions	
	Beverage Product Category (I^1)	Non-Beverage Product Category (I^2)
1	$\sum_{i \in I^1} \sum_{j \in I^1} \beta_{ij} = 0$	$\sum_{i \in I^2} \sum_{j \in I^2} \beta_{ij} = 0$
2	$\sum_{s=1}^3 \gamma_{1s} = 0$	$\sum_{s=1}^3 \gamma_{2s} = 0$
3	$\sum_{i \in I^1} \sum_{j \in I^1} \beta_{ij} = 0$	$\sum_{s=1}^3 \gamma_{2s} = 0$
4	$\sum_{s=1}^3 \gamma_{1s} = 0$	$\sum_{i \in I^2} \sum_{j \in I^2} \beta_{ij} = 0$

Table 2: Unconditional Own-Price and Expenditure Elasticities for Models 1 through 4.

Good ^a	Own-Price Elasticities				Expenditure Elasticities			
	Model 1 ^b	Model 2	Model 3	Model 4	Model 1	Model 2	Model 3	Model 4
q_1	0.00 (0.13) ^c	-1.66 (0.49)	-0.04 (0.14)	0.06 (0.13)	-0.37 (0.10)	1.32 (0.40)	0.07 (0.06)	-0.26 (0.39)
q_2	-0.81 (0.12)	-5.42 (1.27)	-0.91 (0.12)	-0.84 (0.17)	0.01 (0.09)	1.70 (0.39)	0.45 (0.04)	0.12 (0.38)
q_3	-0.37 (0.16)	-4.02 (1.02)	-0.45 (0.16)	-0.36 (0.20)	-0.07 (0.09)	1.63 (0.41)	0.37 (0.04)	0.05 (0.39)
q_4	-0.17 (0.09)	-4.21 (1.11)	-0.26 (0.08)	-0.21 (0.16)	-0.18 (0.09)	1.56 (0.42)	0.27 (0.04)	-0.03 (0.40)
q_5	-1.30 (0.07)	-7.21 (2.51)	-1.78 (0.07)	-1.26 (0.06)	-0.10 (0.09)	3.49 (0.19)	0.53 (0.19)	0.08 (0.02)
q_6	-0.65 (0.12)	-7.33 (2.90)	-1.07 (0.12)	-0.61 (0.11)	-0.15 (0.09)	3.41 (0.20)	0.45 (0.20)	-0.03 (0.02)
q_7	-1.28 (0.16)	-5.66 (1.93)	-1.50 (0.11)	-1.25 (0.15)	-0.05 (0.09)	3.47 (0.19)	0.51 (0.19)	0.13 (0.03)
q_8	-1.72 (0.15)	-5.98 (2.06)	-1.54 (0.08)	-1.69 (0.14)	-0.31 (0.09)	3.29 (0.21)	0.33 (0.20)	-0.13 (0.02)
q_9	-0.14 (0.12)	-4.64 (1.32)	-1.80 (0.18)	-0.12 (0.12)	-0.10 (0.09)	5.69 (0.31)	2.73 (0.31)	0.08 (0.03)
q_{10}	-1.04 (0.00)	-8.49 (0.02)	-1.02 (0.00)	-1.04 (0.00)	1.03 (0.01)	1.00 (0.01)	1.01 (0.01)	1.03 (0.01)

^a The subscripts denote product categories, i.e. (1) coffee, tea and creamer, (2) soft drinks and bottled water, (3) juices, (4) milk products, (5) condiments, sauces and dressings, (6) baking products, bread and pasta, (7) deserts, (8) fruits and vegetables, (9) cheese products, and (10) all other goods.

^b Model definitions based on parametric restrictions given in Table 1.

^c Values in parentheses are standard errors. Standard errors were calculated using a Monte Carlo method. The estimates obtained from each system of equations estimated were assumed to be distributed multivariate normal with mean equal to the estimated parameters and covariance matrix equal to the estimated covariance matrix from SAS. Based on these assumptions, 5000 sets of parameters were randomly generated and the corresponding elasticities computed and saved. The standard errors are the standard errors of the saved computed elasticity estimates.

Table 3: Likelihood Ratio Test Results for Parametric Restriction for Models 1 through 4

Systems of Equations				
Model	Beverage Conditional Share	Non-Beverage Conditional Share	Group Share Equations	
	Equations	Equations		
1	Null Hypothesis	$H_0 : \sum_{i \in I^1} \sum_{j \in I^1} \beta_{ij} = 0$	$H_0 : \sum_{i \in I^2} \sum_{j \in I^2} \beta_{ij} = 0$	---
	Test Result	2.638 (0.451)	24.416 (0.000)	---
2	Null Hypothesis	---	--	$H_0 : \sum_{s=1}^3 \gamma_{1s} = 0$ and $\sum_{s=1}^3 \gamma_{2s} = 0$
	Test Result	---	---	2319.0 (0.000)
3	Null Hypothesis	$H_0 : \sum_{i \in I^1} \sum_{j \in I^1} \beta_{ij} = 0$	---	$H_0 : \sum_{s=1}^3 \gamma_{2s} = 0$
	Test Result	2.638 (0.451)	---	0.000 (1.000)
4	Null Hypothesis	---	$H_0 : \sum_{i \in I^2} \sum_{j \in I^2} \beta_{ij} = 0$	$H_0 : \sum_{s=1}^3 \gamma_{1s} = 0$
	Test Result	---	24.416 (0.000)	0.315 (0.854)

Notes: ‘---’ indicates that a test was not conducted, because the parametric restrictions being examined were not imposed. The number in parentheses is the associated p-value