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A General Index of Inherent Risk

By

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Abstract

We extend the pioneering work of Aumann and Serrano by presenting an index of inherent *riskiness* of a gamble having the desirable properties of their index, while being applicable to gambles with either positive or negative expectations. As such, our index provides a measure of riskiness which is of use for both risk lovers and risk aversive gamblers, and is defined for *all* discrete and a large class of continuous gambles. We analyze abstract properties of our index, and present in addition three empirical applications - roulette, horse betting market and US options traded on financial stocks between 2005 and 2007.

Introduction

What is Inherent Risk? It is the riskiness of a gamble defined independently of either the utility or the wealth of the individual contemplating taking the gamble. In other words, the index is unconcerned with the attitude of the individual towards risk, but attempts to capture that risk which is inherent to the gamble itself. The first such index has been presented by Aumann and Serrano (2007)[2] (hereafter [AS]), but it is restricted to gambles which a risk-averse agent would accept. Thus, while it is an index of inherent risk, it is restricted in its applicability to a particular type of agent and does not even cater to a risk-averse investor who wishes to sprinkle his portfolio with some potentially high return, high risk gambles. For [AS], risk aversion applies to every component of the portfolio and is not an "on average" notion. Further, a gamble with a

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[2] For the origins of this index, see Palacios-Huerta, I., R. Serrano and O. Volij (2004).

negative expectation need not be of relevance only for risk lovers. Thus, if a speculator buys a quantity cacao futures with negative expectations, her behavior is clearly risk-loving. And yet, were a large chocolate company to buy the identical gamble for hedging purposes, this would be a risk-reducing measure! In light of the current world-wide financial meltdown, it is difficult to entertain seriously the assumption that risk-loving behavior is restricted to those on the fringes of financial markets such as casino goers, casual horse bettors and compulsive gamblers.

However, the [AS] index has two features which should be true of any index of risk and yet which are surprisingly hard to find among other measure of risk. First is what [AS] term the Duality Axiom: As they put it, "Duality says that if the more risk-averse of two agents accepts the riskier of two gambles, then a fortiori the less risk-averse agent accepts the less risky gamble." The second characteristic is monotonicity with respect to first-order stochastic dominance; "in particular, if gamble *g* is *sure* to yield no more than another gamble *h*, and with positive probability actually yields less, then *g* should be riskier than *h*.

It should be noted that the prevalent definition of risk in finance does not possess these properties. According to a recent statement, it runs "Risk, in a financial sense, is defined as variance about some forecasted value."[3] As [AS] note, all measures of risk based on dispersion alone fail with respect to the monotonicity property. "Thus if g and $g + c$ are gambles, and c is a positive constant, then (variance as a measure of risk will) rate $g + c$ precisely as risky as g, in spite of the fact that it is sure to yield more than g."

Other measures of risk have been introduced but these often depend upon the wealth of the investor and are almost invariably restricted to risk averse agents. From the psychological literature, [AS] note surveys of families of indexes by Brachinger (2002) and Brachinger and Weber (1997). The papers surveyed include Coombs (1969), Pollatsek and Tversky (1970), Fishburn (1977, 1982 and 1984), Luce (1980), Sarin (1987), Luce and Weber (1988), and Jia, Dyer and Butler (1999). In the economic literature, Foster and Hart (2008a and 2008b) present a measure of riskiness based upon the wealth of the investor and defined for risk averse agents only.

In this paper, we build upon the pioneering contribution of [AS] by presenting an index which has the desirable properties of their index while being applicable to all discrete and a large class of continuous gambles. Thus, the index is able to rank gambles ranging from gold kept under the bed in a guarded bomb shelter to buying a lottery ticket and is thus of potential relevance to all individuals regardless of attitude towards risk or area of investment interest. We

[3] Weaver and Michelson (2008). See also Welch, I. (2005).

also provide some simple empirical applications of our index, thereby moving beyond the self-confessed strictly "in principle" approach of [AS].

We proceed as follows. In section 1, we provide a formal extension of the Aumann and Serrano index of riskiness to include the case of risk-loving agents. We do so by introducing the risk-loving equivalent to the constant absolute risk-aversion (CARA) utility function, the constant absolute risk-loving (CARL) function. We use both functions (which are really different forms of the same function) to define an index of inherent risk, which takes on values defined on the positive real line. A gamble which would be acceptable to a risk-averse agent has a value less than one, while risk-lovers might also accept gambles for which the value of the index is greater than one. When the expected return to the gamble is zero, the index takes on a value of one. Our index is analyzed in section 2, where further properties for the simple case of a binary gamble are developed and some empirically testable hypotheses are derived. These hypotheses are tested in section 3, where three simple applications of our index are presented. The first is the example of roulette, where anecdotal evidence is used to support the hypothesis that, for a given (negative) expected return, higher odds roulette bets are less inherently risky than those at lower odds. In the second example, data from the horse betting market at the Happy Valley track in Hong Kong are used, first to estimate winning probabilities for the horses and thereby to calculate the index of inherent risk for each horse. Given that this is a pari-mutuel market, it may be surprising to find that nearly one third of all horses represent bets with a priori net positive expected returns. We use the index to show that risk averse bettors, who bet only when expectations are positive, are inclined to bet on favorites rather than longshots, while risk lovers prefer to bet on longshots. Our final example is taken from the US equity options market. Here we show how difficult it is even to construct a gamble, let alone measure its riskiness! We need to estimate probabilities of outcomes as well as the outcomes themselves. This is done for a large sample of options traded on financial stocks between 2005 and 2007. Using a very simple example of a set of plausible gambles thus constructed, we show that the riskiness index moves, on average, in the same direction as the traditional measure of risk, the variance of the constructed gamble.

1 Generalization of the Aumann and Serrano Index of Riskiness

Following [AS][4], we introduce and discuss the notion of a *generalized index of inherent riskiness*, with no a priori assumptions about attitudes toward risk. A *utility function* is a strictly monotonic twice continuously differentiable function u defined over the entire line. We normalize u so that

$$u(0) = 0 \quad \text{and} \quad u'(0) = 1 \tag{1}$$

If u is concave then an agent with a utility function u is risk averse, while if u is convex, then an agent with a utility function u is risk lover.

The following definition is due to Arrow (1965 and 1971) and Pratt (1964):

Definition 1.1 *The coefficient of absolute risk of an agent i with utility function u_i and wealth w is given by:*

$$\rho_i(w) = \rho_i(w, u_i) = -u_i''(w)/u_i'(w)$$

Note $u_i(x)$ is concave in a neighborhood of w if and only if $\rho_i(w) > 0$, while if it is convex if and only if $\rho_i(w) < 0$.

Along the lines of [AS, Lemma 2,3] we show :

Lemma 1.2 *Let agents i and j have normalized utility functions u_i and u_j and Arrow-Pratt coefficients ρ_i and ρ_j of absolute risk aversion. Then*

1. *For each $\delta > 0$, suppose that $\rho_i(w) > \rho_j(w)$ at each w with $|w| < \delta$. Then $u_i(w) < u_j(w)$ whenever $0 \neq |w| < \delta$.*
2. *If $\rho_i(w) \leq \rho_j(w)$ for all w, then $u_i(w) \geq u_j(w)$ for all w.*

Proof. 1. Let $|y| < \delta$. If $y > 0$. Then by (1),

$$\ln u_i'(y) = \ln u_i'(y) - \ln u_i'(0) = \int_0^y (\ln u_i'(z))' dz = \int_0^y u_i''(z)/u_i'(z) dz$$

$$= \int_0^y -\rho_i(z) dz < \int_0^y -\rho_j(z) dz = \ln u_j'(y)$$

If $y < 0$ then the inequality is reversed by the same arguments. Thus, $\ln u_i'(y) \gtrless \ln u_j'(y)$

[4] For an alternative approach to the derivation of the [AS] index, see Hart (2008).

whenever $y \gtrless 0$, so also $u_i'(y) \gtrless u_j'(y)$ whenever $y \gtrless 0$. So if $w > 0$ then by (1)

$$u_i(w) = \int_0^w u_i'(y)dy < \int_0^w u_j'(y) = u_j(w)$$

and similarly by using the reverse signs, when $w < 0$.

2. In parallel to the first part, with i and j interchanged, strict inequalities are replaced by weak inequalities and the restriction to $|w| < \delta$ is eliminated. QED

Let agent i have utility function u_i, and let w be a real number. We say that i *accepts* g *at* w if $Eu_i(w+g) > u_i(w)$, where E stands for ``expectation", otherwise, i *rejects* g *at* w. We show:

Proposition 1.3 *Let agents i and j have normalized utility functions u_i and u_j and Arrow-Pratt coefficients ρ_i and ρ_j of absolute risk aversion. If $\rho_i(w_i) > \rho_j(w_j)$ then there is a gamble g that j accepts at w_j and i rejects at w_i.*

Proof. Without loss of generality assume $w_i = w_j = 0$, so $\rho_i(0) > \rho_j(0)$. Since u_i and u_j are twice continuously differentiable it follows that there is a $\delta > 0$ so that $\rho_i(w) > \rho_j(w)$ for all $|w| < \delta$. Moreover, for δ small enough u_i and u_j are each either concave or convex in the interval $(-\delta, \delta)$. It follows from Def.1.1 that if u_j is concave then $\rho_i(w) > \rho_j(w) > 0$ and so u_i is concave as well, and if u_i is convex so is u_j. Now, by Lemma 1.2.1,

$$u_i(w) < u_j(w) \quad \text{whenever } 0 \neq |w| < \delta \tag{2}$$

<u>(I) Assume first that u_j is concave</u>**.** Choose ε with $0 \le \varepsilon \le \delta/2$. For $0 \le x \le \varepsilon$ and $k = i, j$, set

$$f_k(x) := \frac{1}{2}u_k(x-\varepsilon) + \frac{1}{2}u_k(x+\varepsilon)$$

By (2),

$$f_i(x) < f_j(x) \text{ for all } x \tag{3}$$

By (3), concavity of u_j, and (1),

$$f_i(0) < f_j(0) < u_j(0) = 0.$$

On the other hand, by monotonicity of the utilities,

$$f_i(\varepsilon) = \frac{1}{2}u_i(2\varepsilon) > \frac{1}{2}u_i(0) = 0$$

Since f_i is continuous, it follows that $f_i(y) = 0$ for some $0 < y < \varepsilon$ and so by (3), $f_j(y) > 0$. So if $\eta > 0$ is sufficiently small then $y - \eta > 0$ and

$$f_i(y-\eta) < 0 < f_j(y-\eta)$$

Let g be the half-half gamble yielding $-\varepsilon + y - \eta$ or $\varepsilon + y - \eta$ Then

$$Eu_i(g) = f_i(y-\eta) < 0 < f_j(y-\eta) = Eu_j(g)$$

Hence j accepts g at 0 while i rejects it.

(II) Now assume that u_i is convex**.** For $k = i, j,$ define $\tilde{u}_k$ by

$$\tilde{u}_k(x) = -u_k(-x)$$

Then

$$\tilde{u}_k(0) = 0,\ \tilde{u}'_k(x) = u'(-x) > 0,\ \tilde{u}'_k(0) = 1 \text{ and } \tilde{u}''_k(x) = -u''(-x).$$

Moreover, by (2),

$$\tilde{u}_j(w) < \tilde{u}_i(w) \quad \text{whenever} \quad 0 \neq |w| < \delta$$

Since $\tilde{u}_i$ is concave in the interval $(-\delta, \delta)$, we are in the same situation as in (I), with i and j interchanged, thus there is a gamble g so that

$$E\tilde{u}_j(g) < 0 < E\tilde{u}_i(g)$$

Take the gamble $-g$, then

$$Eu_i(-g) = -E\tilde{u}_i(g) < 0 < -E\tilde{u}_j(g) = Eu_j(-g)$$

Hence j accepts $-g$ at 0 while i rejects it.

(III) Finally assume u_i is concave and u_j is convex**.** This case is the simplest. Let g be the half-half gamble yielding $-\varepsilon$ or ε. Then

$$Eu_i(g) < 0 < Eu_j(g)$$

Thus i rejects g and j accepts it.

For arbitrary w_i and w_j, define $u_i(x)$ by $u_i^*(x + w_i)$ and $u_j^*(x) = u_j(x + w_j)$ and apply the following to u_i^* and u_j^*. QED

Definition 1.4 *Call i at least risk averse or no more risk loving than j (written $i \trianglerighteq j$) if for all levels w_i and w_j of wealth, j accepts at w_j any gamble that i accepts at w_i. Call i more risk averse or less risk loving than j (written $i \triangleright j$) if $i \trianglerighteq j$ and $j \not\trianglerighteq i$.*[5]

As a corollary of Prop.1.3 we have:

Corollary 1.5 *Given agents i and j, then*

$$i \trianglerighteq j \Leftrightarrow \rho_i(w_i) \geq \rho_j(w_j)$$

for all w_i and w_j.

Proof. Assume $i \trianglerighteq j$ and assume there are w_i and w_j with $\rho_i(w_i) < \rho_j(w_j)$. By Prop.1.3, there is a gamble g that i accepts and j rejects, a contradiction. So $\rho_i(w_i) \geq \rho_j(w_j)$ for all w_i, w_j.

Assume now $\rho_i(w_i) \geq \rho_j(w_j)$ for all w_i, w_j. We wish to show that for all w_i and w_j and any gamble g, if i accepts g at w_i then j accepts g at w_j. Without loss of generality assume $w_i = w_j = 0$. Then Lemma 1.2.2 with i and j interchanged implies $u_j(w) \geq u_i(w)$ for all w. Hence $Eu_j(g) \geq Eu_i(g)$ for all g implying the desired result. QED

Definition 1.6 *An agent is said to have Constant Absolute Risk (CAR) utility function if his normalized utility function $u(x)$ is given by*

$$u_\alpha(x) = \begin{cases} \alpha^{-1}(1 - e^{-\alpha x}), & \alpha \neq 0 \\ x & \alpha = 0 \end{cases}$$

If $\alpha > 0$ then the agent is risk-averse with a CARA utility function, while if $\alpha < 0$ then the agent is risk-loving with a CARL - Constant Absolute Risk-Loving - utility function . If $\alpha = 0$ then the agent is risk neutral. The notion of ``CAR'' is justified since for any α, the

[5] Note that in [AS] the above is defined for risk averse agents only, and is denoted by ``i is at least as risk averse as j''.

coefficient of absolute risk ρ defined in Def.1.1, satisfies $\rho(w) = \alpha$ for all w, that is, the Arrow-Pratt coefficient is a constant that does not depend on w. We have thus a sheaf of functions u_α satisfying for all x:

$$u_\alpha(x) \text{ is continuous at } \alpha = 0.$$

To see this, we need to show that for all x, $\lim_{\alpha\to 0} u_\alpha(x) = x$, where we consider the two sided limit. Now, if $x = 0$ then for all α, $u_\alpha(x) = x$ and if $x \neq 0$ then

$$\lim_{\alpha\to 0} \alpha^{-1}(1 - e^{-\alpha x}) = \lim_{\alpha\to 0} xe^{-\alpha x} = x$$

Given any β, observe that

$$Eu_{-\beta}(g) = -Eu_\beta(-g) \tag{4}$$

Indeed, assume g results in $\{x_1, \dots x_n\}$ with respective probabilities $p_1, \dots p_n$. Then

$$Eu_{-\beta}(g) = \sum p_i u_{-\beta}(x_i) = \sum p_i(-\beta)^{-1}(1 - e^{\beta x_i}) = -Eu_\beta(-g)$$

The CARA version of following proposition is proved in [AS, Prop.4.1]. We state here the general case.

Proposition 1.7 *An agent i has CAR utility function if and only if for any gamble g and any two wealth levels, i either accepts g at both wealth levels, or rejects g at both wealth levels.*

Proof. Any CAR utility function u_α accepts g if and only if $-Ee^{\alpha(g+w)} > -e^{-\alpha w}$, that is if and only if $Ee^{-\alpha g} > 1$ which is independent of w. Conversely, assume an agent i so that $\rho_i(w_*) < \rho_i(w_*)$ for wealth levels w, w_*. If $\rho_i(w) > \rho_i(w_*) > 0$ then we can follow the proof at [AS]. The proof there is based on the formula $\rho(w) = \lim_{\delta\to 0}(p_\delta(w) - 1/2)/\delta$ where $p_\delta(w)$ is that p for which i is indifferent at w between taking and not taking the gamble yielding ±δ with probabilities p and 1−p respectively. (This formula can be found in e.g. Aumann and Kurz (1977), Section 6). It is then used to construct another gamble which is rejected by i at w but accepted at w_*.

If $\rho_i(w) < 0$, then as in 1.3, define $\tilde{u}_i(x) = -u_i(-x)$. Then $\rho_i(\tilde{u}_i, w)$ is positive and we have a gamble g accepted at one level and rejected at the other for $\tilde{u}_i$. Replacing g by $-g$ concludes the proof for this case.

If $\rho_i(w_*) < 0 < \rho_i(w\ )$ then for δ small enough a half-half gamble resulting $\pm\delta$ will be accepted at w_* and rejected at w. QED

The next theorem verifies the existence of the general index for the following class of gambles. A gamble g is *gameable* if it results in possible losses and possible gains. If g has a continuous distribution function, then it is gameable if it is bounded from above and below, that is, its distribution function is truncated.

Theorem 1.8 *Let* g *be a gameable gamble. Then there exists a unique number* α*, so that, for any wealth, a person with utility function* u_α *is indifferent between taking and not taking* g*. In other words, the CAR utility function* u_α *satisfies for all* x*,*

$$Eu_\alpha(g+x) = u_\alpha(x).$$

Moreover, α is positive (negative) if and only if Eg is positive (resp. negative),

Proof. Define a map $f(\alpha)$ by

$$f(\alpha) = 1 - Ee^{-\alpha g} \tag{5}$$

Since g is gameable it follows first that $f(\alpha)$ is defined for all α, and then, since it results positive and negative values, we have $\lim_{\alpha\to-\infty} f(\alpha) = -\infty$ and $\lim_{\alpha\to\infty} f(\alpha) = -\infty.$

Now,

$$(i)\ \ f(0) = 0 \quad (ii)\ \ f'(0) = Eg \quad (iii)\ \ f''(\alpha) < 0\ \forall\alpha$$

By (iii) *f* is concave, hence has at most two roots, one of which is zero. If $Eg > 0$ then f increases at 0, hence the second root α is positive. If $Eg < 0$ then f decreases at 0, hence the second root is negative. If $Eg = 0$ then $\alpha = 0$ is the only root.

To show the last part note that if $\alpha \neq 0$ we have by definition

$$Eu_\alpha(g) = \alpha^{-1}(1 - Ee^{-\alpha g}) = 0$$

It follows that for all x,

$$Eu_\alpha(x+g) = \alpha^{-1}\left(1 - Ee^{-\alpha(g+x)}\right) = \alpha^{-1}(1 - e^{-\alpha x}Ee^{-\alpha g}) = u_\alpha(x)$$

Also if $\alpha = 0$, then by the proof above necessarily $Eg = 0$ and so $Eu_0(g) = Eg = 0$ and $Eu_0(g+x) = x = u_0(x)$. QED

Remark As pointed out by Schulze (2008), [hereafter Sc], for an unbounded distribution function u(x), the map $f(\alpha)$ is not necessarily defined for all α . (In [Sc, Ex 3] it is defined for $\alpha = 0$ only). In this case we cannot apply the proof of Th. 1.8. In [Sc] it is shown that $Ee^{-\alpha g}$ is the Laplace transform of u(x). Since we consider both positive and negative values of α , we use the two-sided Laplace transform. Thus $f(\alpha)$ is defined for all real α in the region of convergence of u(x). If this region of convergence is wide enough, then the proof is still applicable. The question, which distributions admit the appropriate range of convergence, is beyond the scope of this paper.

Definition 1.9 *Given a gamble* g, *denote the number* α *obtained in Th.1.8 by the upper limit of taking* g.

Note that if we replace g by Ng then the upper limit of Ng is the corresponding root of (5) where g is replaced by Ng, and thus equals $N^{-1}\alpha.$

Remark 1.10 *Given a gamble* g *where* $Eg > 0,$ *let its upper limit* α *be as in Def.1.9. Then* α^{-1} *is the index of riskiness of* g *as defined in [AS].*

The notation upper limit is justified by the following corollary.

Corollary 1.11 *Let* α *be the upper limit of taking a gamble* g. *Then:*

1. If $Eg > 0$ *then all CARL accept* g *and a CARA person with a utility function* u_β *accepts* g *if and only if*

$$0 < \beta < \alpha$$

2. If $Eg < 0$ *then all CARA reject* g *and a CARL person with a utility function* u_β *accepts* g *if and only if*

$$\beta < \alpha < 0$$

3. If $E(g) = 0$ *the all CARA people reject* g *while all CARL people accept* g.

Proof. 1. Assume $Eg > 0$ and let $0 < \beta < \alpha.$ Note that for all $w,$ $\alpha = \rho(w,u_\alpha)$ and $\beta = \rho(w,u_\beta)$ as defined in Def.1.1. By Lemma 1.2.2, $\beta < \alpha$ implies $u_\beta(x) > u_\alpha(x)$ for all

$x > 0$. Hence by definition of the upper limit α,

$$Eu_\beta(g) > Eu_\alpha(g) = u_\alpha(0) = 0$$

This implies that a CARA person with a utility function u_β accepts g. Similarly, if $\beta > \alpha$ then a β-CARA person will not accept g.

2. If $Eg < 0$ then by Th.1.8, $\alpha < 0$, and for $\beta < \alpha < 0$ we have $0 < -\alpha < -\beta$. Since $E(-g) > 0$ this implies by (4) and part 1:

$$-Eu_\beta(g) = Eu_{-\beta}(-g) < Eu_{-\alpha}(-g) = -Eu_\alpha(g) = 0$$

Hence $Eu_\beta(g) > 0$ and a CARL person with a utility function u_β accepts g. QED

We propose here a general index of inherent riskiness. Given a gamble g and its upper limit α, define $Q(g)$ by

$$Q(g) = e^{-\alpha} \tag{6}$$

It is straightforward to check the following properties:

Corollary 1.12 *The generalized index $Q(g)$ given in (6) satisfies:*

1. $Q(g) > 0$ for all g.

2. If $Eg > 0$ then $Q(g) < 1$ and if $Eg < 0$ then $Q(g) > 1$. When $Eg = 0$ then $Q(g) = 1$.

3. $Q(Ng) = Q(g)^{1/N}$. In particular

$$Q(-g) = Q(g)^{-1}$$

Proof. 1. is clear. 2. follows directly from Th.1.8.

3. By Remark 1.9, the upper limit of taking Ng is $N^{-1}\alpha$, where α is the upper limit of taking g. Hence by (7),

$$Q(Ng) = e^{-N^{-1}\alpha} = Q(g)^{1/N}$$

QED

Remark 1.13 *Unlike the case of the [AS]- index, homogeneity of degree* 1 *does not hold. However, when* $E(g)>0$ *then it is replaced by (increasing) monotonicity. This follows since in this case* $Q(g)<1$. *Hence if* $t<1$ *then* $Q(tg)=(Q(q))^{1/t}<Q(g)$, *while if* $t>1$ *then* $(Q(q))^{1/t}>Q(g)$.

If $E(g)<0$ *then* $Q(g)>1$ *and* Q *is monotonically decreasing by the same argument as above, with the reverse inequalities. The intuition for this result will be demonstrated in Comment 3 after Application 3.1.*

We wish to show now that our index $Q(g)$ satisfies duality. Along the lines of [AS]:

Theorem 1.14 *Let* g *be a gamble and let* i *and* j *be agents so that* $i \trianglerighteq j$. *If* i *accepts* g *at* w_i *and* $Q(g)>Q(h)$ *then* j *accepts* h *at* w_j.

Proof. Without loss of generality, assume $w_i=w_j=0$. Let γ be the upper limit of taking g and η the upper limit of taking h, as defined in Def.1.9. Thus $Eu_\gamma(g)=Eu_\eta(h)=0$. By assumption, $Q(g)>Q(h)$ hence by (6),

$$\eta>\gamma. \tag{7}$$

Set $\alpha_i=\inf_w \rho_i(w), \alpha_j=\sup_w \rho_j(w)$. Since $i \trianglerighteq j$, it follows from Cor.1.5 that

$$\alpha_i \ge \alpha_j \tag{8}$$

By Lemma 1.2.2,

$$u_i(x)\le u_{\alpha_i}(x) \quad \text{and} \quad u_{\alpha_j}(x)\le u_j(x) \quad \text{for all } x. \tag{9}$$

If $Eu_i(g)>0$ then by (9), also $Eu_{\alpha_i}(g)>0$, hence by Cor.1.11 (part 1 or part 2),

$$\alpha_i<\gamma \tag{10}$$

Combining (7), (8) and (10) we have $\alpha_j<\eta$. Hence again by Lemma 1.2.2,

$$0=Eu_\eta(h)<Eu_{\alpha_j}(h)\le Eu_j(h)$$

Hence j accepts h. This concludes the proof of the theorem. QED

The results above give a characterization of the inherent risk index.

Theorem 1.15 *Any index* satisfying *duality and properties (1)-(3) of Cor.1.12 is a positive power of the index defined in (6).*

Proof. Let Q' be an index satisfying the hypothesis of the theorem. Then $R' = -(\ln Q')^{-1}$ satisfies duality, as an increasing function of Q'. By Cor. 1.12.2, R' is homogeneous of degree 1, hence, when considering gambles with positive mean, we have by uniqueness of the [AS]-index R, that R'=aR where a is a positive number. It follows that $Q' = e^{-\frac{1}{R'}} = (e^{-\frac{1}{R}})^{\frac{1}{a}} = Q^{\frac{1}{a}}$. If g is a gamble with a negative expectation, then –g has a positive expectation and the result follows from Cor.1.12.3.

Other properties of the general index of inherent risk follow almost directly from the corresponding properties stated and proved in [AS]. In what follows we list two of them.

Corollary 1.16 *1. [AS,5.3] The general index Q is first and second order monotonic.*

2. [AS, 5.4] The index Q is (uniformly) continuous in the sense that $Q(g_n) \to Q(g)$ whenever $g_n \to g$ uniformly.

Proof. 1. Assume $g > g_*$. Consider the map $f_g(\alpha)$ defined in (2). Then for all $\alpha > 0, f_g(\alpha) > f_{g_*}(\alpha)$, while for all $\alpha < 0, f_g(\alpha) < f_{g_*}(\alpha)$. If $Eg < 0$ then by Th.1.8 the nonzero roots α, α_* of f_g, f_{g_*} respectively are negative and hence satisfy $\alpha > \alpha_*$. It follows that $Q(g) < Q(g_*)$. If $Eg > 0$ then α is positive and the proof is as in [AS].

2. The proof is as in [AS], moreover, the continuity of the general index is uniform since the general index $Q(g)$ never approaches ∞. QED

2 Analytical discussion

In this section we discuss further properties of the index of inherent risk. We start with the binary case. Let g be a gamble that results in a gain of M with probability p and a loss of L with probability $q = 1 - p$. We assume M and L are positive real numbers. Following Th.1.8 we need to solve $Eu_\alpha(g) = 0$. That is:

$$0 = pu_\alpha(M) + qu_\alpha(-L) = \alpha^{-1}\left(p(1 - e^{-\alpha M}) + q(1 - e^{\alpha L})\right)$$

Set: $$x = e^{-\alpha}$$

Then we are looking for positive roots of the function:

$$f(x) = px^M + qx^{-L} - 1 \tag{11}$$

where $x = 1$ is one root and the index of inherent risk $Q(g)$ is another root of that function.

We discuss first some analytical properties of $Q(g)$. Assume L=1 and consider the following special cases:

<u>$M = 1$</u>: In this case we need to solve $px - q = 0$, so $x = q/p$. When $p = 0.5$ then $E(g) = 0$ and $Q(g) = 1$. This agrees with the discussion in the proof of Th.1.8.

<u>$M = 2$</u>: We need to solve $px^2 + px - q = 0$ which yields the positive solution $x = \dfrac{-1 + \sqrt{1 + 4t}}{2}$, where $t = q/p$. Then $x > 1$ if and only if $t > 2$, that is $p < 1/3$ and $Eg < 0$. When $p = 1/3$ then $E(g) = 0$ and $Q(g) = 1$.

<u>M very large</u>: In this case we solve the original equation $px + qx^{-M} - 1 = 0$. As $M \to \infty$ then $x \to 1/p$.

In order to facilitate the empirical examples discussed in the next section, we summarize partial relations between expected utilities, expectations of gambles, chances to win and riskiness. We start with expected utilities:

Proposition 2.1 Assume g results in a gain of M with probability p and a loss of L otherwise. Consider $Eu_\alpha(g)$ as a function of the independent variables L, M and Eg; then:

1. $\dfrac{\partial Eu_\alpha(g)}{\partial Eg} > 0$

2. If $\alpha > 0$ then $\frac{\partial Eu_\alpha(g)}{\partial M} < 0$ and if $\alpha < 0$ then $\frac{\partial Eu_\alpha(g)}{\partial M} > 0$.

Proof. 1. Since $Eg = pM + (1-p)(-L) = p(M+L) - L$, we have

$$p = \frac{Eg + L}{M + L} \ . \tag{12}$$

Hence

$$Eu_\alpha(g) = \alpha^{-1}(1 - pe^{-\alpha M} - (1-p)e^{\alpha L}) = \alpha^{-1}(1 - \frac{Eg+L}{M+L}(e^{-\alpha M} - e^{\alpha L}) - e^{\alpha L}) \tag{13}$$

implying

$$\frac{\partial Eu_\alpha(g)}{\partial Eg} = \alpha^{-1} e^{-\alpha M} (L+M)^{-1} (e^{\alpha(L+M)} - 1)$$

If $\alpha > 0$ then the last factor above is positive, while if $\alpha < 0$ then it is negative, thus the value of the product above is positive in both cases.

2. By (13),

$$\begin{aligned}
\frac{\partial Eu_\alpha(g)}{\partial M} &= \\
&= \alpha^{-1}\left(\frac{Eg+L}{(M+L)^2}(e^{-\alpha M} - e^{\alpha L}) + \alpha \frac{Eg+L}{M+L} e^{-\alpha M} \right) \\
&= \frac{\alpha^{-1}(Eg+L)}{(M+L)^2}(e^{-\alpha M} - e^{\alpha L} + \alpha(M+L)e^{-\alpha M}) \\
&\overset{\text{by (12)}}{=} \frac{e^{-\alpha M}\alpha^{-1} p}{L+M}\left(1 + \alpha(L+M) - e^{\alpha(L+M)}\right)
\end{aligned}$$

We claim that $f(\alpha) = 1 + \alpha(L+M) - e^{\alpha(L+M)}$ is negative for all $\alpha \neq 0$. Indeed,

$$f'(\alpha) = L + M - (L+M)e^{\alpha(L+M)} = (L+M)(1 - e^{\alpha(L+M)})$$

If $\alpha > 0$ then $f'(\alpha) < 0$ while If $\alpha < 0$ then $f'(\alpha) > 0$. Since $f(0) = f'(0) = 0$, our claim follows. Since $Eu_\alpha(g) = f(\alpha)$ multiplied by a positive value, the desired result follows.

We consider now how Q = Q(g) is related to the other variables .

Proposition 2.2 *Let g be gambles that result in a gain M with probability p and a loss L otherwise. Then:*

1. $\dfrac{\partial Q(g)}{\partial p} < 0$

2. $\dfrac{\partial Q(g)}{\partial Eg} < 0$

3. If $Eg < 0$ then $\dfrac{\partial Q(g)}{\partial M} < 0$. If $Eg > 0$ then $\dfrac{\partial Q(g)}{\partial M} > 0$. Finally if $Eg = 0$ then $\dfrac{\partial Q(g)}{\partial M} = 0$

Proof. 1. If $p_1 < p_2$ then $f_{p_1}(x) < f_{p_2}(x)$ for all $x > 1$, and $f_{p_1}(x) > f_{p_2}(x)$ for all $x > 1$, where $f_{p_i}(x)$ is the concave function defined in (11) with $p = p_i$. Recall $Q(g_i)$ is the root of this function other than 1. If $Eg_i > 0$ then $Q(g_i) > 1$ for $i = 1, 2$, hence we must have $Q(g_1) > Q(g_2)$. If $Eg_i < 0$ then $Q(g_i) < 1$ for $i = 1, 2$, hence again $Q(g_1) > Q(g_2)$. If $Eg_1 < 0 < Eg_2$ then $Q(g_1) > 1$ while $Q(g_2) < 1$.

2. Clearly $\dfrac{\partial Eg}{\partial p} > 0$, hence by part 1, $\dfrac{\partial Q}{\partial Eg} = \dfrac{\partial Q}{\partial p}\dfrac{\partial p}{\partial Eg} < 0$.

3. Assume $M_1 < M_2$. Let g_1 be the gamble resulting in M_1 and g_2 resulting in M_2. Let $\alpha_1 = -\ln(Q(g_1))$. By (6), α_1 is the upper limit of taking g_1 and $Eu_{\alpha_1}(g_1) = 0$. If $Eg < 0$ then $\alpha_1 < 0$ by Cor.1.11.2, so by Prop. 2.1.2, $0 = Eu_{\alpha_1}(g_1) < Eu_{\alpha_1}(g_2)$. Hence α_1 accepts g_2. This implies by Cor.1.11.2 again that $\alpha_1 < \alpha_2$, where $\alpha_2 < 0$ is the upper limit of taking g_2. Hence $Q(g_1) > Q(g_2)$ and we are done. When $Eg > 0$ then by Cor.1.11.1 $\alpha_1 > 0$, and by Prop. 2.1.2, $0 = Eu_{\alpha_1}(g_1) > Eu_{\alpha_1}(g_2)$. Hence α_1 rejects g_2 and thus $\alpha_2 < \alpha_1$ and $Q(g_1) < Q(g_2)$. If $Eg = 0$ then by Cor.1.12.2, Q(g) = 1 and the result follows. QED

Prop. 2.1.2 and 2.2.3 can be summarize as follows :

Corollary 2.3 $\frac{\partial Eu_\alpha(g)}{\partial Q(g)} < 0$ if $\alpha Eg > 0$ and $\frac{\partial Eu_\alpha(g)}{\partial Q(g)} > 0$ if $\alpha Eg < 0$. *This follows since* $\frac{\partial Eu_\alpha(g)}{\partial Q(g)} = \frac{\partial Eu_\alpha(g)}{\partial M}\frac{\partial M}{\partial Q(g)}$.

Since increasing M is the same as increasing the Variance and the Standard Deviation of g, we could restate Prop. 2.2.3 as follows: Set $R = \frac{\sigma}{Eg}$, where σ = std(g), then:

$$\frac{\partial Q}{\partial R} > 0 .$$

One could ask whether the above hold for any gamble. The answer is negative as will be demonstrated in the following counter examples. Recall that R violates monotonicity with respect to first order dominance, as shown in [AS, 7.2], while Q does not. Nevertheless, in the next section we show that for the empirical results this is true "on average". This justifies the use of R as a measure of risk (It is the reciprocal of the *Sharpe Ratio*[6]).

Example 2.4 The following two tables are examples of gambles g with fixed Eg, where the last two columns demonstrate how even though Q increases, $E(g^2)$ and thus Var(g) and R, neither decrease nor increase.

Assume first that g results in **{**-3,-2, 0, 1, 2} with probabilities $\{p_1, p_2, p_3, p_4, p_5\}$ respectively. We have:

p1	p2	p3	p4	p5	Eg	E(g²)	Q
0.05	0.15	0.20	0.35	0.25	0.4	2.4	0.727857
0.00	0.30	0.00	0.40	0.30	0.4	2.8	0.753716
0.10	0.10	0.10	0.50	0.20	0.4	2.6	0.753805

Even if we fix the p's and change the M's, we do not see unambiguous relations. Assume g results in $\{M_1, M_2, M_3, M_4, M_5\}$ with corresponding fixed probabilities {0.1, 0.3, 0.2, 0.25, 0,15}. Then we have:

---

[6] See Bodie, Kane and Marcus (2002) and Welch (2005).

M_1	M_2	M_3	M_4	M_5	Eg	$E(g^2)$	Q
-1.00	-1.00	0.00	1.00	3.67	0.4	2. 67	0.667933
-2.00	-1.00	2.00	1.10	1.50	0.4	2.14	0.675817
-2.00	-1.00	1.00	1.00	3.00	0.4	2.5	0.698304

3 Applications

In this section we present three examples of possible applications of our index in order of increasing complexity. We begin with a case of a binary gamble, where all probabilities and payouts are known: roulette. In this simple case, all investors are risk lovers, since expected returns are uniformly negative.

Next, we consider a more complex example of a binary bet: horse race betting. Here payouts are more-or-less known, depending upon whether the betting is via pari-mutuel or bookmakers, but the probabilities of different outcomes must be estimated. The inherent riskiness of different horse bets are calculated using estimates of horses' winning probabilities derived from data drawn from the Happy Valley horse betting market in Hong Kong.

Finally, we move on to a case in which neither probabilities nor returns are known with certainty and the gambles are no longer binary: the Wall Street stock options market. Using a simple example, we estimate probabilities and payouts and calculate the inherent risk index of a large range of option bets. Both here and in the case of the horse betting market, we use our calculated indexes to test hypotheses derived in the previous section.

Application 3.1:

We begin with the simplest practical example of the inherent risk index, the casino game of roulette. In this case, every possible bet is a binary gamble where the return to a losing bet is always the outlay and both the probability of success and the concomitant payout are known. There is thus no uncertainty here, merely risk. Table I provides complete details for the different kinds of bets available in the American version of the game[7].

[7] In the European version, the setup of the wheel is slightly different.

Table I: The Inherent Risk Index (Q) For American Roulette

Bet name	Winning spaces	Payout = M	Odds p=1/(odds+1)	Expected value (on a $1 bet) = Eg	Q(g)	-ln(Q) = upper limit of taking g
0	0	35 to 1	37 to 1	−$0.053	1.003065	-0.00306
00	00	35 to 1	37 to 1	−$0.053	1.003065	-0.00306
Straight up	Any single number	35 to 1	37 to 1	−$0.053	1.003065	-0.00306
Row 00	0, 00	17 to 1	18 to 1	−$0.053	1.006318	-0.0063
Split	any two adjoining numbers vertical or horizontal	17 to 1	18 to 1	−$0.053	1.006318	-0.0063
Trio	0, 1, 2 or 00, 2, 3	11 to 1	11.667 to 1	−$0.053	1.00978	-0.00973
Street	any three numbers horizontal	11 to 1	11.667 to 1	−$0.053	1.00978	-0.00973
Corner	any four adjoining numbers in a block	8 to 1	8.5 to 1	−$0.053	1.013457	-0.01337
Six Line	any six numbers from two horizontal rows	5 to 1	5.33 to 1	−$0.053	1.02138	-0.02116
1st Column	1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34	2 to 1	2.167 to 1	−$0.053	1.05467	-0.05323
2nd Column	2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35	2 to 1	2.167 to 1	−$0.053	1.05467	-0.05323
3rd Column	3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36	2 to 1	2.167 to 1	−$0.053	1.05467	-0.05323
1st Dozen	1 through 12	2 to 1	2.167 to 1	−$0.053	1.05467	-0.05323
2nd Dozen	13 through 24	2 to 1	2.167 to 1	−$0.053	1.05467	-0.05323
3rd Dozen	25 through 36	2 to 1	2.167 to 1	−$0.053	1.05467	-0.05323
Odd	1, 3, 5, ..., 35	1 to 1	1.111 to 1	−$0.053	1.111	-0.02116
Even	2, 4, 6, ..., 36	1 to 1	1.111 to 1	−$0.053	1.111	-0.02116
Red	1, 3, 5, 7, 9, 12, 14, 16, 18, 19, 21, 23, 25, 27, 30, 32, 34, 36	1 to 1	1.111 to 1	−$0.053	1.111	-0.02116
Black	2, 4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35	1 to 1	1.111 to 1	−$0.053	1.111	-0.02116
1 to 18	1, 2, 3, ..., 18	1 to 1	1.111 to 1	−$0.053	1.111	-0.02116
19 to 36	19, 20, 21, ..., 36	1 to 1	1.111 to 1	−$0.053	1.111	-0.05323
Five Number	0, 00, 1, 2, 3	6 to 1	6.6 to 1	−$0.079	1.027295	-0.02693

The initial bet is returned in addition to the mentioned payout. Note also that 0 and 00 are neither odd nor even in this game.

Comments:

1. We have by Prop. 2.2.3, that $Eg < 0 \Rightarrow \frac{\partial Q(g)}{\partial M} < 0$. This is demonstrated in the table (except for the "Five Number Bet", which has a different expectation).

2. There are risk loving gamblers that will not take "higher risk" gambles, but will take "lower risk" ones. Here "higher" and "lower" are with respect to the table only, and NOT immediately intuitive. If their utility function has a coefficient of absolute risk aversion α, and -0.05323 < α < -0.00306 then they will never take the 1-1 gamble, but can take the 1-35 gamble. This follows since the 1-1 gamble has its upper limit -0.05323, and by Cor.1.11 gamblers with higher α never take it. It is riskier in this sense.

3. An intuitive explanation to higher and lower values of riskiness for negative expectation could be the following. Gamblers who put money on gambles with negative expectations are all risk lovers, which means that they get thrills from higher values of money. Thus to love risk means to love thrills. Such a gambler risks losing more utility by taking 2 to 1 bets, because he gets fewer thrills.

4. If a gambler has a utility function with a coefficient of absolute risk aversion α, and α < -0.05323, then he can take any of the gambles. This DOES NOT IMPLY that he will bet on the higher or on the lower risk gamble. This can vary for different values of α. For example: If α =-0.01 then he will bet on one of the following three gambles, 35 to 1, 17 to 1 or 11 to 1.

5. Based upon these observations, we would predict that more roulette players choose to play 35 to 1 gambles and fewest would chooses even money gambles. Unfortunately, we have no data that would permit us to test this hypothesis formally, but we have been told that the following holds in casinos operated by HIT in Slovenia and elsewhere in Southern Europe.[8] First, less than 5 percent of all gamblers play 2 to 1 or even money gambles. Second, in most instances there are multiple bets on one spin of the wheel. Thus, most of the gamblers choose 17 to 1 or 35 to 1 gambles, but most of the customers will cover, with such bets, approximately 12 of the available numbers (out of 37) on one roulette spin. Finally, following winning bets, gamblers will proceed to cover more numbers in a subsequent bet. There is no observable trend following losing bets.

[8] This information was provided by Igor Rus of HIT.

Application 3.2:

As a second application of the inherent risk index, consider the Hong Kong horse betting market at the Happy Valley racetrack. Using data on 16068 horses who ran at Happy Valley between September 2000 and October 2006, we ran the following conditional logit regression (Mc. Fadden 1974) to predict winning probabilities as a function of the following variables, all of which would be known prior to the particular race and which might be considered as relevant to a horse's performance: the horse's age, the weight it is to carry in the race, its starting barrier position, its opening and starting odds and the odds five minutes before the start of the race at its last start, the distance of its last race, the margin by which it lost its last race (zero if it won), its barrier draw, weight and finishing position at its last start, the change in class from this race to its last race and the winning strike rate of the horse's jockey in the sample.

This regression was then used to predict the winning probability of the each horse in the sample. The approximate tote odds 5 minutes before the race (and thus also known to bettors before the race) were used to calculate an estimate of the expected return for each horse. Given the loss of observations entailed by the use of lagged variables in the above regression, we were left with 7522 horses with negative expected returns and 7522 with positive expected returns. Note that, unlike the roulette case, where the index of inherent risk for all bets is known with certainty, in horse betting, there is uncertainty regarding the calculated Q for two reasons: (a) The true winning probabilities of the horses are unknown and are thus estimated with error. (b) The final payouts (and thus M) are unknown before the race is run and thus bettors must use estimates provided by the track. Uncertainty notwithstanding, we proceeded to test three hypotheses from Proposition 2.2:

1. $\partial Q(g)/\partial Eg < 0$.

2. If $Eg < 0$ then $\dfrac{\partial Q(g)}{\partial M} < 0.$

3. If $Eg > 0$ then $\dfrac{\partial Q(g)}{\partial M} > 0.$

The results are shown in Table II.

Table II: The Impact of Expected Returns and Payouts on Inherent Risk

Dep. variable Eg>0	Q Yes	Q No
M	.001 (.0000)	-.0073 (.0005)
Eg	-.0367 (.0006)	-.8555 (.0293)
Constant	.948 (.0007)	.9994 (.0139)
N	5422	7522
Adjusted $\mathbf{R}^2$	0.4801	0.1459

Notes: Standard errors are in parentheses. M is the profit per unit bet on a winning horse and Eg is the estimated expected return to a bet on the horse. These regressions were also run with horse fixed effects, but the latter were statistically significant only in the case of Eg < 0 and the coefficients of M and Eg and the intercept were virtually identical in size and significance to those shown in the table.[9] All coefficients are statistically significant at better than the 0.1% level.

Note that, as expected, in the case of both regressions, the greater is Eg, the lower will be Q. Thus, our perhaps counterintuitive results hold here for both horses with positive and negative expected returns: risk averse bettors, who only bet when expectations are positive are inclined to bet on favorites rather than longshots (inherent risk rises as M rises), while for risk lovers we see that inherent risk falls as M rises, suggesting that risk lovers will prefer to bet on longshots. It should be noted that the division of data set according to the sign of Eg does not provide us with populations of exclusively risk averse and risk loving bettors, since risk lovers will also bet on horses for which Eg>0. However, to that extent that our estimates of Eg are at least reasonable, all the bettor in the sub-sample for which Eg<0 will be risk lovers.

Application 3.3: Finally, we present an example from the stock exchange and consider the gambles provided by options. In example 2.4 above, we showed that, given expected return, for non-binary gambles, an increase in inherent risk does not necessarily imply either an increase or a decrease in the variance of a gamble. Our purpose here is to show that, this mathematical result notwithstanding, the gambles represented by options may behave, at least on average, as if $\frac{\partial Q}{\partial R} > 0$. Our data set comprises all the call and put options traded on all financial stocks traded

[9] Full results are available upon request.

on Wall Street between the first trading day of 2005 and the last day of May, 2007.[10] After removal of observations with missing data and accommodation of the data set to the investment strategy described below, we were left with 508,284 observations. While in the example of the horse betting, L and M were more or less known (and would be known in a bookmakers' market), it remained to estimate p as the betting market may be characterized as a series of binary bets. Thus, there was an added complication that did not exist in the roulette example, where the index of inherent risk could be calculated accurately. Moving to options, none of the components of a gamble are known a priori with certainty and all must be estimated and the gambles are no longer binary.

Since our purpose here is to provide a simple demonstration of the way in which the index of inherent risk is related the more usual measure of risk in finance, variance, we conducted the following mental experiment: An investor wishes to buy an option, hold it for 21 days and then sell it. Since we are unable to predict winning options in advance, we calculated the *ex post* rate of return that accrued to each such investment in the data set. The options were then ranked by rate of return and the sample was divided into ten more or less equal-sized groups (subject to the constraint that all investments in any group yielded either exclusively positive or exclusively negative returns. This yielded three groups with positive returns and seven with negative returns. The ten means of the returns were used as the returns to the gamble thus generated.

Two multinomial logit regressions (one for call options and one for puts) were then run to predict the probabilities that the returns to investing in one of the options would fall into each of the profit groups. The explanatory variables used in the regression were the closing price of the underlying stock, the implicit volatility of the option relative the mean over the relevant industry group, the volumes traded in both the stocks and options, the spread of the option price relative to the best bid and the gamma, delta, vega and theta of the option. These variables were all statistically significant at better than 1 percent for all profit groups except volume of stock traded, which for statistically significant at 5 percent in at least one group.[11]

Having thus estimated the various components of the gamble, we proceeded to calculate Q and regressed it on two explanatory variables, "$\frac{\sigma(g)}{Eg}$" and "outness", and fixed effects for the individual options in the sample. $\frac{\sigma(g)}{Eg}$ is the ratio of the standard error of the specific option to its expected return, being a measure of the riskiness of the option in standard finance terms

[10] The data were provided by OptionMetrics.

[11] Full details available upon request.

corrected for the expected return. This means that the variable is positive for gambles with positive expected returns and we would expect a positive coefficient if volatility and inherent risk are positively correlated, while $\frac{\sigma(g)}{Eg} < 0$ for gambles that would be taken by risk lovers only and we would thus predict a positive coefficient again. However, for negative expected returns, the coefficient indicated that inherent riskiness falls with a rise in volatility.

The second variable, outness, is the ratio of strike price to closing price of the underlying and thus measures, as the name suggests, the extent to which the option is out-of-the-money. Given that the standard error of the option has been taken into account, we would expect the coefficient of outness to be negative, since the odds payable to a successful bet for a given outlay on an option increase as outness increases. With volatility accounted for, increased outness will be less inherently risky for both risk lovers and the risk averse. The results are given in Table 3 and the conform to our *a priori* expectations.

Table III: The Impact of Volatility and Outness on Inherent Riskiness

Dep. variable	*Q*	*Q*	*Q*	*Q*
Option Type	*Call*	*Call*	*Put*	*Put*
Eg>0	*Yes*	*No*	*Yes*	*No*
$\frac{\sigma(g)}{Eg}$	.2483 (.0173)	.0474 (.0087)	.3254 (.0481)	4.806 (1.098)
Outness	-.2804 (.0031)	-.4469 (.013)	-.5545 (.0223)	-9.435 (.0542)
Constant	.9477 (.0033)	1.697 (.0129)	1.412 (.0248)	13.35 (.0579)
N	191241	97531	15977	202515
Adjusted $\mathbf{R}^2$	0.7294	0.4488	0.4002	0.5709

Notes: Standard errors are in parentheses. All regressions include option fixed effects. SE/Eg is the ratio of standard error to expected yield divided by one million. Outness is defined as the ratio of the underlying stock closing price to the option strike price for a put option and the ratio of the option strike price to the underlying stock closing price. All coefficients are statistically significant at better than the 0.1% level.

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