Generalized solutions for the joint replenishment problem with correction factor

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Abstract

In this paper we give a complete analysis of the joint replenishment problem (JRP) under constant demands and continuous time. We present a solution method for the JRP when a correction is made for empty replenishments, and we test the solution procedures with real data. We show that the solutions obtained differ from the standard JRP when no correction is made in the cost function. We further show that the JRP with correction outperforms independent ordering. Additional numerical experiments are presented.

Keywords: inventory, joint replenishment, correction factor

1. Introduction

During the last two decades much attention has been given in the literature to the deterministic joint replenishment problem (JRP) in continuous time. In this problem it is assumed that a major ordering cost is charged at a basic cycle time T, and that the ordering cycle of each item j is some integer k_j multiple of T, which is called a (k_j,T) policy. Furthermore, it is generally assumed that a minor set-up cost is charged for each item j included in a single order. In this paper we want to give a comprehensive analysis of the JRP. First of all we present a complete theory proving some claims made in earlier papers; second we consider the inclusion in the objective function of a correction factor for empty replenishments, and third we make a comparison with independent ordering.

Although many heuristics and exact methods have been proposed to solve the JRP efficiently, no paper so far incorporates the correction factor in the analysis. In this paper we show that the inclusion of the correction factor in the cost function may yield very different optimal solutions in terms of the basic cycle time and the values of the k_j. We show by experimentation that this is often the case for large values of the minor set-up costs and moderate major set-up cost. This may have a large impact on the quality of the solution from an implementation point of view, especially when using the deterministic JRP as an approximation to the stochastic case. Moreover, a major theoretical shortcoming of the standard JRP is that it does not indicate whether an independent ordering solution with EOQ applied to the individual lot sizes is better. We show that the inclusion of the correction factor in the cost function of the JRP yields a solution that always outperforms independent ordering. Numerical results support this theoretical finding.

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In this paper we present a method similar to van Eijs [2], but modified to be suitable for general integer policies (GI) with a correction factor in the objective function. We show that the objective function with correction factor is still piecewise convex with respect to T but discontinuous. Additionally, we substantiate the claim that the methods provided in the literature to solve the JRP are optimal algorithms.

The method presented in this paper is similar to the one given in Porras and Dekker [8], who studied the JRP under minimum order quantities for the lot sizes of individual items included in the replenishment order.

The set-up of the paper is as follows: In the next section we give the definition of the problem and the relevant literature review. In section 3 we present the algorithms to solve the JRP using GI policies with and without correction factor. In section 4 numerical experiments are presented and further theoretical results are discussed. The final conclusions are included in section 5. For clarity of the ideas presented, we include all the theoretical details of the methods proposed in the appendices at the end of the paper.

2. Problem definition

Consider the problem of ordering M items that can be jointly replenished against a major set up cost S. The demand D_j for item j is assumed constant and known. No backorders are allowed. In the base formulation one seeks an interval T and a vector $\mathbf{k}$ of k_j's which minimize the total holding and ordering costs, given by:

$$TC(T,\mathbf{k}) = \frac{S}{T} + \sum_{j=1}^{M}\left(\frac{s_j}{k_j T} + \frac{1}{2}h_j k_j D_j T\right)$$

where s_j and h_j are the minor set-up cost and unit holding cost of item j.

The formulation for the JRP without correction factor is given below:

JRP (*standard formulation*)

$$(P)\ \min\left\{TC(T,\mathbf{k})\ \middle|\ T>0,\ k_j \geq 1\ \textit{integers for}\ j=1,2,\ldots,M\right\}$$

Note that a solution (k_j,T) may have empty replenishments, e.g. $\mathbf{k}$ = (2,3), for which the major set-up cost is still charged. Therefore, a correction factor $\Delta(\mathbf{k})$ should be included in the objective function, as follows:

$$TC^{(c)}(T,\mathbf{k}) = \frac{S\Delta(\mathbf{k})}{T} + \sum_{j=1}^{M}\left(\frac{s_j}{k_j T} + \frac{1}{2}h_j k_j D_j T\right)$$

The factor $\Delta(\mathbf{k})$ in the previous equation is the fraction of non-empty replenishments per year. When such a correction for empty replenishments is made in the objective function, we denote this by problem ($P^{(c)}$).

The following formula can be derived using the principle of inclusion and exclusion for the evaluation of $\Delta(\mathbf{k})$ (Dagpunar [1]):

$$\Delta(\mathbf{k}) = \sum_{i=1}^{M} (-1)^{i+1} \sum_{\{\alpha \subset \{1,\dots,M\} : |\alpha| = i\}} (lcm(k_{\alpha_1},\dots,k_{\alpha_i}))^{-1} \tag{1}$$

$$= \sum_{i=1}^{M} \frac{1}{k_i} - \sum_{(i,j)\subseteq\{1,\dots,M\}} \frac{1}{lcm(k_i,k_j)} + \sum_{(i,j,k)\subseteq\{1,\dots,M\}} \frac{1}{lcm(k_i,k_j,k_k)} - \cdots + (-1)^{M+1} \frac{1}{lcm(k_1,\dots,k_M)}$$

where $lcm(k_{\alpha_1},\dots,k_{\alpha_i})$ denotes the least common multiple of the integers $k_{\alpha_1},\dots,k_{\alpha_i}$.

Note that if $k_j = 1$ for some j, $\Delta(\mathbf{k}) = 1$. In other cases it is more difficult to compute $\Delta(\mathbf{k})$. In Appendix A we present an algorithm for the evaluation of $\Delta(\mathbf{k})$.

Both formulations (P) and ($P^{(c)}$) are non-linear mixed-integer programming problems, which are difficult to solve especially for large number of items. Goyal [5] and other authors argue that the cost improvement gained by the inclusion of the correction factor is only of few percentage points, and hence it should be left out. However, Goyal does not consider any possible effect that the correction factor may have on the optimal values of T and k_j. Wildeman *et al*. [12] showed that the solution of a relaxation of the JRP can be used as a lower bound for the optimal solution of the JRP with correction factor. Jackson *et al*. [6] and Roundy [10] proposed the use of the so-called power-of-two (PoT) policies, by letting $k_j = 2^p$, $p \geq 0$ (p: integer). The former showed that a PoT policy for the JRP produces 94%-effective solutions with respect to the optimal value of the objective function when using GI policies. Fang-Chuan and Ming-Jong [3] provided a global optimization procedure for the JRP using a PoT policy, and they showed that the optimal PoT solution contains at least one of the k_j's equal to one. Such a solution has an associated correction factor of one. Consequently (as pointed out by Goyal), its optimal objective value will be either equal or slightly worse than the optimal objective value of a GI policy solution with correction factor. In the next section we give the solution methods for (P) and ($P^{(c)}$).

3. Solution methods

3.1. Solution method for problem (P)

The function $TC(T,\mathbf{k})$ is not jointly convex with respect to T and $\mathbf{k}$. However, for a fixed vector $\mathbf{k}$ the function $TC(T)$ is convex in T, with optimal T given by:

$$T^*(k_1,\dots,k_M) = \sqrt{\frac{2\left(S + \sum_{j=1}^{M} \frac{s_j}{k_j}\right)}{\sum_{j=1}^{M} h_j D_j k_j}} \tag{2}$$

Substituting (2) back in $TC(T,\mathbf{k})$ we get the optimal TC for a fixed $\mathbf{k}$:

$$TC(k_1,\dots,k_M) = \sqrt{2\left(S + \sum_{j=1}^{M} \frac{s_j}{k_j}\right)\left(\sum_{j=1}^{M} h_j D_j k_j\right)} \tag{3}$$

Now consider the following equivalent formulation of problem (P) as suggested by Wildeman *et al*. [12]:

$$(P)\ \text{Min}\ TC(T) = \frac{S}{T} + \sum_{j=1}^{M} z_j(T)$$

s.t. $\quad T > 0$

where the functions $z_j(T)$ are given by:

$$z_j(T) = \min_{k_j}\left\{\frac{s_j}{k_j T} + \frac{1}{2} h_j D_j k_j T\right\} : k_j \geq 1 \text{ integers for } j = 1,\ldots, M. \tag{4}$$

Wildeman *et al*. [12] showed that for a fixed T the optimal value of k_j is given by:

$$k_j(T) = \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_j}{T^2}}\right\rceil, \tag{5}$$

where $b_j = \dfrac{s_j}{h_j D_j}$.

Let $T^{(i-1)}$ and corresponding $\mathbf{k}^{(i-1)} = \mathbf{k}(T^{(i-1)})$ be given. Now let I_i be the interval $[T^{(i)}, T^{(i-1)})$ associated to $\mathbf{k}^{(i-1)}$. Next observe that for $T \in [T^{(i)}, T^{(i-1)})$ the arguments in (5) increase as $T \to T^{(i)}$. The optimal vector $\mathbf{k}$ will change when one (or more) of its elements increases by one unit just below $T^{(i)}$. Therefore, $T^{(i)}$ can be calculated from:

$$T^{(i)} = \max_j \left\{T_j^{(i)}\right\} \tag{6}$$

where

$$T_j^{(i)} = \sqrt{\frac{2s_j}{h_j D_j k_j^{(i-1)}(k_j^{(i-1)} + 1)}} \quad \text{for } j = 1,\ldots, M. \tag{7}$$

The elements of the vector $\mathbf{k}$ just below $T^{(i)}$, say $\mathbf{k}^{(i)}$, are given by:

$$k_j^{(i)} = \begin{cases} k_j^{(i-1)} + 1 & \textit{for} \quad j \in J^{(i)} \\ k_j^{(i-1)} & \textit{for} \quad j \notin J^{(i)} \end{cases} \tag{8}$$

where $J^{(i)}$ is the set of all elements of $\mathbf{k}$ for which the maximum in (6) is attained.

The previous analysis allows us to make a partition on the set $I = (0,\infty)$ of T values using equation (6), with optimal $\mathbf{k}$ vectors given by (5). Note that we do not need a full enumeration on $\mathbf{k}$, since we only consider the vectors $\mathbf{k}$ that minimize the total cost for a given T. Therefore, if we can establish lower and upper bounds on T, say

T_{low} and T_{upp}, we only need to evaluate a finite number of intervals. We can obtain the local minima of TC with formula (3) inside each interval of such a partition, and compute the best solution among all intervals.

The above procedure was first proposed by Goyal [5]. However, he did not explicitly show that the optimality of **k** given by (5) determines the number of **k** vectors that should be considered between T_{low} and T_{upp}. Therefore, in his procedure one needs to enumerate all optimal vectors starting in $\mathbf{k} = (1,\ldots,1)$. In the method we propose in this paper, by using (5) we can directly start the searching procedure in the optimal vector **k** associated to T_{upp}. Another pitfall of Goyal's method, as pointed out by van Eijs [2], is that the lower bound that he used could only guarantee optimal solutions for strict cyclic policies, where the smallest $k_j = 1$.

Now notice that for a given **k**, the optimal value of T given by (2), say $T^*_{(i-1)}$, does not necessarily belongs to the interval $[T^{(i)}, T^{(i-1)})$ where the vector $\mathbf{k}^{(i-1)}$ minimizes TC. However, the overall optimal solution for TC has an associated optimal T, say T_{opt}, equal to some $T^*_{(i-1)}$ (see Figure 1). Therefore, we need to evaluate TC only in the intervals for which $T^*_{(i-1)} \in [T^{(i)}, T^{(i-1)})$. We formalize this result in the following theorem.

Theorem 1. Let $\mathbf{k}_{\text{opt}}$ be the vector of k_j^* values that minimize the function $TC(T,\mathbf{k})$ among all possible T values as given by equation (5). Let $[T^{(i)}_{opt}, T^{(i-1)}_{opt})$ be the interval associated with $\mathbf{k}_{\text{opt}}$. Then $T_{opt} = T^*_{(i-1)}(k_1^*, k_2^*, \ldots, k_M^*) \in [T^{(i)}_{opt}, T^{(i-1)}_{opt})$.

Proof. See appendix B.

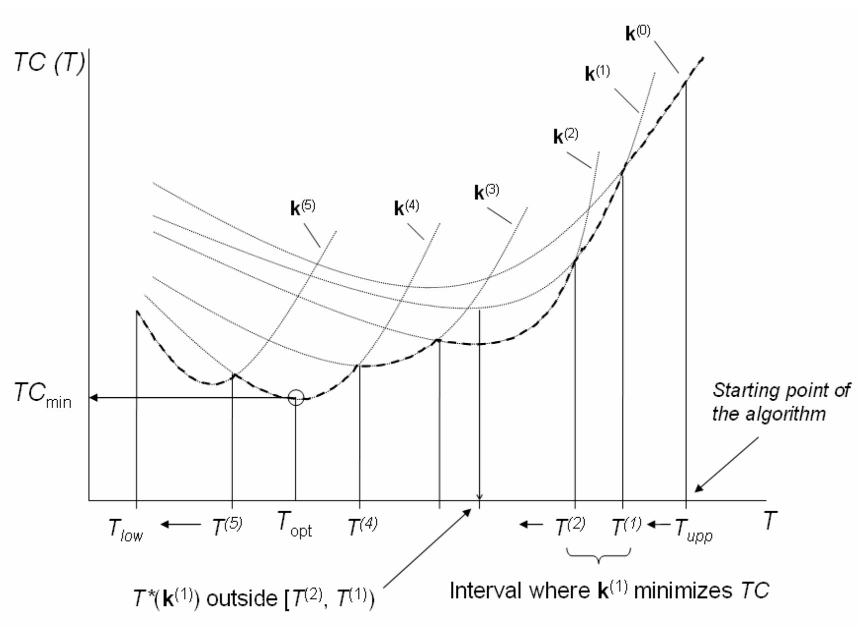

Figure 1. Schematic representation of the searching procedure for the JRP.

The above result was not provided in previous papers to show that Goyal's algorithm and modified versions of it (including the one presented in this paper) provide indeed the optimal solution for the JRP. Moreover, this result may not hold for extended versions of the JRP, e.g. when a constraint is imposed on the lot sizes for individual items [8] or when the correction factor is included in the objective function. For these cases the optimal T may be on the boundary of a given interval in the partition of T values.

Bounds on T

Before giving the algorithm to solve the JRP, we need to establish bounds on T. In order to overcome the problem associated with Goyal's lower bound, Van Eijs [2] proposed the following lower bound to ensure an optimal solution for GI policies:

$$T_{low}^{(VE)} = 2S / TRC^{(f)}$$

where $TRC^{(f)}$ is the total cost associated with a feasible solution for the JRP.

Although $T_{low}^{(VE)}$ can be improved iteratively by inserting in the last equation the best TC found so far in the algorithm, for high values of the major set-up cost the resulting lower bound can be very small.

Van Eijs uses the same upper bound as the one proposed by Goyal, given by:

$$T_{upp}^{(VE)} = T^{*}(1,...,1)$$

Viswanathan [11] presented an iterative method to obtained tighter bounds on T for the JRP. Starting with the van Eijs lower bound, Viswanathan uses iteratively formulas (2) and (5) to obtain a local optimal solution of TC, in say T_{low}^{*}. He shows that the function TC is monotonically decreasing between $T_{low}^{(VE)}$ and T_{low}^{*}. A similar procedure is used to find an upper bound on T, say T_{upp}^{*}. The Viswanathan bounds are appropriate for GI policies and therefore we can use them in our method.

Wildeman *et al*. [12] uses a relaxation of problem (P), say (R), for which the optimal solution of $TC^{(\mathrm{R})}$ is found in, say, $T(\mathrm{R})$. Then a feasible solution for the JRP is obtained using $T(\mathrm{R})$ and formula (5). Finally, by determining the intersection between the level line corresponding to the feasible TC and the $TC^{(\mathrm{R})}$ curve, a lower and an upper bound on T, say $T_{low}^{(W)}$ and $T_{upp}^{(W)}$, are obtained using bisection. This procedure can yield tighter bounds on T with respect to the ones in Viswanathan [11] for a number of problem configurations, namely for moderate major set-up costs and relatively high minor set-up costs. Moreover, the initial Wildeman lower bound can be further improved by repeating the bisection procedure using the best value of TC found so far in the algorithm, whenever $TC(T)<TC(T(R))$. Notice that given the initial Wildeman bounds, tighter bounds on T can be obtained by the Viswanathan procedure described above (for a numerical comparison on the performance of these procedures see Porras and Dekker [9]). Based on this analysis, we proposed the following algorithm.

Algorithm to solve (*P*)

Step 0. Initialization

Evaluate Wildeman bounds $T_{low}^{(W)}$ and $T_{upp}^{(W)}$, and improve them using Viswanathan iterative procedure.

Set $\mathbf{k}^{(0)} = \mathbf{k}(T_{upp}^{(W)})$ using equation (5).

Set $TC_{\min}^{(0)} = \infty$, $T^{(0)} = \infty$ and $n = 1$.

Evaluate $T_j^{(1)}$ for $j = 1, \ldots, M$ using formula (7).

Step 1. For $\mathbf{k}^{(n-1)}$ determine $T^{(n)}$ using (6) and set $J^{(n)} = \{ j\colon \max_j \{T_j^{(n)}\}\}$.

For $\mathbf{k}^{(n-1)} = (k_1^{(n-1)}, \ldots, k_M^{(n-1)})$ evaluate T_{n-1}^* using equation (2).

$$\text{Set: } TC_{\min}^{(n)} = \begin{cases} \min\{TC_{\min}^{(n-1)}, TC(k_1^{(n-1)}, \ldots, k_M^{(n-1)})\} & \textit{if } T_{n-1}^* \in [T^{(n)}, T^{(n-1)}) \\ \infty & \textit{otherwise} \end{cases}$$

Obtain the elements of the new vector $\mathbf{k}^{(n)}$ according to (8)

and set $T_j^{(n+1)} = \sqrt{\dfrac{2b_j}{k_j^{(n)}(k_j^{(n)}+1)}}$ if $j \in J^{(n)}$. Otherwise $T_j^{(n+1)} = T_j^{(n)}$.

Step 2. If $T^{(n)} \le T_{low}$ STOP with $TC_{\min}(T, \mathbf{k}) = TC_{\min}^{(n)}$ and $T_{opt} = T_{n-1}^*$.

Otherwise set $n = n + 1$ and GOTO step 1.

END of the algorithm.

The above algorithm is similar to the one proposed by van Eijs [2], although implemented in a slightly different way and with tighter bounds on the basic cycle time. Notice that in each round of the algorithm we check whether the optimal T lies inside the interval for which the associated vector $\mathbf{k}$ minimizes TC. If not, no evaluation of TC is done, which may save some computation time, especially for a large number of items.

Computational complexity of the proposed algorithm

An additional result of our algorithm comes from the use of formula (5), which was not previously incorporated in algorithms to solve the JRP. Using (5) and noting that the k_j's change in step sizes of one, we can evaluate the maximum number of intervals needed to obtain the optimal solution. Thus, given lower and upper bounds on T we provide the following formula:

$$\textit{Maximum \# of steps} = \sum_{i:b_i \neq b_j} k_i(T_{low}) - k_i(T_{upp})$$

For fixed T_{low} and T_{upp} this number increases linearly in the number of items, M. This has been unnoticed in the literature, as most papers give no explicit expression for the optimal k_j-values, like equation (5). Next assume that the initial list of $T_j^{(1)}$-values is sorted before entering *Step* (1) of the algorithm. Since the items change their k_j values one by one at each step of the algorithm with only one T_j-value updated in each round, it follows that the number of computation steps of the algorithm is $O(M \log M)$ under constant upper and lower bounds.

In the remainder of the complexity analysis, we need to set bounds on the s and hD values. This comes from a practical reason, since we assume that in reality there is always an effort associated with the handling or receiving of an item. Similarly, items are assumed to cause holding costs when kept on stock. Thus, for $s_j \in [s_{\min}, s_{\max}]$ and $h_jD_j \in [hD_{\min}, hD_{\max}]$ we distinguish the following cases:

a) S fixed.

First notice that $T_{low,VE}$ is proportional to $1/M$, since the total cost TC adds up M positive terms in s_j and h_jD_j, plus a constant term in S. For the Wildeman bounds, since we take the intersection of a relaxation of (P) with the TC curve, it follows by a similar reasoning that $T_{low}^{(W)}$ and $T_{upp}^{(W)}$ are proportional to $1/M$ (for a complete analysis see Porras and Dekker [9]). Therefore, from (5) we have that $k_j(T_{low}^{(W)})$ and $k_j(T_{upp}^{(W)})$ are proportional to M. From this it follows that the number of steps in the algorithm is proportional to $M^2 \cdot \log(M)$. It can also be shown that the complexity to obtain the solution $T(R)$ of the relaxation is $O(M \log M)$, since M derivatives of TC(R) need to be sorted in the procedure [12]. Therefore the complexity of the overall algorithm is $O(M^2 \log M)$ under Wildeman bounds.

b) S increases in M but M/S is bounded.

In this case we have that $T_{low}^{(W)}$ and $T_{upp}^{(W)}$ remain bounded as M increases. Therefore the number of steps in the algorithm increases linearly in M. It follows that the algorithm complexity is $O(M \log M)$.

For $S \downarrow 0$, the number of steps of the algorithm increases more than in the previous cases, however it is not such an interesting case since a practical lower bound on T can be used. Moreover, for small values of S the JRP is less relevant, and independent ordering for the items should be applied.

3.2. Solution method for problem ($P^{(c)}$)

Now we consider the JRP when a correction factor is included in the cost function. As before we consider the following alternative formulation of the function $TC^{(c)}(T,\mathbf{k})$:

$$TC^{(c)}(T) = \frac{S\Delta(\mathbf{k})}{T} + \sum_{j=1}^{M} z_j(T)$$

where the function $z_j(T)$ are defined in the same way as for $TC(T)$.

The problem ($P^{(c)}$) is generally more difficult to solve than problem (P), since the inclusion of the correction factor makes the function $TC^{(c)}$ discontinuous in T. Similar to TC, the function $TC^{(c)}$ is not jointly convex with respect to T and **k**. As before, for a fixed vector **k** the function $TC^{(c)}$ is convex in T, with optimal T given by:

$$T^*(k_1,...,k_M) = \sqrt{\frac{2\left(S\Delta(k) + \sum_j \frac{s_j}{k_j}\right)}{\sum_j h_j D_j k_j}} \tag{9}$$

Substituting (9) back in $TC^{(c)}(T, \mathbf{k})$ we get the optimal $TC^{(c)}$ for a fixed **k**:

$$TC^{(c)}(k_1,...,k_M) = \sqrt{2\left(S\Delta(k) + \sum_{j=1}^{M} \frac{s_j}{k_j}\right)\left(\sum_{j=1}^{M} h_j D_j k_j\right)}$$

Moreover, the inclusion of $\Delta(\mathbf{k})$ requires the evaluation of equation (1) in every step of the search algorithm. In addition to that, numerical experiments presented in the next section suggest that the function $TC^{(c)}$ tends to fluctuate around a certain value as T goes to zero, rather than going to infinity as in the case of TC (see Porras and Dekker [8] for a detailed description). Therefore, the traditional lower bounds on T presented in the literature are not valid anymore, and a completely new analysis is necessary. On the other hand, since $\Delta(\mathbf{k})$=1 if at least one k_j equals one, problem ($P^{(c)}$) and problem (P) are the same for large values of T, and therefore the traditional upper bounds on T presented in the literature ([2], [11]) to solve (P) are still valid, as long as $\Delta(\mathbf{k}) = 1$.

Lower bound on T for problem ($P^{(c)}$)

We will derive a lower bound on T for problem ($P^{(c)}$) in a similar way as in Porras and Dekker [8]. Here we provide the main results and the reader is referred to Appendix B for details and proofs. As in that paper, we use the following proposition for our analysis:

Proposition 1. Given M products with associated vector **k**, the following holds:

$$\sum_{i=1}^{M} \frac{1}{T \cdot k_i(T)} - \sum_{i \neq j} \frac{1}{T \cdot lcm(k_i(T), k_j(T))} \leq \frac{\Delta(\mathbf{k})}{T} \leq \sum_{i=1}^{M} \frac{1}{T \cdot k_i(T)} \tag{10}$$

Proposition 1 will be used to establish upper and lower limits on the function $TC^{(c)}(T)$ as T goes to zero. We give first the following definition.

Definition 1. Let $a_i \equiv \sqrt{2b_i}$ for i =1,…, M. For two items i, j with $a_j/a_i \in Q$, where Q denotes the set of rational numbers, let $m_0^{(i,j)}, n_0^{(i,j)}$ be the smallest integers for which the following equality holds:

$$\frac{a_i}{a_j} = \frac{m_0^{(i,j)}}{n_0^{(i,j)}}$$

For ease of notation, in the sequel we drop the super index (i, j).

Theorem 2. Given M products with demands $D_1,\ldots,D_M$, and minor set-up costs $s_1,\ldots,s_M$. If $a_j/a_i \in Q\ \ \forall i, j$, then the following holds:

$$\liminf_{T\to 0} TC^{(c)}(T) = S\left(\sum_{j=1}^{M}\frac{1}{\sqrt{2b_j}} - \sum_{(i,j):a_i\neq a_j}\frac{1}{n_0\sqrt{2b_i}}\right) + \sum_{j=1}^{M}\sqrt{2s_j h_j D_j}$$

and

$$\limsup_{T\to 0} TC^{(c)}(T) = S\left(\sum_{j=1}^{M}\frac{1}{\sqrt{2b_j}}\right) + \sum_{j=1}^{M}\sqrt{2s_j h_j D_j}$$

Note: Alternatively, the terms $(n_0\sqrt{2b_i})^{-1}$ inside the second summation of lim inf, can be replaced by $(m_0\sqrt{2b_j})^{-1}$.

Theorem 2 can be extended for the case where $a_j/a_i \in \Re\backslash Q$, the set of irrational numbers. As in Porras and Dekker [8] we give the following theorem:

Theorem 3. Given M products with demands $D_1,\ldots,D_M$, and minor set-up costs $s_1,\ldots,s_M$. If $a_j/a_i \in \Re\backslash Q\ \ \forall i, j$, then the following holds:

$$\lim_{T\to 0} TC^{(c)}(T) = S\left(\sum_{j=1}^{M}\frac{1}{\sqrt{2b_j}}\right) + \sum_{j=1}^{M}\sqrt{2s_j h_j D_j}$$

Although for most real applications the ratios a_j/a_i are often truncated to rational numbers, for which Theorem 2 is the one of practical interest, note that for arbitrary numbers a_i, a_j the values of m_0 and n_0 are likely to be large. In that case, the gap between the two limits in Theorem 2 is likely to be small.

The analysis used to establish Theorem 2, also yields a lower bound on T. First we give the following lemma.

Lemma 1. If $a_j/a_i \in Q\ \forall i, j$, then there exists a time value $T_{low}^{(c)}$, s.t. for any $T \leq T_{low}^{(c)}$ the following holds:

$$\sum_{i=1}^{M}\frac{1}{T\cdot k_i(T)} - \sum_{i\neq j}\frac{1}{n_0\sqrt{2b_i}} \leq \frac{\Delta(\mathbf{k})}{T} \leq \sum_{i=1}^{M}\frac{1}{T\cdot k_i(T)}$$

In the proof of lemma 1 (see Appendix B), we find that $T_{low}^{(c)}$ is obtained from:

$$T_{low}^{(c)} = \min_{(i,j):a_i \neq a_j} \left\{ T_{i,j}^{*} \right\}$$

where $T_{i,j}^{*}$ is a time value for which the following holds (lemma 4 of Appendix B):

$$T_{i,j}^{*} = \max\left\{ T_0 \middle| T \cdot lcm(k_i(T), k_j(T)) \geq n_0\sqrt{2b_i} = m_0\sqrt{2b_j} \ \ \forall T < T_0 \right\}$$

Remark 1. The above procedure to evaluate a lower bound on T can yield very small values (e.g. less than 1 hour), which may not be useful for practical purposes. In such a case, a practical lower bound on the basic cycle time T should be established, say one day or one hour, which is a reasonable assumption for most inventory tracking systems. Note that T is just a multiplier in the deterministic JRP as the replenishment interval for each item is giving by k_jT. However, if we use the deterministic solution of the JRP as an approximation to the stochastic case, T may well be considered as a real review time for the inventory system and the value k_jT a guide on the ordering time of item j. Moreover, for the deterministic JRP, T actually represents the precision of the ordering interval for the items. Therefore, using a lower bound of one day on T, means that the items are ordered with a precision of 1 day.

Given the previous results, we now formulate the following algorithm.

Algorithm to solve ($P^{(c)}$)

Replace the function TC by $TC^{(c)}$ in the algorithm to solve (P) presented in section 3.1 and use the new lower bound $T_{low}^{(c)}$, or a practical lower bound on T. Use formula (9) for the evaluation of T_{n-1}^{*} in step 1 of the algorithm.

As we can see in the numerical experiments presented in section 3, the inclusion of the correction factor $\Delta(\mathbf{k})$ makes the function $TC^{(c)}$ discontinuous in T at the points where the vector $\mathbf{k}$ changes. In this case Theorem 1 does not apply and we have to check the extreme points of the intervals as well. Therefore, the following formula should be used in each round of the algorithm for the evaluation of $TC_{\min}$:

$$TC_{\min}^{(n)} = \begin{cases} \min\{TC_{\min}^{(n-1)}, TC^{(c)}(k_1^{(n-1)},\ldots,k_M^{(n-1)})\} & \textit{if } T_{n-1}^{*} \in \left[T^{(n)}, T^{(n-1)}\right] \\ \min\left\{TC_{\min}^{(n-1)}, TC^{(c)}(T^{(n)}, \mathbf{k}^{(n-1)}), TC^{(c)}(T^{(n-1)}, k^{(n-1)})\right\} & \textit{otherwise} \end{cases}$$

3.3. Independent ordering

We now consider the general case of independent ordering of M items, where each item pays the major set-up cost S in addition to its minor set up cost s_j, but can be scheduled at an own time T_j, as follows:

$$(P_{IO}) \quad \min_{T_j} \sum_{j=1}^{M} \frac{(S+s_j)}{T_j} + \frac{1}{2} h_j D_j T_j \quad \text{, s.t. } T_j > 0$$

The optimal solution of (P_{IO}) is given by:

$$T_j^* = \sqrt{\frac{2(S+s_j)}{h_j D_j}} \qquad j = 1,\ldots, M \tag{11}$$

Substituting (11) back into the total cost function yields the optimal cost for independent ordering:

$$TC_{IO}^* = \sum_{j=1}^{M} \sqrt{2(S+s_j)h_j D_j}$$

We now would like to prove that the solution of problem ($P^{(c)}$) always outperforms the independent ordering solution given by (11), while this does not need to be the case for problem (P).

Let T_j^* be given by equation (11) and let $\varepsilon > 0$. By continuity of the term h_jD_jT it follows that there exists $\delta_j > 0$, $j = 1,\ldots, M$, such that for all T satisfying $\left|T - T_j^*\right| < \delta_j$ we have:

$$-\frac{1}{2}h_j D_j T_j^* + \frac{1}{2}h_j D_j T < \frac{\varepsilon}{M}$$

Next realize that if we choose the basic cycle time T_b in ($P^{(c)}$) small enough we can find integers k_j, $j = 1,\ldots, M$, s.t. $T_j^* < k_j T_b < T_j^* + \delta_j \;\; \forall j$. Hence, using inequality (10):

$$\begin{aligned} \frac{S\cdot\Delta(\mathbf{k})}{T_b} + \sum_{j=1}^{M}\frac{s_j}{k_jT_b} + \frac{1}{2}\sum_{j=1}^{M} h_jk_jT_bD_j &\le S\sum_{j=1}^{M}\frac{1}{k_jT_b} + \sum_{j=1}^{M}\frac{s_j}{k_jT_b} + \frac{1}{2}\sum_{j=1}^{M} h_jk_jT_bD_j \\ &< S\sum_{j=1}^{M}\frac{1}{T_j^*} + \sum_{j=1}^{M}\frac{s_j}{T_j^*} + \frac{1}{2}\sum_{j=1}^{M} h_jD_jT_j^* + \varepsilon \end{aligned}$$

Notice that the combination T_b, k_j, $j = 1,\ldots, M$ is a feasible solution to ($P^{(c)}$) and hence $TC_{\min}^{(c)}$ is less than the first term of the previous inequality. Since the previous calculation can be done for all $\varepsilon>0$, we have proved that $TC_{\min}^{(c)} \le TC_{IO}^*$. A similar result can be obtained for ordering only a subset of all items independently. We formalize this result in the following theorem.

Theorem 4. For the standard formulation of the JRP the following holds:

$$TC_{\min}^{(c)} \le TC_{IO}^*.$$

As a side result, notice that from Theorem 3 it follows that the limit as $T\to 0$ of $TC^{(c)}(T)$ gives the total cost of the system assuming that the items are ordered according to their EOQ (evaluated with s_j alone) and that each item pays an additional set-up cost S in every replenishment. That is, each item is replenished every

$\sqrt{2s_j/(h_j D_j)}$ units of time and pays an additional annual cost of $S/\sqrt{2s_j/(h_j D_j)}$. Note however that this result differs from $\lim_{T\to 0} TC(T)$, which yields infinite costs.

4. Numerical experiments

In this section we will show by numerical experimentation that the solutions of problem (P) and problem ($P^{(c)}$) given by the above algorithms can be radically different. We apply the algorithms to the following data taken from a real case [7] (that case did not provide s_j's, since the minor set-up costs where replaced by a minimum order quantity for the lot size of each item j included in the order):

M = 8 items
S = 950 *euros*
h_j = 0.325 *euros/unit·year* for all j

Table 1 shows the demand rates for the items, where values for demand set 1 were taken from the case and values for demand set 2 were randomly generated from [5000, 50000]. For the above data we performed a set of experiments for values of the minor set-up costs ranging from 5 to 50,000 euros, as shown in Tables 2-3. This choice of values of s_j allows us to analyze the effect on the optimal solution of the JRP under two conditions: 1) when there is an incentive to include the items in every replenishment opportunity (low values of s_j) and 2) when ordering the items in a multiple time of the basic cycle time makes more economical sense (for large values of s_j). The average time between replenishments (T_{avg}) is also reported for the optimal solution, where $T_{avg} = T/\Delta(\mathbf{k})$.

Table 1. Demands per year, D_j

Item j	Set 1	Set 2
1	18,304	17,906
2	20,176	5,203
3	16,796	13,368
4	10,140	45,376
5	21,216	43,449
6	10,140	22,460
7	25,428	35,872
8	25,428	9,567

Table 2. Solutions for demand set 1

	Solutions with correction factor					Solutions without correction factor				
sj: j=1,…,8	$TC^{(c)}$ min	$T^{(c)}_{opt}$ (weeks)	$k^{(c)}$ opt	Δ(k)	T_{avg}	TC min	T_{opt} (weeks)	k opt	T_{avg}	%Diff. TC min
5	9,747	11	(1,1,…,1)	1	11	9,746.72	11	(1,1,…,1)	11	0.00
50	11,382	12	(1,1,…,1)	1	12	11,381.72	12	(1,1,…,1)	12	0.00
150	14,364	16	(1,1,…,1)	1	16	14,363.50	16	(1,1,…,1)	16	0.00
350	18,970	21	(1,1,…,1)	1	21	18,969.53	21	(1,1,…,1)	21	0.00
700	25,070	27	(1,1,…,1)	1	27	25,070.43	27	(1,1,…,1)	27	0.00
1,050	29,953	32	(1,1,…,1)	1	32	29,953.45	32	(1,1,…,1)	32	0.00
1,500	35,251	38	(1,1,…,1)	1	38	35,251.36	38	(1,1,…,1)	38	0.00
2,000	40,330	44	(1,1,…,1)	1	44	40,329.78	44	(1,1,…,1)	44	0.00
3,000	48,846	25	(2,2,2,3,2,3,2,2)	0.67	37	48,930.12	53	(1,1,1,1,1,1,1,1)	53	0.17
4,000	56,018	28	(2,2,2,3,2,3,2,2)	0.67	43	56,210.40	54	(1,1,1,2,1,2,1,1)	54	0.34
5,000	62,370	32	(2,2,2,3,2,3,2,2)	0.67	47	62,638.57	60	(1,1,1,2,1,2,1,1)	60	0.43
10,000	87,465	44	(2,2,2,3,2,3,2,2)	0.67	67	87,835.75	45	(2,2,2,3,2,3,2,2)	67	0.42
20,000	123,168	62	(2,2,2,3,2,3,2,2)	0.67	94	123,431.73	63	(2,2,2,3,2,3,2,2)	94	0.21
40,000	173,709	21	(9,9,9,12,8,12,8,8)	0.25	84	173,999.83	88	(2,2,2,3,2,3,2,2)	132	0.17
50,000	194,081	24	(9,9,9,12,8,12,8,8)	0.25	94	194,412.49	99	(2,2,2,3,2,3,2,2)	148	0.17

Table 3. Solutions for demand set 2

sj: j=1,…,8	Solutions with correction factor					Solutions without correction factor				
	$TC^{(c)}$ min	$T^{(c)}_{opt}$ (weeks)	$k^{(c)}$ opt	Δ(k)	T_{avg}	*TC* min	T_{opt} (weeks)	k opt	T_{avg}	%Diff. *TC* min
5	11,150	9	(1,1,…,1)	1	9	11,150.11	9	(1,1,…,1)	9	0.00
50	13,021	11	(1,1,…,1)	1	11	13,020.52	11	(1,1,…,1)	11	0.00
150	16,358	13	(1,2,1,1,1,1,1,1)	1	13	16,358.40	13	(1,2,1,1,1,1,1,1)	13	0.00
350	21,439	16	(1,2,1,1,1,1,1,2)	1	16	21,438.65	16	(1,2,1,1,1,1,1,2)	16	0.00
700	28,121	22	(1,2,1,1,1,1,1,2)	1	22	28,121.34	22	(1,2,1,1,1,1,1,2)	22	0.00
1,050	33,445	24	(1,2,2,1,1,1,1,2)	1	24	33,445.36	24	(1,2,2,1,1,1,1,2)	24	0.00
1,500	39,227	28	(1,3,2,1,1,1,1,2)	1	28	39,227.37	28	(1,3,2,1,1,1,1,2)	28	0.00
2,000	44,774	29	(2,3,2,1,1,1,1,2)	1	29	44,773.62	29	(2,3,2,1,1,1,1,2)	29	0.00
3,000	54,114	18	(3,6,3,2,2,3,2,4)	0.67	27	54,143.72	35	(2,3,2,1,1,1,1,2)	35	0.06
4,000	61,953	20	(3,6,4,2,2,3,2,4)	0.67	30	62,116.08	41	(2,3,2,1,1,1,1,2)	41	0.26
5,000	68,898	22	(3,6,4,2,2,3,2,4)	0.67	34	69,175.66	45	(2,3,2,1,1,1,1,2)	45	0.40
10,000	96,389	31	(3,6,4,2,2,3,2,4)	0.67	47	96,914.44	31	(3,6,4,2,2,3,2,4)	47	0.54
20,000	135,568	44	(3,6,4,2,2,3,2,4)	0.67	66	135,941.58	44	(3,6,4,2,2,3,2,4)	66	0.28
40,000	191,191	62	(3,6,4,2,2,3,2,4)	0.67	93	191,456.32	62	(3,6,4,2,2,3,2,4)	93	0.14
50,000	213,639	69	(3,6,4,2,2,3,2,4)	0.67	104	213,876.66	69	(3,6,4,2,2,3,2,4)	104	0.11

In Fig. 2 we show a plot of the functions $TC(T)$ and $TC^{(c)}(T)$ for demand set 1 and s_j = 40,000 for all items. As one can see from the plot, the function TC is smoother than $TC^{(c)}$, since the latter exhibits discontinuities for each interval $[T^{(i)}, T^{(i-1)})$ with associated constant vector $\mathbf{k}^{(i-1)}$. Furthermore, we can see that the function $TC^{(c)}$ does not go to infinity as T approaches to zero, as stated in Theorem 2.

From the results presented in Tables 2-3 several conclusions are drawn. First note that for small values of s_j ($s_j \leq$ 2,000), the optimal solutions for (P) and ($P^{(c)}$) are exactly the same. This is due to the fact that for these values of s_j the optimal **k**'s are given by $k_j = 1$ for some j, with the corresponding correction factor equal to one. From equation (6) it follows that as the s_j's decrease, so does the time T_i in which the vector **k** changes its last coordinate(s) from 1 to 2 (step 1 of the algorithms). In other words, for small s_j's the optimal value of the function TC or $TC^{(c)}$ is likely to lie in the region in which at least one element of **k** equals one.

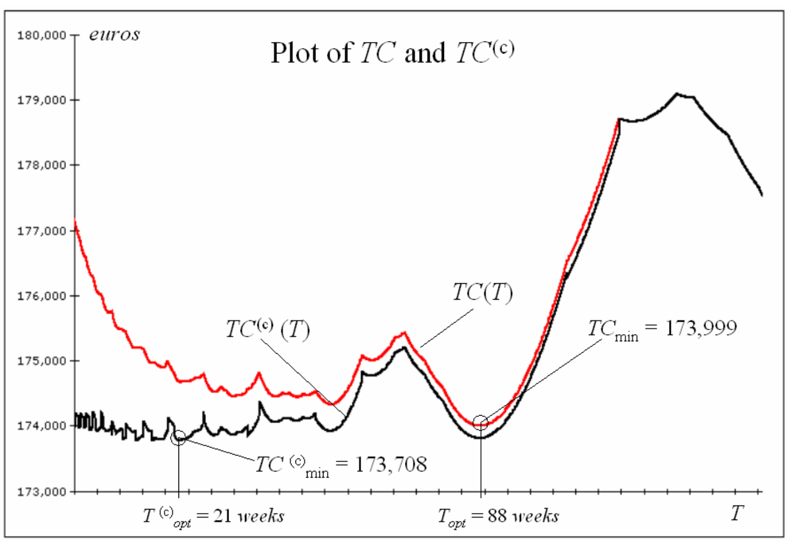

Figure 2. Plot of $TC^{(c)}$ and $TC(T)$

For large values of s_j ($s_j \geq$ 3,000) the optimal values of the objective function for problems (P) and ($P^{(c)}$) differ no more than 0.54% in all problems solved for both demand sets. Nevertheless, for most of the cases, the optimal T and **k** differ significantly. In all problems solved, the optimal T obtained by using formulation

($P^{(c)}$) is lower or equal than the one corresponding to problem (P). The inclusion of the correction factor prevents the optimal T to reach high values, and the vector $\mathbf{k}$ takes on values that allow more frequent replenishment occasions. For instance consider the solutions for demand set 1 with $s_j = 40{,}000$. The optimal review time for problem ($P^{(c)}$) is $T^{(c)}_{opt} = 21$ *weeks*, with an average review time of 84 *weeks*, whereas for problem (P) these values equal 88 and 132 *weeks*, correspondingly. We can observe this behavior graphically in the plots of Fig. 2. If we use the deterministic JRP as an approximation for the stochastic case, when the minor set-up costs are high, using formulation ($P^{(c)}$) the system gets more opportunities to be reviewed. This may have a great impact on the performance of the system.

In addition to the previous experiments, we carried out a large set of experiments for 5, 10, 15 and 20 items using the real case as base but with expanded ranges for the demand and holding costs. Accordingly, for each problem size we considered 7 different values of the major set up cost S (1, 25, 75, 150, 500, 1000 and 5000). Therefore we considered 28 different problem instances. For each of them we solved 100 problems using both algorithms plus independent ordering, with demands randomly generated from [5000, 50000], holding costs randomly generated from [0.1,1] and minor set-up costs randomly generated from [50, 500]. Thus, 2800 different problems were solved with each algorithm. We present the numerical results in Table 4, where the average values are reported over the 100 demand realizations. To be fair in the comparison, we implemented an additional step in the algorithm for problem (P) where the function TC is corrected with the correction factor associated with the optimal vector $\mathbf{k}_{\text{opt}}$ (P_{corr}). Both values of TC are reported. For the independent ordering solution (with optimal total cost TC^*_{IO}), the lot size for each item j is evaluated using formula (10) together with D_j. Values of $T_{low,pract}$ between 0.001 and 0.008 years were used in the algorithm for problem ($P^{(c)}$). The average number of intervals evaluated and the average CPU time in seconds is reported for each algorithm. The percentage difference between TC values is calculated from:

$$\%Diff.(TC) = \frac{TC_{corr} - TC^{(c)}}{TC} \times 100\%$$

A similar formula was used for *%Diff.* TC_{IO}, *%Diff.* T_{opt} and *%Diff.* T_{avg}.

From the numerical results presented in Table 4 we derive the following conclusions:

1. As the major set-up cost S decreases, T_{opt} also decreases and the savings with respect to independent ordering become smaller. Eventually it does not pay off anymore to apply the joint replenishment policy and therefore applying EOQ suffices. Note however that for some problem instances, even for low S, important savings with respect to independent ordering can still be achieved. E.g. for 20 items and S in the range 25~75, savings of 5.6~11.8% are achieved w.r.t independent ordering. For these problem instances the percentage difference in average replenishment time can be as high as 26.8% (for S = 25). Note that the solution of ($P^{(c)}$) always outperforms the independent ordering solution, while this is not always the case for (P) (see the result for 5 items and S = 1).

Table 4. Comparison of algorithms for the standard JRP

		Av. no. of intervals		Average *TC* optimal				%Diff.	%Diff.	Average T_{opt} (1)		%Diff.	Average T_{avg}		%Diff.	Average T_{low}		Average T_{upp}		Av. CPU	
No. of		Algorithm for		indep.	Algorithm for			TC_{opt}	TC^*_{IO}	Algorithm		T_{opt}	Algorithm		T_{avg}	Algorithm		Algorithm		time in sec.	
items	*S*	(*P*)	($P^{(c)}$)	order.	(*P*)	(P_{corr})	($P^{(c)}$)	(P_{corr} - $P^{(c)}$)		(*P*)	($P^{(c)}$)		(*P*)	($P^{(c)}$)		(*P*)	($P^{(c)}$)	(*P*)	($P^{(c)}$)	(*P*)	($P^{(c)}$)
5	1	84.0	252.4	12,212	12,220	12,216	12,109	0.88%	0.8%	0.072	0.024	67.3%	0.108	0.071	34.5%	0.008	0.003	0.170	0.170	3.3	13.7
	25	7.9	191.3	14,226	13,908	13,902	13,875	0.20%	**2.5%**	0.135	0.108	**20.1%**	0.139	0.126	**9.2%**	0.072	0.005	0.176	0.176	0.5	8.4
	75	4.5	190.7	15,368	14,281	14,281	14,259	0.15%	**7.2%**	0.154	0.139	**9.6%**	0.154	0.144	**6.4%**	0.105	0.005	0.189	0.189	0.4	8.4
	150	2.9	191.2	16,917	14,774	14,774	14,774	0.0%	12.7%	0.175	0.175	0.0%	0.175	0.175	0.0%	0.131	0.005	0.201	0.201	0.3	8.5
	500	2.2	171.3	22,719	16,592	16,592	16,592	0.0%	27.0%	0.219	0.219	0.0%	0.219	0.219	0.0%	0.174	0.006	0.229	0.229	0.3	7.3
	1,000	1.6	138.3	29,021	18,782	18,782	18,782	0.0%	35.3%	0.250	0.250	0.0%	0.250	0.250	0.0%	0.200	0.007	0.263	0.263	0.2	5.7
	5,000	1.5	139.1	58,674	30,791	30,791	30,791	0.0%	47.5%	0.429	0.429	0.0%	0.429	0.429	0.0%	0.299	0.007	0.431	0.431	0.2	5.6
10	1	422.4	828.3	28,965	28,964	28,951	28,688	0.91%	1.0%	0.034	0.011	67.7%	0.063	0.038	39.0%	0.003	0.002	0.184	0.184	36.9	364.5
	25	26.1	315.6	27,621	26,815	26,787	26,719	0.26%	**3.3%**	0.096	0.070	**27.3%**	0.111	0.093	**16.5%**	0.042	0.005	0.164	0.164	1.99	69.5
	75	10.9	314.2	29,933	27,260	27,260	27,214	0.17%	**9.1%**	0.133	0.111	**16.3%**	0.133	0.121	**8.7%**	0.079	0.005	0.172	0.172	1.59	68.4
	150	7.0	314.0	33,059	27,822	27,822	27,822	0.0%	15.8%	0.147	0.147	0.0%	0.147	0.147	0.0%	0.104	0.005	0.177	0.177	1.5	68.6
	500	4.1	247.2	44,694	30,043	30,043	30,043	0.0%	32.8%	0.176	0.176	0.0%	0.176	0.176	0.0%	0.136	0.006	0.203	0.203	0.7	54.1
	1,000	3.4	179.2	57,271	32,681	32,681	32,681	0.0%	42.9%	0.208	0.208	0.0%	0.208	0.208	0.0%	0.159	0.007	0.223	0.223	0.6	41.0
	5,000	1.7	179.9	116,219	47,843	47,843	47,843	0.0%	58.8%	0.318	0.318	0.0%	0.318	0.318	0.0%	0.234	0.007	0.325	0.325	0.3	40.5
15	1	808.8	1633.4	37,976	37,951	37,935	37,568	0.97%	1.1%	0.024	0.005	79.1%	0.039	0.012	68.3%	0.002	0.001	0.123	0.123	92.9	974.5
	25	41.4	465.3	41,407	39,994	39,932	39,740	0.48%	**4.0%**	0.088	0.061	**31.2%**	0.090	0.070	**22.2%**	0.037	0.004	0.121	0.121	7.0	328.8
	75	26.5	466.1	45,037	40,616	40,616	40,527	0.22%	**10.0%**	0.095	0.071	**25.2%**	0.095	0.079	**17.6%**	0.056	0.004	0.150	0.150	6.3	328.6
	150	17.4	466.2	49,915	41,377	41,377	41,377	0.0%	17.1%	0.127	0.127	0.0%	0.127	0.127	0.0%	0.073	0.004	0.165	0.165	4.2	328.0
	500	8.3	338.1	67,947	43,890	43,890	43,890	0.0%	35.4%	0.160	0.160	0.0%	0.160	0.160	0.0%	0.111	0.005	0.182	0.182	3.4	195.4
	1,000	6.1	331.2	87,341	46,896	46,896	46,896	0.0%	46.3%	0.173	0.173	0.0%	0.173	0.173	0.0%	0.133	0.006	0.205	0.205	3.1	156.1
	5,000	1.7	311.7	177,891	64,408	64,408	64,408	0.0%	63.8%	0.269	0.269	0.0%	0.269	0.269	0.0%	0.200	0.007	0.282	0.282	2.3	117.0
20	1	1368.4	2910.8	52,157	52,109	52,096	51,082	1.95%	2.1%	0.024	0.004	83.9%	0.035	0.009	75.6%	0.002	0.001	0.103	0.103	189.1	1396.4
	25	103.2	654.0	54,764	52,746	52,634	51,703	1.77%	**5.6%**	0.059	0.037	**36.8%**	0.081	0.059	**26.8%**	0.021	0.004	0.164	0.164	13.4	608.1
	75	44.2	653.4	59,645	53,253	53,226	52,581	1.21%	**11.8%**	0.092	0.065	**29.8%**	0.093	0.074	**20.4%**	0.042	0.004	0.169	0.169	7.0	606.1
	150	21.7	653.8	66,118	54,692	54,692	54,102	1.08%	**18.2%**	0.125	0.107	**14.7%**	0.125	0.115	**8.4%**	0.075	0.004	0.168	0.168	6.2	606.7
	500	9.6	458.3	89,389	57,145	57,145	57,145	0.0%	36.1%	0.156	0.156	0.0%	0.156	0.156	0.0%	0.114	0.005	0.184	0.184	4.1	446.6
	1,000	7.4	430.8	114,543	60,210	60,210	60,210	0.0%	47.4%	0.172	0.172	0.0%	0.172	0.172	0.0%	0.134	0.006	0.202	0.202	3.8	366.4
	5,000	3.1	358.2	232,438	78,350	78,350	78,350	0.0%	66.3%	0.258	0.258	0.0%	0.258	0.258	0.0%	0.196	0.007	0.267	0.267	2.9	318.6

(1) All time units are in years

2. For moderate values of the major set-up cost (S = 150), both algorithms yield the same solution for 5, 10 and 15 items. However for 20 items the solutions are different, with the algorithm for ($P^{(c)}$) achieving a lower average replenishment time (difference of 8.4%). For this problem instance the savings w.r.t. independent ordering are of 18%.

3. For large values of the major set-up cost (S > 500) both algorithms yield the same solution in all problems solved. However, the effect of increasing S becomes less important as the number of items increases, as can be seen for moderate values of S.

4. Although we only present in Table 4 summary information for the experiments, i.e. average values over 100 random demand realizations, we pair-checked the statements in the solutions of both algorithms for all individual problem instances. The statements $T_{opt} \geq T_{opt}^{(c)}$, $T_{avg} \geq T_{avg}^{(c)}$ and $TC_{\min}^{(c)} < TC_{IO}^{*}$ were always confirmed.

5. Although the computation time associated with the solution of ($P^{(c)}$) is much higher than that of (P), we believe that for the cases discussed in the previous paragraph the algorithm with correction factor is relevant and can yield better solutions, especially when we use the deterministic JRP as approximation in stochastic environments. Moreover, the JRP is related to tactical managerial decisions, as the JRP is solved only once over a certain period of time (months up to a year). In this respect the high difference in computation time between both algorithms becomes less relevant.

The above numerical observations can be generalized in the following empirical observations.

Empirical observation 1. The general shape of $\Delta(\mathbf{k})$ follows a decreasing pattern in $\mathbf{k}$ as T decreases, as can be seen in the plot of Fig. 3.

Explanation. From equation (1) and the principle of inclusion and exclusion, we can establish an upper bound on $\Delta(\mathbf{k})$ by replacing the least common multiple with the multiplication of the integers k_j, as follows:

$$UB_{\Delta(k)} = \sum_{i=1}^{M} \frac{1}{k_i} - \sum_{(i,j)\subseteq\{1,\ldots,M\}} \frac{1}{k_i \cdot k_j} + \sum_{(i,j,k)\subseteq\{1,\ldots,M\}} \frac{1}{k_i \cdot k_j \cdot k_k} - \cdots + (-1)^{M+1} \frac{1}{k_1 \cdot \ldots \cdot k_M}$$

From equation (5) it follows that as T decreases, the upper bound on $\Delta(\mathbf{k})$ decreases, and therefore the general shape of $\Delta(\mathbf{k})$ will be decreasing in $\mathbf{k}$.

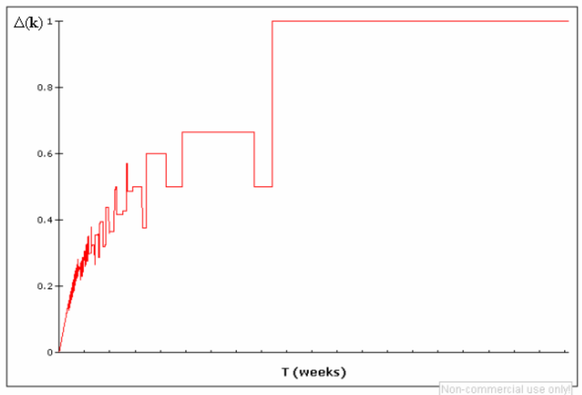

Fig. 3. Plot of $\Delta(\mathbf{k})$

Note: It should be pointed out that often the k_j values in the optimal solution are obtained when the value of $\Delta(\mathbf{k})$ observes a drop. This result is not surprising since this will happen when the values of the k_j's allow a better coordination between the orders for the different items. In this case, some of the k_j values will be either equal to each other or multiple of each other. For these values of the k_j's, $\Delta(\mathbf{k})$ will be smaller than when no coordination is observed.

Empirical observation 2. Let T_{opt} and $T_{opt}^{(c)}$ be the optimal basic cycle times for problems (P) and (P_c), with corresponding optimal vectors $\mathbf{k}_{opt}$ and $\mathbf{k}_{opt}^{(c)}$. Then the following has been observed in the numerical experiments:

If $\Delta(\mathbf{k}_{opt}^{(c)}) = 1$ then $T_{opt}^{(c)} = T_{opt}$ and $TC^{(c)}(T_{opt}^{(c)}) = TC(T_{opt})$. For $\Delta(\mathbf{k}_{opt}^{(c)}) < 1$, in all cases we observed that $T_{opt}^{(c)} \leq T_{opt}$ and $TC^{(c)}(T_{opt}^{(c)}) < TC(T_{opt})$. Moreover, if $\mathbf{k}_{opt}^{(c)} \neq \mathbf{k}_{opt}$ then $T_{opt}^{(c)} < T_{opt}$.

Note: We failed to find a formal proof of this finding. One can prove that the derivative with respect to T of $TC^{(c)}$ is larger than that of TC, implying that any minimum of TC has a minimum of $TC^{(c)}$ left of it. Moreover, if $\Delta(\mathbf{k})$ were monotonic in T, then the result could be shown. However, there are some cases where it is not.

Remark 2. For large values of the minor set-up costs s_j, it follows from equation (5) that the optimal k_j's are likely to be large. Therefore, by the result of the previous observations it follows that in this case often $T_{opt}^{(c)} < T_{opt}$. This is observed in the numerical results presented in Tables 2-3. The equivalent result is found in Table 4 for small values of S. This can also be seen graphically in the plots of Fig. 2.

In addition to the above results, we investigated the behavior of the system when S is very small in comparison with s_j for $j = 1,\ldots, M$. In order to gain some theoretical insight in this respect, we let $S\to 0$ and observe that problem (P) (or equivalently ($P^{(c)}$)) becomes:

$$(P^{(s)})\ \text{Min } TC^{(s)}(T) = \sum_{j=1}^{M} \min_{k_j}\left\{\phi_j(k_jT)\right\} \ \text{ s.t. } T > 0$$

where $\phi_j(k_jT) = \dfrac{s_j}{k_jT} + \dfrac{1}{2}h_jD_jk_jT$ for $j = 1,\ldots, M$.

It is not difficult to verify that $\phi_j(k_jT)$ is strictly convex in (k_jT), with a minimum attained in: $(k_jT)^* = \sqrt{2b_j}$. On the other hand, in Appendix B we show that

$$\lim_{T\to 0} T\cdot k_j(T) = \sqrt{2b_j}\ .$$

Notice that the smaller T is, the closer $T\cdot k_j(T)$ is towards $\sqrt{2b_j}$. Accordingly, for $S\to 0$ we expect $T^* \to 0$ as well. This coincides with the observation that for very low major set-up cost, the optimal solution is not to use joint replenishment at all. In other words, it is optimal to check the system in a continuous fashion, and to order each item j independently every $(k_jT)^*$ units of time. This theoretical result is illustrated in the numerical example shown in Table 5 (it can also be observed in the results corresponding to $S = 1$ in Table 4).

As we can see in Table 5, as S goes to zero, the vector of optimal replenishment times defined by $((k_1T)^*,(k_2T)^*,\ldots, (k_MT)^*)$ tends to its limit given by:

$$\lim_{T\to 0}(\mathbf{k}(T)\cdot T)^* = \left(\sqrt{2b_1}, \sqrt{2b_2}, \ldots, \sqrt{2b_M}\right)$$

and the objective function goes to its limit given by:

$$\lim_{T\to 0}\phi_j(k_jT) = \frac{s_j}{\sqrt{2b_j}} + \frac{1}{2}h_jD_j\sqrt{2b_j}$$

Table 5. Demand set 2 with varying *S* and fixed s_j = 2,000

S	$TC^{(c)}$min	T_{opt} (weeks)	k_{opt}	Δ(k)	T_{avg}
1000	44,860.3	32	(1,3,2,1,1,1,1,2)	1.000	32
500	43,867.2	15	(3,6,3,2,2,3,2,4)	0.666	22
100	42,882.7	14	(3,6,4,2,2,3,2,4)	0.666	21
50	42,757.8	14	(3,6,4,2,2,3,2,4)	0.666	21
10	42,611.2	2.5	(18,32,20,11,11,16,12,24)	0.253	10
1	42,557.9	1	(38,71,44,24,24,34,27,52)	0.163	7
0.1	42,550.8	0.9	(48,90,56,30,31,43,34,66)	0.149	6
0.0001	42,549.4	0.7	(61,113,71,38,39,54,43,83)	0.135	5

5. Conclusions

In this paper we presented a complete analysis for the JRP, by showing that the optimal methods found in the literature to solve the JRP provided indeed optimal solutions. Furthermore, we provided an efficient optimal solution method to solve the JRP when a correction is made in the cost function. We showed that although the cost improvement when using the correction for empty replenishments is only of few percentage points, the quality of the solution in terms of optimal T and $\mathbf{k}$ is higher. Particularly this proves to be the case for large values of the minor set-up costs and moderate major set-up costs. We further showed that the solution with correction factor outperforms the solution given by applying independent ordering using EOQ's. This is not the case for the formulation of the problem without correction factor, which proves formally that this is a particular case of the model with correction factor.

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Appendix A

In this appendix we provide an algorithm for the evaluation of the correction factor $\Delta(\mathbf{k})$. Notice that the amount of work for the evaluation of (1) increases exponentially with the number of items. Nevertheless, we show that in many cases the size of the vector $\mathbf{k}$ can be reduced and therefore also the number of terms in (1). First observe that for a given a vector $\mathbf{k}$ with some of its elements being equal or multiples of each other, it is easy to verify the following: formula (1) will give the same numerical value for $\Delta(\mathbf{k})$ if we apply it to a reduced vector, say $\mathbf{k}_{\mathbf{new}}$, with its elements extracted from the original $\mathbf{k}$ and satisfying: $k_i/k_j \notin \mathrm{N}$ for all related-pairs i,j. Since $\Delta(\mathbf{k}_{\mathbf{new}})$ counts the fraction of effective replenishments, it is clear that it will also include the replenishments in which products with k_j element of $\mathbf{k}$ take place. In such a case, we can reduce considerably the amount of work needed to evaluate (1).

Given a vector $\mathbf{k} = (k_1, k_2,\ldots, k_M)$ the following algorithm is used to evaluate the value of $\Delta(\mathbf{k})$ using formula (1).

Algorithm for the evaluation of $\Delta(\mathbf{k})$

Step 1. If $k_j = 1$ for any $j = 1,\ldots, M$, then $\Delta(\mathbf{k}) = 1$. STOP.
Step 2. Re-arrange the elements of $\mathbf{k}$ s.t. $k_1 \le k_2 \le \cdots \le k_M$ and define the set
$K = \{k_1, k_2,\ldots, k_M\}$.
Set $R^{(0)} = K$.
Set $\mathrm{D}(0) = \dim(K)$ and $n = 1$.
Step 3. For $j = n$ to $\mathrm{D}(n-1)-1$ do
if $k_{j+1}/k_n \in \mathrm{N}$ then $R^{(j)} = R^{(j-1)} \setminus \{k_{j+1}\}$ else $R^{(j)} = R^{(j-1)}$
Next j
Set $K = R^{(j)}$ and $\mathrm{D}(n) = \dim(R^{(j)})$.
Step 4. If $\mathrm{D}(n) = n$ GOTO Step 5.
Else set $n = n + 1$ and $R^{(n-1)} = K$. GOTO Step 3.
Step 5. Apply formula (1) to the new vector $\mathbf{k}_{\mathbf{new}}$ with elements given by K.

Appendix B

Proof of Theorem 1

First note that from (2) it follows that $T^*(k_1,\ldots,k_M)$ is monotone decreasing in $\mathbf{k}$. Now let $\mathbf{k}^{(i)}$ be the adjacent locally optimal vector to $\mathbf{k}_{\mathrm{opt}}$ for $T > T_{opt}^{(i-1)}$ and suppose that $T^*(\mathbf{k}_{opt}) > T_{opt}^{(i-1)}$. By the convexity of $TC(T)$ it follows that TC is decreasing in $[T_{opt}^{(i)}, T_{opt}^{(i-1)})$ which implies that the minimum of TC is found in $T_{opt}^{(i-1)}$. It follows that $TC(T)$ is increasing for $T > T_{opt}^{(i-1)}$. Again by the convexity of TC this implies that $T^*(\mathbf{k}^{(i)}) < T_{opt}^{(i-1)} \Rightarrow T^*(\mathbf{k}^{(i)}) < T^*(\mathbf{k}_{opt})$, which is a contradiction by the monotonicity of T^*. Therefore, $T^*(\mathbf{k}_{opt}) < T_{opt}^{(i-1)}$ and the minimum of TC is to the left of $T_{opt}^{(i-1)}$. Proceed in a similar way to show that $T^*(\mathbf{k}_{opt}) \ge T_{opt}^{(i)}$, implying $T_{opt} = T^*(\mathbf{k}_{opt})$. □

Proof of proposition 1
First note that the inequality in proposition 1 is equivalent to:

$$\sum_{i=1}^{M}\frac{1}{k_i}-\sum_{(i,j)\subseteq\{1,\ldots,M\}}\frac{1}{lcm(k_i,k_j)}\leq\Delta(\mathbf{k})\leq\sum_{i=1}^{M}\frac{1}{k_i}$$

Next notice that since the fraction of replenishments of item i per year is $(1/k_i)$ the RHS of the above inequality holds. Now realise that through the principle of inclusion-exclusion the number of non-empty replenishments due to item i is larger than the number of replenishments of item i minus the joint replenishments of pairs of products including item i. Hence, the LHS of the inequality holds. □

Note: If at least one of the $k_i = 1$, then $\Delta(\mathbf{k}) = 1$ and $TC^{(c)}(T,\mathbf{k})$ coincides with $TC(T,\mathbf{k})$. We use proposition 1 to establish a lower bound on T for problem $(P^{(c)})$ and the basis for Theorem 2 and Lemma 1. First we have:

$$k_j(T)=\left\lceil-\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8b_j}{T^2}}\right\rceil\Rightarrow-\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8b_j}{T^2}}\leq k_j(T)\leq\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8b_j}{T^2}}$$

Multiplying both sides of the above inequality by T and taking the limit as T goes to zero yields:

$$\lim_{T\to 0}T\cdot k_j(T)=\sqrt{2b_j}\equiv a_j$$

In the following analysis, we will see that the behaviour of the second term in the LHS of (10) as $T\to 0$ is very much determined by the nature of the ratios a_j/a_i. Although for practical purposes these ratios can be considered as rational numbers, we found an interesting behaviour of the product $T\cdot lcm(k_i(T),k_j(T))$ for $T\to 0$ when the ratios are regarded as irrational numbers, as is the case when demands are continuous variables, rather than discrete (see Porras and Dekker [8]). Therefore, we consider both cases in our analysis.

Accordingly, we first consider the case for which the a_j's are rational numbers, and we proceed in a similar way as in Porras and Deker [8]. Let $\Re\setminus Q$ denote the set of irrational numbers, where $\Re$ is the set of real numbers and Q the set of rational numbers.

We first try to construct a subsequence of T going to zero, say $T^{(N_l)}$, $l = 1,2,\ldots$ s.t.

$$k_i(T^{(N_l)})=\left\lceil-\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8b_i}{T^{(N_l)2}}}\right\rceil=N_l\cdot m$$

and

$$k_j(T^{(N_l)})=\left\lceil-\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8b_j}{T^{(N_l)2}}}\right\rceil=N_l\cdot n$$

where m, n are given integers with gcd(m, n) = 1 and $N_l \in Z^+$.

Hence, $lcm(k_i(T^{(N_l)}), k_j(T^{(N_l)})) = N_l \cdot m \cdot n$

Such a subsequence of T should satisfy the following system for N_l, $l = 1,2,\ldots$

$$N_l m - 1 < -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_i}{T^{(N_l)2}}} \le N_l m$$

$$N_l n - 1 < -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_j}{T^{(N_l)2}}} \le N_l n$$

Or equivalently:

$$\sqrt{\frac{8b_i}{(2N_l m + 1)^2 - 1}} \le T^{(N_l)} < \sqrt{\frac{8b_i}{(2N_l m - 1)^2 - 1}}$$

and

$$\sqrt{\frac{8b_j}{(2N_l n + 1)^2 - 1}} \le T^{(N_l)} < \sqrt{\frac{8b_j}{(2N_l n - 1)^2 - 1}}$$

From the previous system it follows that we can find such a sequence of T's if and only if

$$\sqrt{\frac{8b_i}{(2N_l m - 1)^2 - 1}} \ge \sqrt{\frac{8b_j}{(2N_l n + 1)^2 - 1}}$$

and

$$\sqrt{\frac{8b_j}{(2N_l n - 1)^2 - 1}} \ge \sqrt{\frac{8b_i}{(2N_l m + 1)^2 - 1}}$$

Letting $N_l \to \infty$ we obtain:

$$a_j \le \sqrt{\frac{N_l n^2 + n}{N_l m^2 - m}} a_i \ \forall N_l \Rightarrow a_j \le \frac{n}{m} a_i$$

and

$$a_j \ge \sqrt{\frac{N_l n^2 + n}{N_l m^2 - m}} a_i \ \forall N_l \Rightarrow a_j \ge \frac{n}{m} a_i$$

The above inequalities yield: $a_j = \frac{n}{m} a_i$ (12)

This implies that the only m, n for which

$$\left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_i}{T^{(N_l)2}}} \right\rceil = N_l \cdot m \text{ and } \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_j}{T^{(N_l)2}}} \right\rceil = N_l \cdot n$$

have an infinite number of solutions N_l is given by $\frac{a_i}{a_j} = \frac{m}{n}$. Let us call these values $m_0^{(i,j)}$, $n_0^{(i,j)}$. For simplicity of notation, in the sequel we drop the super index (i, j).

Next note that for $T^{(N)} = \sqrt{\frac{2b_i}{Nm_0(Nm_0 + 1)}}$, N = 1,2,… we have such a sequence for which:

$$\lim_{N_l \to \infty} (T^{(N_l)} \cdot lcm(k_i(T^{(N_l)}), k_j(T^{(N_l)})))$$

$$= \lim_{N_l \to \infty} \left\{ \sqrt{\frac{2b_i}{N_l m_0 (N_l m_0 + 1)}} \cdot N_l \cdot m_0 \cdot n_0 \right\} = \lim_{N_l \to \infty} \left\{ \sqrt{\frac{2b_i N_l m_0}{N_l m_0 + 1}} \cdot n_0 \right\} = n_0 \sqrt{2b_i}$$

Observe that we can also select $T^{(N)} = \sqrt{\frac{2b_j}{Nn_0(Nn_0 + 1)}}$, N =1,2,… Using this value and equation (12) we have:

$$\lim_{N_l \to \infty} (T^{(N_l)} \cdot lcm(k_i(T^{(N_l)}), k_j(T^{(N_l)}))) = m_0 \sqrt{2b_j} = n_0 \sqrt{2b_i}$$

Now consider an arbitrary m, n given with gcd(m,n) = 1 and a time T for which

$$\left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_i}{T^2}} \right\rceil = N \cdot m \text{ and } \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_j}{T^2}} \right\rceil = N \cdot n \text{ for some integer } N > 0.$$

Note that N satisfies the following system:

$$\begin{aligned} &\frac{1}{2m}\left(-1 + \sqrt{1 + \frac{8b_i}{T^2}}\right) \le N < \frac{1}{2m}\left(1 + \sqrt{1 + \frac{8b_i}{T^2}}\right) \\ &\frac{1}{2n}\left(-1 + \sqrt{1 + \frac{8b_j}{T^2}}\right) \le N < \frac{1}{2n}\left(1 + \sqrt{1 + \frac{8b_j}{T^2}}\right) \end{aligned} \tag{13}$$

System (13) implies that

$$\frac{1}{2m}\left(-1+\sqrt{1+\frac{8b_i}{T^2}}\right)\le\frac{1}{2n}\left(1+\sqrt{1+\frac{8b_j}{T^2}}\right) \text{ and } \frac{1}{2n}\left(-1+\sqrt{1+\frac{8b_j}{T^2}}\right)\le\frac{1}{2m}\left(1+\sqrt{1+\frac{8b_i}{T^2}}\right)$$

Rewrite the above system as follows:

$$\frac{\sqrt{T^2+8b_i}-T}{\sqrt{T^2+8b_j}+T}\le\frac{m}{n}$$

$$\frac{\sqrt{T^2+8b_j}-T}{\sqrt{T^2+8b_i}+T}\le\frac{n}{m}$$

Solving the above system for T yields:

$$T\ge\sqrt{\frac{2\left(\frac{n}{m}b_i-\frac{m}{n}b_j\right)^2}{\left(\frac{m}{n}+1\right)\left(\frac{n}{m}b_i+b_j\right)}}\equiv T_{m,n}$$

Using the root $T = 0$ in the above system of inequalities yields that $T_{m,n}$ is nonzero if $\frac{m}{n}\ne\frac{\sqrt{b_i}}{\sqrt{b_j}}=\frac{a_i}{a_j}$, similarly as the result found in Porras and Dekker [7].

Let $T^*_{i,j}=\min_{\{(m,n)\in\mathbf{N}:m<m_0,n<n_0\}}\{T_{m,n}\}$.

Now consider a $T<T^*_{i,j}$ and let w.l.o.g.

$k_i(T)=N^{(T)}\cdot m$, $k_j(T)=N^{(T)}\cdot n$ for some $m,n,N^{(T)}\in\mathbb{N}$

with gcd(m,n) = 1, and from (13) we have that:

$$N^{(T)}\ge\max\left\{\frac{1}{2m}\left(-1+\sqrt{1+\frac{8b_i}{T^2}}\right),\frac{1}{2n}\left(-1+\sqrt{1+\frac{8b_j}{T^2}}\right)\right\}.$$

Then,

$$T\cdot lcm(k_i(T),k_j(T))=T\cdot N^{(T)}\cdot m\cdot n$$

$$\ge\begin{cases} n\sqrt{2b_i} & \text{if} \quad \frac{1}{2m}\left(-1+\sqrt{1+\frac{8b_i}{T^2}}\right)>\frac{1}{2n}\left(-1+\sqrt{1+\frac{8b_j}{T^2}}\right) \\ m\sqrt{2b_j} & \text{otherwise}\end{cases} \tag{14}$$

Since $T < T^*_{i,j}$ we cannot have by (14) both $m < m_0$ and $n < n_0$. Hence, either $m \geq m_0$ or $n \geq n_0$. In both cases $T \cdot N^{(T)} \cdot m \cdot n \geq n_0 \sqrt{2b_i} = m_0 \sqrt{2b_j}$.

From the previous analysis we have established the following lemma:

Lemma 4. If $\frac{a_j}{a_i} \in Q \ \forall i, j$, then the following holds:

$$0 \leq \frac{1}{T \cdot lcm(k_i(T), k_j(T))} \leq \frac{1}{n_0 \sqrt{2b_i}} = \frac{1}{m_0 \sqrt{2b_j}} \quad \text{for any } T \leq T^*_{i,j} .$$

Proof of Lemma 1
By lemma 4 and equation (10) the result follows. □

Lemma 5. If $\frac{a_j}{a_i} \in Q \ \forall i, j$, then the following holds:

$$\limsup_{T \to 0} \frac{1}{T \cdot lcm(k_i(T), k_j(T))} = \frac{1}{n_0 \sqrt{2b_i}} = \frac{1}{m_0 \sqrt{2b_j}}$$

and

$$\liminf_{T \to 0} \frac{1}{T \cdot lcm(k_i(T), k_j(T))} = 0$$

Proof. A large part of this lemma follows from lemma 4 and the analysis preceding it. What remains to be proved is that the liminf as $T \to 0$ is indeed 0. For this part suppose w.l.o.g. that $a_i > a_j$, implying that $k_i(T) > k_j(T)$ for T small. Note that as $T \to 0$, $k_i(T)$ takes all possible integers 1,2,… Let α_1, α_2, α_3,… be an increasing sequence of prime numbers and let $T^{(\alpha_l)}$ be the T-values for which $k_i(T^{(\alpha_l)}) = \alpha_l$, $l = 1,2,...$ Now note that for T small enough $lcm(k_i(T^{(\alpha_l)}), k_j(T^{(\alpha_l)})) = k_i(T^{(\alpha_l)}) \cdot k_j(T^{(\alpha_l)})$ and hence,

$$\lim_{T^{(\alpha_l)} \to 0} \frac{1}{T^{(\alpha_l)} \cdot lcm(k_i(T^{(\alpha_l)}), k_j(T^{(\alpha_l)}))} = \lim_{T^{(\alpha_l)} \to 0} \frac{1}{T^{(\alpha_l)} \cdot k_i(T^{(\alpha_l)}) \cdot k_j(T^{(\alpha_l)})}$$

$$= \frac{1}{\lim\limits_{T^{(\alpha_l)} \to 0} (T^{(\alpha_l)} \cdot k_i(T^{(\alpha_l)})) \cdot \lim\limits_{T^{(\alpha_l)} \to 0} (k_j(T^{(\alpha_l)}))}$$

$$= \frac{1}{\sqrt{2b_i} \cdot \lim\limits_{T^{(\alpha_l)} \to 0} \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8b_j}{T^{(\alpha_l)2}}} \right\rceil} = 0 . \ □$$

Proof of Theorem 2

By lemma 5 we can evaluate the limit as $T \to 0$ on the LHS of inequality (10) and since the limit exists, the first part of the theorem follows. For the second part take the limit on the RHS of (10), and since this limit exists and it is independent of a_j/a_i $\forall i, j$, the claim of the theorem follows. □

Before giving the proof of Theorem 3, we need first the following lemma.

Lemma 6. If $a_j / a_i \in \Re \setminus Q$ $\forall i, j$, then

$$\lim_{T \to 0} \frac{1}{T \cdot lcm(k_i(T), k_j(T))} = 0.$$

Proof. Let $lcm(k_i(T), k_j(T)) = n(T)k_i(T) = m(T)k_j(T)$ for some integers $n(T)$, $m(T)$. Suppose that there is a bounded subsequence $n(T^{(r)}), m(T^{(r)})$, r = 1,2,…, such that $T^{(r)} \downarrow 0$ as $r \to \infty$ and $n(T^{(r)}) \le K$, $m(T^{(r)}) \le K$ for some $K > 0$. Since this implies that there are only finitely many different values of $n(T^{(r)}), m(T^{(r)})$, there exists a second subsequence $n(T^{(s)}), m(T^{(s)})$ of integers such that $n(T^{(s)}) = n$, $m(T^{(s)}) = m \in \mathbb{N}$ and

$$\frac{n}{m} = \lim_{s \to \infty} \frac{n(T^{(s)})}{m(T^{(s)})} = \lim_{s \to \infty} \frac{k_j(T^{(s)})}{k_i(T^{(s)})} = \lim_{s \to \infty} \frac{T^{(s)} k_j(T^{(s)})}{T^{(s)} k_i(T^{(s)})} = \frac{a_j}{a_i}$$

However, we assumed that a_j/a_i was irrational, so there can be no bounded subsequence, hence $n(T)$, $m(T) \to \infty$ as $T \to 0$ and

$$\lim_{T \to 0} \frac{1}{T \cdot lcm(k_i(T), k_j(T))} = \lim_{T \to 0} \frac{1}{(k_i(T)T) \cdot n(T)} = 0 \text{ as required. } □$$

Proof of Theorem 3

By Lemma 6 we can take limits on both sides of inequality (10) and since both limits exist, the limit of $\Delta(\mathbf{k})/T$ exists and is equal to the stated value. □