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## Utility functions for life years and health status: An additional remark

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**Utility Functions for  
Life Years and Health Status  
- an additional remark  
by Michael Happich**

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**Zusammenfassung:** Nutzen-basierte Maße für gesundheitsbezogene Lebensqualität gewinnen für Kosten-Effektivitäts-Analysen immer mehr an Bedeutung. Ihre axiomatische Fundierung qualifiziert sie im Gebrauch des QALY Konzeptes. Aber die Nutzung ist problematisch, da sie von verschiedenen Annahmen abhängt. Pliskin et al. (1980) haben gezeigt, welche Annahmen plausibel sind, Lebensqualität mit Lebenslänge zu kombinieren, jenen Attributen, die grundlegend für das QALY Konzept sind. Eine ist die sogenannte „konstante proportionale Austausch“ Annahme. Sie besagt, dass Menschen immer eine proportionale Anzahl an Lebensjahre für bessere Gesundheit opfern würden. Diese Annahme beschränkt die zugrunde liegenden Nutzenfunktionen für Lebensjahre auf solche, die mit konstanter proportionaler Risikoeinstellung übereinstimmen, das heißt Potenz- und Logarithmusfunktion oder eine lineare Funktion. Dennoch könnten diese Funktionstypen zu restriktiv sein, da sie nicht „konstanten absoluten Austausch“ darstellen können. Damit ist gemeint, dass Menschen eventuell immer dieselbe Anzahl von Lebensjahren aufgeben, unabhängig ihrer verbleibenden Lebenserwartung. Pliskin et al. erwähnten diesen Fall bereits und schlugen die Exponentialfunktion als geeigneten Funktionstyp vor, die zugrunde liegende konstante absolute Risikoeinstellung wiederzugeben. Ich werde den Beweis liefern. Eine Befragung unter Tinnitus Patienten könnte darüber hinaus die Gültigkeit dieser Funktionen unter Beweis stellen.

**Abstract:** Utility-based measures for health-related quality of life gain more and more importance in cost-effectiveness analysis. The axiomatic foundation qualifies them as decision weights in use of the QALY concept. But their use is strained for they are loaded with assumptions to make them work. Pliskin et al. (1980) have impressively shown which assumptions might be reasonable to combine quality of life with length of life, those attributes fundamental to the QALY concept. One of those assumptions is the so called „constant proportional tradeoff“. It states that people will always sacrifice the same proportion of remaining life years in order to gain better health. This assumption restricts the underlying utility functions for life years to those consistent with constant proportional risk posture, i.e. power, logarithmic and linear function. However, these types of function might be too restrictive for they do not reflect „constant absolute tradeoff“. That means people might rather exchange the same number of life years for better health, independent of remaining life expectancy. Pliskin et al. mentioned that case already and suggested the exponential function as a proper function to reflect the underlying constant absolute risk posture. I will deliver its proof. In addition, a survey among Tinnitus patients is mentioned that could further stress the validity of those functions.

## Introduction

Nowadays, cost-effectiveness plays an ever more important part in the evaluation of different health care policies.<sup>1</sup> But to measure **effectiveness**, it is not seen as sufficient any more just to measure gains in life expectancy. In times of growing concerns about chronic diseases, quality of life becomes emancipated as a valid predictor of effective medical interventions. Both together are seen as approximately sufficient to guide decisions in health care.<sup>2</sup>

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<sup>1</sup> Schöffski, „Einführung“ in Schöffski et al. (1998).

<sup>2</sup> Garber et al., „Theoretical Foundations of Cost-Effectiveness Analysis“ in Gold et al. (1996).

To avoid „rough and intuitive manner“<sup>3</sup>, Pliskin et al. (1980) propose the use of multiattribute utility functions to – as they say – „force one to be explicit about preferences and tradeoffs“<sup>3</sup>. They first derive separate utility functions for life years and health states and finally combine both resulting in a fundamental theorem about the very nature of utility functions concerning life years.

Although QALYs, which relate length and quality of life multiplicatively, can now be used on the basis of decision analysis, they rely on quite restrictive assumptions: The first assumption is that Y (life years) and Q (health status) are mutually utility independent.

**Mutual utility independence** holds if and only if <sup>4</sup> there are utility functions,  $U_Y(y)$  for remaining life years and  $U_Q(q)$  for health status, and constants a and b such that:

$$U(y, q) = a \cdot U_Y(y) + b \cdot U_Q(q) + (1 - a - b) \cdot U_Y(y) \cdot U_Q(q). \quad (1)$$

Pliskin et al. call that a quasi-additive form (P. 210). Mutual utility independence indicates that the utility of any attribute does not depend on the particular level of any other attribute. For example, a Tinnitus patient does not judge her own health state differently because she has either two or rather twenty years still to live. In the context of this article, this is taken as a reasonable approximation of behavior.

The second assumption considers certain preference pattern. People are assumed to sacrifice always the same proportion of remaining life years in order to gain better health. For example, if asked how many years of life expectancy one would exchange for a health state free of symptoms, a person that gives up 10 years out of 40 remaining life years, would equally give up 5 out 20 or 2,5 out of 10 years, i.e. in that case always a quarter of remaining life years. The term „constant proportional tradeoff“ is supposed to describe such behavior.

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<sup>3</sup> Pliskin et al. (1980), page 206.

<sup>4</sup> Keeney and Raiffa (1976).

What if this assumption does not hold? Suppose people would sacrifice a certain number of life years for better health independent of remaining life expectancy. The person already mentioned would always give up, for example, 5 years independent of 40, 20 or 10 remaining life years. The result is the following definition:

**Definition 1:** The **constant absolute tradeoff** assumption of life years for health status is said to hold if the number of remaining life years that one is willing to give up for an improvement in health status from any given level  $q_1$  to any other level  $q_2$  does not depend on the absolute number of remaining life years.

These two assumptions – utility independence and constant absolute tradeoff – restrict the form of the bivariate utility function  $U(y; q)$  to those consistent with (1). In addition, they considerably influence the form of the component utility for life years  $U_Y(y)$ . However, the form of the component utility function for life years is not the one suggested by Pliskin et al. since they referred to constant proportional tradeoff. They finally derived power and logarithmic functions as utility functions for longevity to correctly reflect mutual utility independence and constant proportional tradeoff. These functions have the characteristic of constant proportional risk posture as defined by Pratt (1964). This is in contrast to:

**Definition 2:** A twice-differentiable utility function  $U(y)$  exhibits **constant absolute risk posture** if  $c \equiv -\frac{U''(y)}{U'(y)}$  is constant.<sup>5</sup>

The following theorem states that the constant absolute tradeoff assumption and utility independence imply that the utility function for life years alone exhibits constant absolute risk posture:

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<sup>5</sup> Pratt (1964) as well. Constant proportional risk posture means  $c \equiv -y \cdot \frac{U''(y)}{U'(y)}$  is constant.

**Theorem:** If mutual utility independence and the constant absolute tradeoff assumption of Y and Q both hold then a „well-behaved“  $U_Y(y)$  exhibits constant absolute risk posture. ( By „well-behaved“ is meant  $U_Y(y)$  is twice differentiable, and

that 
$$\lim_{y \rightarrow 0} \left( - \frac{U_Y''(y)}{U_Y'(y)} \right) \text{ exists.}^6$$

The proof is given in the appendix and follows narrowly the path Pliskin et al. have used.

## Why all that?

Which type of utility function to use will, in future, depend heavily on the underlying preference pattern. That is obvious. But which pattern is more reasonable? At first sight, it might be tempting to assume some kind of proportionality in answers to question about length of life. That would promote the use of a power function  $U_Y(y) = y^r$ . Since the relative **curvature of utility functions**<sup>7</sup> can be interpreted as attitude towards risk,<sup>8</sup> to find the exponent r of the power function means to define the risk parameter. The „certainty equivalence“ method is a proper way to calculate r.<sup>9</sup> It compares two states of the world, an uncertain with a certain one: Given a lottery of life years, what is the certain amount of life years one would accept for sure in exchange for avoiding the risk involved? The difference in expected value of life years and the certain number determines r.<sup>10</sup>

The problem is that any curvature is interpreted as attitude towards risk. That neglects a whole domain of decision analysis, the analysis of value functions. A value function is, per definitionem, a function representing preferences under certainty.<sup>11</sup>

<sup>6</sup> The attentive reader perceives the similarities to Pliskin et al. in approach.

<sup>7</sup> Utility functions are defined as functions representing preferences under uncertainty. Barron et al. (1984), P. 233.

<sup>8</sup> Miyamoto et al. (1985), P. 193.

<sup>9</sup> Miyamoto (1985).

<sup>10</sup> Eisenführ und Weber (1999), P. 227.

<sup>11</sup> Barron (1984), P. 233.

Those functions can be curved as well, for example concave, but no risk interpretation is possible. That curvature just displays diminishing marginal value, i.e. a decision maker prefers ever more of a good, but with less and less sensation. The increase in happiness between having zero or one refrigerator might be different to the one in having ten or eleven.

There has been much debate about the difference of value and utility functions. Still, two positions coexist:<sup>12</sup> The first considers both to be equivalent or virtually the same. The second states that there might be a difference. The curvature consists of two parts – a riskless component and a risky one, the so called „**intrinsic risk attitude**“, but to separate both does not actually matter. It makes no sense to distinguish between components because a measurement method like the certainty equivalence captures both.

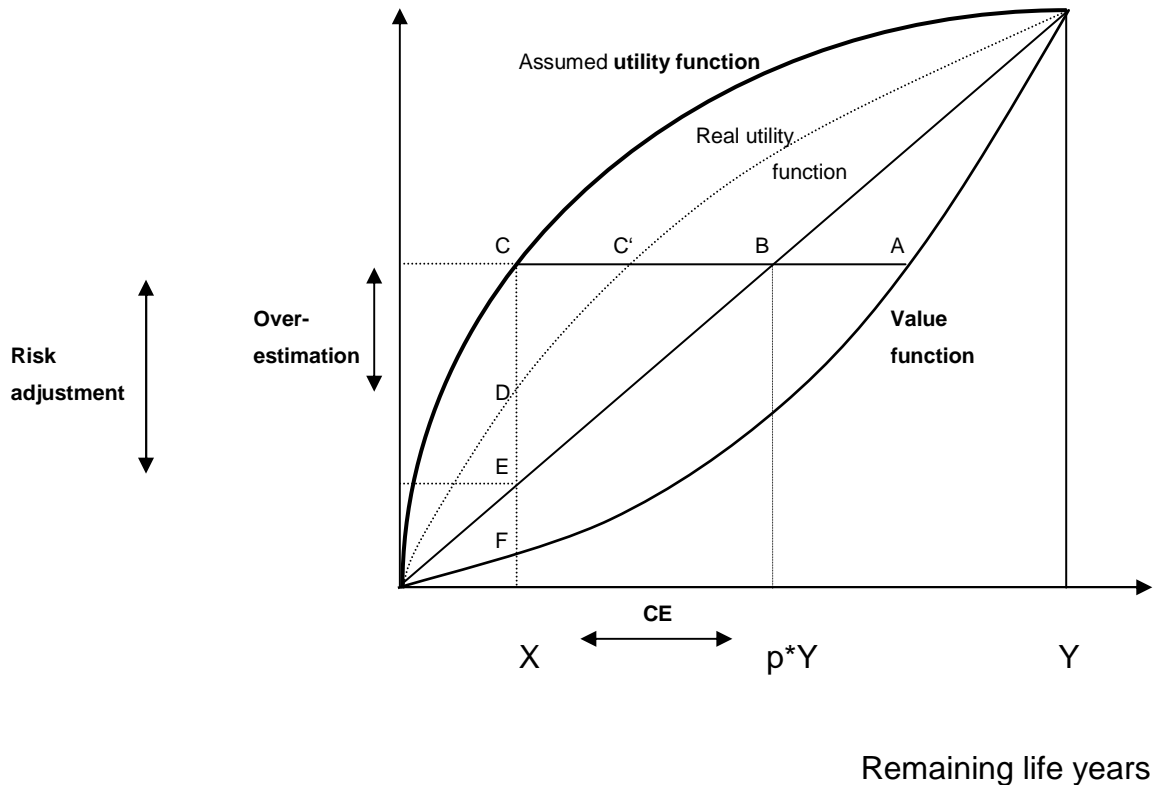
I say, it does matter. Risk parameters  $r$  are evaluated to adjust riskless measures of health states to situations in an uncertain environment. For example, the time tradeoff approach is a riskless measure<sup>13</sup> that is quite extensively used. Those measures will lead to biased results in medical decision making if they are applied to operation situations where risks about life and death are involved. But any risk adjustment will lead to over- or underestimations as well if those time tradeoff values do not lie on a straight value curve (See figure 1, next page).

Smidts (1997) analysed empirically value and utility function. He could show significant differences between both and described their relationship with an exponential function. Hence, an **exponential representation** of (intrinsic) risk attitudes in the health domain might be reasonable as well since the other component of curvature is already covered by riskless measures.

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<sup>12</sup> Bamberg and Coenenberg (1996), P. 93.

## Utility of life years



**Figur 1: Effect of risk adjustment under the assumption of a (non-)linear value function.**

Suppose that a patient expects  $Y$  years still to live. Asked how many years would leave her indifferent to a symptom freeing operation with a survival-probability of  $p$ , she might indicate  $X$  years. That means the certainty equivalence of  $(p \cdot Y - X)$  years generates a certain risk parameter that allows for adjustment. This parameter encloses two parts of curvature: The actually risk free part  $B$  to  $A$ , and the „intrinsic“ risk part  $A$  to  $C'$  which corresponds to the distance from  $B$  to  $C$ . Suppose further that a patient might judge  $X$  years without symptoms of a particular health condition to be equivalent to  $Y$  years with those symptoms. Under the assumption of a linear value function  $E$  would be determined as the „right“ value for  $X$  years. This obviously neglects the convex curvature of the value function and leads to an overestimation to  $C$  instead of  $D$  if risk adjusted. The „correct“ risk free value lies at  $F$ .

<sup>13</sup> Torrance (1986).



## Empirical assessment

### What can we expect?

Life expectancy and the maximum number of life years one would be willing to give up allows to analyse time tradeoff conditions. If the number of sacrificed life years depends on remaining life years one should be able to observe a strong relationship between both variables. We can conclude that constant proportional tradeoff and, hence, constant proportional risk posture holds. Is there no strong relationship one can assume constant absolute tradeoff with the corresponding conclusion.

### Methods

Tinnitus patients were asked to answer a questionnaire-based interview to evaluate the relationship between life expectancy and willingness to exchange life years for better health. The illness „**Tinnitus**“ is characterized by an undebilitating sound in the ear subjectively perceived by patients. After a year that condition is chronic and can only be cured by chance. Patients are forced to cope with that sound.<sup>14</sup>

210 patients were interviewed between september and december 2000, 110 women and 100 men between 16 and 85 years old, on average 53,8 years. Almost two third of them were married (146), 24 were singles, 14 lived as widows, and 26 were divorced or seperated. Patients have been met at four different places in Berlin: 21 at the Tinnitus-League, a self-help association, 21 at the Heinrich-Heine-Hospital, a hospital with a focus on psychosomatic conditions, 63 at the ear, nose, and throat department of the Charité, the hospital connected to the Humboldt-University, and 105 patients at Dr. Berndt, a leading medicin in Tinnitus treatment.

All patients were asked how old they guess to become. This procedure allowed to define individual life expectancy to avoid reference point biases.<sup>15</sup> The difference between the individual life expectancy and actual age can be defined as remaining life years. Another question asked the maximum number of years patients would be willing to give up in order to free themselves of the symptoms of Tinnitus. A medicament was given as example that would have that effect but had an influence on life expectancy. The number of years to give up were successively risen until the respondent were indifferent between taking the medicine or living with that condition. To analyse the linear relationship the curve fitting procedure of the statistical software „SPSS“ has been used.

## Results

The relationship between individual remaining life years and the number of sacrificed years is weak at best:

Dependent Mth		<b>Rsq</b>	d.f.	F	<b>Sigf</b>	b0	b1
sacrific	LIN	<b>,070</b>	190	14,23	<b>,000</b>	1,3406	,1191

**Table 1:** Linear regression: Remaining life years – sacrificed years

Although the relationship between both variables is significant (Sigf)<sup>16</sup>, the independent variable „remaining life years“ explains about 7% of total variance (Rsq= $R^2$ ) only. Considering that  $R^2 = 0$  means there is no relationship between both variables whatsoever and  $R^2 = 1$  is the closest connection possible,  $R^2 = 0,07$  is pretty close to the interpretation of zero correlation, i.e. constant absolut tradeoff. In addition, one has to keep in mind that the spread of the dependent variable is bounded since one cannot sacrifice more years than one has still to expect. That means, points in the diagram can only lie between the horizontal axis and the 45°-line. That provokes a quasi-dependency.

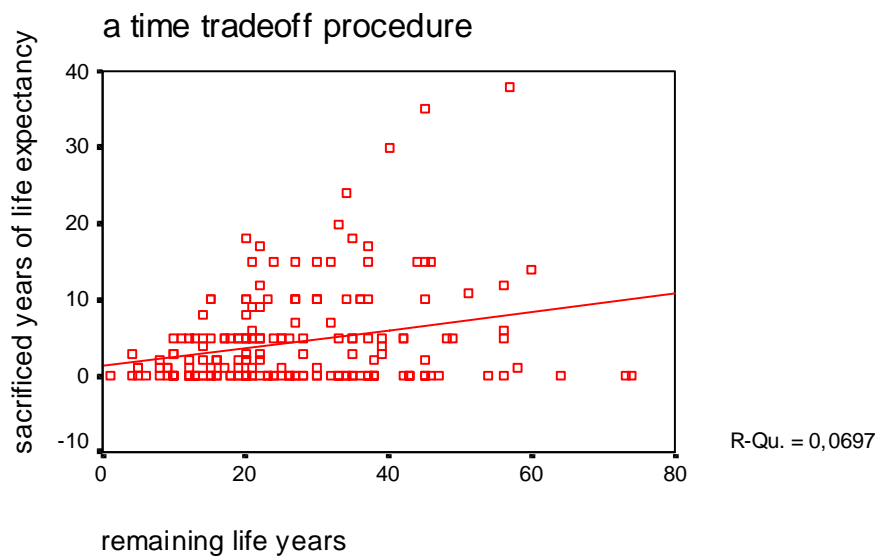
<sup>14</sup> Feldmann (1998).

<sup>15</sup> Verhoef et al. (1994).

<sup>16</sup> Sigf = 0,000 indicates that the prediction error to assume a connection is less than half percent .

**Figur 2:**

Tinnitus: Relationship between remaining life years and sacrificed years for better health



## Conclusion

Constant absolute risk posture can be considered as an **appropriate representation** of preferences under uncertainty. Its mathematical form is the exponential function. Although those functions have been widely used in medical decision making, its axiomatic foundation has been missed so far. In addition, more reasonable justifications for its use can be founded on insights of the very nature of utility functions. Since the relationship of value and utility function has been neglected in the past, exponential functions might better reflect „intrinsic“ risk posture. Finally, which type of function to use will always depend on the constant absolute or proportional tradeoff assumption. Future analysis in medical decision making will have to answer that question first before working with a specified type.

## Appendix

**Proof:** First, I will show that there exists a constant-absolute-risk-posture functional form  $\frac{1 - e^{-\lambda y}}{1 - e^{-\lambda}}$ , which is consistent with the constant absolute tradeoff assumption.

If  $q_*$  and  $q^*$  denote the lowest and highest health status levels, respectively, with its utilities set to 0 and 1 for convenience, then the constant absolute tradeoff assumption states that

$$U(y, q_*) = U(y - p, q^*) \quad \text{for } p \text{ (} 0 < p < y \text{) and for all } y \geq 0.$$

Substituting  $(y, q_*)$  and  $(y - p, q^*)$  into equation (1), recalling that  $U_Q(q_*) = 0$  and  $U_Q(q^*) = 1$ , and equating the two utilities thus obtained, we find that

$$\begin{aligned} a \cdot U_Y(y) &= a \cdot U_Y(y - p) + b + (1 - a - b) \cdot U_Y(y - p) \\ \Leftrightarrow a \cdot U_Y(y) &= b + (1 - b) \cdot U_Y(y - p). \end{aligned} \tag{A-1}$$

If  $U_Y(y) = \frac{1 - e^{-\lambda y}}{1 - e^{-\lambda}}$ , then follows - with some algebra - in (A-1)

$$e^{-\lambda y} \cdot [(1 - b) \cdot e^{\lambda p} - a] = 1 - a - b \cdot e^{-\lambda}$$

For the equality to hold, one condition (among others) for parameter a and b can be:

$$\Rightarrow \quad (1-b) \cdot e^{\lambda \cdot p} - a = 0 \quad \Leftrightarrow \quad a = \frac{1 - e^{-\lambda}}{1 - e^{-\lambda \cdot (1+p)}} \quad (\text{A-2})$$

$$\wedge \quad 1 - a - b \cdot e^{-\lambda} = 0 \quad \wedge \quad b = \frac{e^{\lambda \cdot p} - 1}{e^{\lambda \cdot p} - e^{-\lambda}} \quad (\text{A-3})$$

The following holds: (1.)

$$\lim_{\lambda \rightarrow -\infty} a = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} a = 1, \quad \text{and}$$

$$\lim_{\lambda \rightarrow -\infty} b = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} b = 1,$$

(behavior of terms in infinity),

$$\lim_{\lambda \rightarrow -0} a = \lim_{\lambda \rightarrow +0} a = \frac{1}{p+1} \quad \text{for } p \geq 0,$$

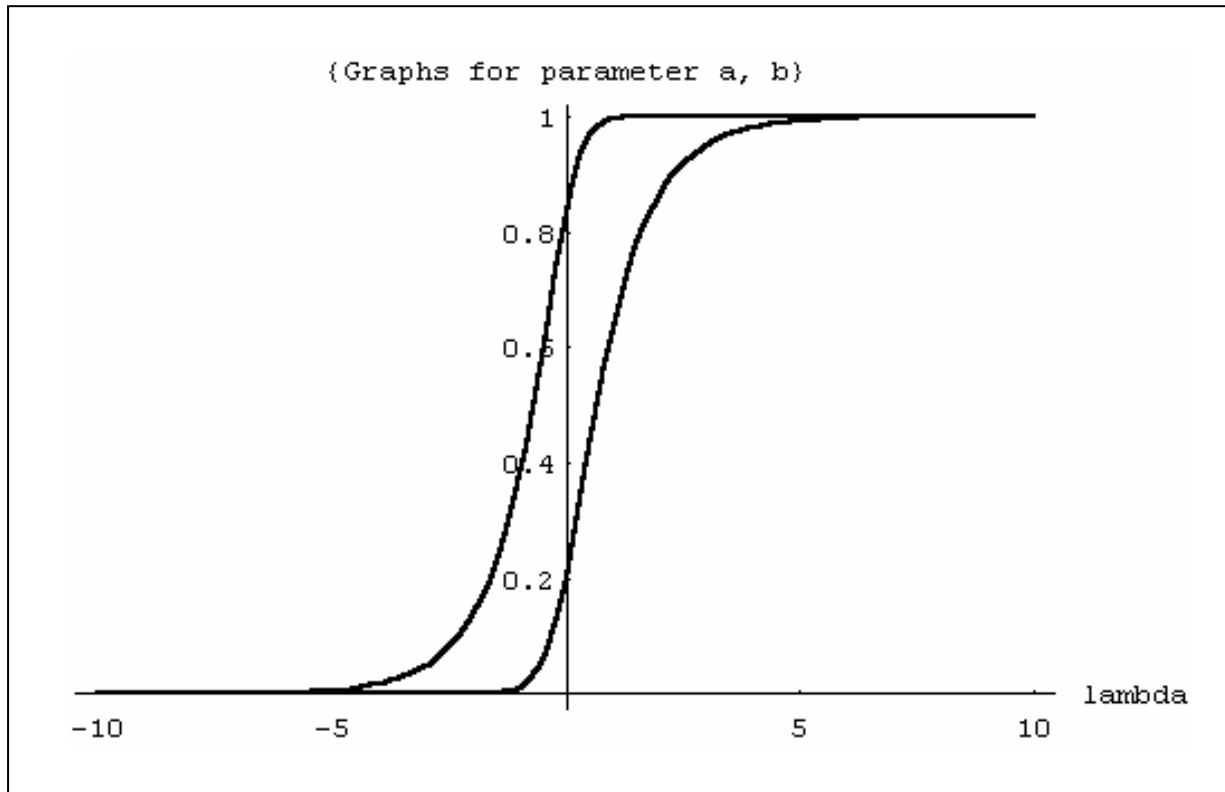
$$\lim_{\lambda \rightarrow -0} b = \lim_{\lambda \rightarrow +0} b = \frac{p}{p+1} \quad \text{for } p \geq 0,$$

(behavior of terms at the domain gap),

and (2.) parameter a and b increase monotonically in  $\lambda$  for  $-\infty < \lambda < 0$   
and  $0 < \lambda < \infty$ , i.e.

$$\frac{\partial a}{\partial \lambda} \geq 0 \quad \text{and} \quad \frac{\partial b}{\partial \lambda} \geq 0 .$$

For example, if  $p = 4$ , the following parameter curves can be observed :



**Figure A-1: Graphs for parameter a and b depending on  $\lambda$ .** The upper curve corresponds to b, the lower one to a.

Therefore, parameter a and b in (A-2) and (A-3) are consistent with (A-1), because  $0 < a, b < 1$  as required.<sup>17</sup>

It has been shown that functional forms exist with constant absolute risk posture which are consistent with the constant absolute tradeoff assumption when two particular health states  $q_*$  and  $q^*$  are involved in the tradeoffs.

It remains to be shown that the utility functions with constant absolute risk posture are the only ones consistent with the constant absolute tradeoff assumption.

The function for the constant absolute risk posture is  $R_Y(y) = -\frac{U''(y)}{U'(y)}$ .

If one differentiates (A-1) twice and divides the second derivative of each side by the first derivative gives

$$\frac{U''(y)}{U'(y)} = \frac{U''(y-p)}{U'(y-p)}.$$

Expressed with the risk posture function

$$R_Y(y) = R_Y(y-p).$$

Since this equation holds for all  $p$  ( $0 < p < y$ ) and for all  $y \geq 0$ , i.e, the risk posture function is independent of the argument of the function,

$$R_Y(y) = \text{const.} \quad \text{can be concluded.}$$

Thus,  $U_Y(y)$  exhibits constant absolute risk posture.

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<sup>17</sup> All calculations have been carried out with software package „mathematica“.

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