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"Common Knowledge of Rationality and Market Clearing in Economies with Asymmetric Information"

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Common Knowledge of Rationality and Market Clearing in Economies with Asymmetric Information[1]

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Abstract:

Consider an exchange economy with asymmetric information. What is the set of outcomes that are consistent with common knowledge of rationality and market clearing?

To address this question we define an epistemic model for the economy that provides a complete description not only of the beliefs of each agent on the relationship between states of nature and prices but also of the whole system of interactive beliefs. The main result, theorem 1, provides a characterization of outcomes that are consistent with common knowledge of rationality and market clearing (henceforth, $CKRMC$ outcomes) in terms of a solution notion - $Ex - Post\ \ Rationalizability$ - that is defined directly in terms of the parameters that define the economy. We then apply theorem 1 to characterize the set of $CKRMC$ outcomes in a general class of economies with two commodities. $CKRMC$ manifests several intuitive properties that stand in contrast to the full revelation property of *Rational Expectations Equilibrium*. In particular, we obtain that for a robust class of economies: (1) there is a continuum of prices that are consistent with $CKRMC$ in every state of nature, and hence these prices do not reveal the true state, (2) the range of $CKRMC$ outcomes is monotonically decreasing as agents become more informed about the economic fundamentals, and (3) trade is consistent with common knowledge of rationality and market clearing even when there is common knowledge that there are no mutual gains from trade.

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[1] This is a major revision of the working paper "Rationalizable Expectations".

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1 Introduction

We study the implications of the assuming common knowledge of rationality and market clearing in economies with asymmetric information.

The starting point is the concept of *rational expectations equilibrium* (REE). REE extends the classical concept of a competitive equilibrium to economies with asymmetric information (i.e., economies in which different agents might have different information). When each agent has only partial information on the value of a commodity or an asset he can deduce additional information from the prices because prices reflect the information that other agents have. REE is a solution concept that is based on the assumption that agents make these inferences. However, the concept of REE is based on an additional strong assumption that agents know (and therefore agree on) the function that specifies the prices in each state of nature. (A state of nature specifies the real variables of the economy, i.e., preferences and endowments.) As Radner (1979) has shown this strong assumption leads to the strong result that in a generic economy with a finite number of states the only REE is a fully revealing equilibrium, i.e., an equilibrium in which each agent can infer from the prices all the information that any other agent has. This conclusion is at odds both with intuition and real-world practice. To take just one example, the daily volume of trade in foreign exchange is significantly larger than the value of international trade, indicating that much of the former is speculative and based on non-unanimous evaluation of the information embedded in prices.

In the current research the assumption that players know the price function is relaxed, that is, we consider a situation where each agent may have a different theory on how the vector of prices which is observed has materialized and on what would have happened in other states of nature. However, the assumption is maintained that each agent makes inferences from the observed prices and furthermore assumes that other agents are doing likewise. More precisely we ask: what is the set of outcomes that are consistent with *common knowledge of rationality and market clearing* ?

To address this question we define an epistemic model of an economy with asymmetric information. This model provides a complete description of the beliefs of each agent not only on the relationship between states of nature and prices but also on the beliefs of other agents. In this model consistency with common knowledge of rationality and market clearing can be defined in a precise way. Our main result, theorem 1, establishes that under a mild qualification an outcome (s, p), where s is a state of nature and p a vector of prices, is consistent with common knowledge of rationality and market clearing iff it is an *Ex-Post Rationalizable* outcome. Ex-Post Rationalizability is a solution notion that is defined directly in terms of the parameters that define the economy and does not involve type spaces. The main advantage of Ex-Post Rationalizability is that it is easier to compute. However, we view it as a derived notion, the fundamental concept being consistency with common knowledge of rationality and market clearing (henceforth, we will often abbreviate and refer to this solution concept as *CKRMC).*

We use the characterization of theorem 1 to compute the set of prices that are consistent with common knowledge of rationality and market clearing in a general class of economies with two commodities. In this class of economies $CKRMC$ manifests several properties which stand in contrast to the full revelation property of REE. In particular, we obtain that for a robust subset of these economies:

(a) There is a whole range of prices that are consistent with common knowledge of rationality and market clearing in all the states of nature and therefore these prices do not reveal any information.

(b) Refining the knowledge of a positive measure of agents strictly shrinks the set of $CKRMC$ prices.

(c) Trade is consistent with common knowledge of rationality and market clearing even when there is common knowledge that there are no mutual gains from it.

The general motivation for our work is related to the seminal contributions of Bernheim (1984) and Pearce (1984). Bernheim and Pearce pointed out that common knowledge of rationality does not preclude the possibility that players have heterogenous beliefs about the outcome of a game. In particular, common knowledge of rationality does not imply a Nash equilibrium outcome. Bernheim and Pearce proposed the solution concept of *Rationalizability* which is the outcome of a procedure in which strategies that are not best responses to any beliefs are iteratively deleted. The study of the implications of rationalizability in competitive economies was pioneered by Guesnerie (1992)[4]. Guesnerie analyzes a two-period economy with complete information that is based on the classical model of Muth (1961) and identifies conditions under which the only Rationalizable outcome is the Rational Expectations Equilibrium outcome. (In Muth's model Rational Expectations Equilibrium means that each agent has a correct expectation about the prices in the second period. This is very different from the interpretation of REE in our model where the object of analysis is a static economy with incomplete information and the focus is on the information that prices reveal on the state of nature.)

There is some previous work which examines the implications of the assumptions of rationality and market clearing in economies with asymmetric information where players may have different beliefs about the relationship between states of the economy and prices. However the solution concepts that are proposed in these papers are different from consistency with common knowledge of rationality and market clearing. MacAllister (1990) and Dutta and Morris (1997) propose a solution concept, Belief Equilibrium, which is stronger than $CKRMC$ as it assumes that in addition to common knowledge of rationality and market clearing there is also common knowledge of the belief of each player on the joint distribution of prices and states of nature. As we show in section 3 this additional assumption restricts in a significant way the set of possible outcomes. Desgranges

[4] See also Guesnerie (2005) for additional papers on the *eductive* approach to the analysis of competitive economies.

and Guesnerie (2000)[5] examine iterative deletion of weakly dominated demand strategies in a simple example which is similar to example 1 in the current paper. The solution set that they obtain is equal to the set of $CKRMC$ and Ex-Post Rationalizable outcomes that is obtained in the current paper. Desgranges (2004) also studies the implications of Ex-Post Rationalizability (Desgranges calls it common knowledge equilibrium)[6]. The focus in his work is on determining conditions under which Ex-Post Rationalizability implies the REE outcome. In contrast our main interest is in characterizing the set of Ex-Post Rationalizable outcomes in a general class of economies and demonstrating that in contrast to the predictions of REE the properties (a)-(c) that were mentioned above are consistent with common knowledge of rationality and market clearing. However, the main difference between Desgranges and the current paper is that in Desgranges the starting point is Ex-Post Rationalizability while in our work Ex-Post Rationalizability is a derived notion which is justified only because it is a useful characterization of the fundamental solution concept - common knowledge of rationality and market clearing. This is the content of our main result (theorem 1). To define common knowledge of rationality and market clearing in a rigorous way we construct an epistemic model. Thus in terms of the methodology of the analysis our work is related to the literature on the epistemic foundations of solution concepts in game theory. The general goal of this literature is to clarify the assumptions that underlie different solution concepts. So for example, Tan and Werlang (1988),[7] used an epistemic model to establish that the set of outcomes that are consistent with common knowledge of rationality in a strategic game is equivalent to the set Rationalizable outcomes defined by Bernheim and Pearce[8].

Despite the fact that a system of interactive beliefs is at the heart of the eductive approach there is only one other paper - Morris (1994)- that we are aware of which applies the epistemic approach to the analysis of competitive economies. Morris shows that if there is a common prior on the set of states of the world, where a state of the world specifies not only the fundamentals of the economy (preferences and endowments) but also the whole system of interactive beliefs, then common knowledge of rationality and market clearing implies that the correspondence between states of the *world* and prices is a Rational Expectations Equilibrium. By contrast we do not assume a common prior on the states of the world and our interest (like in the rest of all the papers that were cited) is in the correspondence between states of *nature* and prices (where a state of nature specifies only the fundamentals).

We now present a simple example that motivates the discussion .

Example 1:

There are two commodities in the economy, X and M (money).

[5] See also Guesnerie (2002).

[6] A first draft of Desgranges paper was written before ours. We have developed the concept of $EXPR$ independently, before we learned of his work.

[7] See also Hu (2007).

[8] The literature which applies the epistemic approach to the analysis of game-theoretic solution concepts is by now fairly extensive. Dekel and Gul (1997) and Battigalli and Bonanno (1999) provide excellent overviews.

The set of states is $S = \{1, 3\}$.

The probability of each state is 0.5.

The set of agents is the interval $[0, 1]$. There are two types of agents, I_1 and I_2. Agents in I_1 know the true state; agents in I_2 do not know it. $I_1 = [0, \delta]$ and $I_2 = (\delta, 1]$. All the agents have the same utility function and the same initial bundle. The utility function is:

(1.1) $$u(x, m, s) = s \times \log(x) + m$$

where x and m are the quantities of X and M respectively and s is the state.

The initial bundle consists of one unit of X and $\overline{m}$ units of M where $\overline{m} \geq 3$.

Let p be the price of a unit of X in units of M. It follows from the definition of the utility function in (1.1) that the demand for X of an agent who knows the true state is:

$$x = \frac{s}{p}$$

More generally, the demand of an agent i who assigns to the state s probability $\gamma(s)$ is

(1.2) $$x = \frac{\gamma(1) \times 1 + \gamma(3) \times 3}{p}$$

In this example for every $\delta > 0$ there is only one REE, f^* ,where $f^*(s) = s$. To see that we, first, note that if f is a REE then $f(1) \neq f(3)$. This follows because if $f(1) = f(3) = p$ then agents in I_2 do not obtain any information about the true state and therefore their demand in both states is the same:

$x = \frac{0.5 \times 1 + 0.5 \times 3}{p} = \frac{2}{p}$

However, the demand of agents from I_1 in state 1 is different than their demand in state 3 and therefore the aggregate demands are different as well. Since the aggregate amount of X is fixed this means that the market doesn't clear in at least one of the states and therefore f is not a REE. Thus, if f is a REE then $f(1) \neq f(3)$. In this case agents in I_2 infer the state from the price and it follows from (1.2) that $f(1) = 1$ and $f(3) = 3$. Thus, the only REE is a fully revealing equilibrium in which the price reveals the state. Alternatively put, the only outcomes (p, s) (where p is a price and $s \in S$ is a state) that are consistent with REE are $(1, 1)$ and $(3, 3)$.

We now show that if we relax the assumption that players know the price function (and therefore agree on it) then there are other outcomes which are consistent with common knowledge of rationality and market clearing. We call such outcomes $CKRMC$ *outcomes*.

Assume that the fraction of informed agents in the economy is $\delta = \frac{1}{6}$. Define two price functions f and g as follows:

$$f(1) = 2 \qquad g(1) = 1$$
$$f(3) = 3 \qquad g(3) = 2$$

We will show that f and g are consistent with common knowledge of rationality and hence the outcomes $(2,1)$ and $(2,3)$ are $CKRMC$ outcomes.

Suppose that a fraction β of the agents in I_2 assign probability $\frac{3}{4}$ to the event that f is the price function and a probability $\frac{1}{4}$ to the event that g is the price function, call this belief 'theory A'. Assume that the other agents in I_2 think that g is more likely, they assign probability $\frac{1}{4}$ to the event that f is the price function and probability $\frac{3}{4}$ to the event that the price function is g, call this belief 'theory B'.

What are the beliefs of different agents in I_2 about the true state when they observe the price 2 ?

Since the prior assigns probability 0.5 to each state it is easy to see that agents in I_2 who believe in theory A assign probability $\frac{3}{4}$ to the state 1 and probability $\frac{1}{4}$ to the state 3.[9] Similarly, agents who believe in theory B assign probability $\frac{1}{4}$ to the state 1 and probability $\frac{3}{4}$ to the state 3.

It follows from (1.2) that the demand for X at price 2 of agents who believe in theory A is $\left(\frac{3}{4} \times 1 + \frac{1}{4} \times 3\right)/2 = \frac{3}{4}$ while the demand of agents who believe in theory B is $(\frac{3}{4} \times 3 + \frac{1}{4} \times 1)/2 = \frac{5}{4}$.

Let $x(\beta, s, p)$ denote the aggregate demand for X in state s at price p when a proportion β of the agents in I_2 believe in theory A and the rest of I_2 believe in theory B. We have

$$x(\beta, 1, 2) = (1-\delta) \times \beta \times \tfrac{3}{4} + (1-\delta) \times (1-\beta) \times \tfrac{5}{4} + \delta \times \tfrac{1}{2}$$
$$x(\beta, 3, 2) = (1-\delta) \times \beta \times \tfrac{3}{4} + (1-\delta) \times (1-\beta) \times \tfrac{5}{4} + \delta \times \tfrac{3}{2}$$

Let β_f and β_g be the numbers which equate demand and supply at price 2 in the states 1 and 3 respectively, that is,

$x\left(\beta_f, 1, 2\right) = 1$ and $x\left(\beta_g, 3, 2\right) = 1$. For $\delta = \frac{1}{6}$ we obtain $\beta_f = 0.3$ and $\beta_g = 0.7$.

Now we observe that when β_f of the agents in I_2 believe in theory A and $1 - \beta_f$ of them believe in B then the function f specifies prices which clear the market. (We have just seen that the price 2 clears the market in $s = 1$ and when the price is 3 everyone assigns probability 1 to the state 3 and therefore the price 3 clears the market.) Similarly, when β_g of the agents in I_2 believe in theory A (and the rest in B) the function g specifies prices which clear the market.

In section 2 we present an epistemic model for an exchange economy with asymmetric information and use it to define common knowledge of rationality and market clearing. We then show how to formalize the analysis of example 1. With this formalization, it will become explicit how there can be common knowledge that each agent in I_2 entertains either of the theories A and B, and that the price $p = 2$ is consistent with common knowledge of rationality and market clearing in both states $s = 1, 3$.

In section 3 we define the concept of Ex-Post Rationalizability and present theorem 1, which establishes that under a mild qualification an outcome is consistent with common

[9] Let $P_A(s\,|p=2)$ denote the posterior that an agent who believes in theory A assigns to the state s upon observing the price 2. Then $P_A(1\,|p=2) = \frac{0.75 \cdot 0.5}{0.75 \cdot 0.5 + 0.25 \cdot 0.5} = 0.75$

knowledge of rationality and market clearing iff it is an Ex-Post Rationalizable outcome. In section 4 we characterize the set of Ex-Post Rationalizable outcomes in a general class of economies with two commodities and then use theorem 1 to derive implications on $CKRMC$ outcomes. The characterization which we obtain manifests the properties (a)-(c) that were mentioned above. Section 5 concludes. All the proofs are delegated to the appendix.

2 The Model

In this section we review the definitions of an exchange economy with asymmetric information and the concept of *Rational Expectations Equilibrium*. We then present an epistemic model of the economy and use it to define the concept of *Consistency with Common Knowledge of Rationality and Market Clearing* (which we will often abbreviate to *CKRMC)*. An *outcome* (p, s), where p is a vector of prices and s is a state of nature, is a $CKRMC$ outcome if it is consistent with common knowledge of rationality and market clearing. We then demonstrate how the epistemic model can be used to make the analysis of example 1 complete and precise.

An economy with asymmetric information is defined by:

1. $I = [0, 1]-$ The set of players (consumers).
2. $X_1, ..., X_K-$ K commodities.
3. $S = \{s_1, ..., s_n\}-$ The set of states of nature.
4. π_i- A partition on S that describes the information of player i.
   $\pi_i(s) \subseteq S$ is the information that player i gets at the state s.
5. $\alpha_i \in \triangle(S)-$The prior probability of agent i on S.[10]
6. $u_i : R^K \times S \to R -$ A V.N.M utility function for player i.
   $u_i(x, s)$ is the utility of player i from a bundle
   $x \in R^K$ in the state s.
7. $e_i : S \to R^K - e_i(s)$ is the initial bundle of player i at state s.
   We assume that e_i is measurable w.r.t π_i and that
   $\forall s \in S,\ \int_i e_i(s)\mathrm{d}i-$ the aggregate supply in state $s-$exists.

A *price* p is a vector $p = (p_1, ...p_{K-1})$ where p_k is the price of X_k. The price of X_K is normalized to be 1.

A *price function* $f, f : S \to R^{K-1}$, assigns to every state s a price $f(s)$.

We let L_i denote the set of signals of agent i. So, $L_i \equiv \{\pi_i(s) : s \in S\}$.

A *demand function* for player i is a function z_i, $z_i : L_i \times R^{K-1} \to R^K$, such that $z_i(l_i, p)$ is in the budget set defined by the price p and the initial endowment $e_i(l_i)$. ($e_i(l_i)$ is well defined because e_i is measurable w.r.t π_i.)

[10] If $\alpha_i = \alpha$ for every $i \in I$ then there is a common prior. However, our results do not require such an assumption.

It is assumed that the agents do not observe the supply and demand of the commodities, but only the prevailing price vector $p \in R^{K-1}$. The standard solution notion for economies with asymmetric information is Rational Expectations Equilibrium, REE. A REE is a price function f such that for each state s the price $f(s)$ clears the market when every agent i exhibits a demand which is optimal w.r.t the price $f(s)$ and the information that is revealed by his private signal $\pi_i(s)$ and the fact that the price is $f(s)$. Formally,

Definition: A price function f is a REE if there exists a profile of demand functions, $\{z_i\}_{i \in I}$, that satisfies :

1. Rationality, $\forall s \in S$, $z_i(\pi_i(s), f(s))$ is optimal w.r.t the price $f(s)$ and the posterior $\alpha_i(\cdot \,|\pi_i(s) \cap f^{-1}(f(s)))$.
2. Market clearing, $\forall s \in S$, $\int_i z_i(\pi_i(s), f(s))\mathrm{d}i = \int_i e_i(s)\mathrm{d}i$.

A price function f is a *fully revealing REE* if $f(s) \neq f(s')$ when $s \neq s'$.

As we have pointed out in the introduction the concept of Rational Expectations Equilibrium refers to a situation where all the agents know the price function. In particular, all the agents have the same belief regarding the relationship between the prices and the states of nature. We are interested here in a solution concept that is akin to the concept of Rationalizability in game theory. That is, we ask what is the set of outcomes in the economy when agents may have heterogenous beliefs concerning the relationship between prices and states and yet there is common knowledge of rationality and market clearing ?

To address this question we now define an epistemic model in the spirit of Harsanyi (1967-68) where we represent the choice and belief of each agent i by a *type* t_i in a measurable space T_i and the entire economic situation by a state of the world

$$\omega = (s, p, (t_i)_{i \in I}) \in \Omega \subseteq S \times R_+^{K-1} \times \prod_{i \in I} T_i$$

Each type $t_i \in T_i$ of agent $i \in I$ is associated with

1. its demand function $z_i[t_i]$. That is, for each observed price vector $p' \in R_+^{K-1}$ for which there is a state of the world $\left(s', p', \left(t'_j\right)_{j \in I}\right) \in \Omega$ with $t'_i = t_i$, $z_i[t_i](\pi_i(s'), p')$ is a bundle which is feasible for agent i given the prices p':

$$z_i[t_i](\pi_i(s'), p') \cdot p' \leq e_i(\pi_i(s')) \cdot p'$$

2. its ex-ante belief $b_i[t_i] \in \Delta(\Omega)$ about the states of the world, having the property that its marginal on the space of states of nature S is the agent's prior –

$$\mathrm{marg}_S b_i[t_i] = \alpha_i$$

and that it knows its own type:

$$b_i\left[t_i\right](E) = 1 \quad \text{whenever} \quad \left\{\left(s', p', \left(t'_j\right)_{j\in I}\right) \in \Omega : t'_i = t_i\right\} \subseteq E$$

We denote by $b_i\left[t_i \middle| \pi_i\left(s'\right), p'\right]$ the interim belief of type t_i for each combination of an observed price $p' \in R^{n-1}_{++}$ and a private signal about the state of nature $s' \in S$ in a state of the world $\left(s', p', \left(t'_j\right)_{j\in I}\right) \in \Omega$ with $t'_i = t_i$.

A model $\mathcal{M}$ of the economy is a collection $\mathcal{M} = \left(\left(T_i, b_i, z_i\right)_{i\in I}, \Omega\right)$. The model $\mathcal{M}$ satisfies Common Knowledge of Market Clearing if for each state of the world $\left(s, p, \left(t_i\right)_{i\in I}\right) \in \Omega$ the aggregate demand is well defined and equals the aggregate supply, which is well defined as well:

$$\int_I z_i\left[t_i\right]\left(\pi_i\left(s\right), p\right)\mathrm{d}i = \int_I e_i\left(\pi_i\left(s\right)\right)\mathrm{d}i$$

Indeed, in a model $\mathcal{M}$ with this property not only does the market clear in each state of the world, but also each agent is certain (i.e. assigns probability 1 to the event) that the markets clear, each agent is certain that all agents are certain that the markets clear, and so on *ad infinitum*.

Similarly, the model $\mathcal{M}$ satisfies Common Knowledge of Rationality if for each state of the world $\left(s, p, \left(t_i\right)_{i\in I}\right) \in \Omega$ the bundle consumed by each agent maximizes the agent's expected utility, i.e.

$$z_i\left[t_i\right]\left(\pi_i\left(s\right), p\right) \in \arg \max_{x_i \cdot p \leq e_i\left(\pi_i\left(s\right)\right)\cdot p} \int_\Omega u_i\left(x_i, s\right)\mathrm{d}b_i\left[t_i \left|\pi_i\left(s\right), p\right.\right]$$

[11]

When both properties are satisfied we say that $\mathcal{M}$ satisfies Common Knowledge of Rationality and Market Clearing.

Definition 1 *The price vector p is consistent with common knowledge of rationality and market clearing (CKRMC) at the state of nature $s \in S$, if there exists a model $\mathcal{M}$ satisfying common knowledge of rationality and market clearing which contains a state of the world $\left(s, p, \left(t'_j\right)_{j\in I}\right) \in \Omega$. In such a case we also say that (p, s) is a CKRMC outcome.*

Notice that this definition is *epistemic*, in the sense that it relies on the existence of a *type space* Ω with particular properties, and not in terms of properties of the basic

[11] Clearly, what is relevant for the maximization problem is the marginal belief on S. So we could equivalently write
$z_i\left[t_i\right]\left(\pi_i\left(s'\right), p'\right) \in \arg\max_{x_i \cdot p \leq e_i\left(\pi_i\left(s\right)\right)\cdot p} \sum_{s\in S} u_i(x_i, s)\mathrm{marg}_S b_i\left[t_i \left|\pi_i\left(s'\right), p\right.\right](s)$

asymmetric-information economy $\left(I, S, (\pi_i(\cdot), e_i(\pi_i(\cdot), u_i(\cdot,\cdot)))_{i\in I}\right)$. In section 3 our aim will be to provide a characterization of Rationalizable Expectations prices in terms of properties of the basic economy.

It is useful to see how the concepts of REE and Belief Equilibrium of Dutta and Morris (1997) can be represented in our framework:

1. An REE f, $f : S \to R^{K-1}$, can be represented by a model where each agent i has a single type $\widehat{t_i}$, $\Omega = \left\{(s, f(s), (\widehat{t_i})_{i\in I}) \,|s \in S\right\}$ and $b_i\left[\widehat{t_i}\right](s, f(s), (\widehat{t_i})_{i\in I}) = \alpha_i(s)$.

2. A Belief Equilibrium (Dutta and Morris 1997) is defined by a profile of functions $(\delta_i)_{i\in I}$, $\delta_i : S \to \triangle(R^{K-1})$, where $\delta_i(s)$ is the conditional probability distribution on *prices* given s of agent i. It is assumed that for $s \in S$ the distributions $(\delta_i)_{i\in I}$ have a common support. The profile $(\delta_i)_{i\in I}$ is a Belief Equilibrium if for every $s \in S$ and $p \in R^{K-1}$ such that $\delta_i(s)(p) > 0$ for every $i \in I$, the price p clears the market at the state s (when each agent i chooses an optimal bundle w.r.t. his conditional probability on S given his private signal and the price p.) Thus, a Belief Equilibrium is a more permissive solution concept than REE because it allows different agents to have different beliefs on the relationship between prices and states. On the other hand Belief Equilibrium is more restrictive than $CKRMC$ because it (implicitly) assumes that the beliefs of each agent are common knowledge. In terms of our framework a Belief Equilibrium corresponds to a model where each agent has just one type. To see that we now define a model that corresponds to a Belief Equilibrium, $(\delta_i)_{i\in I}$. Let $Q(s) = \{p\,|\delta_i(s)(p) > 0, \forall i \in I\}$. Define now $T_i = \left\{\widehat{t_i}\right\}$,

$\Omega = \left\{(s, p, (\widehat{t_i})_{i\in I}) \,|p \in Q(s)\right\}$ and $b_i\left[\widehat{t_i}\right](s, p, (\widehat{t_i})_{i\in I}) = \alpha_i(s) \cdot \delta_i(s)(p)$.

Thus, the definition of $CKRMC$ is more general than the definition of Belief Equilibrium because it allows different types of a given agent to have different beliefs over types of the other agents. In particular agent i is uncertain about the beliefs of the other agents. This is in line with our motivation to define the most permissive model that is consistent with common knowledge of rationality and market clearing. Our approach is consistent with the concept of Rationalizability that is defined in the game theoretic literature (Bernheim 1984 and Pearce 1984) . In particular, Rationalizability does not assume that the beliefs of each player are known to the other players.

2.1 A formalization of example 1

We now describe example 1 in terms of our epistemic model.

The set of agents who are informed about the state of nature is $I_1 = [0, \delta]$, where $\delta = \frac{1}{6}$. The demand of an agent who knows that state of nature is determined by the price and is not affected by his beliefs on the behavior of other agents. Thus, we can omit the definition of these beliefs from the description of our formal model and focus only on the beliefs and demands of the uninformed agents.

The set of uninformed agents is $I_2 = (\delta, 1]$. The agents in I_2 are divided into 10 cohorts $(\delta, \delta + 0.1\,(1-\delta)], (\delta + 0.1\,(1-\delta)\,, \delta + 0.2\,(1-\delta)], \ldots, (\delta + 0.9\,(1-\delta)\,, 1]$. In each state of the world the types of all the agents within a given cohort are identical and completely correlated, so we can effectively speak of 10 (representative) agents $j = 1, \ldots, 10$, where each agent $r \in (\delta, 1]$ is represented by a representative agent from her cohort.

Each agent $j = 1, \ldots, 10$ has two possible types

$$t_j^A, t_j^B$$

Before we proceed with the formal definitions we make some general comments on the structure of the model. The beliefs of type t_j^A and t_j^B correspond respectively to theories A and B that were described in the introduction. That is, type t_j^A assigns probability $\frac{3}{4}$ to the event that the profile of types induces the price function f and a probability of $\frac{1}{4}$ to the event that the profile of types induces the function g; type t_j^B assigns probability $\frac{3}{4}$ to the event that the profile of types induces the price function g and a probability of $\frac{1}{4}$ to the event that the profile of types induces the function f. In addition each type $t_j^x, x \in \{A, B\}$,of each player assigns a probability $\frac{1}{2}$ to each one of the two states of nature, and the beliefs of t_j^x on the state of nature are independent of his beliefs over the possible profile of types. In particular,

at the price $p = 2$,

- the (conditional) belief of type t_j^A assigns marginal probability $\frac{3}{4}$ to the state of nature $s = 1$ and marginal probability $\frac{1}{4}$ to the state of nature $s = 3$;
- the (conditional) belief of type t_j^B assigns marginal probability $\frac{3}{4}$ to the state of nature $s = 3$ and marginal probability $\frac{1}{4}$ to the state of nature $s = 1$.

In contrast, at the price $p = s$ (for each of the two states of nature $s = 1, 3$) each type is certain that the state of the world is s.

The demand $z_j\left[t_j^x\right](p)$[12] of type t_j^x maximizes his expected utility w.r.t. his conditional beliefs. Thus, following the calculation in the introduction we define the demand of each type for the commodity X at the prices $p = 1$ and $p = 3$ to be one unit. The demand of a type t_j^A for X at the price $p = 2$ is $\frac{3}{4}$ while type t_j^B demands $\frac{5}{4}$ units of X.

We now explain how to embed these types in a model which satisfies common knowledge of rationality and market clearing.

In the model there are altogether 20 type profiles denoted $\bar{t}^{k,f}, \bar{t}^{k,g}$ for $k = 1, \ldots, 10$. (As will soon become clear, the profile $\bar{t}^{k,f}$ ($\bar{t}^{k,g}$) induces the price function $f(g)$.) Each

[12] Since the agnets have no information about the state of nature we omit reference to the informational signal in the definition of the demand function.

type profile contains one type for each agent $j = 1, \ldots, 10$, in a way that will be specified below. In the state of nature $s = 1$ the possible prices are $p = 1, 2$; in the state of nature $s = 3$ the possible prices are $p = 2, 3$. The set of states of the world in the model is

$$\Omega = \left\{ \left(1, 2, \bar{t}^{k,f}\right), \left(1, 1, \bar{t}^{k,g}\right), \left(3, 2, \bar{t}^{k,g}\right), \left(3, 3, \bar{t}^{k,f}\right) \right\}_{k=1,\ldots,10}$$

Denote by $\oplus$ and $\ominus$ addition and subtraction modulo 10, respectively.

For each agent j the type t_j^A is a member of 10 out of the 20 type profiles, and the other type t_j^B of agent j is a member of the remaining 10 profiles:

$$\begin{array}{l} t_j^A \text{ is in the type profile } \bar{t}^{k,f} \text{ iff } 0 \le |k \ominus j| \le 1 \\ t_j^A \text{ is in the type profile } \bar{t}^{k,g} \text{ iff } 2 \le |k \ominus j| \le 8 \end{array}$$

and similarly

$$\begin{array}{l} t_j^B \text{ is in the type profile } \bar{t}^{k,g} \text{ iff } 0 \le |k \ominus j| \le 1 \\ t_j^B \text{ is in the type profile } \bar{t}^{k,f} \text{ iff } 2 \le |k \ominus j| \le 8 \end{array}$$

In other words, for $k = 1, \ldots 10$

$$\begin{aligned} \bar{t}^{k,f} &= \left\{t_{k\ominus1}^A, t_k^A, t_{k\oplus1}^A, t_{k\oplus2}^B, \ldots, t_{k\oplus8}^B\right\} \\ \bar{t}^{k,g} &= \left\{t_{k\ominus1}^B, t_k^B, t_{k\oplus1}^B, t_{k\oplus2}^A, \ldots, t_{k\oplus8}^A\right\} \end{aligned}$$

Thus, in a profile $\bar{t}^{k,f}$($\bar{t}^{k,g}$) 30% of the population have beliefs that correspond to theory $A(B)$ and the other 70% have beliefs that correspond to theory $B(A)$. As we have seen in the introduction the profile of demands that corresponds to a profile $\bar{t}^{k,f}$ induces the function f and the profile of demands that correspond to a profile $\bar{t}^{k,g}$ induces the function g.

We turn now to the definition of the beliefs. Recall that we want a type $t_j^A(t_j^B)$ to: (1) assign a probability $\frac{3}{4}$ to the event that the profile of types induces the function $f(g)$, (2) to assign a probability $\frac{1}{2}$ to each state of nature and, (3) the two beliefs (1) and (2) are to be independent of one another. Consider a type t_j^A; a simple way of defining his beliefs so that he assigns a probability $\frac{3}{4}$ to profiles of types that induce f and a probability $\frac{1}{4}$ to profiles of types that induce g is to have him assign to each one of the three profiles that induce f to which he belongs a probability of $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$ and similarly to have him assign to each one of the seven profiles that induce g to which he belongs a probability $\frac{1}{4} \times \frac{1}{7} = \frac{1}{28}$. The beliefs of a type t_j^B will be defined in a similar way. With this in mind we now define the beliefs as follows:

$$
\begin{aligned}
b_j\left[t_j^A\right]\left(1,2,\bar{t}^{k,f}\right) &= b_j\left[t_j^A\right]\left(3,3,\bar{t}^{k,f}\right)=\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}=\frac{1}{8} \quad \text{for} \quad 0\leq|k\ominus j|\leq 1 \\
b_j\left[t_j^A\right]\left(3,2,\bar{t}^{k,g}\right) &= b_j\left[t_j^A\right]\left(1,1,\bar{t}^{k,g}\right)=\frac{1}{2}\times\frac{1}{4}\times\frac{1}{7}=\frac{1}{56} \quad \text{for} \quad 2\leq|k\ominus j|\leq 8
\end{aligned}
$$

and

$$
\begin{aligned}
b_j\left[t_j^B\right]\left(3,2,\bar{t}^{k,g}\right) &= b_j\left[t_j^B\right]\left(1,1,\bar{t}^{k,g}\right)=\frac{1}{2}\times\frac{3}{4}\times\frac{1}{3}=\frac{1}{8} \quad \text{for} \quad 0\leq|k\ominus j|\leq 1 \\
b_j\left[t_j^B\right]\left(1,2,\bar{t}^{k,f}\right) &= b_j\left[t_j^B\right]\left(3,3,\bar{t}^{k,f}\right)=\frac{1}{2}\times\frac{1}{4}\times\frac{1}{7}=\frac{1}{56} \quad \text{for} \quad 2\leq|k\ominus j|\leq 8
\end{aligned}
$$

This completes the definition of the beliefs of the agents' types and hence the definition of Ω. In every state of the world $\omega\in\Omega$ markets clear and each agent is choosing a bundle which is optimal w.r.t. its conditional beliefs. It follows that the model $\mathcal{M}$ satisfies common knowledge of rationality and market clearing and therefore the price 2 is a $CKRMC$ price in both states of nature.

3 A Characterization.

In this section we provide a characterization of $CKRMC$ outcomes in terms of properties of the basic economy. Specifically we present the concept of $Ex-Post\ Rationalizability$ [13] and then show (theorem 1) that every $CKRMC$ outcome is an Ex-Post Rationalizable outcome and that under a mild qualification the opposite implication is also true. We then use this characterization to compute the set of $CKRMC$ outcomes in example 1.

Definition: A price p is $Ex-Post\ Rationalizable$ w.r.t to a set of states $\widehat{S}\subseteq S$ if for every $s\in\widehat{S}$ there exists a profile of probabilities $\{\gamma_i^s\}_{i\in I}$ on $\widehat{S}$, $\gamma_i^s\in\triangle(\widehat{S}\cap\pi_i(s))$, and a profile of demands $\{x_i^s\}_{i\in I}$, $x_i^s\in R^K$, such that:

1. For every $i\in I$ x_i^s is an optimal bundle at the price p w.r.t γ_i^s.
2. Markets clear, that is, $\int_i x_i^s \mathrm{d}i=\int_i e_i^s \mathrm{d}i$.

We will say that the price p can be supported in the state s by the beliefs $\gamma^s=\{\gamma_i^s\}_{i\in I}$ on $\widehat{S}$ if there exists a profile of demands $x^s=\{x_i^s\}_{i\in I}$ such that the conditions 1. and 2. above are satisfied.

[13] As we have pointed out in the introduction the concept of Ex-Post Rationalizability was first studied in Degranges (2004).

The idea that underlies the concept of Ex-Post Rationalizability is that if p is Ex-Post Rationalizable w.r.t $\widehat{S}$ then $\widehat{S}$ is a set of states in which p could be a clearing price because for every $s \in \widehat{S}$ there is a profile of beliefs on $\widehat{S}$, $\{\gamma_i^s\}_{i \in I}$, which is consistent with the private information of the players and which rationalizes demands that clear the markets at p. (The belief γ_i^s, in turn, is possible for player i because p can be a clearing price in every $s \in \widehat{S}$.)

Definition: An outcome (p, s) is $Ex-Post\ Rationalizable$ (alternatively, p is $Ex-Post\ Rationalizable$ in s) if there exists a set of states $\widehat{S}$ such that $s \in \widehat{S}$ and p is $Ex-Post\ Rationalizable$ w.r.t $\widehat{S}$.

Let $S(p)$ be the set of all states such that (p, s) is Ex-Post Rationalizable. It is easy to see that p is Ex-Post Rationalizable w.r.t $S(p)$ and that $S(p)$ is the maximal set of states w.r.t which p is Ex-Post Rationalizable.

The concept of Ex-post Rationalizability does not specify a complete description of the beliefs of the agents. In particular it does not specify (as REE and Belief Equilibrium do) the joint probability distribution of an agent on the state of nature and prices. It also does not specify the interactive beliefs, in particular, what agent i believes about the beliefs of other agents. Thus, one cannot tell whether and under what conditions an Ex-Post Rationalizable outcome is an outcome that is consistent with common knowledge of rationality and market clearing. Indeed, as we will see (example 2), there are economies where there exist Ex-Post Rationalizable outcomes that are not $CKRMC$ outcomes. Despite these fundamental differences in the definitions of the two concepts our main result, theorem 1, establishes that under a mild qualification an outcome is a $CKRMC$ outcome iff it is an Ex-Post Rationalizable outcome.

Theorem 1:

a. If (p, s) is a $CKRMC$ outcome then (p, s) is Ex-Post Rationalizable.

b. Let E be an economy in which there is a fully revealing Rational Expectations Equilibrium, $\widetilde{f}$. Let p be a price such that $p \notin \cup_{s \in S} \widetilde{f}(s)$. Then (p, s) is a $CKRMC$ outcome iff (p, s) is Ex-Post Rationalizable.

Remarks:

1. The set of economies in which there exists a fully revealing Rational Expectations Equilibrium is generic. For this set theorem 1 provides a characterization of $CKRMC$ outcomes modulo outcomes which involve prices that are in the range of every fully revealing Rational Expectation Equilibrium.[14]

2. The concept of Ex-Post Rationalizability is defined in a way which is independent of the subjective priors - $(\alpha_i)_{i \in I}-$ on S. Similarly, if a price function f is a fully revealing Rational Expectations Equilibrium for some profile of subjective priors on S it is a fully

[14] There are economies in which the set of fully revealing Rational Expectations Equilibrium is not a singleton.

revealing Rational Expectations Equilibrium for every profile of priors. It follows that under the condition specified in part b of theorem 1 if (p,s) is a $CKRMC$ outcome for some profile of subjective priors with full support on S then it is a $CKRMC$ outcome for *every* such profile of priors. In particular, if (p,s) is a $CKRMC$ outcome for some profile of subjective priors with full support then for every $\alpha \in \triangle(S)$ with full support (p,s) is a $CKRMC$ outcome for the economy where α is a common prior.

3. There can be $CKRMC$ outcomes which do not satisfy the condition formulated in part b. of theorem 1. (See example 3 in the appendix.) We do not have a general necessary and sufficient condition for an Ex-Post Rationalizable outcome to be a $CKRMC$ outcome. It is clear that such a condition would be cumbersome. However, the ideas that underlie the proof of theorem 1 can be used to establish theorem 2, which will be useful for our discussion in section 4.

Theorem 2:

Let E be an economy such that in each state $s \in S$ there are at least $2n$ prices that are Ex-Post Rationalizable. Then an outcome (p,s) is a $CKRMC$ outcome iff it is an Ex-Post Rationalizable outcome.

We now use theorem 1 to compute the set of $CKRMC$ outcomes in example 1. Let P_s , $s=1,3$, denote the set of prices that are Ex-Post Rationalizable in s. We will compute P_s and conclude, using theorem 1, that P_s is also the set of $CKRMC$ prices in the state s. Let $P(\widehat{S})$ denote the set of prices that are Ex-Post Rationalizable w.r.t the set of states $\widehat{S}$, $\widehat{S} \subseteq S$. It follows from the definitions that: $P_s = \cup_{\widehat{S}, s\in\widehat{S}} P(\widehat{S})$. In our example:

(3.1) $\quad P_1 = P(\{1\}) \cup P(\{1,3\})$

(3.2) $\quad P_3 = P(\{3\}) \cup P(\{1,3\})$

$P(\{1\}) = 1$ and $P(\{3\}) = 3$ because 1 and 3 are the prices which clear the markets in the states 1 and 3 respectively when everyone knows the state. We now compute $P(\{1,3\})$. Let $P_s(\{1,3\})$ denote the set of prices that can clear the markets in state $s, s=1,3$, when players in I_2 may have any profile of beliefs on $\{1,3\}$. It follows from the definition of $P(\{1,3\})$ that

(3.3) $\quad P(\{1,3\}) = P_1(\{1,3\}) \cap P_3(\{1,3\})$.

We claim that $P_1(\{1,3\}) = [1, 3-2\delta]$. This follows because the price 1 clears the market when every agent in I_2 assigns probability 1 to the state 1(every agent in I_1 <u>knows</u> that the state is 1.) Clearly, the aggregate demand for X and therefore its price are minimal when everyone assigns the state 1 probability 1. Similarly, the price $3-2\delta$ clears the market when every agent in I_2 assigns probability 1 to the state 3 and therefore the maximal point in $P_1(\{1,3\})$ is $3-2\delta$. It is easy to see that for every $1 \leq p \leq 3-2\delta$ there is a probability $\gamma(p)$ such that if every agent in I_2 assigns probability $\gamma(p)$ to the state 3 then p clears the market. The set $P_3(\{1,3\})$ is computed in a similar way. When each agent in I_2 assigns the state 1 probability 1 the clearing price is $1+2\delta$. When agents in I_2 assign the state 3 probability 1 the clearing price is 3. It follows that $P_3(\{1,3\}) = [1+2\delta, 3]$. From (3.3) we obtain that for $\delta \leq 0.5$ $P(\{1,3\}) = [1+2\delta, 3-2\delta]$. For $\delta > 0.5$

$P(\{1,3\}) = \emptyset$. From (3.1) and (3.2) we have that for $\delta \leq 0.5$ $P_1 = \{1\} \cup [1+2\delta, 3-2\delta]$ and $P_3 = \{3\} \cup [1+2\delta, 3-2\delta]$ and for $\delta > 0.5$ $P_1 = \{1\}$ and $P_3 = \{3\}$. It follows from theorem 1 that the difference between the set P_s and the set of $CKRMC$ prices in s, $s = 1, 3$, is at most the price s. Now, s is the Rational Expectations Equilibrium price in the state s and therefore s is a $CKRMC$ price at s. It follows that the sets P_s, $s = 1, 3$, that we have computed are the sets of $CKRMC$ prices in the respective states.

The solution of the example is interesting in several ways: First, when δ is smaller than 0.5 there is a whole range of prices that are $CKRMC$ prices in both states. Second, the set of $CKRMC$ prices (i.e., P_1 and P_3) depends on δ (the fraction of agents who know the true state) in an intuitive way. As δ increases the set P_s shrinks and when more than 0.5 of the population is informed ($\delta > 0.5$) the only $CKRMC$ price at a state s is the Rational Expectation Equilibrium price. Thus, when $\delta > 0.5$ the assumption of rationality and knowledge of rationality is sufficient to select the Rational Expectations Equilibrium[15] (without assuming a-priori that the price function is known).

Consider now the case where all the agents have the same initial endowment and $\delta < 0.5$. In this case consistency with common knowledge of rationality and market clearing allows for trade despite the fact that it is common knowledge that there are no gains from trade (all the agents have the same utility function and the same initial endowment) and furthermore it is common knowledge that trade benefits agents in I_1 at the expense of some of the agents in I_2. The point is that when agents may have different beliefs and when the fraction of agents who are uninformed is high enough, common knowledge of rationality does not preclude the possibility that each uninformed agent is optimistic and believes that he is making a profit at the expense of other uniformed agents. The result that speculative trade is consistent with common knowledge of rationality and market clearing hinges on the following two properties of $CKRMC$: (1) Different agents may have different beliefs on the set of price functions. (2) Each agent does not know the beliefs of the other agents. Property (2) distinguishes $CKRMC$ from the solution concept that is studied by MacAllister(1990) and Dutta and Morris(1997) and which is based on the assumption that the beliefs of the players are common knowledge. To appreciate the importance of property (2) we note that when $\delta > 0$ it is impossible to obtain trade, even with different beliefs, if these beliefs are common knowledge. The reason for this impossibility is that the beliefs of the uninformed agents determine their demands. So if an agent i in I_2 knows these beliefs he knows the aggregate demand of the uninformed agents. Since the aggregate amount of X is known, agent i can infer the aggregate demand of the informed agents. However, the aggregate demand of the informed agents reveals the state. Thus, if an uninformed agent i knows the beliefs of the other agents and observes the price p he can infer the true state and if everyone infers

[15] Since there are just two states in our example the set of outcomes that is consistent with (just) rationality and knowledge of rationality equals the set of outcomes that are consistent with common knowledge of rationality. In particular, $P_s(\{1,3\})$ is the set of prices that are consistent with rational behavior in state s. When $\delta > 0.5$ $P_1(\{1,3\})$ and $P_3(\{1,3\})$ are disjoint and therefore an agent who knows that all the other agents are behaving rationally can infer the state from the price.

the true state there is no trade. Indeed for every $\delta > 0$ the unique REE is the only belief equilibrium in the models of MacAllister and Dutta and Morris.

We now turn to an example demonstrating two issues: First, the possibility of non-existence of a $CKRMC$ outcome. Second, the possibility of a difference between the set of $CKRMC$ outcomes and the set of Ex-Post Rationalizable outcomes. The example is similar to examples of non-existence of Rational Expectations Equilibrium that were given by Kreps(1977) and Allen(1986). However, the argument which establishes non-existence of a $CKRMC$ outcome is somewhat more involved. In particular, example 3 in the appendix demonstrates that non-existence of a Rational Expectations Equilibrium does not imply non-existence of $CKRMC$ outcomes.

Example 2: The example is a variation on example 1.

There are two states, $S = \{1, 2\}$. The probability of each state is 0.5. The set of agents is $I = [0, 1]$ where agents in $I_1 = [0, \delta]$ know the true state and agent in $I_2 = (\delta, 1]$ don't know it. The utility of an agent in I_1 is $u_1(x, m, s) = a_s \cdot \log(x) + m$. The utility of an agent in I_2 is $u_2(x, m, s) = b_s \cdot \log(x) + m$. The aggregate amount of X is 1 and the number of units of M that each agent has exceeds $\text{Max}\{a_s, b_s : s = 1, 2\}$. All this implies that if p is the price of X in units of M then the demand for X of an agent in I_1 in state s is $\frac{a_s}{p}$ and the demand of an agent $i \in I_2$ who assigns probability $\gamma_i(s)$ to the state s is $\frac{\gamma_i(1)\cdot b_1 + \gamma_i(2)\cdot b_2}{p}$.

We make the following assumptions:

(3.4) $a_1 > a_2$ and $b_1 < b_2$

(3.5) There exists a number $\widehat{p}$ such that

$$a_1 \cdot \delta + b_1(1 - \delta) = a_2 \cdot \delta + b_2(1 - \delta) = \widehat{p}$$

We claim that under these assumptions the set of $CKRMC$ outcomes is empty.

To prove this we compute, first, the set of outcomes that are Ex-Post Rationalizable outcomes. Let $\gamma = \{\gamma_i\}_{i \in I_2}$ be a profile of probabilities on S, (agents in I_1 assign probability 1 to the true state), and let $x_s^p(\gamma)$ denote the aggregate demand for X in the state s at the price p when the profile is γ. Since $b_1 < b_2$ the demand of each agent in I_2 is increasing in the probability which he assigns to the state 2. It follows that for every profile of probabilities γ, the aggregate demand in state $s = 1$ satisfies

(3.6) $x_1^p(\gamma) \geq \frac{a_1 \cdot \delta + b_1(1-\delta)}{p}$

and the aggregate demand in state $s = 2$ satisfies

(3.7) $x_2^p(\gamma) \leq \frac{a_2 \cdot \delta + b_2(1-\delta)}{p}$.

Now we claim that (3.5)-(3.7) and the fact that the aggregate supply of X is 1 imply that the only outcomes that are Ex-Post Rationalizable are $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$. To see that we, first, observe that $\widehat{p}$ is the clearing price in state s when every agent in I_2 assigns the state s probability 1 and therefore $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$ are Ex-Post Rationalizable. Now, assume by contradiction that there exists $p \neq \widehat{p}$ such that $(p, 1)$ is Ex-Post Rationalizable. It follows from (3.6) that if $p < \widehat{p}$ then for every profile of probabilities γ it is the case that $x_1^p(\gamma) > 1$,but this is impossible because the aggregate amount of X is 1. If $p > \widehat{p}$ then (3.7) implies that for every profile γ it is the case that $x_2^p(\gamma) < 1$ which means that p cannot be a clearing price in state 2. It follows that p cannot be Ex-Post Rationalizable w.r.t the set S. Clearly, p cannot be Ex-Post Rationalizable w.r.t $\{1\}$ and therefore we have obtained a contradiction. A similar argument establishes that $\widehat{p}$ is the only price that is Ex-Post Rationalizable in the state 2. It follows from part a. of theorem 1 that the only possible $CKRMC$ outcomes are $(\widehat{p}, 1)$ and $(\widehat{p}, 2)$.

We now show that $(\widehat{p}, 1)$ is not a $CKRMC$ outcome. (The proof that $(\widehat{p}, 2)$ is not a $CKRMC$ outcome is identical.) Assume by contradiction that there exists a model $\mathcal{M}$ that is consistent with common knowledge of rationality and market clearing and a state $\omega \in \Omega$ such that $\omega = (1, \widehat{p}, (t_i)_{i \in I})$. It follows that for almost every[16] $i \in I_2$, $\mathrm{marg}_S b_i\,[t_i]$ assigns probability 1 to the state 1. (To see that recall that the demand of an agent $i \in I_2$ equals $\frac{\gamma_i(1) \cdot b_1 + \gamma_i(2) \cdot b_2}{\widehat{p}} \geq \frac{b_1}{\widehat{p}}$ where $\gamma_i(s)$ is the posterior probability that i assigns to the state s. If $\mathrm{marg}_S b_i\,[t_i]$ assigns a positive probability to state 2 then the demand of agent i for X at $\widehat{p}$ is greater than $\frac{b_1}{\widehat{p}}$. If there is a positive measure of such agents then it follows from (3.6) and (3.5) that the aggregate demand for X is higher than one unit, but that is impossible because the aggregate amount of X is one unit.) Let $\omega' \in \Omega$ be another state of the world such that $\omega' = (2, p, (t'_i)_{i \in I})$. Since $\widehat{p}$ is the only $CKRMC$ price we must have $p = \widehat{p}$ but now again we obtain that for almost every $i \in I_2$ $\mathrm{marg}_S b_i\,[t'_i]$ assigns probability 1 to the state 2. (Otherwise, an argument which is similar to the one we just gave implies that at the state $s = 2$ the aggregate demand for X at the price $\widehat{p}$ is smaller than one unit.) However, this implies that for almost every $i \in I_2$ $\mathrm{marg}_S b_i\,[t_i] \neq \mathrm{marg}_S b_i\,[t'_i]$, but this is impossible because we must have

$$\mathrm{marg}_S b_i\,[t_i] = \mathrm{marg}_S b_i\,[t'_i] = \alpha_i.$$

4 Common Knowledge of Rationality and Market Clearing in economies with two commodities

In this section we characterize the set of Ex-Post Rationalizable outcomes in a class of economies with two commodities, X and M, in which the utility function of each agent is quasi-linear w.r.t M. We then apply theorems 1 and 2 to obtain implications regarding the set of $CKRMC$ outcomes. The class of economies that are studied includes example 1. Here, however, we allow for any finite number of states of nature and any finite number of types of agents where a type is characterized by a utility function and an

[16] That is, for every $i \in I_2$ except possibly for a set of agents of measure zero.

information partition. We provide a characterization of the set of prices, P_s,that are Ex-Post Rationalizable in each state $s \in S$. This characterization is useful in several ways. First, it extends the qualitative results which were obtained for example 1 to this more general class of economies. In particular, the characterization implies that for a robust[17] class of economies there is a whole segment of prices that are Ex-Post Rationalizable in every state and therefore the observation of a price in this segment does not exclude any state. Second, we derive a corollary on the effect of refining the knowledge of agents on the set of Ex-Post Rationalizable prices in a given state. Finally, the characterization result makes it possible to solve the system, i.e., compute P_s for every $s \in S$, by a simple procedure which involves (only) n^2 calculations of Walrasian equilibrium prices in complete information economies, where n is the number of states. We turn now to the formal description.

The commodities are denoted by X and M.

The set of states is $S = \{1, ..., n\}$.

The set of agents is $I = [0, 1]$. There are L types of agents, so $I = \cup_{l=1}^{L} I_l$ and $I_j \cap I_k = \emptyset$ for $j \neq k$. We let λ_l denote the measure of the set I_l. All the agents in I_l, $1 \leq l \leq L$, have the same utility function $\overline{u}_l(x, m, s) = u_l(x, s) + m$, the same initial bundle $e_l = (\overline{x}_l, \overline{m}_l)$, and the same information partition π_l. We make the following assumptions:

(1) The function u_l, as a function of x, is strictly monotonic, strictly concave, and twice continuously differentiable. For every $s \in S$ $\lim_{x \to 0} u_l'(x, s) = \infty$ and $\lim_{x \to \infty} u_l'(x, s) = 0$. Also, $\overline{m}_l > 0$ for every $l = 1, ...L$.

(2) For every $x \in R$, $s, s' \in S$, such that $s > s'$ $u'(x, s) > u'(x, s')$. That is, the marginal utility from X increases in s.

(3) The elements of Π_l are segments of states. That is if $\pi \in \Pi_l$ then there exist $\overline{s}$ and $\underline{s}$, $\overline{s} > \underline{s}$, such that $\pi = \{s : \underline{s} \leq s \leq \overline{s}\}$.

Let $p \in R$ denote the price of a unit of X in terms of units of M.

For every $\widehat{s} \in S$ and $\widehat{S} \subseteq S$ such that $\widehat{s} \in \widehat{S}$ define $P(\widehat{s}, \widehat{S})$ to be the set of all the prices p with the following property: there exists a profile of probabilities $\gamma = (\gamma_i)_{i \in I}$, $\gamma_i \in \triangle(\pi_i(\widehat{s}) \cap \widehat{S})$, such that p is an equilibrium price w.r.t γ. Thus, $P(\widehat{s}, \widehat{S})$ is the set of equilibrium prices that can be generated in $\widehat{s}$ when the support of the probability distribution of an agent i is contained in $\widehat{S}$ and in his information set in the state $\widehat{s}$. Define:

$P(\widehat{S}) \equiv \cap_{\widehat{s} \in \widehat{S}} P(\widehat{s}, \widehat{S})$ and

$P_{\widehat{s}} \equiv \cup_{\widehat{s} \in \widehat{S}} P(\widehat{S})$

Thus, $P(\widehat{S})$ is the set of prices that are Ex-Post Rationalizable w.r.t the set $\widehat{S}$ and $P_{\widehat{s}}$ is the set of prices that are Ex-Post Rationalizable w.r.t the state $\widehat{s}$.

We now characterize the set $P(\widehat{s}, \widehat{S})$.

Define $\overline{p}(\widehat{s}, \widehat{S})$ to be the equilibrium price implied by the profile $\overline{\gamma} = \{\overline{\gamma}_i\}_{i \in I}$ where

[17] A set of economies is robust if it contains an open set. In the appendix we define a metric on the space of economies.

$\overline{\gamma}_i$ assigns probability 1 to the maximal state in the set $\pi_i(\widehat{s}) \cap \widehat{S}$.[18] Similarly, we define $\underline{p}(\widehat{s}, \widehat{S})$ to be the equilibrium price implied by the profile $\underline{\gamma}= \left\{\underline{\gamma}_i\right\}_{i\in I}$ where $\underline{\gamma}_i$ assigns probability 1 to the minimal state in the set $\pi_i(\widehat{s}) \cap \widehat{S}$.

Proposition 1: Let $\widehat{s} \in S$ and $\widehat{S} \subseteq S$ such that $\widehat{s} \in \widehat{S}$ then $P(\widehat{s}, \widehat{S}) = \left[\underline{p}(\widehat{s}, \widehat{S}), \overline{p}(\widehat{s}, \widehat{S})\right]$.

The intuition behind this result is very simple. Since the marginal utility from X increases with s (assumption 2) then for any price p the demand of each agent i for X increases when i assigns a higher probability to a higher state. It follows that for any price p the maximal aggregate demand for X, in the state $\widehat{s}$ when beliefs are restricted to the set $\widehat{S}$, is obtained when each agent i assigns probability 1 to the maximal state in $\pi_i(\widehat{s}) \cap \widehat{S}$. It follows that $\overline{p}(\widehat{s}, \widehat{S})$ is the maximal price in $P(\widehat{s}, \widehat{S})$. Similarly $\underline{p}(\widehat{s}, \widehat{S})$ is the minimal price in $P(\widehat{s}, \widehat{S})$. The result that every price p, $\underline{p}(\widehat{s}, \widehat{S}) \leq p \leq \overline{p}(\widehat{s}, \widehat{S})$, can be obtained as an equilibrium price for some profile of probabilities γ_p follows from the continuity of the equilibrium price in the beliefs. The formal proof of the proposition is given in the appendix.

Let $\underline{s}$,$\overline{s} \in S$, $\underline{s}<\overline{s}$. Define $[\underline{s}, \overline{s}] \equiv \{s : \underline{s} \leq s \leq \overline{s}\}$.

The following proposition is a consequence of proposition 1.

Proposition 2:

(a) Let $\underline{s}$,$\overline{s} \in S$, $\underline{s}<\overline{s}$. Then $P([\underline{s}, \overline{s}]) = \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right]$ [19]

In words, $P([\underline{s}, \overline{s}])$ is the segment of prices where the lowest (highest) price is the lowest (highest) equilibrium price in the maximal (minimal) state in $[\underline{s}, \overline{s}]$ when the support of the probability of agent i is contained in $\pi_i(\overline{s})\cap$ $[\underline{s}, \overline{s}]$ ($\pi_i(\underline{s})\cap$ $[\underline{s}, \overline{s}]$).

(b) $P_s = \cup_{s\in[\underline{s},\overline{s}]} P\left[\underline{s}, \overline{s}\right]$.

The proof of the proposition is in the appendix.

The characterization of Ex-Post Rationalizable outcomes in proposition 2 has several implications for $CKRMC$ outcomes: First, it follows from theorem 1 that for a generic set of economies proposition 2 provides a characterization of $CKRMC$ outcomes modulo outcomes which involve the REE equilibrium prices. Second, it follows from proposition 2 that P_s contains a segment of prices whenever there exist states $\underline{s}$ and $\overline{s}$, $\underline{s}\leq s \leq \overline{s}$, such that $\overline{p}(\underline{s}, [\underline{s}, \overline{s}]) > \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. In the appendix we use this result to prove that the set of economies in which there is a segment of prices that are Ex-Post Rationalizable in every state is robust. Theorem 2 implies that for an economy in this subclass the set of Ex-Post Rationalizable outcomes equals the set of $CKRMC$ outcomes without any qualification.

The characterization that is obtained in proposition 2 extends the qualitative properties of the solution of example 1. First, as we have pointed out, there exists a robust subclass of economies in which there is a whole segment of prices that are Ex-Post Rationalizable in every state. Thus, the observation of a price in this segment does not

[18] The existence and uniquness of $\overline{p}(\widehat{s}, \widehat{S})$ is proved in the sequel.

[19] If $\underline{p}(\overline{s}, [\underline{s}, \overline{s}]) < \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$ then the RHS is the empty set.

exclude any state. Another implication of proposition 2 has to do with the effect of refining the knowledge of agents on the set of Ex-Post Rationalizable prices. To describe this implication we need to introduce some additional notation. Given a subset of agents I' we want to consider the economy $E^{I'}$ which is obtained from the original economy E by refining the knowledge of agents in I' so that each agent in I' has complete information on S. We will denote different terms which refer to $E^{I'}$ by adding a superscript I'. In particular, the set of prices that are Ex-Post Rationalizable in a state s in the economy $E^{I'}$ will be denoted by $P_s^{I'}$.

Proposition 3: Let $s \in S$ be a state such that P_s strictly contains the (singleton) set $P(\{s\})$. For any number $\mu > 0$ there exists a set of agents I' of a measure that is smaller than μ such that $P_s^{I'}$ is strictly contained in P_s. [20]

The proof of the proposition is in the appendix.

Finally, the characterization in proposition 2 implies that it is possible to solve the economy, i.e., compute P_s for every $s \in S$, by a procedure which involves only n^2 calculations of Walrasian equilibrium prices. To see this we note that it follows from the first part of proposition 2 that for every $\underline{s}, \overline{s} \in S$, $\underline{s} < \overline{s}$, the computation of the set $P([\underline{s}, \overline{s}])$ involves two calculations of Walrasian equilibrium prices (i.e., $\underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ and $\overline{p}(\underline{s}, [\underline{s}, \overline{s}])$). In addition for every $s \in S$ we calculate $P(\{s\}) = P([s, s])$, that is, the Walrasian equilibrium price in the complete information economy that is defined by the state s. All this involves $2 \times \frac{n(n-1)}{2} + n = n^2$ calculations of Walrasian equilibrium prices. Now, by the second part of proposition 2 every set P_s is just a union of the sets $P([\underline{s}, \overline{s}])$ for $\underline{s}, \overline{s} \in S$ such that $s \in [\underline{s}, \overline{s}]$.

5 Conclusion

This research was motivated by the following question: What is the set of outcomes that are consistent with common knowledge of rationality and market clearing in an exchange economy with asymmetric information ? To address this question we have defined an epistemic model for the economy. This model provides a complete description not only of the beliefs of each agent on the relationship between states of nature and prices but also of the whole system of interactive beliefs. The main result, theorem 1, provides a characterization of outcomes that are consistent with common of rationality and market clearing in terms of a solution notion - $Ex - Post \;\; Rationalizability$ - that is defined directly in terms of the parameters that define the economy. We then applied theorem 1 to characterize the set of $CKRMC$ outcomes in a general class of economies with two commodities. We have pointed out several properties of $CKRMC$ that stand in contrast to the full revelation property of REE. In particular, propositions 1 and 2 imply that in a robust class of economies, (a) There is a whole range of prices that are $CKRMC$ in every state (and therefore do not reveal the true state). (b) The set of $CKRMC$ outcomes

[20] It is easy to see that weak containement is implied by any refinement of the knowledge of agents.

is sensitive to the amount of information in the economy. (c) Trade is consistent with $CKRMC$ even when there is common knowledge that there are no mutual gains from it.

6 Appendix

6.1 The Proof of Theorem 1

We start with part a.

Let $(\widehat{p}, \widehat{s})$ be a $CKRMC$ outcome. Let $\mathcal{M}$ be a model that satisfies common knowledge of rationality and market clearing and $\widehat{\omega} \in \Omega$ a state of the world such that $\widehat{\omega} = (\widehat{s}, \widehat{p}, (\widehat{t_i})_{i \in I})$. We have to show that $(\widehat{p}, \widehat{s})$ is an Ex-Post Rationalizable outcome. Define

$\widehat{S} = \{s \,|\exists \omega \in \Omega \text{ s.t. } \omega = (s, \widehat{p}, (t_i)_{i \in I}\}$

We claim that $\widehat{p}$ is Ex-Post Rationalizable w.r.t the set $\widehat{S}$. The proof is simple: let $\omega \in \Omega$ be a state of the world such that $\omega = (s, \widehat{p}, (\widehat{t_i})_{i \in I})$. Clearly, for every $i \in I$ the support of the probability distribution $\text{marg}_S b_i \left[t_i \,|\pi_i(s), \widehat{p}\right]$is contained in $\widehat{S}$. Also, since $\mathcal{M}$ satisfies common knowledge of rationality and market clearing the demand $z_i\left[t_i\right](\pi_i(s), \widehat{p})$ of each t_i is optimal w.r.t. $\text{marg}_S b_i \left[t_i \,|\pi_i(s), \widehat{p}\right]$ and the aggregate demand equals the aggregate supply. It follows that the profile of probabilities $\{marg_S b_i \left[t_i \,|\pi_i(s), \widehat{p}\right]\}_{i \in I}$ supports the price $\widehat{p}$ at the state s. Since this holds for every $s \in \widehat{S}$, $\widehat{p}$ is Ex-Post Rationalizable w.r.t. $\widehat{S}$. Since $\widehat{s} \in \widehat{S}$ we obtain that $\widehat{p}$ is an Ex-Post Rationalizable price at $\widehat{s}$.

We turn now to part b.

Let $(\widehat{p}, \widehat{s})$ be an Ex-Post Rationalizable outcome. For $s \in S(\widehat{p})$ we let $\gamma^s = \{\gamma_i^s\}_{i \in I}$ and $x^s = \{x_i^s\}_{i \in I}$ denote respectively the profiles of beliefs and demands that support the price $\widehat{p}$ w.r.t. $S(\widehat{p})$. For $s \in S$ we let p^s and $y^s = (y_i^s)_{i \in I}$ denote respectively the price vector $\widetilde{f}(s)$ and a profile of demands for the agents that constitute a Walrasian equilibrium in the complete information economy where the state s is common knowledge. For $\overline{s} \in S(\widehat{p})$ and $i \in I$ we define a type $t_i^{\overline{s}}$ for agent i which has a demand function that is defined as follows:

$$z_i\left[t_i^{\overline{s}}\right](\pi_i, p) = \begin{cases} x_i^{\overline{s}} & p = \widehat{p} \\ y_i^s & p = p^s \text{ for some } s \in S \end{cases}$$

We note that the profile of demand functions $(z_i\left[t_i^{\overline{s}}\right])_{i \in I}$ induces the price function $f^{\overline{s}}$ defined by:

$$f^{\overline{s}}(s) = \begin{cases} \widehat{p} & s = \overline{s} \\ p^s & s \neq \overline{s} \end{cases}$$

That is, for every $s \in S$ $\quad \int\limits_i z_i\left[t_i^{\overline{s}}\right](\pi_i(s), f^{\overline{s}}(s))di = \int\limits_i e_i(\pi_i(s))di.$

Thus, a state of the world $(\overline{s}, \widehat{p}, (t_i^{\overline{s}})_{i \in I})$ is a state of the world where the state of nature is $\overline{s}$ and the demands of the agents clear the market at the price $\widehat{p}$. We now construct Ω

so that we can assign to each type $t_i^{\overline{s}}$, $\overline{s} \in S(\widehat{p}), i \in I$, beliefs that rationalize his demand function, $z_i\left[t_i^{\overline{s}}\right]$. To do this we need the following definitions and lemma 1.

Define: $F = \left\{f^{\overline{s}}\,|\overline{s} \in S(\widehat{p})\right\}$. Let α be a probability distribution on S and let μ be a probability distribution on F. We let $\alpha \times \mu$ denote the product probability distribution over $S \times F$. For $S' \subseteq S$ and a price p such that the event $(S', p) = \left\{(s,f)\,|s \in S', f \in F, f(s) = p\right\}$ has a positive $\alpha \times \mu$ probability we let $\alpha \times \mu(\cdot\,|S', p)$ denote the conditional of $\alpha \times \mu$ on (S', p). Finally, we remind that α_i denotes the prior probability of agent i on S and that α_i has a full support.

<u>**Lemma 1:**</u>

For every $\overline{s} \in S$ and $i \in I$ there exists a probability distribution on $F, \mu_i^{f^{\overline{s}}}$, such that:

(a) For every $s \in S$ $\mathrm{marg}_S\alpha_i \times \mu_i^{f^{\overline{s}}}(\cdot\,|(\pi_i(s), p^s))$ assigns probability 1 to the state s and $\mathrm{marg}_S\alpha_i \times \mu_i^{f^{\overline{s}}}(\cdot\,|(\pi_i(\overline{s}), \widehat{p})) = \gamma_i^{\overline{s}}$

(b) For every $s \notin \pi_i(\overline{s})$ $\mu_i^{f^{\overline{s}}}(f^s) = 0$.

The proof of lemma 1 is given in the end of the section

Note that property (a) implies that the beliefs $\mu_i^{f^{\overline{s}}}$ rationalize the demand function $z_i\left[t_i^{\overline{s}}\right]$. That is, for every event $(\pi_i(s), p)$ that has a positive $\alpha_i \times \mu_i^{f^{\overline{s}}}$ probability the bundle $z_i\left[t_i^{\overline{s}}\right](\pi_i(s), p)$ is optimal w.r.t. $\mathrm{marg}_S\alpha_i \times \mu_i^f(\cdot\,|(\pi_i, p)$. We now construct Ω so that we can assign to each type $t_i^{\overline{s}}$ beliefs that are induced by the beliefs $\alpha_i \times \mu_i^{f^{\overline{s}}}$ that satisfy the properties defined in lemma 1. These beliefs rationalize his demand - $z_i\left[t_i^{\overline{s}}\right]$.

For $\overline{s} \in S(\widehat{p})$ and $s' \in \pi_i(\overline{s})$ we let $(t_{-i}^{s'}, t_i^{\overline{s}})$ denote the profile of types where the type of agent i is $t_i^{\overline{s}}$ and the type of an agent $j \neq i$ is $t_j^{s'}$. We note that the profile $(t_{-i}^{s'}, t_i^{\overline{s}})$ induces the demand function $f^{s'}$. To see that observe that:

(a) Since there is a continuum of agents the aggregate demand in the economy is not effected by the demand of the single agent i.

(b) The profile of demand functions $\left(z_j\left[t_j^{s'}\right]\right)_{j \neq i}$ clears the markets at the prices specified by $f^{s'}$.

With this in mind we turn to the definition of Ω. Define,

$$\Omega_1 = \left\{(s', \widehat{p}, (t_{-i}^{s'}, t_i^{\overline{s}})\,|\overline{s}, s' \in S(\widehat{p}), s' \in \pi_i(\overline{s}), i \in I\right\}$$
$$\Omega_2 = \left\{(s, p^s, (t_{-i}^{s'}, t_i^{\overline{s}})\,|\overline{s}, s' \in S(\widehat{p}), s' \in \pi_i(\overline{s}), s \neq s', i \in I\right\}$$

Now, define

$$\Omega = \Omega_1 \cup \Omega_2.$$

We note that Ω_1 is a set of states of the world in which the price is $\widehat{p}$ while Ω_2 is a set of states of the world in which the price fully reveals the state of nature.

The demand function of type $t_i^{\overline{s}}-$ $z_i\left[t_i^{\overline{s}}\right]-$ has already been defined, to define his beliefs define first:

$\Omega_1^{\overline{s},i}=\left\{(s',\widehat{p},(t_{-i}^{s'},t_i^{\overline{s}})\,|s'\in S(\widehat{p})\text{ and }s'\in\pi_i(\overline{s})\right\}$
$\Omega_2^{\overline{s},i}=\left\{(s,p^s,(t_{-i}^{s'},t_i^{\overline{s}})\,|s'\in S(\widehat{p}),s'\in\pi_i(\overline{s})\text{ and }s\neq s'\right\}$ and

$\Omega^{\overline{s},i}=\Omega_1^{\overline{s},i}\cup\Omega_2^{\overline{s},i}$

We note that $\Omega_1^{\overline{s},i}\subseteq\Omega_1$ and $\Omega_2^{\overline{s},i}\subseteq\Omega_2$. Thus, $\Omega_1^{\overline{s},i}$ is a set of states of the world in which the price is $\widehat{p}$ while $\Omega_2^{\overline{s},i}$ is a set of states of the world in which the price fully reveals the state of nature.

Now define

$$b_i\left[t_i^{\overline{s}}\right](s,p,(t_{-i}^{s'},t_i^{\overline{s}}))=\begin{cases}\alpha_i(s)\times\mu_i^{f^{\overline{s}}}(f^{s'}) & \text{if }\ (s,p,(t_{-i}^{s'},t_i^{\overline{s}}))\in\Omega^{\overline{s},i}\\ 0 & \text{Otherwise}\end{cases}$$

To understand this construction we observe that the belief of $t_i^{\overline{s}}$ on S is α_i and his beliefs on the types of the other players is defined by $\mu_i^{f^{\overline{s}}}$. Specifically, $t_i^{\overline{s}}$ assigns probability $\mu_i^{f^{\overline{s}}}(f^{s'})$ to the event that the profile of types of the other players is $t_{-i}^{s'}$. Now, since the profile of types $(t_{-i}^{s'},t_i^{\overline{s}})$ induces the price function $f^{s'}$ the posterior probability of $t_i^{\overline{s}}$ on S at an event $(\pi_i(s),p)$ that is consistent with Ω is $\text{marg}_S\alpha_i\times\mu_i^{f^{\overline{s}}}(\cdot\,|(\pi_i(s),p))$ and therefore the beliefs of $t_i^{\overline{s}}$ rationalize his demand function. We have thus both explained the construction of $\mathcal{M}$ and proved that $\mathcal{M}$ satisfies common knowledge of rationality and market clearing. If follows that $(\widehat{p},\widehat{s})$ is a $CKRMC$ outcome.

We have completed the proof of theorem 1 given lemma 1. The proof of lemma 1 relies on lemma 2.

Lemma 2:

Let $\beta_1,...,\beta_m$ be m positive numbers and let $\gamma=(\gamma_1,...,\gamma_m)$ be a probability vector. There exists a probability vector $\delta=(\delta_1,...,\delta_m)$ which solves the following system of equations:

$$\gamma_k=\frac{\beta_k\cdot\delta_k}{\sum_{j=1}^m\beta_j\cdot\delta_j}\quad k=1,...,m$$

Proof of lemma 2:

First, we assume w.l.o.g that $\gamma_k>0$ for every k because if this is not the case we define $\delta_j=0$ if $\gamma_j=0$ and proceed to prove the lemma for the set $\{k:\gamma_k>0\}$.

Second, multiplying the equations by the denominator and subtracting the RHS from the LHS gives a system of m homogeneous linear equations in $\delta_1,..,\delta_m$ that are linearly dependent (the sum of all the equations is zero). Therefore there exists a solution to this system, $\overline{\delta}=(\overline{\delta_1},....,\overline{\delta_m})$ that is different from zero.

Third, if $\overline{\delta}$ is a solution and c is a constant then $c \cdot \overline{\delta}$ is also a solution.

Finally, since $\gamma_k > 0$ for all $k = 1, ..., m$ then if $\overline{\delta}$ is a solution then $\overline{\delta_1}, ..., \overline{\delta_m}$ all have the same sign which is the sign of the denominator.

It follows from all this that there is a solution $\widehat{\delta}$ to the system that is a probability vector because if $\overline{\delta}$ is some solution there is a constant c such that $c \cdot \overline{\delta}$ is a probability vector.

Proof of lemma 1:

We map lemma 2 to the proof of lemma 1 as follows: Suppose that $S(\widehat{p}) \cap \pi_i(\overline{s})$ is the set $\{1, .., m\}$. For $s \in \{1, .., m\}$ define $\beta_s \equiv \alpha_i(s)$, the prior probability that agent i assigns to the state s, and $\gamma_s \equiv \gamma_i^{\overline{s}}(s)$, the probability of the state s according to $\gamma_i^{\overline{s}}$. We claim that if for $s \in \{1, .., m\}$ we define $\mu_i^{f^{\overline{s}}}(f^s)$ to be δ_s so that the equations in the statement of lemma 2 are satisfied then property (a) is satisfied (property (b) is satisfied as well because the probability of a price function different from $f^1, ..., f^m$ is zero.) To see that (a) is satisfied we observe that $\text{marg}_S \alpha_i \times \mu_i^{f^{\overline{s}}}(\cdot \, |\pi_i(s), p^s)$ assigns probability 1 to the state s[21] and that Bayesian updating implies that for $s = 1, ..., m$

$$\text{marg}_S \alpha_i \times \mu_i^{f^{\overline{s}}}(s \ \ |\pi_i(s), \widehat{p}) = \frac{\alpha_i(s) \cdot \mu_i^{f^{\overline{s}}}(f^s)}{\sum_{s'=1}^{m} \alpha_i(s') \cdot \mu_i^{f^{\overline{s}}}(f^{s'})}$$ [22]

and therefore by lemma 2 $\text{marg}_S \alpha_i \times \ \mu_i^{f^{\overline{s}}}(s \ |\pi_i(s), \widehat{p}) = \gamma_i^{\overline{s}}(s)$.

6.2 Proof of theorem 2

Part (a) of theorem 1 establishes that if (p, s) is a $CKRMC$ outcome then it is also an Ex-post Rationalizable outcome.

Consider now the other direction. We will prove the theorem by constructing a model that is consistent with common knowledge of rationality and market clearing such that for every price function f, $f : S \to R^{K-1}$, that satisfies:

(1) $f(s) \in P_s$ for every $s \in S$ and (2) $f(s) \neq f(s') \ \forall s, s' \in S$ there exists a profile of types t^f which induces the price function f. (We remind that P_s denotes the set of prices that are Ex-Post Rationalizable at the state s.)

First, we need some notation and definitions. For every $s \in S$ and $p \in P_s$ we let $\gamma^{s,p} = (\gamma_i^{s,p})_{i \in I}$ and $x^{s,p} = (x_i^{s,p})_{i \in I}$ denote respectively the profiles of beliefs and demands that support the price p at the state s. For $i \in I$ and $p \in P_s$ we let $y_i^s(p)$ denote an optimal bundle for agent i at the state s (i.e., when i assigns probability 1 to the state s) at the price p. Define now,

[21] Because (s, p^s) is the only element in the set $\left\{(s', f^{s^*}(s')) \, |s' \in S, s^* \in \{1, .., m\}\right\}$ with the price p^s.

[22] The fact that Bayesian updating is given by the equation above relies on the assumption that $\widehat{p} \neq p^s$ for every $s \in S$. This is the only point where this assumption is used.

$F = \{f \,|f(s) \in P_s \text{ and } f(s) \neq f(s') \;\forall s \neq s'\}$

With every $f \in F$ associate a function $f^* \in F$ such that $f(s) \neq f^*(s')\;\; \forall s, s' \in S$. We note that the assumption that $|P_s| \geq 2n \;\forall s \in S$ ensures the existence of such a function f^*. Lemma 3 below plays a key role in the proof of the theorem.

Lemma 3: For every $f \in F$ and $i \in I$ there exists a probability measure μ_i^f on F such that for every $s \in S$:

(a) $m\arg_S \alpha_i \times \mu_i^f\; (\cdot\,|\pi_i(s), f(s)) = \gamma_i^{s,f(s)}$ and

(b) $m\arg_S \alpha_i \times \mu_i^f\; (|\pi_i(s), f^*(s))$ assigns probability 1 to the state s.

Proof of lemma 3: Let $f \in F$. For every tuple $(f, \overline{s}, \widetilde{s}, i)$ s.t. $\widetilde{s} \in \pi_i(\overline{s}) \cap S(f(\overline{s}))$ define a function $(f, \overline{s}, \widetilde{s})$ as follows:

$$(f, \overline{s}, \widetilde{s})(s) = \begin{cases} f(\overline{s}) & s = \widetilde{s} \\ f^*(s) & s \neq \widetilde{s} \end{cases}$$

Define $F_i(f, \overline{s}) = \{(f, \overline{s}, \widetilde{s})\,|\widetilde{s} \in S(f(\overline{s})) \cap \pi_i(\overline{s})\}$

It is easy to see that since $f^*(s) \neq f^*(s')$ for $s \neq s'$ and since for every $s, s' \in S$ $f(s) \neq f^*(s')$ the argument that was presented in the proof of lemma 1 implies that there exists a probability measure $\mu_i^{f,\overline{s}}$on F with a support in $F_i(f, \overline{s})$ such that:

(a) $m\arg_S \alpha_i \times \mu_i^{f,\overline{s}}\; (\cdot\,|\pi_i(\overline{s}), f(\overline{s})) = \gamma_i^{\overline{s},f(\overline{s})}$.

(b) For every $s \in S$ $m\arg_S \alpha_i \times \mu_i^{f,\overline{s}}\; (\cdot\,|\pi_i(s), f^*(s))$ assigns probability 1 to the state s.

Define now,

$F_i(f) = \cup_{s\in S} F_i(f, s)$ and define a probability measure μ_i^f with a support on $F(f)$ by $\mu_i^f = \frac{1}{n} \sum\limits_{s\in S} \mu_i^{f,s}$

Since $f(s) \neq f(s')$ for $s \neq s'$ it is easy to see that (a) and (b) above imply that for every $s \in S$ we have:

$m\arg_S \alpha_i \times \mu_i^f\; (\cdot\,|\pi_i(s), f(s)) = \gamma_i^{s,f(s)}$.

$m\arg_S \alpha_i \times \mu_i^f\; (\cdot\,|\pi_i(s), f^*(s))$ assigns probability 1 to the state s.

Thus, the proof of lemma 3 is complete.

For every $f \in F$ and $i \in I$ we define a type t_i^f for agent i which has the following demand function:

$$(6.1)\;\; z_i\left[t_i^f\right](\pi_i(\overline{s}), p) = \begin{cases} x_i^{\overline{s},p} & p = f(\overline{s}) \\ y_i^{\overline{s}} & p = f^*(\overline{s}) \end{cases}$$

We note that the profile of demand functions $(z_i\left[t_i^f\right])_{i\in I}$ induces the price function f. Define $T_i = \left\{t_i^f\,|f \in F\right\}$. We now construct the model $\mathcal{M}$ so that we can assign to each type t_i^f, $f \in F, i \in I$, beliefs that rationalize his demand. For every $f, g \in F$ and $i \in I$ we let (t_{-i}^g, t_i^f) denote the profile of types where the type of agent i is t_i^f and the type of every agent $j \neq i$ is t_j^g. Since there is a continuum of agents the demand of agent i does not effect the aggregate demand and therefore the profile (t_{-i}^g, t_i^f) induces the price function g. With this in mind we now define the model $\mathcal{M}$ as follows:

$\Omega = \left\{ (s, g(s), (t^g_{-i}, t^f_i)) \,|s \in S, f \in F, g \in F_i(f) \right\}$

The demand function of a type t^f_i, $f \in F, i \in I$, has been defined in (6.1). The beliefs of t^f_i are:

$b_i \left[t^f_i \right] (s, g(s), (t^g_{-i}, t^f_i)) = \alpha_i(s) \times \mu^f_i(g).$

Lemma 3 implies that the beliefs $b_i \left[t^f_i \right]$ of type t^f_i rationalize his demand, $z_i \left[t^f_i \right]$. It follows that $\mathcal{M}$ satisfies common knowledge of rationality and market clearing. We now complete the proof by showing that for every $\overline{s} \in S$ and $\overline{p} \in P_{\overline{s}}$ there exist a state of the world $\omega \in \Omega$ in which the state of nature is $\overline{s}$ and the price is $\overline{p}$. So let $f \in F$ be a price function such that $f(\overline{s}) = \overline{p}$. Define the price function $g \in F$ as follows:

$g(s) = \left\{ \begin{array}{ll} f(s) & s = \overline{s} \\ f^*(s) & s \neq \overline{s} \end{array} \right\}$

$(\overline{s}, \overline{p}, (t^g_{-i}, t^f_i))$ is a state of the world in Ω and hence $(\overline{p}, \overline{s})$ is a $CKRMC$ outcome. The proof of theorem 2 is now complete.

6.3 Example 3

This is an example of an economy with two states in which there is no REE and yet there is a segment of prices that are $CKRMC$ in both states.

There are two states, $S = \{1, 2\}$, and each one of them has a probability 0.5. There are three sets of agents : $I_1 = [0, \delta]$, $I_2 = \left(\delta, \frac{1+\delta}{2}\right]$, $I_3 = \left(\frac{1+\delta}{2}, 1\right]$. Agents in I_1 know the true state. The others don't know it. The utilities of the agents are similar to those defined in the previous examples, so $u_i(x, m, s)$ is $a_s \log(x) + m$ if $i \in I_1$ it is $b_s \log(x) + m$ if $i \in I_2$ and it is $c_s \log(x) + m$ if $i \in I_3$. Also, the aggregate amount of X is 1 and each agent has enough money.

We assume:

(6.2) $a_1 > a_2$; $b_1 < b_2$; $c_1 > c_2$.

(6.3) $\widehat{p} \equiv a_1 \cdot \delta + (b_1 + c_1) \cdot \frac{(1-\delta)}{2} = a_2 \cdot \delta + (b_2 + c_2) \cdot \frac{(1-\delta)}{2}$

The equality in (6.3) implies non-existence of a REE. The argument is familiar: Full revelation would imply that the price which clears the market is $\widehat{p}$ in both states. However, if that is the case then $\widehat{p}$ does not reveal the true state. On the other hand there cannot be a non-revealing REE $f, f(1) = f(2)$ because the demands of agents in I_1 for X in states 1and 2 are different so the same price cannot clear the market in both states.

We now compute the set of prices that are Ex-Post Rationalizable w.r.t S and then use theorem 2 to conclude that this is also the set of prices that are $CKRMC$ in both states. The computation is similar to the one in example 1 in the main text. Let $P_s, s = 1, 2$, be the set of prices that clear the market in state s when agents in $I_2 \cup I_3$ may have any profile of probabilities $\widehat{\gamma} = \{\widehat{\gamma}_i\}_{i \in I_2 \cup I_3}$ on S (agents in I_1 assign , of course, probability 1 to the true state.) We claim that :

$P_1 = \left[a_1 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_1 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

$P_2 = \left[a_2 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_2 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

To see this we note that the extreme points in each set are clearly the lowest and highest prices in the respective states (for example, the demand of agents in $I_2 \cup I_3$ is minimal when agents in I_2 assign probability 1 to the state 1and agents in I_3 assign probability 1to the state 2. When these are the beliefs the clearing prices in states 1 and 2 are the respective minimal points in P_1 and P_2.) Any price p between these points can be obtained as a clearing price by having a fraction $\beta = \beta(p)$ of the agents in I_2 and I_3 assign probability 1 to the states 1 and 2 respectively and a fraction $1 - \beta(p)$ assign probability 1 to the states 2 and 1 respectively. The set of prices that are Ex-Post Rationalizable w.r.t S is:

$P \equiv P_1 \cap P_2 = \left[a_1 \cdot \delta + (b_1 + c_2) \cdot \frac{(1-\delta)}{2}, a_2 \cdot \delta + (b_2 + c_1) \cdot \frac{(1-\delta)}{2}\right]$

It follows from (6.2) and (6.3) that this is a non-empty segment. Theorem 2 implies that P is also the set of prices that are $CKRMC$ in both states.

6.4 Proof of proposition 1

The proof of proposition 1 relies on lemma 4 below. To state the lemma we need the following notation. Let τ be a number $0 \leq \tau \leq 1$. Denote by $\gamma(\tau)$, $\gamma(\tau) = \{\gamma(\tau)_i\}_{i \in I}$, the profile of probabilities where $\gamma(\tau)_i$ assigns probabilities τ and $1 - \tau$ respectively to the maximal and minimal states in the set $\pi_i(\widehat{s}) \cap \widehat{S}$.

Lemma 4**:**

(a) For every $0 \leq \tau \leq 1$ there exists a single equilibrium price w.r.t $\gamma(\tau)$. Denote this price by $p(\tau)$.

(b) $p(\tau)$ is continuous in τ.

(c) Let $\gamma = (\gamma_i)_{i \in I}$, $\gamma_i \in \triangle(\pi_i(\widehat{s}) \cap \widehat{S})$, be a profile of probabilities such that there exists a price p which is an equilibrium price w.r.t γ then $\underline{p}(\widehat{s}, \widehat{S}) \leq p \leq \overline{p}(\widehat{s}, \widehat{S})$.

Proof of lemma 4

Let $i \in I_l$ for some $l \in \{1, ..L\}$, $\gamma \in \triangle(S)$, and $p \in R_+$. We let $x_l(\gamma, p)$ denote the demand of i for X at the price p when the probability distribution of player i on S is γ. (We note that for a given γ and p all the agents in I_l have the same demand hence the notation $x_l(\gamma, p)$.) Our assumptions on the utility function u_l imply that $x_l(\gamma, p)$ is an internal solution and therefore satisfies the first order condition i.e.,

(6.4) $\sum_{s \in S} \gamma(s) \cdot u_l'(x_l(\gamma, p), s) = p$

Let $0 \leq \tau \leq 1$. Recall that $\gamma(\tau)$ is a profile of probabilities $\gamma(\tau) = \{\gamma_i(\tau)\}_{i \in I}$ where for $i \in I_l$ $\gamma_i(\tau)$ is the probability distribution that assigns probabilities τ and 1-τ, respectively, to the maximal and minimal states in $\pi_l(\widehat{s}) \cap \widehat{S}$. Thus, in the profile $\gamma(\tau)$ all the agents of the same type have the same probability distribution on S. Therefore, for every price p there exists an aggregate demand $x(\gamma(\tau), p)$, which equals:

(6.5) $x(\gamma(\tau), p) = \sum_{l=1}^{L} \lambda_l \cdot x_l(\gamma_l(\tau), p)$

where $x_l(\gamma_l(\tau), p)$ is the demand of an agent in I_l. The FOC (6.4) plus the assumption that u_l' has a negative derivative imply that for every agent $i \in I_l$ and every probability $\gamma \in \triangle(\pi_l(\widehat{s}) \cap \widehat{S})$ the demand of i for X, $x_l(\gamma, p)$,is: 1. Continuous in p. 2. Strictly decreasing in p. 3. $\lim_{p \to \infty} x_l(\gamma, p) = 0$ and $\lim_{p \to 0} x_l(\gamma, p) = \infty$. It follows that for every τ the aggregate demand $x(\gamma(\tau), p)$ has properties 1., 2. and 3. as well. These properties imply that for every τ there exits a unique price $p, p = p(\tau)$, such that $x(\gamma(\tau), p) = p$. Thus, part (a) of the lemma is established.

Consider now part (b). We will show that $p(\tau)$ is differentiable. It follows from (6.5) and part (a) that for every $0 \leq \tau \leq 1$ we can write:

(6.6) $\sum_{l=1}^{L} \lambda_l \cdot x_l(\gamma_l(\tau), p(\tau)) - \overline{x} = 0$

where $\overline{x}$ is the aggregate amount of X in the economy. If each function x_l has continuous partial derivatives w.r.t τ and p and if $\sum_{l=1}^{L} \lambda_l \cdot \frac{\partial x_l}{\partial p} \neq 0$ then we can apply the implicit function theorem and obtain that p is differentiable w.r.t τ. Specifically, $\frac{dp}{d\tau} = -\frac{\sum_{l=1}^{L} \lambda_l \cdot \frac{\partial x_l}{\partial \tau}}{\sum_{l=1}^{L} \lambda_l \cdot \frac{\partial x_l}{\partial p}}$

We will now show that $\frac{\partial x_l}{\partial p}$and $\frac{\partial x_l}{\partial \tau}$ indeed exist, are continuous, and $\frac{\partial x_l}{\partial p}$<0.

Equation (6.7) below is the FOC (4.1) of the optimization problem of an agent in I_l w.r.t the probability distribution $\gamma_l(\tau)$ and the price p,

(6.7) $\tau \cdot u_l'(x_l(\gamma_l(\tau), p), \overline{s}_l) + (1-\tau) \cdot u_l'(x_l(\gamma_l(\tau), p), \underline{s}_l) - p = 0$

where $\overline{s}_l$ and $\underline{s}_l$ are the maximal and minimal states in the set $\pi_l(\widehat{s}) \cap \widehat{S}$, respectively. Since u_l' has a continuous derivative and since u_l''<0 we can apply the implicit function theorem w.r.t the variables x_l and τ (holding p fixed) and obtain that:

(6.8) $\frac{\partial x_l(\gamma(\tau), p)}{\partial \tau} = \frac{u_l'(x_l(\gamma_l(\tau), p), \underline{s}_l) - u_l'(x_l(\gamma_l(\tau), p), \overline{s}_l)}{\tau \cdot u_l''(x_l(\gamma_l(\tau), p), \overline{s}_l) + (1-\tau) \cdot u_l''(x_l(\gamma_l(\tau), p), \underline{s}_l)}$

So $\frac{\partial x_l}{\partial \tau}$ exists and since u_l' and u_l'' are continuous it is continuous as well.

Using again the equation (6.7) and applying the implicit function theorem, this time, w.r.t the variables x_l and p (holding τ fixed) we obtain that:

(6.9) $\frac{\partial x_l(\gamma(\tau), p)}{\partial p} = \frac{1}{\tau \cdot u_l''(x_l(\gamma_l(\tau), p), \overline{s}_l) + (1-\tau) \cdot u_l''(x_l(\gamma_l(\tau), p), \underline{s}_l)}$

So $\frac{\partial x_l}{\partial p}$ exists, it is continuous, and since u_l''<0 it is different from zero.

This completes the proof that $p(\tau)$ is differentiable.

We turn now to part (c). Since $u_l'(x, s)$ is increasing in s and decreasing in x it is easy to see that (6.4) implies that $x_l(\gamma_l(1), p) > x_l(\gamma, p)$ for any $\gamma \in \triangle(\pi_l(\widehat{s}) \cap \widehat{S}), \gamma \neq \gamma_l(1)$ and any p. It follows that for any profile of probabilities $\gamma = \{\gamma_i\}_{i \in I}$, $\gamma_i \in \triangle(\pi_i(\widehat{s}) \cap \widehat{S})$, and any price p, if the aggregate demand $x(\gamma, p)$ exists it is smaller or equal to $x(\gamma(1), p)$. Since for every agent $i \in I_l$ the demand $x_l(\gamma, p)$ is strictly decreasing in p an equilibrium price w.r.t the profile $\gamma, p(\gamma)$, cannot be higher than $p(1)$. A similar argument establishes that $p(\gamma)$ cannot be smaller than $p(0)$. Thus, the proof of part (c) is complete.

The proof of proposition 1 from lemma 4 is simple. We have $p(1) = \overline{p}(\widehat{s}, \widehat{S})$ and $p(0) = \underline{p}(\widehat{s}, \widehat{S})$. The continuity of $p(\tau)$ implies that for every p, $p(0) \leq p \leq p(1)$, there exists a τ such that $p = p(\tau)$. Therefore, $P(\widehat{s}, \widehat{S}) \supseteq \left[\underline{p}(\widehat{s}, \widehat{S}), \overline{p}(\widehat{s}, \widehat{S})\right]$. On the other hand part (c) of lemma 4 implies that $P(\widehat{s}, \widehat{S}) \subseteq \left[\underline{p}(\widehat{s}, \widehat{S}), \overline{p}(\widehat{s}, \widehat{S})\right]$.

6.5 Proof of proposition 2

(a) By definition $P([\underline{s}, \overline{s}]) = \cap_{s \in [\underline{s}, \overline{s}]} P(s, [\underline{s}, \overline{s}])$. Therefore, $p \in P([\underline{s}, \overline{s}]) \Rightarrow p \in P(s, [\underline{s}, \overline{s}])$ for every $s \in [\underline{s}, \overline{s}]$. Since by lemma 4 $\underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ is the minimal element in $P(\overline{s}, [\underline{s}, \overline{s}])$ we obtain that $p \geq \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. In a similar way we obtain that $p \leq \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$ and therefore $P([\underline{s}, \overline{s}]) \subseteq \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right]$. For the other direction we observe that since the information sets of each agent are segments (assumption 3) then for every $s \in [\underline{s}, \overline{s}]$ and $i \in I$ the maximal state in the set $\pi_i(s) \cap [\underline{s}, \overline{s}]$ is greater or equal to the maximal state in the set $\pi_i(\underline{s}) \cap [\underline{s}, \overline{s}]$. It follows that for every $s \in S$ $\overline{p}(s, [\underline{s}, \overline{s}]) \geq \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$. Similarly, we obtain that for every $s \in S$ $\underline{p}(s, [\underline{s}, \overline{s}]) \leq \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. It follows that for every $s \in S$ $P(s, [\underline{s}, \overline{s}]) \supseteq \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right]$ and therefore the result follows.

(b) By definition $P_s \equiv \cup_{s \in \widehat{S}} P(\widehat{S})$. Clearly, $\cup_{s \in [\underline{s}, \overline{s}]} P([\underline{s}, \overline{s}]) \subseteq \cup_{s \in \widehat{S}} P(\widehat{S})$ because the union in the RHS is taken over a larger set of sets. We will establish the claim by showing that for every set $\widehat{S}$ $P(\widehat{S}) \subseteq P([\underline{s}, \overline{s}])$ where $\underline{s}$ and $\overline{s}$ are the minimal and maximal states respectively in $\widehat{S}$. To see this recall that $P(\widehat{S}) \equiv \cap_{s \in \widehat{S}} P(s, \widehat{S})$ and therefore, in particular, $P(\widehat{S}) \subseteq P(\overline{s}, \widehat{S})$ and $P(\widehat{S}) \subseteq P(\underline{s}, \widehat{S})$. It follows that $P(\widehat{S}) \subseteq \left[\underline{p}(\overline{s}, \widehat{S}), \overline{p}(\underline{s}, \widehat{S})\right]$. Now since $\widehat{S} \subseteq [\underline{s}, \overline{s}]$ we have $\underline{p}(\overline{s}, \widehat{S}) \geq \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$ and $\overline{p}(\underline{s}, \widehat{S}) \leq \overline{p}(\underline{s}, [\underline{s}, \overline{s}])$. It follows that $P(\widehat{S}) \subseteq \left[\underline{p}(\overline{s}, [\underline{s}, \overline{s}]), \overline{p}(\underline{s}, [\underline{s}, \overline{s}])\right] = P([\underline{s}, \overline{s}])$ where the last equality is established by part (a) of the proposition.

6.6 Proof of proposition 3

Since P_s strictly contains $P(\{s\})$ then either there exists $p < P(\{s\})$ such that $p \in P_s$ or there exists $p > P(\{s\})$ such that $p \in P_s$. Assume w.l.o.g the former case. Let $\underline{p}$ be the minimal element in P_s. Assume, first, that there is a single set $[\underline{s}, \overline{s}]$, $s \in [\underline{s}, \overline{s}]$, such that $\underline{p} \in P([\underline{s}, \overline{s}])$. Since $\underline{p}$ is the minimal element in $P([\underline{s}, \overline{s}])$, it follows from part (a) of proposition 2 that $\underline{p} = \underline{p}(\overline{s}, [\underline{s}, \overline{s}])$. Consider the state $\overline{s}$. Clearly, there exists a set of agents I_l, $l \in \{1, .., L\}$, such that $\pi_l(\overline{s}) \cap [\underline{s}, \overline{s}]$ strictly contains $\overline{s}$, because if that were not the case then $\underline{p}(\overline{s},\ [\underline{s},\overline{s}]) > \overline{p}(\underline{s},\ [\underline{s},\overline{s}])$ implying that $P([\underline{s},\overline{s}]) = \emptyset$ in contradiction to the assumption that $\underline{p} \in P([\underline{s},\overline{s}])$. Let I' be a set of agents of a positive measure in I_l (the measure of I' can be as small as we wish.) Since the minimal state in the set $\pi_l(\overline{s}) \cap [\underline{s},\overline{s}]$, $\widetilde{s}$, is strictly smaller than $\overline{s}$ the demand of agents in I' at $\widetilde{s}$ is smaller than their demand at the state $\overline{s}$. Since the set I' has a positive measure it follows that $\underline{p}(\overline{s},[\underline{s},\overline{s}])$ is strictly smaller than $\underline{p}^{I'}(\overline{s},[\underline{s},\overline{s}])$. (Recall that the superscript I' refers to the economy $E^{I'}$that is obtained form the original economy E by refining the knowledge of agents in I'.) So the point is that $\underline{p}(\overline{s},[\underline{s},\overline{s}])$ is an equilibrium price for a profile of probabilities $\widetilde{\gamma}$ in which agents in I' assign probability 1 to the state $\widetilde{s}$ while $\underline{p}^{I'}(\overline{s},[\underline{s},\overline{s}]$ is an equilibrium price for a profile of probabilities $\overline{\gamma}$ where agents in I' assign probability 1to $\overline{s}$. (Agents that do not belong to I' have the same beliefs in $\overline{\gamma}$ and $\widetilde{\gamma}$.) In particular, $\underline{p} = \underline{p}(\overline{s},[\underline{s},\overline{s}]) \notin P^{I'}([\underline{s},\overline{s}])$. Since by assumption there was no segment of states $[\underline{s}^*,\overline{s}^*]$, $s \in [\underline{s}^*,\overline{s}^*]$, $[\underline{s}^*,\overline{s}^*] \neq [\underline{s},\overline{s}]$, such that $\underline{p} \in P(\ [\underline{s}^*,\overline{s}^*])$ and since for any set $[\underline{s}^*,\overline{s}^*]$, $P^{I'}($ $[\underline{s}^*,\overline{s}^*]) \subseteq P([\underline{s}^*,\overline{s}^*])$[23], it follows that $\underline{p} \notin P_s^{I'}$ and therefore $P_s^{I'}$ is strictly contained in P_s. We have thus proved the proposition for the case where $\underline{p}$ belongs to a single set $P\,[\underline{s},\overline{s}]$. It is easy to see that if $\underline{p}$ belongs to a collection of sets we can find for each such set, $P(\ [\underline{s}^*,\overline{s}^*])$,a set of agents I', $I' = I'(\ [\underline{s}^*,\overline{s}^*])$, such that $\underline{p} \notin P^{I'}([\underline{s}^*,\overline{s}^*])$. In the economy which is obtained by *all* these refinements $\underline{p}$ is not a price that is Ex-Post Rationalizable w.r.t s.

6.7 Robustness of economies in which there is a segment of prices that are Ex-Post Rationalizable in every state

We start with a definition of a metric on the space of economies.

Let $E(n,\mathcal{L},\pi)$ denote the set of economies where $S = \{1,..,n\}$ is the set of states, $\mathcal{L} = \{I_1,..,I_L\}$ is the set of types of agents, and $\pi = (\pi_1,..,\pi_L)$ is the profile of information partitions of the different types. An economy E in $E(n,\mathcal{L},\pi)$ is characterized by a vector $(\lambda_l,\overline{m}_l,\overline{x}_l,u_l)_{l\in\mathcal{L}}$. The distance between two vectors E and E' is defined as the maximal distance among the coordinates of the two vectors where the distance between the different coordinates is defined as follows: The distance between two numbers is the absolute value of their difference. Since the demand function of an agent depends on the derivative of his utility we define the distance between utility functions u and v as follows:

$d(u,v) \equiv \max\{\|u-v\|, \|u'-v'\|, \|u''-v''\|\}$

where $\|\cdot\|$ is the supremum norm, i.e.,

$\|u\| \equiv \sup\left\{u(x,s) : x \in R_+^K, s \in S\right\}.$

[23] Any refeinment of the knowledge of any set of agents can only shrink the set of prices that are Ex-Post Rationalizable w.r.t a given set of states $\widehat{S}$. Therefore, $P^{I'}(\ [\underline{s}^*,\overline{s}^*])$ which is the set of prices that are Ex-Post Rationalizable w.r.t $[\underline{s}^*,\overline{s}^*]$ in the economy $E^{I'}$ is contained, in the weak sense, in $P($ $[\underline{s}^*,\overline{s}^*]$.

We now show that the set of economies in which there is a segment of prices that are Ex-Post Rationalizable in every state is open. Let E be an economy such that $\underline{p}(n,[1,n]) < \overline{p}(1,[1,n])$. To economize in notation we let $\underline{p}$ and $\overline{p}$ denote the prices $\underline{p}(n,[1,n])$ and $\overline{p}(1,[1,n])$ respectively. Proposition 4.2 implies that (in the economy E) the segment $\left[\underline{p},\overline{p}\right]$ is Ex-Post Rationalizable in every state. We will show that a 'small' change in the utilities of the agents induces a 'small' change in the prices $\underline{p}$ and $\overline{p}$, so that if the change in the utilities is sufficiently small the inequality $\underline{p} < \overline{p}$ is maintained. It will then be easy to see that a small change in the other parameters of the economy $(\{\lambda_l, \overline{m}_l, \overline{x}_l, \}_{l\in\mathcal{L}})$ also leads to a small change in $\underline{p}$ and $\overline{p}$. This will establish the existence of a neighborhood of E, $N(E)$, such that for every $E' \in N(E)$ there exists a segment of prices that are Ex-Post Rationalizable in every state.

Consider the price $\overline{p}$ (the argument for the price $\underline{p}$ is similar.) The price $\overline{p}$ is generated by a profile of beliefs $\gamma \equiv \{\gamma_i\}_{i\in I}$ where all players of the same type assign probability 1 to the same state. We fix the profile γ and examine the change in the clearing price when there is a small change in the utilities of the agents. Let u_l, $l \in \mathcal{L}$, denote the utility of an agent of type l in E. First, we note that for every $l \in \mathcal{L}$ and closed positive segment in which the parameter x may vary $\left|u_l^{''}(x,s)\right|$ is bounded from above and below by positive numbers M_l and m_l,respectively. Let $v = (v_1,...,v_L)$ be a vector of utilities such that $d(u_l, v_l) < \epsilon$ for every $l \in \mathcal{L}$. We have $m_l - \epsilon \leq |v_l''| \leq M_l + \epsilon$. Define $C \equiv \max_{l\in\mathcal{L}} \frac{M_l}{m_l-\varepsilon}$and let $\overline{p_v}$ denote the clearing price for the utilities v at the beliefs γ.

<u>Claim:</u> $|\overline{p_v} - \overline{p}| \leq 2C\varepsilon$.

<u>Proof:</u> Let u be a utility, l a type, and p a price. We let $x_l(p,u)$ denote the demand of an agent who has the belief $\gamma_l, \gamma_l \equiv \gamma_i$ for $i \in l$, and the utility u at the price p. We have

(6.10) $|x_l(\overline{p},u_l) - x_l(\overline{p_v},v_l)| \geq |x_l(\overline{p_v},u_l) - x_l(\overline{p},u_l)| - |x_l(\overline{p_v},u_l) - x_l(\overline{p_v},v_l)|$.

(6.11) $|x_l(\overline{p_v},u_l) - x_l(\overline{p},u_l)| \geq \frac{|\overline{p}-\overline{p_v}|}{M_l}$.

(6.12) $|x_l(\overline{p_v},u_l) - x_l(\overline{p_v},v_l)| \leq \frac{\varepsilon}{m_l-\epsilon}$.

The inequality (6.10) is the triangle inequality. Let s_l denote the state to which the belief γ_l assigns probability 1. To see (6.11) we note that the first-order condition implies that $|u_l'(x_l(\overline{p_v},u_l),s_l) - u_l'(x_l(\overline{p},u_l),s_l)| = |\overline{p_v} - \overline{p}|$. Since $\left|u_l^{''}(x,s)\right| \leq M_l$ (6.11) follows. The inequality (6.12) is established in a similar way; the first-order condition implies that $u_l'(x_l(\overline{p_v},u_l),s_l) = v_l'(x_l(\overline{p_v},v),s_l) = \overline{p_\nu}$. Since v_l'' is bounded from below by $m_l - \epsilon$ the fact that $\|u_l' - v_l'\| \leq \varepsilon$ as well implies (6.12). The inequalities (6.10)-(6.12) imply that if $\overline{p_v} - \overline{p} > 2C\varepsilon$ then $x_l(\overline{p},u_l) > x_l(\overline{p_v},v_l)$ for every $l \in \mathcal{L}$ which means that when the vector of utilities is v there is excess supply of X at the price $\overline{p_v}$ and hence it cannot be a clearing price . Similarly, $\overline{p_v}$ cannot be a clearing price if $\overline{p} - \overline{p_v} > 2C\varepsilon$. The claim follows.

We have, thus, shown that a small change in the utilities of the agents induces a small change in the prices $\overline{p}$ and $\underline{p}$. The argument that a small change in the other parameters of the economy also induces a small change in $\overline{p}$ and $\underline{p}$ is simpler. We omit the details.

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