A note on Herbert Gintis' "Emergence of a Price System from Decentralized Bilateral Exchange "

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Abstract

In two recent contributions, Herbert Gintis introduces agent-based imitation models built upon evolutionary bargaining games where agents use private prices as strategies. He reports surprising convergence results for simulations performed in exchange economies where goods are strict complement. We investigate analytically these results using the notion of stochastic stability.

Key Words: Exchange economies, Bargaining Games, Equilibrium Selection, Stochastic Stability.

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1 Introduction

In two recent contributions, Gintis (2006) and Gintis (2007), Herbert Gintis introduces agent-based imitation models (see Dawid (2007)) built upon evolutionary bargaining games where agents use private prices as strategies. He reports surprising convergence results for simulations performed in exchange economies where goods are strict complement. In Gintis (2007), he focuses on the Scarf economy (see Scarf (1960)) and his simulations show convergence to equilibrium, whereas this equilibrium is well-known to be unstable for the tâtonnement mechanism. In Gintis (2006), he studies a variant of the Scarf economy with Leontief preferences and corner endowments. In this setting, the equilibrium price is completely indeterminate. Still, Gintis' simulations show convergence to an unique equilibrium. According to Gintis (2007), these results suggest, first that *"a highly decentralised Walrasian economy, under a wide range of plausible conditions, has a unique, stable steady state in which the economy is reasonably close to Pareto efficient"* , second that *"the stability of a market system depends on the fact that prices are private information"* and finally[1] that *"a major mechanism leading to convergence of economic behaviour is imitation in which poorly performing agents copy the behaviour of better-performing agents."* The aim of this note is to illustrate and reinforce these claims by studying analytically a simplified version of Gintis (2006).

The analysis is performed using the notion of stochastic stability (see the exposition of stochastic stability in Ellisson (2000) or Peyton-Young (1993) and references therein). Stochastic stability methods have previously been put forward in dynamic models of exchanges economies as refinement tools providing foundations for competitive outcomes (see Alos-Ferrer and Kirschteiger (2009), Serrano and Volij (2008) and references therein). We illustrate here their potential as equilibrium selection devices. Indeed, we prove that among a completely indeterminate set of equilibria, stochastic stability selects, as Gintis' simulations do, the equilibrium which requires the trading of the least share of initial endowments. Minimizing the quantities traded seems a fairly appealing equilibrium selection mechanism: it strongly echoes the principle of minimum energy in thermodynamics, it would be compatible with the presence of transaction cost and implies a form of maximum stability towards rationing. This last point will be crucial in the evolutionary selection process. The less trading an equilibrium involves, the less welfare is affected when deviation from the terms of trade by some agents introduce rationing. In evolutionary terms (using the concepts put forward in Ellisson (2000)), the less trading an equilibrium involves, the larger the number of mutations required to leave its

[1] A third claim stating that *"when even a small fraction of agents are assumed to share the same price system and update in a coordinated manner, as suggested by the tâtonnement mechanism, the price system becomes highly volatile."* is not discussed here.

basin of attraction. This fact will play the key role in the proof of the stochastic stability of the least trade equilibrium which is the central result of this note. The result is proved in the economy with Leontief preferences and corner endowments studied in Gintis (2006), but using a more stylized model for the exchange and learning dynamics. As in Gintis (2006), agents base their exchanges on private prices and equilibria are identified with situations where all the agents adopt the same prices. However, we strengthen the role of private prices, taking as a reference point a situation where agents only trade with peers using similar prices. This simplification allows us to propose a not too involve proof, though partly at the expense of generality. Further work could however lead to the construction of a fairly powerful equilibrium selection device.

The paper is organized as follows. In section 2, we present the characteristics of the exchange economy considered in Gintis (2006) and summarize the results Gintis obtains as convergence to the least trade equilibrium. In section 3, we introduce a class of dynamics in this exchange economy based on the sequential composition of trading, imitation and mutation processes for which we prove that the least trade equilibrium is the only stochastically stable state. Section 4 concludes.

2 An exchange economy with strict complementarity

2.1 The framework

Given an n-dimensional vector of positive elements [2] $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n_{++}$, we consider the exchange economy, denoted by $\mathcal{E}(\omega)$, with the following characteristics.

- There are $n \times m$ agents, where m is an arbitrary positive natural number.
- Each agent has the same utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ defined by:

$$u(x_1, \cdots, x_n) = \min(\frac{x_1}{\omega_1}, \cdots, \frac{x_n}{\omega_n}) \tag{1}$$

- For each $i = 1 \cdots n$, there are exactly m agents (hereafter called the agents of type i) which have as initial endowment ω_i units of good i and zero units of every other good. So that the total initial resources in the economy are equal to $m\omega$.

[2] Notations : in the following, we shall denote by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid \forall i \in \{1 \cdots n\}\ x_i \geq 0\}$ the positive orthant of $\mathbb{R}^n$, and by $\mathbb{R}^n_{++}$ its interior.

The demand of an agent of type i at a positive price $p \in \mathbb{R}^n_{++}$ is given by:

$$d_i(p) := \text{argmax}_{p \cdot x \leq p_i \cdot \omega_i} u(x) = \frac{p_i \omega_i}{p \cdot w} \omega \tag{2}$$

and its excess demand by :

$$z_i(p) := d_i(p) - \omega_i e_i = \frac{p_i \omega_i}{p \cdot w} \omega - (0, \cdots, \omega_i, \cdots, 0) \tag{3}$$

One can then notice that the aggregate excess demand $Z(p) = \sum_{i=1}^n m z_i(p)$ vanishes at any positive price.This yields:

Proposition 1 *In the economy $\mathcal{E}(\omega)$, every price $p \in \mathbb{R}^n_{++}$ is an equilibrium price.*

In the following, we denote by Q the space of goods $\mathbb{R}^n_{+}$. Moreover we consider normalized prices in the simplex $S = \{p \in \mathbb{R}^n_{++} \mid p \cdot \omega = n\}$.

2.2 The minimal trading equilibrium

In order to lift part of the indeterminacy on the outcome of exchange in the economy $\mathcal{E}(\omega)$, Gintis considers agents characterized by private prices engaged in three kind of processes: trading, imitation and mutation. In other words, he studies evolutionary dynamics in a bargaining game where agents use private prices as strategies (see Gintis (2006)).

During the trading process, each agent is engaged in a sequence of bilateral trades. Starting with its initial endowment it tries to obtain via exchange a demand it computes according to its private price. The main constraint put forward on bilateral trades is that their value must be non-negative according to the private prices of both contractors. The trading process itself consists in a randomly determined sequence of bilateral trades which yields a reallocation of the resources $m\omega$ among the agents (see Gintis (2006) for details).

The imitation process takes place after a certain number iterations of the trading process. It randomly implements a sequence of agent pairings during which successful agents see their private price copied by less successful ones. The imitation process hence entails an updating of the private prices distribution on the basis of the utility gained during the trading process.

Finally the mutation process takes place after the imitation process. The private prices then mutate (randomly change) with a low probability called the mutation rate. The mutation process hence entails a random perturbation of the price distribution.

In Gintis' numerical experiments, the iteration of these processes entails convergence of the economy to the equilibrium associated with the price

$$\bar{p} = (\frac{1}{\omega_1}, \cdots, \frac{1}{\omega_n}) \tag{4}$$

As a matter of fact, the corresponding equilibrium is the one in which the smallest fractions of initial endowments have to be traded. Namely, one has:

Proposition 2 *The price $\bar{p}$ is the unique minimizer of*[3]*:*

$$\sum_{i=1}^{n} (\|z_i(p)/\omega\|_2)^2 \tag{5}$$

Proof: *We are looking for the minimum over S of*

$$\phi(p) = \sum_{i=1}^{n} (\sum_{j \neq i} (\frac{p \cdot \omega_j}{p \cdot \omega})^2 + (\frac{p \cdot \omega_i}{p \cdot \omega} - 1)^2)$$

$$\phi(p) = (\sum_{i=1}^{n} (\sum_{j=1}^{n} (\frac{p \cdot \omega_j}{p \cdot \omega})^2) + n - 2$$

It is straightforward to see that this quantity is minimized when all the $p \cdot \omega_j$ are equal, that is if and only if the price is $\bar{p}$.

Remark 1 *One should note that in the preceding proposition the quantities traded are measured in normalized units (in shares of the initial resources). Therefore, the characterization given in proposition 2 is independent of the units of measurement (and of the utility representation as well).*

Hence, Gintis' experiments give raise to a fairly appealing equilibrium selection criterium: evolutionnary mechanisms would tend to an equilibrium satisfying a minimum principle, the "least trade" equilibrium. In the following, we aim at providing some analytical foundations for this principle using the notion of stochastic stability (see e.g Ellisson (2000), Peyton-Young (1993)). Indeed we shall exhibit, for a class of markovian models closely related to Gintis' experiments, sufficient conditions to ensure that the least trade equilibrium is the only stochastically stable state.

[3]The symbol $/$ denotes here the division coordinatewise.

3 A Markovian model

As put forward in Gintis (2007), Gintis' experiments can be modeled by a Markov chain of very large dimension. Indeed, let us restrict attention to prices in an arbitrary finite subset of the simplex containing $\bar{p} : P \subset S$. Let us also identify each agent by a pair $(i, j) \in \{1 \cdots n\} \times \{1 \cdots m\}$, where i is the type of the agent and j indexes the agents within a type. We can then represent the state of a system where each agent is endowed with a private price in P by an element of the finite set $P^{n\times m}$ which we shall call a population. Gintis experiments can then be apprehended as Markovian dynamics on populations.

3.1 Trading process

Let us first focus on the trading process. In Gintis (2006), once a sequence of bilateral trades is chosen, the allocation achieved via exchange is a function of the private prices of the agents only. The trading sequence being chosen randomly, the trading process in fact associates to a population of prices $\pi \in P^{n\times m}$ a probability distribution $\mathcal{T}_\pi$ on the set of allocations $Q^{n\times m}$ (for sake of technical simplicity, we shall assume that $Q^{n\times m}$ is endowed with the Borel σ-algebra and that $\mathcal{T}_\pi$ has finite support). Now mechanisms representing the allocation of goods on the basis of private prices should satisfy specific properties. In Gintis (2006) and Gintis (2007), the adoption of a private price by an agent constrains its demand and restricts the trades it accepts to those having positive values according to its price. Here, aiming at a formal analysis rather than at a numerical implementation, we shall be more concerned with the global properties of the trading process than with the inner structure of the bargaining mechanism (in this rerspect our approach is closer to Alos-Ferrer and Kirschteiger (2009) than to Serrano and Volij (2008)). We shall therefore restrict further than in Gintis (2006) or Gintis (2007) the set of allocation mechanisms in order to obtain a simpler aggregate picture. In particular, we shall strengthen the role of private prices, taking as a reference point a situation where agents only trade with peers using the same price. This can be seen as a stylization of Gintis (2006) but also relates to an alternative "market selection" interpretation in the spirit of Alos-Ferrer and Kirschteiger (2009): one could consider there exists different market institutions, each characterized by a prevailing price and interpret the private price of an agent as indicating which market institution he has chosen to perform his exchanges[4].

[4]In such a framework, the evolution of the population of agents could be seen as agents voting with their feet for a trading post.

Let us now translate those considerations into axiomatic requirements on the probabilistic representation of the trading process. First of all an allocation mechanism, as such, should conserve total quantities:

$$\forall \pi \in P^{n\times m}\ \mathcal{T}_\pi\{\xi \in Q^{n\times m} \mid \sum_{i=1}^{n}\sum_{j=1}^{m} \xi_{i,j} = m\omega\} = 1. \tag{I}$$

Second a useful simplification, which can be seen as a proxy for the averaging that would take place if the trading process was iterated several times as in Gintis (2006), is to consider the allocation mechanism is anonymous, that is deliver the same allocation to agents of the same type using the same prices.

$$\begin{gathered}\forall \pi \in P^{n\times m}\ \forall i \in \{1\cdots n\}\forall j,k \in \{1\cdots m\},\\ \pi_{i,j} = \pi_{i,k} \Rightarrow \mathcal{T}_\pi\{\xi \mid \xi_{i,j} = \xi_{i,k}\} = 1.\end{gathered} \tag{II}$$

Now, the main restriction we put forward on the allocation to an agent is that it satisfies the agent's private budget constraint. This condition strictly holds only if trade is restricted to agents using the same price as put forward above. Nevertheless as the numbers of coexisting prices diminish, this becomes a better approximation of the trading process *à la* Gintis as "lucky" trades increasing the value of one's stock should become exceptional. Let us hence posit:

$$\begin{gathered}\forall \pi \in P^{n\times m}\ \forall i \in \{1\cdots n\}\ \forall j \in \{1\cdots m\}\\ \mathcal{T}_\pi\{\xi \mid \pi_{i,j}\cdot(\xi_{i,j} - \omega_i) \leq 0\} = 1.\end{gathered} \tag{III}$$

Let us finally focus on "uniform" populations in which all agents have the same private price. Given proposition 1, it seems a minimal requirement of efficiency for the allocation mechanisms to then deliver the corresponding equilibrium allocation. One could ensure this is indeed the case by assuming some general property of efficiency for the trading process. It is much less restrictive to simply state the property as such. Hence, denoting by $\upsilon(p) \in P^{n\times m}$, the population such that every agent uses $p \in P$ as a private price, we shall assume:

$$\forall i \in \{1\cdots n\}\ \forall j \in \{1\cdots m\}\ \mathcal{T}_{\upsilon(p)}\{\xi \mid \xi_{i,j} = d_i(p)\} = 1 \tag{IV}$$

Remark 2 *Comparing our conditions to Gintis' trading algoritm in Gintis (2006), we remark that condition (I) is satisfied in Gintis (2006)*[5] *; condition (IV) approximates very well the results obtained there ; condition (II) is a less exact approximation but is asymptotically true as the numbers of trading iterations in Gintis*

[5] Gintis also considers the possibility for an agent to produce extra units of goods when it has traded "optimally". This additional process does not seem crucial in Gintis' simulations. As its embedding also fairly complicates the analysis, we do not take it into account here.

(2006) increases ; finally conditions (III) (and (VIII) introduced below) are much more stylized than Gintis algorithm and as pointed out above, closer to a situation where agents only trade with peers using the same price.

3.2 Imitation process

The imitation process randomly associates to a population and an allocation (via utility evaluation), a new population. In other words, it associates to a pair $(\pi, \xi) \in P^{n\times m} \times Q^{n\times m}$, a discrete probability distribution $\mathcal{I}_{(\pi,\xi)}$ on populations.

The dynamics of populations generated by the sequential iteration of trading and imitation are then represented by a Markov transition matrix $\mathcal{F}$ defining a transition probability on the set of populations according to:

$$\mathcal{F}_{(\pi,\pi')} = \int_{Q^{n\times m}} \mathcal{I}_{(\pi,\xi)}(\pi')\, d\mathcal{T}_{\pi}(\xi) \tag{6}$$

In absence of noise, the asymptotic properties of the dynamics induced on populations are usually determined by the clustering properties of the imitation process (see Dawid (2007)). Indeed, let us first point out that imitation mechanisms should not increase the variety of prices in the population. That is, denoting $s(\pi) := \{p \in P \mid \exists (i,j) \in \{1 \cdots n\} \times \{1 \cdots m\}\ \pi_{i,j} = p\}$, let us posit:

$$\forall \pi \in P^{n\times m}\ \forall \xi \in Q^{n\times m}\ \mathcal{I}_{(\pi,\xi)}\{\pi' \mid s(\pi') \subset s(\pi)\} = 1. \tag{V}$$

It should then suffice that there exists a tendency towards imitation in order to "uniformize" the price distribution. Such a tendency is commonly introduced by considering that, typewise, poorly performing agents copy the characteristics of successful ones (see e.g Gintis (2006), Dawid (2007)). In particular the frequency of the characteristics of the most successful agents should increase. If one assumes independent inertia in the imitation process, there should even be a positive probability that the whole population of a given type adopts in one shot the price of the most successful agent. However, the existence of such a leverage effect in the imitation process would tend to minimize the influence of the trading process by giving too much weight to exceptional events (see Dawid (2007)). Such abrupt transitions can be prevented by delaying the imitation process (as in asynchronous learning) or by introducing further constraints, such as an invasion threshold, on incumbent characteristics. In our framework, the fitness of a price is *in fine* determined by the distribution of prices in the whole population. This creates a feedback loop which leads to rapidly changing environmental conditions. The speed of evolution should be somehow proportionate. Therefore, we discard asynchronous learning for independent inertia though we introduce an "invasion threshold" $\overline{m}$ the frequency of a price must reach in order to start growing with some probability. Namely, denoting by $(i', j') \to (i, j)$ the event where agent (i', j') adopts the price of agent (i, j), we shall first assume that for any (π, ξ) the events $(i', j') \to (i, j)$ and $(i'', j'') \to (i, j)$ are independent under the probability $\mathcal{I}_{(\pi,\xi)}$. Second, we shall assume that the frequency of a price can grow within a type only if it is above the invasion threshold. It can then grow at the expense of prices surely yielding a lower utility than the one currently achieved. That is to say, we compare the utility $u(\xi_{i,k})$ of a challenger k (of type i) to the maximal attainable utility $\overline{u}_{i,j}(\pi) = \max_{\{\xi | \mathcal{I}_\pi(\xi)>0\}} u(\xi_{i,j})$ for the incumbent j. Adopting such a criterion for comparing the fitness of prices will ease considerably latter computations by preventing the need to keep track for the whole set of agents of the utility obtained during a stochastic trading process. It can also be interpreted as a form of resistance to change of incumbent agents.

For sake of consistency, we also consider that prices whose frequency is above the invasion threshold can grow at the expense of prices whose frequency is below the invasion threshold. All together, denoting by $\mu_i(\pi, p)$ the number of agents of type i using price p in the population π, we posit [6]:

$$\begin{gathered} \forall \pi \in P^{n\times m}\, \forall \xi \in Q^{n\times m}\, \forall i \in \{1 \cdots n\}\, \forall j,k \in \{1 \cdots m\}, \\ \mathcal{I}_{(\pi,\xi)}\{(i,j) \to (i,k)\} > 0 \\ \Leftrightarrow \\ \mu_i(\pi, \pi_{i,k}) \geq \overline{m} \wedge (u(\xi_{i,k}) \geq \overline{u}_{i,j}(\pi) \vee \mu_i(\pi, \pi_{i,j}) < \overline{m}). \end{gathered} \tag{VI}$$

Finally, agents should not in general copy prices used outside their type. However, a particular case occurs when the distribution of prices is uniform within types but distinct among types. Indeed, trading then becomes impossible. The permanence of such a situation is highly unlikely if agents have the slightest bit of information about other types. We shall hence assume in this case that there is a positive probability that agents of a given type adopt the price used by another one. Namely:

$$\begin{gathered} \forall \pi \in P^{n\times m}\, \forall \xi \in Q^{n\times m}\, \forall i, i' \in \{1 \cdots n\} \\ \mathcal{I}_{(\pi,\xi)}\{(i,j) \to (i',j')\} > 0 \Leftrightarrow \\ \exists p, q \in P\, \forall k \in \{1 \cdots m\}\; \mu(\pi_{i,k}, p) = m \wedge \mu(\pi_{i',k}, q) = m. \end{gathered} \tag{VII}$$

Remark 3 *In order to compare these conditions to the imitation algoritm in Gintis (2006), let us underline that we have opted for independent inertia while Gintis uses a slower learning process in between independent inertia and asynchronous learning. We also have introduced condition (VII) in order to be consistent with our assumptions (III) and (VIII) (see below) on the trading process.*

[6]Condition (VI) is consistent only if $\frac{m}{\overline{m}} >$ card(P) which we shall implicitly assume in the following.

3.3 Asymptotic dynamics without noise

As suggested above, conditions (I) to (VII) suffice to characterize the asymptotic properties of the dynamics generated by the sequential composition of imitation and trading processes. On the one hand, condition (V) ensures that any uniform population (a population in which every agent uses the same private price) is an absorbing state of Markov chains associated with $\mathcal{F}$. On the other hand, condition (VI) ensures that from every population there is a positive probability to reach a population where the frequency of each price is above the invasion threshold. Therefrom, there is a positive probability that the maximal possible utility (pricewise) be realized and that each agent copies the corresponding price following (VI). This would lead to a situation where the prices are uniform in the population typewise. Condition (VII) finally ensures the possibility to then reach an uniform population. In conclusion, from any population, there is a positive probability of transition to an uniform population in a finite number of periods. Also note that this number of periods can be bounded independently of the the initial population. Denoting by $\mathcal{U} = \{\pi \in P^{n\times m} \mid \exists p \in P \; \pi = \upsilon(p)\}$ the set of uniform populations, we therefore have:

Proposition 3 *For any Markov chain $(\Pi^t)^{t\in\mathbb{N}}$ on $P^{n\times m}$ whose transition matrix is $\mathcal{F}$, we have*

$$lim_{t\to+\infty}\mathbb{P}(\Pi^t \in \mathcal{U}) = 1$$

where $\mathbb{P}$ is the probability on $(P^{n\times m})^{\mathbb{N}}$ induced by the law of Π^0 and $\mathcal{F}$.

3.4 Stochastic stability and equilibrium selection

The previous proposition together with condition (IV) allows us to identify the limit set of Markov processes based on $\mathcal{F}$ with the equilibrium set of the economy $\mathcal{E}(\omega)$. Still, as every price is an equilibrium price in the economy $\mathcal{E}(\omega)$, a lot of indeterminacy remains. Contemplating the possibility of errors in the imitation process or more generally studying the sensitivity of equilibria to random perturbations has become relatively standard in the evolutionary game theory literature under the label of stochastic stability (see Fudenberg and Levine (1998), Kandori and al. (1993), Peyton-Young (1993)). Accordingly, Gintis complements his experiments with a mutation process where agents randomly revise their prices. Let us then consider a mutation process (less specific than the one implemented by Gintis) in which, given a mutation rate $\epsilon > 0$, the price of each agent stays identical with probability $1 - \epsilon$ and with probability ϵ is drawn anew uniformly in P. This yields a square probability transition matrix $\mathcal{M}_\epsilon$ on $P^{n\times m}$ whose element $\mathcal{M}(\epsilon)_{(\pi,\pi')}$ corresponds to the probability of reaching the price distribution π' from the price distribution π.

By construction this matrix is strictly positive, its diagonal elements are equal to $(1-\epsilon)^{n\times m}$ while its element (π, π') is a polynomial in ϵ whose non-zero term of least degree is equal to the number of distinct prices between π and π' (we shall in the following denote this degree by $c(\pi, \pi')$). The dynamics of populations triggered by the sequential iteration of trading, imitation and mutation at rate $\epsilon > 0$ are then represented by the Markov transition matrix $\mathcal{F}(\epsilon) = \mathcal{M}(\epsilon) \times \mathcal{F}$. The triple $(P^{n\times m}, \mathcal{F}, \mathcal{F}(\epsilon))$ then defines a model of evolution with noise in the sense of Ellisson (2000):

1. $\mathcal{F}(\epsilon)$ is ergodic for each $\epsilon > 0$,
2. $\mathcal{F}(\epsilon)$ is continuous in ϵ and $\mathcal{F}_0 = \mathcal{F}$,
3. there exists[7] a function $c : P^{n\times m} \times P^{n\times m} \to \mathbb{N}$ such that for all $\pi, \pi' \in P^{n\times m}$, $\lim_{\epsilon\to 0} \frac{\mathcal{F}(\epsilon)_{(\pi,\pi')}}{\epsilon^{c(\pi,\pi')}}$ exists and is strictly positive.

We are then concerned with the asymptotic properties of equilibrium selection of the model of evolution with noise $(P^{n\times m}, \mathcal{F}, \mathcal{F}(\epsilon))$. Now, ergodicity of $\mathcal{F}^{\epsilon}$ implies that there exists an unique probability distribution $\lambda(\epsilon)$ invariant for $\mathcal{F}(\epsilon)$ towards which the law of every Markov chain associated with $\mathcal{F}(\epsilon)$ converges. This implies in particular that whatever its initial state may be, a population following the dynamics specified by $\mathcal{F}(\epsilon)$ has a limit probability of presence in every state given by $\lambda(\epsilon)$. Moreover, one can check (see also Ellisson (2000)) that as ϵ tends towards 0, $\lambda(\epsilon)$ tends towards a probability distribution $\overline{\lambda}$ whose support is included in the set of absorbing states of $\mathcal{F}$, that is the set of uniform populations. As far as the study of equilibrium selection in the economy $\mathcal{E}(\omega)$ is concerned, a positive result would then be that the support of $\overline{\lambda}$ is reduced to a single element which could then be identified as the stochastically stable equilibrium of $\mathcal{E}(\omega)$. Gintis experiments suggest that the least trade equilibrium should play this role.

Now, the characteristics of $(P^{n\times m}, \mathcal{F}, \mathcal{F}(\epsilon))$ in terms of stochastic stability will be informed by a graph whose nodes are the uniform populations and whose edges are weighted by the number of mutations necessary to transit between two such populations with a positive probability (see e.g Peyton-Young (1993)). We have identified uniform populations with equilibrium allocations using condition (IV). In non-uniform populations which shall appear during a transition , some rationing will occur (following in particular condition (III)). The properties of the rationing schemes will condition the efficiency of prices within mixed populations and hence determine the transition probabilities between uniform populations. In this respect,the principle put forward in Gintis (2007) is that agents acting as sellers

[7] This last point follows form the fact that the coefficients of $\mathcal{M}(\epsilon)$ are polynomia in ϵ.

determine "myopically" the trades they accept on the basis of their value rather than taking strategically into account their own demand (i.e *the general rule of agreeing on a trade as long as the value of one's inventory increases* as Gintis puts it in Gintis (2007)). This echoes the condition of no manipulation of initial endowments implicitly present in most of the general equilibrium literature (but the one on endowment games pioneered by Safra, see Safra (1985)). In our framework where the evaluation of goods is thought to be signaled by the choice of a price, this can be translated by assuming that an agent can acquire as much as it can afford of the stock of agents of other types using a similar price. That is to say agents of type i using price p can acquire at most $\dfrac{\mu_j(\pi,p)}{\mu_i(\pi,p)}\omega_j$ units of good j. Together with condition (III), this yields the following bounds on utilities attainable with some probability:

$$\forall \pi \in P^{n\times m}\, \forall p \in P\, \forall i,j \in \{1\cdots n\}:$$
$$\max_{\{\xi|\mathcal{T}_\pi(\xi)>0\}} u(\xi_{i,j}) = \min(u(d_i(\pi_{i,j})), \min_{j=1\cdots n}(\frac{\mu_j(\pi,\pi_{i,j})}{\mu_i(\pi,\pi_{i,j})})) \tag{VIII}$$

In this setting, transitions to equilibrium prices involving little trading are comparatively easier. Indeed, a group of agents promoting such a transition (whose utility at the target equilibrium is higher) has to be followed by a number of peers of other types (whose utility at the target equilibrium is lower and who therefore have to mutate) proportional to its excess demand. Hence the smallest the excess demand at the target, the lowest the number of followers (mutations) needed. In summary, the resistance of a transition is proportional to the quantity of trading required at the target equilibrium. It is therefore harder to leave the least-trade equilibrium for any other equilibrium than to reach it from a given equilibrium. In the language of stochastic stability, the radius of the least trade equilibrium is greater than its coradius. This will imply using Theorem 1 in Ellisson (2000) that the least trade equilibrium is the only stochastically stable equilibrium of the economy $\mathcal{E}(\omega)$ for the dynamics given by $(P^{n\times m}, \mathcal{F}, \mathcal{F}(\epsilon))$. This is the main result of the paper which is proven in our Theorem 1 below.

Let us however first point out that the no strategic rationing condition (VIII) is a crucial one. Indeed, if agents acting as sellers were strategically restraining their trade in function of their demand, independently of the quantity of trading required, a group of agents promoting a transition would have to be followed by an equivalent numbers of peers of the other types in order to fulfill its demand. This would prevent any distinction between equilibria in terms of stochastic stability.

Theorem 1 *For m sufficiently large, the only stochastically stable state of the system $(P^{n\times m}, \mathcal{F}, \mathcal{F}(\epsilon))$ is the uniform state associated to the least-trade equilibrium price, $\upsilon(\bar{p})$.*

Proof: *Let us first mention that the fact that m is sufficiently large is used on the one hand to ensure that m is substantially greater than $\overline{m}$ (e.g m $\geq 3\overline{m}$) and on the other hand to ensure that the inequalities we put forward below extend to integral parts.*

Let us then introduce some auxiliary definitions:

- *A path from $\pi \in P^{n\times m}$ to $\pi' \in P^{n\times m}$ is a finite sequence of states, $\pi^1, \cdots, \pi^K$ such that $\pi^1 = \pi$ and $\pi^K = \pi'$. The set of paths from π to π' is denoted by $S(\pi, \pi')$. The cost of a path $(\pi^1, \cdots, \pi^K)$ is defined as:*

$$c(\pi^1, \cdots, \pi^K) = \sum_{k=1}^{K-1} c(\pi^k, \pi^{k+1}) \tag{7}$$

  One can then remark that $c(\pi, \pi') = 0$ whenever $\mathcal{F}_{\pi,\pi'} > 0$, that is whenever there is a positive probability to reach π' from π via the unperturbed process and that $c(\pi, \pi')$ is bounded above by the number of distinct prices between π and π'.

- *The basin of attraction of an uniform state $\upsilon(p)$ is the set of initial states from which the unperturbed Markov process (based on $\mathcal{F}$) converges to $\upsilon(p)$ with probability one, that is if one denotes by $(\Pi^t)^{t\in\mathbb{N}}$ a generic Markov chain with transition matrix $\mathcal{F}$:*

$$D(p) = \{\pi \in \mathcal{P}^{n\times m} \mid \mathbb{P}_{\Pi^0=\pi}\{(\Pi^t)^{t\in\mathbb{N}} \mid \exists T \in \mathbb{N},\ \Pi^T = \upsilon(p)\} = 1\} \tag{8}$$

  where $\mathbb{P}_{\Pi^0=\pi}$ is the probability distribution induced by $\mathcal{F}$ and the initial condition $\Pi^0 = \pi$.

- *The radius of an uniform state $\upsilon(p)$ is then defined as the minimal cost of a path leaving $D(p)$ (i.e reaching its complement $D(p)^c$):*

$$r(p) = min c(s) \text{ for } s \in \cup_{\pi\in D(p)^c} S(\upsilon(p), \pi) \tag{9}$$

- *Finally, the coradius of an uniform state $\upsilon(p)$ is defined as the maximal cost of a transition to $\upsilon(p)$:*

$$cr(p) = max_{p'\neq p}\, min_{s\in S(\upsilon(p),\upsilon(\bar{p}))} c(s) \tag{10}$$

Now, the application of Theorem 1 *in Ellisson (2000) to our framework yields that if the radius of* $\upsilon(\bar{p})$ *is greater than its coradius, it is the only stochastically stable of* $(P^{n\times m}, \mathcal{F}, \mathcal{F}(\epsilon))$*. We can hence prove our Theorem* 1 *by showing that for all* $p \neq \bar{p}$ *:*

$$r(\bar{p}) > min_{s\in S(\upsilon(p),\upsilon(\bar{p}))} c(s) \tag{11}$$

Let us then first consider an arbitrary price $p \in P - \{\bar{p}\}$ *and determine an upper bound for* $c(s)$ *with* $s \in S(\upsilon(p), \upsilon(\bar{p}))$*. As* $p \neq \bar{p}$*, there exists* $i \in \{1 \cdots n\}$ *such that* $p_i < \dfrac{1}{\omega_i}$*. According to* (2) *the indirect utility of the corresponding type of agent is strictly smaller than* $\dfrac{1}{n}$*:*

$$u(d_i(p)) = \frac{p_i \cdot \omega_i}{n} < \frac{1}{n} \tag{12}$$

Under condition (III), this also is a bound on the utility the corresponding type of agent can reach with some probability via the trading process. Now, let π *be the population such that* $\overline{m}$ *agents of type* i *and* $\overline{m}\dfrac{p_i\omega_i}{n}$ *agents of type* $i' \neq i$ *use the special price* $\bar{p}$ *while all other agents use price* p*. One has:*

$$c(\nu(p), \pi) = \overline{m}(1 + (n-1)\frac{p_i\omega_i}{n}) < \overline{m}(1 + \frac{n-1}{n}) \tag{13}$$

Now, under condition (VIII), one has for the agents of type i *such that* $\pi_{i,j} = \bar{p}$ *:*

$$\mathcal{T}_\pi\{\xi \in Q^{m\times n} \mid u(\xi_{i,j}) = \frac{p_i\omega_i}{n}\} > 0 \tag{14}$$

Hence agents using price $\bar{p}$ *achieve with some probability an utility at least equal to the maximal utility reachable by agents using price* p*. According to (VI) there then is a positive probability to reach via the imitation process a population* π' *where all the agents of type* i *use price* $\bar{p}$ *while agents of other types use price* p*. That is:*

$$\sum_{\{\pi' \mid \mu_i(\pi',\bar{p})=m \wedge \forall i'\neq i\ \mu_i(\pi',p)=m\}} \mathcal{F}_{\pi,\pi'} > 0 \tag{15}$$

Hence, there exists π' *such that* $c(\pi, \pi') = 0$*,* $\mu_i(\pi', \bar{p}) = m$ *and for all* $i' \neq i$ $\mu_i(\pi', p) = m$*. For any such* π'*, it is clear, using (VII), that there exists a path* $s \in S(\pi', \upsilon(\bar{p}))$ *such that* $c(s) = 0$*. One can then conclude that for all* $p \in P - \{\bar{p}\}$ *:*

$$min_{s\in S(\upsilon(p'),\upsilon(p))} c(s) < \overline{m}(1 + \frac{n-1}{n}) \tag{16}$$

To end the proof using (11), *it remains to show that for* m *large enough:*

$$r(\bar{p}) \geq \overline{m}(1 + \frac{n-1}{n}). \tag{17}$$

We shall prove this by contradiction. Let us assume there exists $\pi \notin D(\bar{p})$ *such that:*

$$c(\upsilon(\bar{p}), \pi) < \overline{m}(1 + \frac{n-1}{n}). \tag{18}$$

Now, for m *large enough, one has for all* $i \in \{1 \cdots n\}$:

$$\mu_i(\pi, \bar{p}) \geq \overline{m}. \tag{19}$$

and according to (VIII) for all i, j *such that* $\pi_{i,j} = \bar{p}$:

$$\mathcal{T}_\pi\{\xi \in Q^{m\times n} \mid u(\xi_{i,j}) = u(d_i(\bar{p}))\} > 0 \tag{20}$$

According to (VI), it must then be if $\pi \notin D(\bar{p})$ *that there exists* i_0 *and* p *such that* $\mu_{i_0}(\pi, p) \geq \overline{m}$ *and that for all* j *such that* $\pi_{i_0,j} = p$:

$$\mathcal{T}_\pi\{\xi \in Q^{m\times n} \mid u(\xi_{i_0,j}) \geq u(d_i(\bar{p}))\} > 0 \tag{21}$$

Now, one has for all $i \in \{1 \cdots n\}$ $u(d_i(\bar{p})) = \dfrac{1}{n}$. *Hence* (21) *requires:*

$$\mathcal{T}_\pi\{\xi \in Q^{m\times n} \mid u(\xi_{i_0,j}) \geq \frac{1}{n}\} > 0 \tag{22}$$

On the other hand, (VIII) yields :

$$\max_{\{\xi \mid \mathcal{T}_\pi(\xi)>0\}} u(\xi_{i_0,j}) \leq \min_{j=1\cdots n}(\frac{\mu_j(\pi,p)}{\mu_{i_0}(\pi,p)}) \tag{23}$$

For (22) *and* (23) *to hold simultaneously, one must have for all* i :

$$\frac{\mu_i(\pi,p)}{\mu_{i_0}(\pi,p)} \geq \frac{1}{n}. \tag{24}$$

As $\mu_{i_0}(\pi, p) \geq \overline{m}$, *this implies for all* i :

$$\mu_i(\pi, p) \geq \frac{1}{n}\overline{m}. \tag{25}$$

And Finally:

$$c(\upsilon(\bar{p}), \pi) \geq \overline{m}(1 + \frac{n-1}{n}). \tag{26}$$

This contradicts (18) *and ends the proof.*

4 Concluding remarks

We have hence proved that evolutionnary mechanisms can select an equilibrium in a setting with a lot of indeterminacy, according to a minimization principle reminiscent of this of thermodynamics. However, though we hope that the economic intuition is conserved, in particular that the role of rationing is crucial, the demonstration we give is in some sense minimal. The axioms are chosen as strong as possible in order to obtain the simplest proof. It is clear that further generalizations are needed in order to obtain a truly operative equilibrium selection mechanism.

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