A Log-linear Homotopy Approach to Initialize the Parameterized Expectations Algorithm

Preliminary Version

Javier J. Pérez*
`centrA`, and
Universidad Pablo de Olavide de Sevilla

Abstract

In this paper I present a proposal to obtain appropriate initial conditions when solving general equilibrium rational expectations models with the Parameterized Expectations Algorithm. The proposal is based on a log-linear approximation to the model under study, so that it can be though of as a particular variant of the homotopy approach. The main advantages of the proposal are: i. it guarantees the ergodicity of the initial time series used as an input to the Parameterized Expectations algorithm; ii. it performs well as regards speed of convergence when compared to some homotopy alternatives; iii. it is easy to implement. The claimed advantages are successfully illustrated in the framework of the Cooley and Hansen (1989) model with indivisible labor and money demand motivated via a cash-in-advance constraint, as compared to a procedure based on the standard implementation of homotopy principles.

Keywords: Parameterized Expectations Algorithm, initial conditions, log-linear approximations, homotopy, rational expectations.
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1 Introduction

Modern dynamic macroeconomic theory builds extensively on the construction of dynamic, stochastic, general equilibrium models with rational expectations. The solutions to these type of very often highly nonlinear models tend to be numerical, as exact solutions are usually unobtainable. See for example Marimón and Scott (1999) for a variety of numerical solution methods.

When solving stochastic, dynamic rational expectations models using any numerical algorithm, it is necessary to search for good initial conditions for a certain set of parameters and variables. The initialization takes the form of adequate initial values for the coefficients of the selected approximating functions, in order to initialize a fine-tuning algorithm. This tends to be an important issue for any numerical method, and it is wise to use some information provided by the actual system one aims at solving in order to have an indication on where and how to initialize a given algorithm. In case non-appropriate initial conditions were chosen, the convergence of the algorithm to a rational expectations equilibrium would not be guaranteed, and in case of convergence the computational time needed might have been enormous.

This remark is specially true for the Parameterized Expectations Algorithm (PEA henceforth), a widely used algorithm in economics. The most popular variant in economics is the one introduced in Marcet (1988), and further developed by Marcet and some coauthors, although there are different versions of the algorithm – see Christiano and Fisher (2000) for a discussion on this issue, and Duffy and McNelis (2001) for a neural network approach. Technical expositions can be found in Marcet (1993), Marcet and Marshall (1994), while some practical issues are discussed in, for example, den Haan and Marcet (1990), or Marcet and Lorenzoni (1999). The PEA scheme involves the approximation of the conditional expectation functions in the Euler equations with certain functions, and the use of a numerical optimization method to determine the parameterization of these functions. PEA tends to be a convenient algorithm, especially when there are a large number of state variables and stochastic shocks in key conditional expectations terms.

PEA users tend to recognize the difficulty in finding appropriate initial conditions for both the PEA and the Gauss-Newton-like algorithm used to solve a nonlinear system of equations at each iteration of the PEA method – see for example Jensen (1997). den Haan and Marcet (1990), Marcet and Marshall (1994) and Marcet and Lorenzoni (1999), propose the use of a systematic approach to obtaining the initial point for the PEA, based on homotopy ideas. These ideas are also applied by Jensen (1997) to solving a nonlinear rational expectations model. The approach is very convenient in general, and is supported by an important stream of the literature on numerical solutions – see for example Eaves and Schmedders (1999). Depending on the complexity of the model at hand, this type of methods may be of difficult and/or cumbersome application.

In this paper I propose basing the obtention of appropriate initial conditions for the parameters of the approximating functions (normally polynomials) on a log-linear approximation about the steady state of the system one wants to solve for. In this sense the proposal can be though of as an homotopy-type method. I present the idea in two variants. The first one uses the log-linear time series to directly estimate initial values for the parameters of the approximating functions. The second variant analytically identifies some of the coefficients of the PEA polynomials, and it is close to some standard practice, as in Christiano and Fisher (2000). When used to simulate artificial time series, both approximations produce stationary and ergodic variables, and in this sense are suitable to obtaining an initial set of parameters with a stationary distribution as the coefficients for the PEA approximating functions. Nevertheless, the first variant is more general, and can be used, for example, in cases with multiple expectations, and expectations at horizons larger than one. Both proposed log-linear methods are simple to implement.

In order to highlight the convenience of basing the homotopy on a log-linear version of the model at hand, a comparison with two standard homotopy-based solutions is provided in the paper. The chosen framework is the Cooley and Hansen (1989) model, with indivisible labor and money demand motivated via a cash-in-advance constraint. The proposed log-linear methods outperform in terms of speed of convergence and ease of implementation the standard homotopy-based alternatives.

The rest of the paper is organized as follows. Section 2 starts by describing the PEA method and discussing why it is important to have appropriate initial conditions when solving nonlinear rational expectations problems using PEA. Section 3 exposes the general principles of homotopy theory, while Section 4 presents the proposed ideas for obtaining initial conditions on the basis of a log-linear approximation to the model one wants to solve for. In Section 5 I present and evaluate the performance of the proposed methods as compared to two standard ways of implementing homotopy ideas, in the framework of the Cooley and Hansen (1989) model. Finally, Section 6 concludes.

2 Parameterized Expectations and initial conditions

Assume that a dynamic, stochastic economy can be described as a set of endogenous state variables, $\{s_t\}$, other endogenous variables, $\{v_t\}$, and exogenous variables, $\{z_t\}$, related among them by means of objective functions and constraints. Let an equilibrium solution for this economy be a vector $\{s_t, v_t, z_t\}$ that fulfils the set of first order conditions and constrains of this problem,

$$L^*(E_t[\phi_t(s_{t+1}, v_{t+1})], s_t, v_t, z_t, s_{t-1}, v_{t-1}) = 0 \tag{1}$$

for all t, given the exogenous process $\{z_t\}$ that is assumed to be a Markov process of order one. The functions L^* and ϕ_t are known functions once the structural parameters of the economy are fixed. Alternatively, let the solution be expressed as a law of motion L such that the vector $\{s_t, v_t\}$ generated by

$$\begin{bmatrix} s_t \\ v_t \end{bmatrix} = L(s_{t-1}, z_t) \tag{2}$$

fulfils (1), given that all past information relevant to forecast $\phi_t(s_{t+1},\ v_{t+1})$ could be summarized in a finite-dimension function of $\{s_{t-1}, z_t\}$.

Obtaining a solution to (1) using PEA consists of finding a flexible function $\psi_t(q; s_{t-1}, z_t)$, such that for a positive integer ν, $q \in D_\nu$, where $D_\nu \subset \{\ q \in R^\infty : i\text{-th element of } q \text{ is zero}$

if $i > \nu$ }, the process $\{s_t(q), v_t(q), z_t\}$ satisfies for all t the set of equations

$$L^*(\psi_t(q; s_{t-1}(q), z_t), s_t(q), v_t(q), z_t, s_{t-1}(q), v_{t-1}(q)) = 0 \tag{3}$$

and the order ν is such that when solving $S(q) = argmin_q\, E_t\left[\phi_t(s_{t+1}(q), v_{t+1}(q), z_{t+1}) - \psi_t(q; s_{t-1}(q), z_t)\right]^2$, then $q = S(q)$. Given these conditions, the stochastic process $\{s_t(q), v_t(q)\}$ is the PEA approximated solution.

Under certain regularity conditions over the functions defining the equilibrium (1), the function $L^*(\bullet)$ is invertible in its second and third arguments, and equation (3) can be written as (see Marcet and Marshall (1994))

$$\begin{bmatrix} s_t(q) \\ v_t(q) \end{bmatrix} = L_q(q; s_{t-1}(q), z_t) \tag{4}$$

for stationary and ergodic processes. Marcet and Marshall (1994) show that under those regularity conditions, fulfilled by standard business cycle models, it is always possible to find an approximated function, L_q, arbitrarily close to the true law of motion of the system, L.

The time series $\{s_t(q), v_t(q)\}_{t=1}^{T}$ generated using L_q, and given initial conditions for the state variables, $\{s_0, v_0\}$, and the exogenous processes, are used to obtain inferences on the stochastic properties of the true stationary equilibrium, $\{s_t, v_t\}_{t=-\infty}^{\infty}$. For the approximation to be acceptable, it is necessary that: (i.) If given $\{s_0, v_0\}$ the process $\{s_t, v_t\}_{t=-\infty}^{\infty}$ verifying (1) is stationary, then, given the initial condition $\{s_0, v_0\}$ and an initial vector q, the resulting process $\{s_t(q), v_t(q)\}_{t=-\infty}^{\infty}$ verifying (3) has to be stationary; (ii.) The closer L_q to L, the closer has to be $\{s_t(q), v_t(q)\}_{t=-\infty}^{\infty}$ to $\{s_t, v_t\}_{t=-\infty}^{\infty}$.

If the first point is guaranteed, the second one is guaranteed as an implication. In practical applications of the PEA method, one of the main difficulties – as it is the case with any non-linear numerical solution – lies in finding appropriate initial conditions for the vector q, such that $\{s_t(q), v_t(q)\}_{t=1}^{T}$ be stationary. This, in turn, makes difficult the convergence of the solution method to the fixed point value for q, and as a consequence, the obtention of an approximated law of motion – parameterized law of motion– L_q, arbitrarily close to L.

The structure of the algorithm to solve for such law of motion L_q, and implied vector q of coefficients can be found in Marcet and Lorenzoni (1999), or den Haan and Marcet (1990), and can be sketched out as follows:

1. Compute the first order conditions of the problem.
2. Substitute the conditional expectations, $E_t(\bullet)$, by parameterized polynomial functions $\psi_t(q; s_t, z_t)$, where q is a vector of parameters. In general terms, ψ_t should approximate the expectation arbitrarily well when increasing the order of the polynomial.
3. Choose an initial value for q. Fix the initial conditions s_0, v_0, and z_0. Draw a series $\{z_t\}_{t=0}^{T}$ that obeys the distribution of z in the model, with T sufficiently large.
4. Use the first order conditions of the problem plus the constraints (with the conditional expectations substituted by $\psi_t(q; s_t(q), z_t)$) to generate time series of all variables in the economy: $s_t(q)$, $v_t(q)$. The initial q should be such that $s_t(q)$ and $v_t(q)$ be stationary.

5. Let $S : \Re^m \rightarrow \Re^m$, where m is of the same dimension as q, and

$$S(q) = argmin_q \, E_t \left[\phi_t(s_{t+1}(q), v_{t+1}(q)) - \psi_t(q; s_t(q), z_t) \right]^2 .$$

   Iterate until $q = S(q)$. This guarantees that if agents use ψ_t as the function with which they make up their expectations, the q vector is the best one they could use in the sense that it minimizes the mean squared error between the true expectation and the approximated expectation. In order to find q^{i+1}, starting from a q^i, a nonlinear least-squares regression has to be performed.

6. Update q using the rule

$$q^{i+1} = q^i + \lambda_q S(q^i)$$

   where λ_q controls the size of the updating in each iteration.

7. Generate a new set of time series using q^{i+1}. Repeat the steps until

$$\|q^{i+1} - q^i\| < Tolerance.$$

Once the algorithm has converged to the fixed point, say q^f, the approximated solution is $\{s_t(q^f), v_t(q^f)\}_{t=0}^{\infty}$, generated using $\psi_t(q^f; s_t(q^f), z_t)$. It is worth noticing that I will focus in this paper on PEA applications that can be characterized by Euler equations. Also, I take for granted the invertibility of the function $L^*(\bullet)$ in its second and third arguments, so that the results in Marcet and Marshall (1994) holds.

In order to use the described algorithm successfully, one needs reasonable good initial values for the vector of coefficients q, in such a way that the initial q is not too far away from q^f. As pointed out in Marcet and Lorenzoni (1999), this is necessary for two reasons: i. only local stability is guaranteed under PEA, so it is necessary that the algorithm is properly initialized in order to achieve convergence; ii. it is necessary that the set $\{s_t(q^i), v_t(q^i)\}_{t=0}^{\infty}$ generated at each iteration i has a stationary distribution.

3 The homotopy approach

In order to solve the class of models we are interested in with this paper, the usual approach to initialize the PEA method is the homotopy approach – see for example den Haan and Marcet (1990), Marcet and Marshall (1994), Jensen (1997), or Marcet and Lorenzoni (1999). For a formal exposition of the homotopy approach applied to fixed point problems see Garcia and Zangwill (1981).

The basic idea behind homotopy is to slowly move from a simple case, where the solution is known or easy to compute, to the desired case where the solution is difficult to solve for and typically unknown. As long as the solutions to the intermediate versions of the model are continuous with respect to the parameter/s that drives the model from the known to the desired solution, one would always be solving models with appropriate initial conditions. In this way, one only needs local stability of the algorithm that solves for the fixed point.

As an illustrative example, let us consider the basic neoclassical growth model, with logarithmic utility and full depreciation of the capital stock. The representative agent chooses capital and consumption paths such that maximizes her expected utility $E_0 \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t)$ subject to her resource constraint, that can be expressed as $z_t k_{t-1}^{\alpha} = c_t + k_t - k_{t-1}$, given k_0, and z_0. $E_t(\bullet)$ denotes the conditional expectation operator. c_t time-t consumption t, k_{t-1} is the beginning of period t capital stock, and z_t is a technology shock. Regarding the parameters, $0 < \beta < 1$ is the subjective discount factor, while α is the capital share in production. The first order condition to this problem involving an expectation term is,

$$\frac{1}{\beta} c_t^{-1} = E_t \left[c_{t+1}^{-1} \alpha z_{t+1} k_t^{\alpha-1} \right] \tag{5}$$

The implementation of the PEA scheme to solve the set of first order conditions including the previous one would amount to approximating the previous conditional expectation by means of a flexible function. Let us assume the selected function is a first order polynomial function such as, $\psi_t(q_1, q_2, q_3; k_{t-1}, z_t) = q_1 \exp(q_2 \log(k_{t-1}) + q_3 \log(z_t))$, so that for the approximated PEA solution the Euler condition in (5) would be written as,

$$\frac{1}{\beta} c_t^{-1} = \psi_t(q; k_{t-1}, z_t)$$

For the selected parameter values the neoclassical model turns out to have an analytical solution of the form,

$$\frac{1}{\beta} c_t^{-1} = \frac{1}{\beta(1-\alpha\beta)} k_{t-1}^{-\alpha} z_t^{-1}$$

which is a first order polynomial function. Then, in this case, the proposed PEA solution would be exact, with a set of coefficients q_1, q_2, and q_3 that are easy to obtain by equating the suggested PEA solution and the true solution,

$$q_1 = \frac{1}{\beta(1-\alpha\beta)}, \quad q_2 = -\alpha, \quad q_3 = -1.$$

Starting from this case with zero capital depreciation, say $\delta = 0$, one could proceed, as den Haan and Marcet (1990) to solve for the case with δ close to zero, and so on until the solution obtained using the desired (calibrated) value of delta is achieved. The standard way of implementing the ideas of homotopy is by moving some key parameters, as for example in Jensen (1997), although one could also move from a simple model to a complex one, as it is the case in, for example, den Haan and Marcet (1994). In both cases, and especially in the second one, the number of necessary steps until the solution to the desired version of the model is achieved can be cumbersome.

The proposal in this paper builds upon the ideas of homotopy. In our case, the homotopy would start from the log-linear version of the model. The parameters would be fixed to the desired ones.

4 A log-linear homotopy approach

We can perform a log-linearization of the necessary equations characterizing the equilibrium of the system, as denoted by (1), in order to make the system of equations approximately linear in the log-deviations of the variables from the deterministic steady state of the model. Let us denote this log-linear system as,

$$\hat{L}^*(E_t[\phi(\hat{s}_{t+1}, \hat{v}_{t+1})], \hat{s}_t, \hat{v}_t, z_t, \hat{s}_{t-1}, \hat{v}_{t-1}) = 0 \tag{6}$$

where the hat symbol on a given variable, say $\hat{s}$, denotes the log-linearized counterpart of the same given variable in the model, s, and $\hat{L}^*(\bullet)$ is a log-linear function approximating $L^*(\bullet)$.

The stable solution to the linear approximation of the system in (6) can be obtained by solving for the desired recursive equilibrium law of motion. This can be done, for example, via the method of undetermined coefficients, as in McCallum (1983), Binder and Pesaran (1996), or Uhlig (1999), among many others. The recursive equilibrium law of motion of the system (6) can be expressed as follows,

$$\begin{aligned} \tilde{\hat{s}}_t &= \Omega_1 \ \tilde{\hat{s}}_{t-1} + \Omega_2 \ \tilde{z}_t \\ \tilde{v}_t &= \Omega_3 \ \tilde{\hat{s}}_{t-1} + \Omega_4 \ \tilde{z}_t \end{aligned} \tag{7}$$

where the matrices Ω_1, Ω_2, Ω_3, and Ω_4, are such that the equilibrium described by these rules is stable, and the symbol tilde over a variable denotes log-deviations of the variable from its deterministic steady state level. The conditions under which the stable solution is characterized and (7) is indeed a stable solution of (6) can be found in, for example, Theorem 3.2 of Uhlig (1999). The solution in (7), by solving for the stable manifold of the system forces the transversality conditions to hold. This, in turn, is a necessary and sufficient condition for the stationarity of the solution. For an explicit discussion of how to impose stationarity when solving rational expectations model it is illustrative the discussion in Novales *et al.* (1999).

By solving (6) and picking up the stable solution, we can develop alternative methods to initialize the PEA method. Basing the obtention of the initial conditions for the non-linear model in a log-linear version about the deterministic steady state of the very model, implicitly makes use of the ideas of homotopy. The log-linear approximation is a local counterpart of the model about the steady state, and at least locally should be close to the nonlinear model.

One can exploit the parallelism between the set of PEA-approximated first order conditions in (3) and the log-linear first order conditions in (6), or between the PEA-law of motion (4) and the log-linear law of motion in (7).

The identification approach By simple matching of the coefficients attached to the same state variables between the laws of motion in (4) and (7), in the cases in which $\psi_t(q; s_t(q), z_t)$ is polynomial, one could get the q parameters of the PEA polynomial attached to the first order terms, as the approximation in (6) is a first order Taylor approximation. This approach is closely related to that suggested in Christiano and Fisher (2000).

The estimation approach Alternatively, one could evaluate (3) for $\hat{s}_t$ and $\hat{v}_t$, and estimate the approximated $L^*(\bullet)$ and the implied approximated $L_q(\bullet)$. Given we are running a regression between stationary variables, the resulting estimated qs have a stationary distribution. On other grounds, if T is long enough the potential multicollinearity problems that might arise are kept to a minimum.

Let us denote by $\hat{\psi}$ the value for ψ obtained directly from the equations in (6). Then one could write a sub-system from (6) in which, given the invertibility property,

$$\hat{\psi} = \psi(q;\ \hat{s}_{t-1},\ z_t) \tag{8}$$

that can be linear in the logarithms of the variables. Based on expression (8) one can write the regression

$$\hat{\psi} = \psi(q;\ \hat{s}_{t-1},\ z_t) + \text{noise variable} \tag{9}$$

The estimation of the q parameters from the preceding regression would give a good starting point for the q coefficients needed to initialize PEA. Standard results in regression analysis – see for example Hamilton (1994) – guarantee a stationary distribution for the resulting, estimated vector of coefficients q.

For example, were (9) linear in the logarithms of the variables, then under the standard assumptions of regression theory of normality and independence of the noise variables, we could state that q follows a Gaussian distribution with known mean and standard deviation.

The algorithm to frame the estimation would be the following: i. Generate $\{z_t\}_{t=0}^{T}$, with T long enough; ii. Perform a log-linear approximation to the original system; iii. Simulate time series paths for all variables in the economy: $\hat{s}_t$ and $\hat{v}_t$; iv. Evaluate (3) at $\hat{s}_t$ and $\hat{v}_t$; v. Estimate the nonlinear regression for q.

The estimation approach is more general than the identification approach, and can be always applied. Paradigmatic cases in which the identification approach would not be applicable are systems of Euler equations that display multiple expectations in the same equation, and those presenting expectations at horizons larger than one.

To sum up: Solving by either the identification or the estimation methods based on the log-linear approximation around the steady state address the two problems when deciding on the appropriate initial conditions: i. they guarantee local stability; ii. each set $\{s_t(q^i), v_t(q^i)\}_{t=0}^{\infty}$ generated at each iteration i has a stationary distribution when based on the log-linear approach.

5 A framework to compare initialization methods

In this section I will try to clarify through an example the proposed methodology, and to check its performance. I have selected the Cooley and Hansen (1989) model for a number of reasons. First, it is a model solved by den Haan and Marcet in its 1994 paper, so that I have a benchmark for comparison. den Haan and Marcet achieve a well-behaved final form for ψ, and give numerical values for the corresponding final vector q. I will take both facts for given, in order to concentrate the discussion on the obtention of initial conditions. Second, the model is complex enough not to make superfluous any comparison, moving the discussion away from the usual one sector neoclassical growth model, and making possible the extrapolation of the results to more complex environments.

In the next subsections, first I will briefly present the model. Next I will explain how to implement the log-linearization-based homotopy methods in the framework of the model. Then, as a means of comparison, I will build two additional, standard ways of implementing the homotopy ideas. Both of them are arbitrary, in the sense of being two possibilities in a hundred of putting homotopy in practice. The two homotopy approaches are based on moving in a smooth way a given parameter. This means that, everything equal, I will start the homotopy from a version of the model in which *one* parameter has a value different from the one in the baseline parameterization: on the one hand I will move the steady state value of the money growth rate, g_{ss}, and on the other hand the depreciation rate δ. Finally, I will check the performance of the log-linear homotopy methods as compared to the standard ones.

5.1 The model

The Cooley and Hansen (1989) model includes a non-convexity, indivisible labor. Money is introduced via a cash-in-advance constraint in consumption. The competitive equilibrium is non-Pareto-optimal, and the second welfare theorem does not apply. The representative firm solves a standard profit maximization problem, while households seek to maximize their time preferences subject to their holdings of money balances and a set of standard budget constraints. There are two sources of uncertainty in this economy: an autoregressive shock to technology, z_t,

$$z_{t+1} = (1 - \rho_z)z_{ss} + \rho_z z_t + \epsilon_{z_{t+1}}.$$

and an autoregressive logged money growth rate,

$$\log(g_{t+1}) = (1 - \rho_g)\log(g_{ss}) + \rho_g \log(g_t) + \epsilon_{g_{t+1}}.$$

In equilibrium, we have the following optimality conditions,

$$\lambda_t = \beta E_t \left[\lambda_{t+1}\left(\alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta\right)\right] \quad (10)$$

$$\lambda_t = \frac{1}{c_t}\beta E_t\left[\frac{1}{g_{t+1}}\right] \quad (11)$$

$$\lambda_t = \frac{A_N}{1-\alpha} z_t^{-1} k_{t-1}^{-\alpha} N_t^{\alpha} \quad (12)$$

$$E_t\left[\frac{1}{g_{t+1}}\right] = e^{\frac{\sigma_{\epsilon_g}^2}{2}} g_{ss}^{\rho_g - 1} g_t^{-\rho_g} \quad (13)$$

$$z_t k_{t-1}^{\alpha} N_t^{1-\alpha} = c_t + (1-\delta)k_{t-1} - k_t \quad (14)$$

where λ_t is the Lagrange multiplier attached to the household's budget constraint, A_N is the labor weight in utility, and N_t denote hours worked. c_t is consumption at time t, k_{t-1} the beginning of period t capital stock, and x_t investment. $0 < \beta < 1$ is the subjective discount factor, $0 < \alpha < 1$ the capital share in production, and $0 < \delta < 1$ the depreciation rate. $0 < \rho_z < 1$ and $0 < \rho_g < 1$ controls for the persistence of the shocks. Along the paper the ss subscript affecting a given variable denotes its deterministic steady state value.

The second expectation arises from the first order conditions for real money balances and consumption, and the budget constraint. Assuming normality of the innovation ϵ_{g_t}, this expectation has a known analytical form, linear in the logs of the variables. This way, the only expectation term that needs to be approximated is that in (10).

5.2 den Haan and Marcet's 1994 PEA solution

den Haan and Marcet (1994) preferred specification for the approximating function ψ to the expectation term in (10) is a third order polynomial such that,

$$\begin{aligned}\beta\psi_t &= q_1 \exp\left(q_2 \log(k_{t-1}) + q_3 \log(z_t) + q_4 \log(g_t) + q_5 (\log(k_{t-1})^2\right) \\ &\times \exp\left(q_6 \log(k_{t-1}) \log(z_t) + q_7 (\log(z_t))^2 + q_8 (\log(z_t))^3\right)\end{aligned} \quad (15)$$

Following den Haan and Marcet I will adopt as a baseline parameterization: $\beta = 0.99$, $\alpha = 0.36$, $A_N = 2.86$, $\rho_z = 0.95$, $\rho_g = 0.48$, $\sigma_{\epsilon_z} = 0.00721$, and $\sigma_{\epsilon_g} = 0.009$. Regarding the parameter g_{ss}, the value in the chosen version of the model I will be solving for will be 1.15. As regards δ, the baseline value will be 0.025.

5.3 Implementation of the different homotopy approaches

The identification approach Solving the Cooley and Hansen (1989) model by means of any undetermined coefficients method, as for example the one in Uhlig (1999), it is not difficult to see that the solution for some variables in the model can be written as,

$$\begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \\ \tilde{N}_t \\ \tilde{\lambda}_t \end{bmatrix} = \begin{bmatrix} \nu_{kk} & \nu_{kz} & \nu_{kg} \\ \nu_{ck} & \nu_{cz} & \nu_{cg} \\ \nu_{Nk} & \nu_{Nz} & \nu_{Ng} \\ \nu_{\lambda k} & \nu_{\lambda z} & \nu_{\lambda g} \end{bmatrix} \begin{bmatrix} \tilde{k}_{t-1} \\ \tilde{z}_t \\ \tilde{g}_t \end{bmatrix}$$

where the ν_{ii} coefficients are non-linear functions of the deep, structural parameters of the model economy. Then it is easy to see that,

$$\lambda_t = \left[\lambda_{ss} \left(\frac{1}{k_{ss}}\right)^{\nu_{\lambda k}} \left(\frac{1}{g_{ss}}\right)^{\nu_{\lambda g}}\right] k_{t-1}^{\nu_{\lambda k}} z_t^{\nu_{\lambda z}} g_t^{\nu_{\lambda g}}$$

Equating the coefficients of this expression with the ones on the right hand side of (15), one gets the following identified initial values for the q set of parameters,

$$\begin{aligned} q_1 &= \left[\lambda_{ss} \left(\frac{1}{k_{ss}}\right)^{\nu_{\lambda k}} \left(\frac{1}{g_{ss}}\right)^{\nu_{\lambda g}}\right]; q_2 = \nu_{\lambda k}; \\ q_3 &= \nu_{\lambda z}; \quad q_4 = \nu_{\lambda g}; q_5 = \cdots = q_8 = 0 \end{aligned}$$

For the baseline parameterization this means in numerical terms an initial vector q such that

$$\begin{aligned} q_1 &= 4.0861; \quad q_2 = -0.5316; \quad q_3 = -0.4703; \quad q_4 = -0.0312 \\ q_5 &= \cdots = q_8 = 0 \end{aligned} \quad (16)$$

The estimation approach The algorithm to implement this approach implies pursuing the following steps,

1. Generate $\{z_t, g_t\}_{t=0}^{T}$, with T large enough.
2. Solve the model using a log-linear method as, for example, the method of undetermined coefficients in Uhlig (1999). Obtain the time series paths for all variables in the economy such as the capital stock, $\{\hat{k}_{t-1}\}_{t=1}^{T}$, consumption, $\{\hat{c}_t\}_{t=1}^{T}$, and the lagrange multiplier, $\{\hat{\lambda}_t\}_{t=1}^{T}$.
3. Generate an auxiliary value for the approximated polynomial from the expression for the expectation in (10),
$$\hat{\psi}_t \equiv \frac{1}{\beta}\hat{\lambda}_t, \quad \forall t$$
4. Estimate by ordinary least squares the regression,

$$\begin{aligned}\log(\hat{\psi}_t) = {} & \varrho_1 + \varrho_2 \log(\hat{k}_{t-1}) + \varrho_3 \log(z_t) + \varrho_4 \log(g_t) \\ & + \varrho_5 (\log(\hat{k}_{t-1}))^2 + \varrho_6 (\log(\hat{k}_{t-1}) \log(z_t)) + \varrho_7 (\log(z_t)^2) \\ & + \varrho_8 (\log(z_t)^3) + \text{noise}.\end{aligned} \tag{17}$$

For each specific realization of the exogenous shocks, $\{z_t, g_t\}_{t=0}^{T}$, the obtained vector ϱ would be slightly different, although not statistically. For a set of 250 draws of the exogenous processes of size $T = 40,000$ each, Table 1 shows the summary statistics mean and the standard deviation of the mean of the estimated coefficients. As it is apparent from the Table, and would have been clear a-priori, the coefficients attached to second and third order terms are statistically non-significant. Had the linear approximation been based on a second order Taylor expansion, as for example in Collard and Juillard (2001) or Sims (2000), this would have been different. In any case, the estimation of the parameters ϱ_5 to ϱ_8 will be useful in properly directing the search algorithm, as realized in the results section. These will be the initial values assigned to the parameters of the PEA polynomial.

INSERT Table 1

A homotopy in g_{ss} As previously mentioned, den Haan and Marcet (1994) solve the Cooley and Hansen model, and reach a specification given by (15), for both the selected baseline parameterization, and a parameterization with $g_{ss} = 1.015$ instead. For this latter case, the mentioned authors report the following *final* conditions:

$$\begin{aligned} q_1 &= 3.0275; \quad q_2 = -0.2293; \quad q_3 = -1.3177; \quad q_4 = -0.0324 \\ q_5 &= -0.0631; \quad q_6 = 0.3553; \quad q_7 = -0.1833; \quad q_8 = -1.3690 \end{aligned} \tag{18}$$

This vector is a reasonable approximation, given the parameterization with $g_{ss} = 1.015$, to the rational expectations equilibrium solution to (10) - (14). Imagine now that we were to solve for the case with $g_{ss} = 1.15$. In this case it is reasonable to initialize the PEA for the case with $g_{ss} = 1.15$, with the *final* values in (18) obtained for the case with $g_{ss} = 1.015$. This exercise would amount to starting the homotopy with a 13% lower value of g_{ss} in order to obtain the solution for the desired parametric case.

To me this is a very favourable way of initializing PEA, and has been chosen on purpose to check the performance of the log-linear homotopy alternatives. Normally, following den Haan and Marcet (1990, 1994) one should have started the homotopy from a primitive version of the model - i.e. the one sector growth neoclassical model, with a basic parameterization. Then, slowly, one could have moved the solution to more complex versions of the simple neoclassical model, following a smooth path, until the desired solution is reached. Notice that we have sidestepped all the intermediate steps - already done by den Haan and Marcet (1994) - and started off the algorithm from the very close to the desired solution initial conditions in (18).

A homotopy in δ This second homotopy is a more elaborated one. It is a homotopy in the parameter δ, and also builds upon a steady state version of the model at hand with an analytical solution. From the optimality conditions (11) to (14) setting $\delta = 0$, it is easy to obtain the following expression,

$$\lambda_t^{\frac{1}{\alpha}} z_t^{\frac{1}{\alpha}} k_{t-1} \left[\frac{1-\alpha}{A_N}\right]^{\frac{1-\alpha}{\alpha}} + \lambda_t(k_{t-1} - k_t) = \beta e^{\frac{\sigma^2_{\epsilon_g}}{2}} g_{ss}^{\rho_g - 1} g_t^{-\rho_g}$$

that in steady state turns out to be,

$$\lambda_{ss}^{\frac{1}{\alpha}} z_{ss}^{\frac{1}{\alpha}} k_{ss} \left[\frac{1-\alpha}{A_N}\right]^{\frac{1-\alpha}{\alpha}} + \lambda_{ss}(k_{ss} - k_{ss}) = \beta e^{\frac{\sigma^2_{\epsilon_g}}{2}} g_{ss}^{\rho_g - 1} g_{ss}^{-\rho_g}$$

so that solving for the lagrange multiplier λ_{ss},

$$\lambda_{ss} = \left[\beta e^{\frac{\sigma^2_{\epsilon_g}}{2}} \left[\frac{A_N}{1-\alpha}\right]^{\frac{1-\alpha}{\alpha}}\right]^{\alpha} k_{ss}^{-\alpha} z_{ss}^{-1} g_{ss}^{-\alpha} \tag{19}$$

Equating the coefficients attached to the state variables k, z, and g between the previous expression and the expression of the polynomial in equation (15), one could get

$$\begin{aligned} q_1 &= \left[\beta e^{\frac{\sigma^2_{\epsilon_g}}{2}} \left[\frac{A_N}{1-\alpha}\right]^{\frac{1-\alpha}{\alpha}}\right]^{\alpha}; \quad q_2 = -\alpha \\ q_3 &= -1; \quad q_4 = -\alpha; \quad q_5 = \cdots = q_8 = 0 \end{aligned}$$

that numerically produces the values: $q_1 = 2.5975$, $q_2 = -0.3600$, $q_3 = -1.0000$, $q_4 = -0.3600$, $q_5 = \cdots = q_8 = 0$. These parameter values are a good approximation for the solution to the Cooley and Hansen model with $\delta = 0$. Once the solution for this case is attained, the *final* values can be used to initialize the case with $\delta = 0.025$, our benchmark case. This intermediate step is unavoidable, as the algorithm did not converge when I tried to directly solve for the case with $\delta = 0.025$ using the analytically-obtained parameters for the case with $\delta = 0$. Using the analytical values computed in the steady state case with

$\delta = 0$ to initialize the PEA to solve the Cooley and Hansen model in the case with $\delta = 0$, the following values for the elements of the vector q were achieved:

$$\begin{aligned} q_1 &= 4.0278; \quad q_2 = -0.3790; \quad q_3 = 0.2511; \quad q_4 = -0.0145 \\ q_5 &= -0.0275; \quad q_6 = -0.1332; \quad q_7 = -0.0795; \quad q_8 = -0.3216 \end{aligned} \tag{20}$$

The coefficients in (20) will be the ones used as initial conditions for the benchmark case with $\delta = 0.025$. This is a more realistic way of grasping the normal cost attached to the standard implementation of the homotopy method. I started the homotopy from a primitive version of the targetted model, and then, in a second step moved a key parameter until getting the values for the desired case.

5.4 Comparative results

The comparative experiment is implemented as follows, and repeated 250 times. (i.) To begin with, extract a draw of size T=40,000 for the exogenous processes $\{z_t, g_t\}$. (ii.) Then take in turn the initial conditions described above for each of the four homotopy alternatives. For the identified log-linear homotopy take (16); for the estimated log-linear is estimated each time a new random draw is available; for the homotopy in g_{ss} take (18), and finally for the homotopy in δ use (20). (iii.) In a third step find the PEA solution to the Cooley and Hansen (1989) model using each set of initial conditions. Compute the convergence time and the number of iterations needed to converge to the fixed point, with a tolerance of four digits. (iv.) Apply the den Haan and Marcet (1994) test 250 times to each obtained solution to check that indeed the obtained equilibrium is a rational expectations one. Reject the solutions that do not pass the test. The test is repeated in turn for *each* simulation 250 times in order to obtain the empirical distribution of the den Haan and Marcet statistic, so that the result is not dependent on a particular realization of the exogenous shocks[1].

Table 2 presents the main results, while Figure 1 summarizes the steps involved in the implementation of each homotopy approach. As the time per iteration may change dramatically depending on the employed programming language and the machine used, the computing times are presented in relative terms, using the log-linear estimation approach as the baseline.

As it is apparent from Table 2, the log-linear approaches required less iterations to converge to the fixed point than the other implemented methods. The homotopy in g_{ss} needed, on average, 1.6 more time per iteration to converge, while the homotopy in δ demanded about

[1]The results are available from the author upon request. The idea of the test lies in checking whether there is some function of the variables in the information set up to time t, say I_t, that could be useful to predict the expectations errors, say ξ_{t+1}. If this were the case, it would imply a violation of the the rationality property. For each simulation, the steps to follow would be: (i.) generate a random draw for the exogenous processes (T =40,000 in this case); (ii.) Regress ξ_{t+1} on I_t; iii. Define $\hat{a} = (\sum I_t^T I_t)^{-1}(\sum I_t^T \xi_{t+1})$ and compute the statistic

$$\hat{a}^T(\sum I_t^T I_t)(\sum I_t^T I_t \xi_{t+1}^2)^{-1}(\sum I_t^T I_t)\hat{a} \sim \chi^2_{m_1 m_2},$$

where m_2 is the number of chosen instruments, and m_1 the number of expectations errors, which is equal to one in our case. The statistic provides a test for the null hypothesis of rational expectations:$E_t\left(\xi_{t+1}\right) = 0$.

two times the time of the log-linear estimation approach. When looking at the maximum and minimum times, the identification approach emerges as the more variable one, presenting the highest maximum and the lowest minimum, the first one being four times the time employed by the estimation method, and the second one 1/4 th of it. A desirable property in the convergence times would be its stability, given that the researcher normally runs the procedure for a single realization of the shocks. This characteristic makes the estimation method superior to the rest. The number of iterations needed to converge displays the same kind of information, requiring both log-linear methods 52 iterations on average, as opposed to 80 of the g_{ss} homotopy and almost a hundred of the δ homotopy.

Table 2 also shows one fixed point q attained, as an example, for a common realization of the exogenous shocks. As it is clear from the table, all homotopy methods reach the same fixed point for the same draw of the shock, starting from different initial conditions. On the other hand, for different realizations of the innovations, even starting from the same initial condition, a somewhat different fixed point is achieved. This connection of the convergence point reached to the draw of $\{z_t, g_t\}$ taken, may give an insight into the better performance of the estimation approach as regards times of convergence and stability in the number of iterations accross simulations. When running the regression involved in the implementation of the estimation method, this approach makes use of the information in each realization of the innovation.

Figure 1 sheds some light into the comparative pros and cons attached to each method. As regards the preparation time before the PEA is run, the log-linear homotopy methods seem to be of easier implementation and present less requirements than the other ones. In a first step the cost of implementing the log-linear homotopy approaches are the ones involved in computing the log-linear approximation of the original model about its steady state. In a second step one has to either identify the coefficients of the laws of motions, or run an ordinary least squares regression. Both steps are easy to automate, and can be performed by standard available packages. On the other hand, the homotopy over g_{ss} needs in a first step the implementation of a smooth homotopy over different versions of the one sector growth neoclassical model, that can take several and hard steps. Finally, the homotopy on δ requires the analytical derivation of the initial conditions for the $\delta = 0$ case, plus the obtention of the initial conditions for $\delta = 0.025$. As regards the convergence times, the log-linear approaches are also superior, as discussed in the previous paragraph.

6 Conclusions

In this paper I propose to obtain appropriate initial conditions for the parameters of the approximating PEA functions on the basis of a log-linear approximation about the steady state of the system one wants to solve for. In this sense the proposal can be though of as an homotopy-type method, as the log-linear approximation of the model is a simplified version of the model at hand.

I present the idea by means of two variants. The first one uses the log-linear time series to directly estimate initial values for the parameters of the approximating functions. The second variant analytically identifies some of the coefficients of the PEA polynomial. Both approximations share the desirable property of producing stationary and ergodic variables, and in this sense are suitable to obtaining an initial set of parameters with a stationary distribution as the coefficients for the PEA approximating functions. Nevertheless, the estimated approach is more general, and can be used, for example, in cases with multiple expectations, and expectations at horizons larger than one.

In the framework of the Cooley and Hansen (1989) model with indivisible labor and money demand motivated via a cash-in-advance constraint, I illustrate the performance of the log-linear approach as compared to a procedure based on the standard implementation of homotopy principles. Both proposed log-linear methods are simpler to implement, and outperform the presented alternatives in terms of speed of convergence.

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Table 1: Log-linear estimation approach. Descriptive statistics for the empirical distribution of the ordinary least squares estimator. Mean and standard deviation of the mean (in parenthesis).

ϱ_1	ϱ_2	ϱ_3	ϱ_4
4.08610571451612	-0.53158780860215	-0.47027449908602	-0.03122343300000
(0.00000003532142)	(0.00000000405140)	(0.00000000516988)	(0.00000000000000)
ϱ_5	ϱ_6	ϱ_7	ϱ_8
-0.00000000004440	0.00000000010273	-0.00000000004220	0.00000000003839
(0.00000000069382)	(0.00000000195405)	(0.00000000133405)	(0.00000000038802)

Table 2: Comparative results: solution to the Cooley and Hansen (1989) model starting from the initial conditions described in the text for each homotopy variant. Summary statistics for 250 draws of the exogenous processes $\{z_t, g_t\}$, each of size T=40,000.

	Log-linear homotopy		Standard homotopy	
	Identified	Estimated	g_{ss} : 1.015/1.15	δ : 0.0/0.025
Average time	1.0797	1.0000	1.6203	2.0166
Maximum time	4.0250	1.0000	2.3537	3.0687
Minimum time	0.2510	1.0000	0.9867	1.1786
Average number of iterations	52.31	52.26	80.02	98.73
Example: a typical simulation	Identified	Estimated(*)	g_{ss} : 1.015/1.15	δ : 0.0/0.025
q_1	3.9809	3.9809	3.9803	3.9803
q_2	-0.5104	-0.5104	-0.5099	-0.5101
q_3	-0.9249	-0.9249	-0.9253	-0.9250
q_4	-0.0316	-0.0316	-0.0316	-0.0316
q_5	-0.0043	-0.0043	-0.0043	-0.0043
q_6	0.1998	0.1998	0.1999	0.1998
q_7	-0.0340	-0.0340	-0.0340	-0.0340
q_8	0.8807	0.8807	0.8809	0.8807
(*) Estimated initial q: q = [4.0861,-0.5316,-0.4703,-0.0312,0.0000, 0.0000, 0.0000,0.0000]				

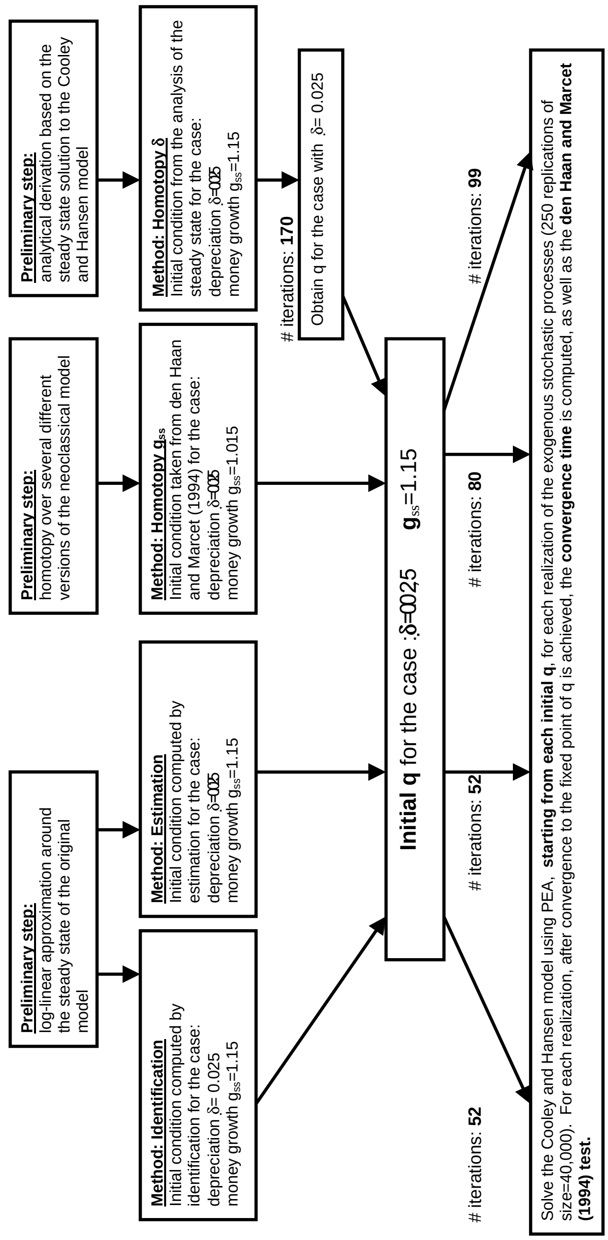

Figure 1: Summary of the experiment. Steps involved in the implementation of each one of the homotopy methods described in the text.