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Is it Worth Refining Linear Approximations to Non-Linear Rational Expectations Models? *

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RESUMEN

En este artículo caracterizamos la senda de crecimiento equilibrado del modelo neoclásico básico de crecimiento usando varios métodos numéricos de solución *casi* lineales, y el método de parametrización de expectativas. También resolvemos el modelo básico incluyendo trabajo indivisible, y también una versión monetaria del modelo con una restricción de *efectivo por adelantado*. En un marco unificado enfrentamos la cuestión de cuánta estructura no lineal del problema original es útil mantener cuando se usa un método de solución *casi* lineal que sea *refinable*. Mostramos que es posible encontrar soluciones *casi* lineales a este conjunto de modelos que son tan exactas como resolver por métodos no lineales más complejos de aplicar. Nuestros resultados muestran la importancia del uso de los logaritmos, así como la conveniencia de *refinar* las soluciones lineales mediante la combinación de estructura del problema no lineal original con las condiciones de estabilidad del sistema aproximado linealmente.

Palabras clave: Aproximaciones lineal-cuadráticas, propiedades de las soluciones numéricas, simulación, métodos numéricos.

ABSTRACT

We characterize the balanced growth path of the basic neoclassical growth economy using standard, *almost* linear numerical solution methods, as well as the parameterized expectations approach, which preserves the nonlinearity in the model. We also apply the same methods after adding *indivisible labor* to the basic model, and to a monetary version of that economy, subject to a cash-in-advance constraint. In a unified framework we tackle the question of how much of the non-linear structure of the original problem is useful to maintain when using an "almost" linear method. We show that it is possible to find an *almost* linear method to solve these models as accurately as by parameterizing expectations. Our results show the importance of performing log-linear approximations, as well as the convenience of refining a linear solution method by mixing some structure of the original non-linear problem with structure of the approximated system.

Keywords: Linear-quadratic approximation, numerical accuracy, simulation, numerical methods.

JEL classification: C63 ; E17.

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1 Introduction

The rising importance of computational methods in economics is fairly evident from simple inspection of any research journal. The interaction between economic theory and computational research is a central aspect of modern economics. This interaction is particularly important in the research agenda outlined in Lucas (1980). The construction of *fully articulated artificial economies* has led to using rational-expectations dynamic stochastic modelling in almost all fields of economics – see for example Marcet (1993), Kydland and Prescott (1996) or Cooley and Prescott (1995) for illustrative reviews.

This generally implies solving a system of stochastic difference equations involving conditional expectations of highly nonlinear functions, or making use of dynamic programming tools when dealing with problems with a recursive structure. The aim is to find the equilibrium solution for all the variables in the economy and to characterize the structure of the decision rules that relate state to decision variables. But the essentially non-linear, stochastic structure embedded in these systems makes generally impossible to obtain analytical solutions and has motivated an explosion of numerical solution methods. Although there is a wide variety of numerical approaches at hand[1], there is not much systematic evidence concerning the consequences of using each one of them when dealing with a particular economic problem, so that the researcher finds itself always at the juncture of what solution method should be using given a particular problem.

Focusing on the basic version of the neoclassical growth model, Taylor and Uhlig (1990) consider fourteen different solution methods. Their analysis was quite rich in terms of the variety of methods compared and the comparison measures used, the general conclusion being that differences among methods turned out to be quite substantial for certain aspects of the model. Nonetheless, their study lacked some homogeneity and robustness given the way it was conducted: for each method they had just one solution realization and the estimated decision rules. In addition, the probability distribution of the technology shock, the single source of dynamics of the artificial economy, was not the same for all the implemented methods.

Another set of papers analyzing the same model are Christiano (1990), who compared a linear quadratic and a log-linear quadratic method with the solution generated by a discrete-grid value-function iteration procedure, closer to the "true" solution, and Christiano and Fisher (2000), who compared a set of weighted residuals and finite element methods, again with the same type of discrete-grid solution. İmrohoroğlu (1994) proposed a forward solution method and compared it with backsolving and with a linear quadratic approximation method in the same context, using the test in den Haan and Marcet (1994) as a measure for comparison. To illustrate the power of the test, these latter authors compared the Parameterized Expectations approach with linear quadratic methods by solving the one sector neoclassical growth model as well as the cash-in-advance monetary model of Cooley and Hansen (1989), in which the decentralized solution is not Pareto optimal. Again in a non-optimal envi-

[1] It is not an objective of this paper to describe the state of the art in this area. For general surveys of existing solution methods see *Computational Methods for the Study of Dynamic Economies*, edited by R. Marimon and A. Scott (1999), the Winter 1990 issue of the *Journal of Business and Economic Statistic*, Cooley and Prescott (1995), Marcet (1993), Danthine and Donaldson (1995) or Judd (1998).

ronment, Dotsey and Mao (1992) compared different linear and log-linear approximations in a modified version of the basic growth model with taxes on production following a five-state Markov chain and no technology shock, using as a criterion for comparison a discrete state space solution to the Euler equations of the model. With a different question in mind Barañano, Iza and Vazquez (2002) compare the performance of an almost lineal method and Parameterized Expectations when solving an endogenous growth model.

In spite of being quite extensive, the picture that emerges from the literature is mixed and scattered. To be brief: regarding the basic neoclassical growth model, linear and log-linear quadratic approximation methods are very similar and perform well, except for the den Haan-Marcet test, linear models having some difficulties in passing the test. In non-optimal settings things change. Weighted residuals-finite element methods seem to behave very similarly, although the Parameterized Expectations approach turns out to be the solution algorithm most often used, and it seems to be quite convenient when there is a large number of state variables.

In our view two questions arising from this literature have not been sufficiently discussed.

First, linear approximation methods are very popular among modelers because they are relatively simple to implement, although there is a perceived loss of accuracy due to the approximation, as compared to more elaborated methods. The implementation of an "almost" linear[2] numerical solution method implies adding some approximation to the model while preserving some of the non-linear structure in the original problem. From this point of view, we can think of refining a linear solution method to increase the accuracy we could get from an "almost" linear solution by either *i*) using second order approximation techniques [see for example Judd (1998), Sims (2001), Collard and Juillard (2001), Schmitt-Grohé and Uribe (2002)], *ii*) combining a linear approximation with some of the original nonlinear structure when obtaining the numerical solution to the model [see Novales et al. (1999)], or *iii*) working in logarithms instead of levels of the variables. In this paper, we focus on refinement alternatives *ii*) and *iii*) to discuss 1) whether refining linear solutions increases the accuracy of the numerical solution, and 2) the extent to which a refined linear solution performs similarly to nonlinear solutions.

The second question refers to the framework in which the different solution methods are usually evaluated, which is generally the basic neoclassical growth model while, most often, solution methods are applied to more complex structures. Hence, a performance analysis of the different methods when departing from the more basic growth model is needed.

In this paper we have tried to answer these questions in a unified and complete framework. We do not attempt to rank different methods or to conclude which one is best. That is the reason we do not use a computationally expensive, very accurate algorithm, against which to compare the alternative solution methods considered. Rather, we evaluate their performance to the light of the two previous questions. This is the novelty of our work, related to previous research in this field. As a by-product, we evaluate two widely used proposals in the literature to solve non-linear rational expectations models, Uhlig (1999) and

[2]We call them "almost" linear, in the Marimon and Scott (1999) terminology, because they mix the stable manifold of the linear/log-linear approximation with parts of the original non-linear problem, like the global resource constraint and the set of equilibrium conditions.

Sims (2002), and provide a user guide to choose among an important set of methods described in Marimón and Scott (1999). We place special emphasis on rationality, since not showing evidence of any violation of rationality should be the first requirement for any solution to a rational expectations model.

Regarding the models considered, we start by analyzing the standard baseline one-sector stochastic growth model, subject to an autoregressive shock to technology leading the dynamics of the economy. Then, we increase the complexity of the model including indivisible labor as in the real business cycle model of Hansen (1985). In a final step, we add money to the previous model via a cash-in-advance constraint on the consumption commodity, as in Cooley and Hansen (1989). This is a non-Pareto optimal setting with an additional exogenous stochastic process, money growth. With this sequence of models, we try to cover a wide range of standard applications.

As regards numerical solution methods, we consider several methods that differ in the degree of non-linearity they preserve. We have implemented: i) the standard linear-quadratic approximation in levels of the variables, as in Hansen (1985) or Díaz-Giménez (1999), and we have simulated, as they do, making use of the original non-linear structure of the problem plus the obtained linear decision rule/s, ii) the undetermined coefficients solution to the log-linear approximated model as proposed by Uhlig (1999), and we have simulated the model using the log-linearized system in state space form, iii) a Blanchard and Kahn (1980)-Sims (2002) approach, either in levels or in logs of the variables, as described in Novales *et al.* (1999), and we have simulated the original non-linear model mixed adding the stability condition of the linearized/log-linearized system. That way, we nearly cover the whole spectrum of "almost" linear methods usage that appears in the literature.

All these methods are very similar in spirit. All search for the stable manifold of a linear or log-linear approximation to the original non-linear problem, and impose stability by selecting the *saddle path equilibrium*. Differences among these methods are mainly related to the way they are usually implemented in practice. A linear solution method can be *refined* by increasing the amount of the original non-linear structure it preserves: one can work directly with a fully linearized system, with a fully log-linearized system, or with a mixture of the original non-linear problem and the stability conditions and decision rules derived from either the linearized or the log-linearized version of the system.

On the other hand, we have also solved the models with a nonlinear approximation method, Parameterized Expectations. This belongs to a class of methods which approximate the solution to functional equations (equations involving conditional expectations, the value function equation, etc.) using flexible combinations of known functions. The advantage of such methods is that, theoretically, one can achieve an approximation arbitrarily good to the true solution while maintaining all the non-linear structure in the original problem. In the Parameterized Expectations approach, each conditional expectation in the model is approximated by a flexible polynomial function, the solution being conditional on the number of elements in the approximation, i.e. the order chosen for the conditional expectation polynomials. Even though a higher order polynomial would be preferred because of providing increased accuracy, estimation quickly loses precision even for moderate degree polynomials due to collinearity among parameters.

We have looked at a wide set of criteria, in the spirit of Taylor and Uhlig (1990), in a unified and consistent framework. On the one hand, we have performed a Monte Carlo simulation of a battery of tests to check the rationality properties of the stochastic Euler equations residuals: den Haan-Marcet tests for correlations with variables in the information set, as well as tests for autocorrelation or a nonzero mean in the estimated rational expectations residual. On the other hand, we have computed a set of statistics (mean, standard deviation, cross correlations with output) for each of the main variables in the model, and we have analyzed the estimated decision rules implied by each method.

We did not use as an evaluation criterion the closeness of the approximations we have implemented to an arbitrarily "exact" solution obtained through a fine discretization of the state space, either based upon Euler equations or dynamic programming, for we did not want to constraint ourselves to a Markov chain distribution with a reduced number of states for the exogenous shocks, as in Christiano (1990) or Dotsey and Mao (1992). Instead, we use a continuous probability distribution function for the technology shock in the first two models, and for the technology shock and the money growth shock in the third model, which turns out to be important when characterizing the statistical properties of a given economy. Besides, we can indirectly check how close are our approximations to the exact solution by testing for fulfillment of the rational expectations hypothesis, as well as for differences with the Parameterized Expectations solution, which can be made to approximate arbitrarily well the "exact" solution.

Our results show that the solutions proposed in Sims (2002)-Novales *et al.* (1999) and Uhlig (1999) applied to the previous models, written in logs of the variables, are almost indistinguishable from those obtained from the Parameterized Expectations approach in all the analyzed dimensions and for all the model economies considered. We also show that it pays to preserve some of the non-linear structure of the original model, specially when solving in levels of the variables. When working with logged variables in these simple models, preserving some non-linearity seems to be relevant just for extreme parametric cases.

The rest of the paper is organized as follows. Section 2 presents the versions of the neoclassical growth model we consider. Section 3 briefly describes the four solution methods we use, while Section 4 sets the basis for the evaluation. In Section 5 we show the results and in Section 6 some concluding remarks. The paper is closed with an Appendix where the decision rules for all methods are shown and some guidance on solving the models is given. A Technical Appendix containing a detailed discussion of the implementation of each method to the three models is available from the authors upon request.

2 Description of models

We focus on several standard versions of the neoclassical, exogenous growth model. The sequence begins with a version of the basic one-sector stochastic growth model. Private agents are assumed to choose capital and consumption sequences to maximize

$$\max_{\{k_t,c_t\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{c_t^{1-\eta}-1}{1-\eta} \right] \tag{1}$$

subject to technological and resource constraints,

$$\begin{aligned}
y_t &= c_t + x_t \\
y_t &= z_t k_{t-1}^{\alpha} \\
k_t &= (1-\delta)k_{t-1} + x_t \\
\log(z_t) &= (1-\rho)\log(z_{ss}) + \rho \log(z_{t-1}) + \epsilon_t \\
\epsilon_t &\sim i.i.d.\, N(0, \sigma_\epsilon^2) \\
& k_t \geq 0, \;\; c_t \geq 0
\end{aligned}$$

given k_0 and z_0, where c_t is consumption at time t, k_{t-1} the beginning of period t capital stock, x_t investment, y_t output, and z_t an exogenous technology shock to output. $0 < \beta < 1$ is the subjective discount factor, $\eta > 0$ is the coefficient of relative risk aversion, $0 < \alpha < 1$ the capital share in production, $0 < \delta < 1$ the depreciation rate and $0 < \rho < 1$ controls for the persistence of the shock. Along the paper the ss subscript affecting a given variable denotes its deterministic steady state value.

The optimality condition of this problem is

$$c_t^{-\eta} = \beta E_t \left[c_{t+1}^{-\eta} R_{t+1} \right] \tag{2}$$

together with the previous constraints, where $R_{t+1} = \alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta$. To perform rationality tests, we are concerned with the properties of the prediction error/s. The one-step ahead rational expectation error associated with (2) is,

$$\xi_{t+1} = \left[c_{t+1}^{-\eta} R_{t+1} \right] - E_t \left[c_{t+1}^{-\eta} R_{t+1} \right] \tag{3}$$

with a theoretical white noise structure: $E_t(\xi_{t+1}) = 0$ so that there is no correlation with any variable contained in the information set available at time t. These are implications of rationality, and we are interested in testing for preservation of these properties as a central issue when evaluating solution methods. Using the time series for consumption and capital that we obtain with each solution method, we will generate time series for the approximated prediction error, ξ_t, as in (3), to test whether it violates rationality.

The second model is proposed in Hansen (1985). It is slightly more non-linear than the previous one in that it includes a non-convexity, indivisible labor. Here the representative household faces the problem,

$$\max_{\{k_t, c_t, N_t\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{c_t^{1-\eta} - 1}{1-\eta} - A_N N_t \right] \tag{4}$$

subject to

$$\begin{aligned}
y_t &= c_t + x_t \\
y_t &= z_t k_{t-1}^{\alpha} N_t^{1-\alpha} \\
k_t &= (1-\delta)k_{t-1} + x_t \\
\log(z_t) &= (1-\rho)\log(z_{ss}) + \rho \log(z_{t-1}) + \epsilon_t \\
\epsilon_t &\sim i.i.d.\, N(0, \sigma_\epsilon^2) \\
& k_t \geq 0, \;\; c_t \geq 0
\end{aligned}$$

given k_0 and z_0. N_t denotes labor and A_N is a parameter that measures the relative weight of labor in the utility function. The remaining parameters are as in the previous model. Again (2) is the single equation involving expectations terms, from the first order condition for capital and consumption, where now $R_{t+1} = \alpha z_{t+1} k_t^{\alpha-1} N_t^{1-\alpha} + 1 - \delta$, and the rational expectations error is defined as in (3).

In addition to (2) and the constraints there is now another optimality condition from maximizing with respect to labor which, using the first order condition for consumption, can be written,

$$A_N = (1-\alpha) c_t^{-\eta} z_t k_{t-1}^{\alpha} N_t^{-\alpha}$$

The last economy considered, Cooley and Hansen (1989), is a version of Hansen (1985), with money introduced via a cash-in-advance constraint in consumption. The competitive equilibrium is non-Pareto-optimal in this case, and the second welfare theorem does not apply. The representative firm solves a standard profit maximization problem, while households seek to maximize their time preferences subject to their holdings of money balances and a set of standard budget constraints. There are two sources of uncertainty in this economy: the autoregressive shock to technology, z_t, and an autoregressive logged money growth rate,

$$\log(g_{t+1}) = (1-\rho_g)\log(g_{ss}) + \rho_g \log(g_t) + \epsilon_{g_{t+1}}.$$

In equilibrium, we have two first order conditions involving expectations terms,

$$\lambda_t = \beta E_t \left[\lambda_{t+1} R_{t+1}\right] \quad (5)$$

$$\lambda_t c_t = \beta E_t \frac{1}{g_{t+1}} \quad (6)$$

where $R_{t+1} = \alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta$ and λ_t is the Lagrange multiplier associated with the household's budget constraint. The first equation is the optimization condition for capital, with an expectation error

$$\xi_{t+1} = \left[\lambda_{t+1} R_{t+1}\right] - E_t \left[\lambda_{t+1} R_{t+1}\right], \quad (7)$$

The second expectation arises from the first order conditions for real money balances and consumption, and the budget constraint. Assuming normality of the innovation ϵ_{g_t}, this expectation has a known analytical form, linear in the logs of the variables[3].

[3] From the process for g_{t+1} and if $\epsilon_{g_{t+1}}$ *i.i.d.* $\sim N(0, \sigma^2_{\epsilon_g})$, we can write,

$$E_t\left[\frac{1}{g_{t+1}}\right] = g_t^{-\rho_g} g_{ss}^{\rho_g - 1} E_t\left[e^{-\epsilon_{g_{t+1}}}\right] = g_t^{-\rho_g} g_{ss}^{\rho_g-1} e^{\frac{\sigma^2_{\epsilon_g}}{2}},$$

so that the corresponding prediction error, ξ_{g_t}, has a known analytical expression

$$\xi_{g_{t+1}} = \frac{1}{g_{t+1}} - E_t\left(\frac{1}{g_{t+1}}\right) = \frac{1}{g_{t+1}} - e^{\frac{\sigma^2_{\epsilon_g}}{2}} g_{ss}^{\rho_g-1} g_t^{-\rho_g}.$$

3 Solution Methods

We evaluate two sets of methods. On the one hand, we use three "almost" linear methods preserving different degrees of the non-linear structure in the original problem that are easy to implement and computationally fast: i) the standard linear-quadratic approximation in levels of the variables (LQA henceforth), ii) the approach proposed in Uhlig (1999) (UHL) and iii) the method proposed by Sims (2002)-Blanchard and Kahn (1980) as described in Novales *et al.* (1999) either in levels or in logs of the variables (SIM / SIL, respectively). The first one is a Value-Function-based method while the other two are Euler-equation-based methods. We discuss the methods as they are usually implemented in practice. Some of the details of their use is what makes them somewhat different. More fundamentally, they all search for the same stable subspace, and can be adapted to become essentially indistinguishable from each other. On the other hand, we also use a nonlinear type method, Parameterized Expectations (PEA), an Euler-equation-based method. We refine our PEA approximation until the prediction error from the stochastic Euler equation passes the den Haan and Marcet (1994) test.

In Figure 1 we show the steps involved in the implementation of each of the selected solution methods, to have an idea of the comparative complexity in implementing the methods. We do not provide in the paper computing times because they depend on the programming language and specific code used. However, the PEA method was clearly the most time consuming.

Insert Figure 1

3.1 "Almost" Linear Methods

LQA uses the non-linear structure of the model, adding linear decision rules for consumption, investment or labor. SIM, also implemented in level variables, only adds linear stability conditions to the original, non-linear model. These conditions guarantee that the numerical solution to the non-linear system of equations is stable. For each of the three models in the paper, just a single stability condition is needed. A comparison between these two solutions will allow us to discuss whether the higher complexity produced by preserving more non-linear structure in the SIM method pays in terms of increased accuracy. We also apply the SIM method to a log-linear approximation to the model around steady-state, which we will denote by SIL. This produces a stability condition linear in logged variables, instead of one such condition linear in the variables. Comparing SIM with SIL we can test whether performing the approximation in logs implies any accuracy gain. Finally, since UHL works with a fully log-linearized system and SIL uses a mixture of the original problem and a log-linear approximation to the original system, we can again evaluate the benefits of preserving some non-linearity. In this sense, SIL is the most refined of the "almost" linear methods, and LQA, as implemented here, the less refined. The LQA method obtains an exact solution to a different but similar model, all the others finding an approximate solution to the original model.

3.1.1 Standard Linear Quadratic Approximation (LQA)

The LQA approach consists in approximating a non-linear problem by one with a linear-quadratic structure, for which the solution is always known. For a detailed description, see Kydland and Prescott (1982), or Hansen and Prescott (1995). In these papers, solving the social-planning problem involves solving a dynamic programming problem of the form:

$$V^{n+1}(\mathbf{z}_t, s_t) = \max_{d_t} \{r(\mathbf{z}_t, s_t, d_t) + \beta E\left[V^n(\mathbf{z}_{t+1}, s_{t+1}|\mathbf{z}_t)\right]\}$$

subject to

$$\begin{bmatrix} \mathbf{z}_{t+1} \\ s_{t+1} \end{bmatrix} = A\varepsilon_{t+1} + B(\mathbf{z}_t, s_t, d_t)$$

where $V^n(\mathbf{z}_t, s_t)$ is the n^{th}-iteration on the optimal value function, β the discount factor, $\mathbf{z}_t$ a vector of exogenous state variables, s_t a vector of endogenous state variables, d_t a vector of decision variables, $r(\mathbf{z}_t, s_t, d_t)$ the return function for the problem, ε_t a vector of exogenous i.i.d. stochastic processes, and the constraints describe the evolution of the state variables. We will maintain this notation across methods. For the exact definition of these vectors in each model, see the Appendix. To solve this problem one can operate directly with the value function. What LQA does is to compute a linear quadratic approximation to the original economy around steady-state and then search for the solution to this approximate linear quadratic economy. Briefly, the steps are:

1. Find the first order conditions and compute the steady state.
2. Substitute the non-linear constraints in the return function, r.
3. Form a second order Taylor approximation of the resulting return function around steady-state. Making use of the certainty equivalence principle, the approximate problem then becomes,

   $$V^{n+1}(\mathbf{z}_t, s_t) = \max_{d_t} \left\{[1, \mathbf{z}_t, s_t, d_t]Q[1, \mathbf{z}_t, s_t, d_t]^T + \beta V^n(\mathbf{z}_{t+1}, s_{t+1}|\mathbf{z}_t)\right\}$$

   subject to

   $$\begin{bmatrix} \mathbf{z}_{t+1} \\ s_{t+1} \end{bmatrix} = B\left[\mathbf{z}_t, s_t, d_t\right]$$

   where Q is a symmetric matrix and ^{T} denotes matrix transposition. Under suitable conditions, the optimal value function exists, it solves this functional equation, and it is quadratic. As a consequence, the associated policy functions are linear.

4. Guess an initial quadratic conjecture for V^0, say $V^0(\mathbf{z}_{t+1}, s_{t+1}) = [1,\ \mathbf{z}_{t+1},\ s_{t+1}]\ L^0\ [1, \mathbf{z}_{t+1},\ s_{t+1}]^T$, where L^0 is a symmetric and negative semi-definite matrix. Then, given the laws of motion for the states (the constraints in the previous problem) we compute $V^0(\mathbf{z}_{t+1}, s_{t+1})$ to get a quadratic expression in $\mathbf{z}_t$, s_t and d_t for $V^1(\mathbf{z}_t, s_t)$.

5. The first order conditions for this approximate problem give us the policy function or decision rule, d_t, as a linear function of $\mathbf{z}_t$ and s_t. Substitute the decision rule into the approximate problem and obtain the optimal value for $V^1(\mathbf{z}_t, s_t)$.
6. Repeat until V^{n+1} is very similar (according to some convergence criterion) to V^n.

The solution to the linear-quadratic problem produces a linear function that maps states into decisions, $d_t = H[1, \mathbf{z}_t, s_t]^T$, with H being a matrix with as many rows as decision variables in d_t. To generate artificial time series we use the original non-linear problem (production function, resource constraint, law of motion of capital) plus the linear decision rule/s. This is the procedure followed to solve the basic stochastic growth model and the Hansen (1985) model. In the first model, the outcome of the algorithm is one linear decision rule for investment as a function of technology and lagged capital. For Hansen (1985) we obtain two linear decision rules, for investment and labor, as functions of technology and lagged capital. For the cash-in-advance model, important changes are needed, due to the distortion introduced by the cash-in-advance constraint. In addition to taking a quadratic approximation to the return function, it is necessary to assume that the perceived law of motion for the inverse of real money balances is linear in the state variables. These changes are described in detail in Kydland (1989) and Cooley and Hansen (1989). To solve this monetary model, we simply take the decision rules provided by Cooley and Hansen (1989) and restrict ourselves to parametric cases considered in that paper, to make our work comparable to the analysis in den Haan and Marcet (1994), who use the same parameters.

3.1.2 Undetermined Coefficients (UHL)

This method consists of log-linearizing the equations characterizing the equilibrium and solving for the recursive laws of motion with the method of undetermined coefficients. We use the approach in Uhlig (1999). Closely related contributions are King, Plosser and Rebelo (2002), Campbell (1994), Binder and Pesaran (1998). The steps to follow are:

1. Find the first order conditions and compute the steady state.
2. Log-linearize the equations characterizing the equilibrium to make the system approximately linear in log-deviations from steady state, and write the system in the form[4],

$$\begin{aligned} 0 &= \Upsilon_1 s_t + \Upsilon_2 s_{t-1} + \Upsilon_3 v_t + \Upsilon_4 \mathbf{z}_t \\ 0 &= E_t\left[\Upsilon_5 s_{t+1} + \Upsilon_6 s_t + \Upsilon_7 s_{t-1} + \Upsilon_8 v_{t+1} + \Upsilon_9 v_t + \Upsilon_{10} \mathbf{z}_{t+1} + \Upsilon_{11} \mathbf{z}_t\right] \\ \mathbf{z}_{t+1} &= \Upsilon_{12} \mathbf{z}_t + \varepsilon_{t+1}, \;\; E_t[\varepsilon_{t+1}] = 0, \end{aligned} \tag{8}$$

   where, again, s_t is a vector with the endogenous states, $\mathbf{z}_t$ contains the exogenous states and v_t is a vector of other endogenous variables of the system. Matrix Υ_{12} has only stable eigenvalues.

[4]For this use the following rules: $\tilde{x}_t \equiv \log(X_t) - \log(X_{ss}) \iff X_t = X_{ss} e^{\tilde{x}_t}$, $e^{\tilde{x}_t + a\tilde{y}_t} \approx 1 + \tilde{x}_t + a\tilde{y}_t$, where $\tilde{x}_t \tilde{y}_t \approx 0$, and $E_t\left[a e^{\tilde{x}_{t+1}}\right] \approx E_t\left[a \tilde{x}_{t+1}\right]$ up to a constant.

3. Let the recursive equilibrium law of motion be those matrices Ξ_1, Ξ_2, Ξ_3 and Ξ_4 that make stable the system

$$\begin{bmatrix} s_t \\ v_t \end{bmatrix} = \begin{bmatrix} \Xi_1 & \Xi_2 \\ \Xi_3 & \Xi_4 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \mathbf{z}_t \end{bmatrix} \tag{9}$$

4. Find estimates for the elements in matrices Ξ_1, Ξ_2, Ξ_3 and Ξ_4 by equating the coefficients of (8) and (9) according to the well-known method of undetermined coefficients, and choosing the values that make (9) stable. For technical details see Uhlig (1999).

One can easily generate time series of size T for all the elements of s_t and v_t using the state-space representation (9) and the law of motion for $\mathbf{z}_t$, given s_0 and $\mathbf{z}_0$.

3.1.3 Eigenvalue/Eigenvector Decompositions (SIM, SIL)

This approach rests heavily on Blanchard and Kahn (1980) and, specially, on Sims (2002), and it is explained in detail and applied to different setups in Novales *et al.* (1999). A related contribution is Klein (1998). Its specific characteristic is that each conditional expectation is considered as an additional variable to solve for, being defined as the realized value of the function inside the expectation, plus a forecast error. The stability conditions associated with the linear approximation to the model are added to the original non-linear problem. In each of the models we consider, the single stability condition takes the form of a highly non-linear function relating the conditional expectation to contemporaneous exogenous states and past endogenous states. The steps to follow are:

1. Find the first order conditions of the problem. Define the conditional expectation as a new variable, W_t, and add to the system of first order conditions and constraints an equation defining each associated expectation error[5]. Find the steady state.

2. Linearize (SIM) the resulting set of equations around steady-state (or log-linearize it in the case of the SIL method) :

$$\Gamma_0 u_{t+1} = \Gamma_1 u_t + \Psi \varepsilon_{t+1} + \Pi \zeta_{t+1}$$

   where u_t is a subset of the vector $\{s_t, v_t, \mathbf{z}_t, W_t\}$, ε_t contains the innovations in the laws of motion of the exogenous states, and ζ_t is the vector of expectations errors.

3. Locate the unstable roots of the pair (Γ_0, Γ_1). There is just one such a root in each of the models we consider. Matrix $\Gamma_0^{-1}\Gamma_1$ has a Jordan decomposition $P\Lambda P^{-1}$, where Λ is a diagonal matrix containing the eigenvalues of $\Gamma_0^{-1}\Gamma_1$ and P^{-1} is the matrix which has as rows the left eigenvectors.

4. Find the stability conditions. A stationary solution to the model requires the time paths of the variables to lie on the stable manifold of the solution space, which can be achieved imposing that some linear combinations of the variables are zero every period. If P^s is

[5] As an example, to solve the basic growth model, define $W_t = E_t[c_{t+1}^{-\eta} R_{t+1}]$. Then, to implement this method substitute equation (2) for $c_t^{-\eta} = \beta W_t$, and rewrite (3) as $\xi_t = [c_t^{-\eta} R_t] - W_{t-1}$.

the row of P^{-1} associated with an unstable eigenvalue, then a stationary equilibrium must satisfy,

$$P^s u_t = 0, \;\; \forall t$$

As explained in Novales *et al.* (1999), this condition can be written to relate the conditional expectation, W_t, to the other variables in u_t in a linear or an exponential way, depending on whether we are using SIM or SIL approximation. Alternatively, it relates the vector of rational expectations errors to the vector of innovations in the exogenous shocks.

To simulate the approximated economy, take the original non-linear problem (Euler equations, production function, resource constraint, law of motion of capital) and solve for the expectation through the stability condition. Mixing the original non-linear structure with the stability condition implies solving a non-linear system of equations in each step of the simulation process, and so the solution method tends to be computationally more demanding than other methods based on linear approximations.

3.2 Parameterized expectations (PEA)

This approach consists in parameterizing the conditional expectation in the stochastic Euler equation. The conditional expectation is specified as a function of the state of the system, and the parameters of that function are estimated before solving the model. For a detailed explanation see den Haan and Marcet (1990), Marcet (1993), Marcet and Marshall (1994) and Marcet and Lorenzoni (1999). The steps to follow are:

1. Find the first order conditions of the problem and compute the steady state.
2. Substitute each conditional expectation, W_t, by a parameterized polynomial function $\psi(q; s_t, \mathtt{z}_t)$, where q is a vector of parameters. Define the residual $\hat{W}_t - \psi_t$, where $\hat{W}_t$ is the realized value of W_t. In principle ψ_t should approximate the conditional expectation arbitrarily well by increasing the order of the polynomial.
3. Choose an initial value for q.
4. Use the first order conditions and constraints of the problem (with the conditional expectation substituted by $\psi(q; s_t(q), \mathtt{z}_t)$) to generate time series paths for the variables of the economy.
5. Define $S : \Re^m \to \Re^m$, where m is the dimension of q, and
$$S(q) = argmin_q \, E_t \left[\hat{W}_t(q) - \psi_t(q; s_t(q), \mathtt{z}_t) \right]^2 .$$
6. Iterate until $q = S(q)$. This guarantees that if agents use ψ_t as their expectation function, then q is the best parameter vector they could use, in the sense that it minimizes the mean squared error to the true expectation. To find each q^{i+1} starting from a previous q^i, take the residual sum of squares from a nonlinear regression of

$\hat{W}_t(q^i)$ on $\psi_t(q^i; s_t(q^i), \mathbf{z}_t)$ as an approximation to $S(q^i)$ and update q according to the rule $q^{i+1} = q^i + \lambda_q S(q^i)$, where λ_q controls the degree of updating in each iteration[6].

4 The evaluation exercise

In this section we describe the parametric cases considered in each of the three models, as well as the tools used in the comparative evaluation of the different solution methods.

In the first two models we analyze the robustness of the results to changes in the relative risk aversion parameter and the variance of the technology shock, suggested in the literature as being the most influential parameters. An increase in risk aversion implies more concavity in the utility function and a more non-linear problem. The technology shock is the main source of dynamics, so an increase in its variance means bigger deviations around steady-state for all the variables, which should deteriorate the performance of methods that use linear approximations around steady-state.

For sensitivity analysis, we consider three values of σ_ϵ: 0.01, which is close to a usual choice in the literature (0.00721), 0.02 and 0.06. Concerning risk aversion, we moved between a lower bound of 0.5 and a highest value of 3.0. The remaining parameter values are standard: $\beta = 0.99, \rho = 0.95, \alpha = 0.36$, and $\delta = 0.025$, and remained constant in all the experiments. For the Hansen model $A_N = 2.86$. Hence, we have nine parametric cases,

CASE	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
σ_ϵ	0.01	0.01	0.01	0.02	0.02	0.02	0.06	0.06	0.06
η	0.5	1.5	3.0	0.5	1.5	3.0	0.5	1.5	3.0

In the Cooley-Hansen economy we focus on the variance of the technology perturbation, as well as on the steady-state money growth rate, analyzing the same cases as in Cooley and Hansen (1989). Parameter values are now $\beta = 0.99, \alpha = 0.36, \delta = 0.025, A_N = 2.86$. To control for persistence of the exogenous shocks, we chose as coefficients of the first-order autoregressive processes for technology and money growth: $\rho_z = 0.95$ and $\rho_g = 0.48$, and as standard deviation for the innovation in the money growth process: $\sigma_{\epsilon_g} = 0.009$. We then changed the money growth rate and the variance of the technology shock, to consider six parametric cases,

CASE	**1**	**2**	**3**	**4**	**5**	**6**
g_{ss}	1.015	1.15	1.015	1.15	1.015	1.15
σ_{ϵ_z}	0.01	0.01	0.02	0.02	0.06	0.06

[6]PEA is substantially more complex than alternative linear methods, due to some practical difficulties. One relates to selecting initial values for the q vector: this generally requires hard computational work, and if one starts to search for the fixed point in q from arbitrary initial conditions, convergence is almost never achieved. Instead of using homotopy techniques to determine initial conditions, as suggested in den Haan and Marcet (1990) and Marcet and Lorenzoni (1999), we estimated adequate initial conditions starting from a log-linear solution method. This proved to be faster and computationally efficient, since stationarity and ergodicity of the time paths obtained under the initial parameterization is guaranteed. In addition, it is very important for the solution to be accurate to select an adequate order of the polynomial, which requires going repeatedly over the steps outlined above.

We solved each model for each parametric case with all the methods. For the sake of robustness, we computed 250 simulations of length $T = 150$ and 250 simulations of length $T = 3000$. Size 150 is representative of a standard quarterly sample length, while a size of 3000 is a more reliable sample length for statistical purposes.

In the basic growth model, results for cases 8 and 9 when $T = 3000$ when solving with SIM are not shown, due to some negative value of k_t arising for every draw of z_t. For the Hansen model, in the high variance cases $7, 8$ and 9 with $T = 3000$, it was not possible to find a solution with the LQA and SIM methods for the same reason. As for the Cooley-Hansen model, the same problem occurred with the SIM method in the high variance cases 5 and 6, with $T = 3000$. When $T = 150$ this method generated negative values for the capital stock for about 30% of the realizations of the shocks in those parametric cases, and we repeated the simulation process until we had 250 valid simulations. We had to make the same exercise in the same cases and sample size when solving this model with LQA, because it generated negative values of the capital stock for about 70% of realizations of the shocks. In contrast to SIM, it was always feasible to achieve a solution using SIL.

For each simulation we calculated two sets of measures, described in the following subsections. The first set has to do with the numerical accuracy of the solution, which we discuss by testing whether the stochastic Euler equation residual ξ_t, defined by (3) for the first two models and (7) for the Cooley-Hansen economy, satisfies the properties implied by rationality. The second set of measures deals with the statistics usually examined in empirical studies to assess the model's responses to meaningful economic questions. It is crucial to analyze whether the answer to these questions depend on the solution method being implemented.

4.1 Expectations error properties

4.1.1 Correlations of ξ_t with the available information set: the den Haan and Marcet accuracy test

The idea of the test proposed in den Haan and Marcet (1994) is to check whether there exists any function of variables dated t or earlier that helps predict ξ_{t+1}. That would be a strong deviation from rationality. To implement the test, the steps to follow are: i) obtain a large number of observations by simulating the model for a long realization of the exogenous processes; ii) run a regression of ξ_{t+1} over I_t, a list of instruments selected from the set of variables in the time t information set; iii) define $\hat{a} = (\sum I_t^T I_t)^{-1}(\sum I_t^T \xi_{t+1})$ and form the statistic:

$$M = \hat{a}^T(\sum I_t^T I_t)(\sum I_t^T I_t \xi_{t+1}^2)^{-1}(\sum I_t^T I_t)\hat{a} \sim \chi^2_{m_1 m_2},$$

where m_2 is the number of instruments chosen and m_1 is the number of Euler equation errors, which is equal to one in our three models. The statistic M provides a test for the rational expectations hypothesis: $E_t(\xi_{t+1}) = 0$. It is worth noting that the alternative hypothesis is that the error is not a martingale; so if the value of the statistic belongs to the *upper* critical region of the $\chi^2_{m_1,m_2}$ distribution, there is evidence against the accuracy of the solution.

The number of observations used can be interpreted as a measure of how stringent the criterion is: that the solution passes the test even for a very large number of data points should be taken as evidence that the solution is very accurate. We have chosen as set of instruments $I_t = [1, k_t, k_{t-1},\ k_{t-2}, \log(z_t), \log(z_{t-1}),\ \log(z_{t-2})]$, so that the test statistic has a χ^2_7 distribution. This is the same set of instruments used by den Haan and Marcet (1994) for the basic model and a standard deviation for the technology shock of 0.02 or 0.06. Even though they could only use a constant as instrument in the low variance case, $\sigma_\epsilon = 0.01$, we were able to use the full set of instruments I_t in all our parametric cases. The better behavior of our PEA solution seems to arise from using as initial conditions for vector q in the expectations polynomial estimates obtained through time series resulting from a the parameters in the expectations polynomials the numerical estimates obtained from the log-linear version of the model. We also used I_t as instruments when testing accuracy of the solutions to the Hansen (1985) and Cooley and Hansen (1989) models.

4.1.2 Time series dependence properties of ξ_t

We also checked for autocorrelation in the conditional expectation residual, ξ_t obtained from each model. We first fitted an AR(1) process with constant to the generated residual ξ_t,

$$\xi_t = \mu + \rho\xi_{t-1} + \epsilon_{\xi_t}, \tag{10}$$

and tested the two null hypothesis $H_0 : \mu = 0$ (zero-mean) and $H_0 : \rho = 0$ (no serial correlation) using conventional t-tests. Under rationality, the conditional expectation residual should have no significant mean and no autocorrelation, since it is a one-period-ahead, rational expectations prediction error. The resulting information on these two issues is complementary to that provided by the den Haan-Marcet test.

4.2 Other characteristics of the implied solutions

4.2.1 Decision Rules

For each solution method and model, we tabulated the values of the decision variables at alternative points in the space of state variables. After building a grid of values for the state variables, we used the decision rules to obtain the implied values for the decision variables. The LQA decision rules arise, as already mentioned, from the linear function $d_t = H[1, \mathbf{z}_t, s_t]^T$. For UHL they are obtained from the log-linear relation $s_t = \Xi_1 s_{t-1} + \Xi_2 \mathbf{z}_t$ in (9), while SIM/SIL's decision rules correspond to the stability conditions $P^s u_t = 0$. Concerning PEA, a system of equations of the kind $F(d_t, \psi_t(q; s_t, \mathbf{z}_t)) = 0$ is used. The reader can see for each model and method the exact definition of the vectors d_t, s_t and $\mathbf{z}_t$, as well as those of H, Ξ_1, Ξ_2, P^s and $\psi_t(q; s_t, \mathbf{z}_t)$ in the Appendix.

Concerning the capital stock, for each of the three models we selected 25 equally spaced values in a ten percent interval around k_{ss}. In relation to the technology shock, for the basic growth model we got again 25 equally spaced values, between 0.4 and 1.6. For Hansen's model the range of variation for the technology shock was narrower, between 0.8 and 1.2, due to numerical problems with the LQA and SIM decision rules. As for the Cooley-Hansen model, we performed two similar exercises: on the one hand, we fixed z_t at its steady state value of 1.0, and selected 25 equally spaced observations for g in a ten percent interval around g_{ss}. On the other hand, we fixed g_t at its steady state, and chose 25 equally spaced data in a 20% interval a around z_{ss}.

4.2.2 Sample cross correlations, standard deviations and means

We compute the autocorrelation function for output, $\rho(y_t, y_{t-j})$ in each simulation. For a given variable X_t, we also obtain its cross-correlation with output, $\rho(y_t, X_{t+j})$, $j \gtrless 0$, standard deviation, σ_X, and sample mean, $\bar{X}$. This way, we produce random samples of size 250 for each statistic.

Since most papers report average values across simulations for some of these statistics, we check whether they differ among solution methods[7]. Dispersion in the sample of N values of a given statistic obtained from a solution method is usually very small for reasonable values of N. This is the main reason why sample means may turn out to be significantly diferent for different methods, since no method produces a systematic bias in any variable. In other cases, a method may have some difficulty in fully capturing the serial correlation in a variable or the correlation between two variables, this test again showing statistically significant differences between average values of the relevant statistics across the set of N simulations. Even though we performed the calculations for a wide set of variables, we only show the results for those variables we deem more representative. In the basic growth model we only look at consumption, $X_t = [c_t]$. As regards Hansen's model we considered employment, given the emphasis placed on the labor market, $X_t = [N_t]$. Finally, for the Cooley-Hansen model, we present statistics for labor and inflation, $X_t = [N_t, \pi_t]$ [8].

5 Results

To compute empirical distributions for each statistic, we repeated the following steps for each of the 250 simulations run with each model, parameter vector, and sample size: i) generate a

[7]Let us denote by γ_i^k a particular statistic obtained from the $i-$th simulation, $1 \leq i \leq N$, with method k . Let μ_{γ^k}denote the population mean for γ_i^k and $a_{\gamma^k}, s_{\gamma^k}$ the sample mean and standard deviation calculated from the sample of N simulations. To test $H_0 : \mu_{\gamma^{k_1}} = \mu_{\gamma^{k_2}}$for any two different methods k_1, k_2, we can use the large sample approximation,

$$\| \frac{a_{\gamma^{k_1}} - a_{\gamma^{k_2}}}{\sqrt{\frac{s^2_{\gamma^{k_1}} + s^2_{\gamma^{k_2}}}{N}}} \| \sim N(0,1)$$

[8]We also implemented non-parametric Kolmogoroff-Smirnov tests, to see whether the empirical *distribution* of a given statistic was the same across the different solution methods. Let the statistic generated with method k in each simulation $i \in N$ be γ_i^k. Let $F_k(X)$ be the probability distribution function of $\{\gamma_i^k\}_{i=1}^N$. For each pair of methods k_1, k_2, we want to test $H_0 : F_{k_1} = F_{k_2}$. The Kolmogorov-Smirnov test is based on the fact that the statistic $KS = sup|F_{k_1} - F_{k_2}|$ has asymptotic distribution,

$$\lim_{N \to \infty} Prob\left(\sqrt{\frac{N}{2}} KS \leq \lambda_{KS}\right) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2j^2\lambda_{KS}^2},\ \lambda_{KS} > 0.$$

The results pointed in the same direction than those obtained with the previous test, and are not reported. Similarly, we used the set of first order conditions and decision rules to generate the response functions of the main variables to a one standard deviation impulse in the shocks. As differences across methods were again negligible, we do not provide the results to save space.

realization of the exogenous shock $\{\mathbf{z}_t\}_{t=1}^{T}$, ii) use it to implement each method (LQA, UHL, SIM, SIL, PEA) to generate time series for all the variables, iii) compute the set of statistics.

We show here a sample of results, selected according to their relevance for the aim of the paper. The whole set of results is available from the authors upon request.

5.1 Basic Neoclassical Growth Model

5.1.1 Expectations error properties

Tables 1 and 2, and figures 2, 3 and 4, summarize the main results for the basic growth model using the five solution approaches.

Insert tables 1 and 2

Insert figures 2, 3,4

In figure 2, we show the results of the den Haan-Marcet test for the linear approximation-based methods: SIM and LQA. The performance of these solutions deteriorates for a large standard deviation of the technology shock, for any sample size, rejecting the null hypothesis of zero correlation between the expectations error and variables in the information set much more often than in 5% of the simulations. This result is intuitive, since a larger deviation from steady-state makes local approximations in levels to be less accurate. When T=150, SIM tends to behave slightly better than LQA, although both solutions fail to pass the test when T=3000, in the sense that the percentage of rejections of the null hypothesis is well above 5%. As already mentioned, the SIM solution could not be obtained for T=3000 and σ_ϵ=0.06. On the other hand, the SIL and UHL "almost" linear solutions are fairly accurate for the nine parametric cases analyzed and both sample sizes, passing the den Haan-Marcet test in about 95% of the realizations. This is the most salient feature in figure 2: when working with logged variables, as in the SIL and UHL methods, an increase in the variance of the technology shock does not deteriorate the statistical properties of the solution, possibly because of the homoskedasticity effect induced by the log-transformation. As regards the effect of the relative risk aversion parameter (still in figure 2), the performance of the SIM and LQA solutions in terms of the den Haan-Marcet test deteriorates for low values of η, i.e. for high values of the elasticity of intertemporal substitution of consumption, while the SIL and UHL solutions are again barely affected[9].

Table 1 shows the results of testing for a significant mean as well as for significant autocorrelation structure in an estimated AR(1) model for the expectations error. There is

[9]Regarding PEA solutions, the statistic associated to the den Haan-Marcet test was precisely the criterion used to accept a particular set of parameters for the polynomial approximation to the expectations equation in each case. So, the PEA solution passes the test at roughly the chosen 5% significance level in all cases.

no evidence of a significant mean in any parametric case and sample size, but there is evidence of a significant autoregressive coefficient under some parameterizations for the LQA and SIM solutions. Rejection of the null hypothesis tends to arise more often than suggested by the 5% significance level for simulations with high elasticity of intertemporal substitution of consumption and high innovation variance. Rejection becomes much more frequent when T=3000. A more frequent evidence of autocorrelation in ξ_t for high elasticity of substitution values may explain the more important failure of the den Haan-Marcet test in those cases. The representative agent then does little smoothing, adjusting consumption to income fluctuations, and the LQA and SIM methods fail to fully capture the higher consumption volatility in these cases. These methods seem to impose more inertia in the expectations mechanism than there actually is in such cases, thereby inducing some spurious autocorrelation in the expectation error.

A very similar effect is produced by an increase in the volatility of the exogenous shock. That will again produce a more volatile decision variable, and methods that impose more inertia in the expectations mechanism will tend to exhibit deviations from rationality. So, it is not surprising that rejections of the den Haan-Marcet test as well as evidence of autocorrelation are more important for high elasticity of intertemporal substitution, as well as for a high variance of the exogenous shock.

For the UHL, SIL and PEA solutions there is no significant evidence of autocorrelation in ξ_t.

Figure 2 and Table 1 refer to possible deviations of rationality. The LQA and SIM solutions tend to produce expectations errors which are correlated with variables in the information set available when forming the conditional expectation, and show some evidence of autoregressive structure. Although they are not independent of each other, these characteristics are very damaging for an interpretation of the time series obtained from the described implementation of these methods as rational expectations solutions. On the other hand, there is essentially no evidence on violation of rationality for the SIL, UHL and PEA solutions.

5.1.2 Other measures

To evaluate the decision rules according to whether decision variables are increasing or decreasing in the state variables we present figure 3 for the LQA, SIM, SIL and UHL solutions, and figure 4 for the PEA solution. In them, the stock of capital takes 25 equally spaced values in a ten percent interval around the deterministic steady-state k_{ss},while z takes 25 values around its steady state value of 1.0, from 0.4 to 1.6. The SIM, SIL,UHL and PEA decision rules are monotonically increasing over the selected values of the state variables, for all parametric cases. The LQA decision rule for consumption is non-monotonic in technology, although it is always increasing in capital. According to this decision rule, for any given level of capital, consumption falls when the value of the technology shock moves from zero to 0.90, ten percent below its deterministic steady state value of 1.0, increasing above the 0.90 threshold. This lack of monotonicity is unlikely to reflect an optimal consumption behavior. If it did, it would be a feature not captured by any other solution approach, which seems unlikely. Solving the basic growth model under full depreciation and a discrete three state first order Markov chain for technology, Christiano (1990) reports the same lack of monotonicity, for a high standard deviation of the technology shock, $\sigma_\epsilon = 0.1$, the anomaly not arising in his work for a low standard deviation, $\sigma_\epsilon = 0.01$.

As for the PEA, a strange non-monotonicity occurred in case 2, even though the polynomial responsible of that parametric case decision rule passed the den Haan-Marcet test. In that case, when k takes its lowest value, consumption declines when the technology shock increases from its lowest value towards $z = 0.55$, increasing with z_t for values above $z = 0.55$. Additionally, when k exceeds its deterministic steady state value by more than ten percent, consumption decreases as z moves from 0.40 to 0.65, increasing from that level on. This result suggests the convenience of using some other criterion, additional to the den Haan-Marcet test, when looking for the fixed point for the polynomial parameterization of the conditional expectations in this solution approach.

There are no significant differences among the means, standard deviations and cross correlations generated with different solution methods in cases 1 to 6. Only in the high variance cases, $\sigma_\epsilon = 0.06$, we can appreciate some deterioration in methods that rely on linear approximations in levels around steady-state: LQA and SIM, in that the statistics they produce are significantly different from those of other methods. From the results of the previous tests, we believe that these two methods are to blame for the differences. Table 2 presents the outcome for case 9: the contemporaneous and lagged correlation of consumption with output, as well as the first two output autocorrelations do not statistically differ across methods. However, the mean of the consumption series generated by the LQA and UHL approximations significantly differs from those generated by the SIL and PEA methods when $T = 3000$. When $T = 150$, the standard deviation of consumption generated with SIM is different from those obtained with SIL, UHL and PEA. When $T = 3000$ the SIM solution could not be computed, but the standard deviation of consumption from the LQA solution is significantly different from those of the UHL, SIL and PEA methods.

To summarize: the performance of the UHL, SIL and PEA solutions is almost identical in all the analyzed dimensions. Linear approximations in levels (LQA, SIM) are less accurate when looking at properties of the prediction error, showing significant deviations from rationality. They also tend to perform slightly worse for high technology shock variances in terms of the mean and variance of decision variables, for which they occasionally produce values significantly different in average from those obtained with the other solution methods. We also observe a non-monotonic behavior in the linear LQA decision rule for consumption that does not appear with any other solution method.

5.2 Hansen (1985) Model

5.2.1 Expectations error properties

The qualitative results emerging from tables 3, 4, and 5, and figures 5, 6 and 7, are similar to those obtained for the basic growth model.

Insert tables 3, 4 and 5

Insert 5, 6 and 7

Figure 5 summarizes the results of the test for the linear approximation-based methods: the SIM method seems now to be more sensitive than LQA to a higher technology shock variance. Both solutions, and specially the former, deteriorate in terms of the den Haan-Marcet test for both sample sizes when the variance increases. As in the more basic model, the SIL and UHL "almost" linear solutions are fairly accurate for the nine parametric cases analyzed and both sample sizes, passing the den Haan-Marcet test in approximately 95% of the simulations. This consistent behavior seems to arise from performing the approximation in logged-variables. As regards the effect of the relative risk aversion parameter, the SIM and LQA solutions again behave worse for low values of η, reaching a very high percentage of rejections of the null hypothesis of lack of correlation between expectations errors and variables in the information set. The performance of SIL and UHL is uniformly good for all values of η.

Figures in table 3 show the results of the tests on the estimated AR(1) model for the expectations error. As in the basic growth model there is nowhere evidence of a significant mean in the expectations error. Statistically significant autoregressive coefficients for the expectation error that emerges from the LQA and SIM solutions tend to be again associated to a high elasticity of intertemporal elasticity of substitution and to a high variance of the technology shock. Jointly with the rejections to the den Haan-Marcet test, this result raises again serious questions regarding the interpretation of the obtained time series as being the rational expectations solution to the model. Reasons for this failure are those described in the basic growth model.

5.2.2 Other measures

SIM, SIL, UHL and PEA decision rules show consumption increasing with both state variables, capital and technology, their values being essentially identical. Figure 6 shows results for "almost" linear methods, while figure 7 shows results for PEA. In these figures, the stock of capital takes 25 equally spaced values in an interval of five percent around its deterministic steady state level k_{ss},while z takes 25 values around its steady-state, from 0.8 to 1.2. We have reduced the range of variation for state variables relative to the basic growth model, because we would otherwise get systematic sign violations for consumption and capital in the LQA and SIM solutions.

Again, the LQA decision rule is non-monotonic in technology, although it is increasingly monotonic in capital. The non-monotonicity effect is less important than in the basic growth model. It shows, for any level of capital, consumption falling when technology moves from zero to 0.90, ten percent below its deterministic steady state value of 1.0, and increasing from that level on.

We do not detect significant differences among the means, standard deviations and cross correlations generated with different methods in cases 1 to 5. But, even with not very large volatility, SIM contemporaneous and lagged correlations of labor with output for $T = 3000$ in case 6 are significantly different from those obtained with the PEA method at the 95% level, and from those obtained with LQA, UHL and SIL at the 90% level [see table 4]. We also present table 5 as an example of results for high technology shock variance cases: SIM correlations between output and labor are significantly different from those obtained from alternative solutions at the 95% level when $T = 150$ (remember we could not solve with SIM for $T = 3000$). The mean of labor from the UHL solution differs from that obtained from SIL and PEA when $T = 3000$.

Summing up, the performance of UHL, SIL and PEA solutions to the "indivisible labor" model is, again, almost indistinguishable in all the dimensions analyzed, except for discrepancies in the mean value of labor in extreme parametric cases among UHL on the one hand, and SIL and PEA on the other. Concerning the den Haan-Marcet accuracy test, LQA and SIM behave badly, showing correlation between the expectations error and variables which were known when the conditional expectation was made. They also tend to present significant autocorrelation in the expectations error for high variance cases and low elasticity of intertemporal substitution of consumption. As in the basic growth model, these failures are related to each other. Again, a strict interpretation of these as being rational expectations solutions is questionable. The non-monotonic performance of the linear LQA decision rule for consumption relative to technology appears again, although it is now weaker.

5.3 Cooley and Hansen (1989) Model

5.3.1 Expectations error properties

The results for the Cooley-Hansen model are shown in tables 6 and 7, and figures 8, 9 and 10.

Insert tables 6 and 7

Insert figures 8, 9 and 10

It is important to point out that the implementation of the LQA method to solve this model is different from that used for the two previous economies, in which the competitive solution was Pareto efficient. Therefore, comments regarding the LQA solution should not be read as a smooth transition from those made when applied to the two non-monetary models.

Regarding the den Haan-Marcet test, the results obtained when solving the previous models also hold for the monetary model. Figure 8 now shows the percentage of rejections as a function of the steady state rate of money growth and the variance of the technology shock, so they are not comparable to those in the previous models. The log-linear SIL and UHL solutions passed the test with an approximate significance level of 5%, and did not deteriorate with an increased variance for the technology shock. The effect of an increased rate of growth of money on the den Haan-Marcet test for these two solutions is also negligible.

On the other hand, when $T = 3000$, LQA and SIM solutions deteriorate for a higher variance of the technology shock, as in the previous models. Moving from $\sigma_\epsilon = 0.01$ to $\sigma_\epsilon = 0.02$ in the LQA solution, the percentage of rejections to the den Haan-Marcet test jumps from 21% to 64%, and from 37% to 96% in the SIM method. Also, for a given variance of the technology shock, the greater the growth rate of money, the worse the performance of the LQA and SIM solutions. This is intuitive since as g_t is log-normal, an increase in g_{ss} implies an increase not only in the mean of g_t, but also in its variance σ_{ϵ_g}. When $T = 150$, the LQA and SIM solutions do not fail to pass the test so often as when $T = 3000$, due to the lack of power of the test for low values of T.

Figures in table 6 show again no evidence of a significant mean in the expectations error, although there is some indication of serial correlation, specially in the higher variance cases, for the LQA and SIM solutions. For the LQA solution, that evidence becomes very clear for T=3000.

5.3.2 Other measures

As regards decision rules, the LQA approach to solving non Pareto optimal problems proposed by Kydland (1989) and Cooley and Hansen (1989) does not present the non-monotonicity problem we obtained for standard social planner problems. Consumption increases with both, technology and capital, and decreases with an increasing money growth. The SIM, SIL, UHL and PEA solutions also have these properties. What is more, the grids are quite similar among all five solution approaches [figures 9 and 10].

Finally, table 7 shows the statistics to test for differences in the mean of sample averages, standard deviations and cross correlations generated with the different solution methods for case 6, the one with a higher variance for the technology shock and a higher money growth: σ_ϵ =0.06 and g_{ss} =1.15. We did not appreciate any significant differences in these tests for cases 1 to 4, while the picture for case 5 is very similar to that for case 6. When T =150, the statistics to compare LQA with the SIM, SIL, UHL and PEA solutions exceed the 5% or the 15% [critical value 1.0364] significance level when applied to the second autocorrelation of output as well as to the contemporaneous and lagged correlations of output with labor and inflation. For that sample size, and at the 5% or 10% level, SIM also tends to differ from SIL, UHL and PEA concerning the mean of labor, the contemporaneous and lagged correlation of labor with output, and the contemporaneous correlation of inflation with output. The same applies when T = 3000: the LQA solution seems to be significantly different from those obtained with SIL, UHL and PEA [remember that it was not possible to implement SIM in this case]. A last comment on table 7 refers to the mean of labor that arises from UHL solution when T = 3000: it is different from those obtained with SIL and PEA, a phenomenon similar to that observed in Hansen's model for extreme parameter values.

Hence, the performance of UHL, SIL and PEA solutions to the Cooley-Hansen cash-in-advance economy is again almost identical in all the analyzed dimensions, except for average labor in the high variance cases. As in the previous non-monetary models, the LQA and SIM solutions violate rationality, since they perform badly in terms of the den Haan-Marcet test and tend to show some significant autoregression coefficients for the expectations error. The non-monotonicity of the linear LQA decision rule for consumption in the non-monetary models disappears in the version of the method designed to cope with non Pareto optimal settings that we have applied here.

6 Concluding remarks

We have evaluated applications of different "almost" linear numerical solution methods for nonlinear rational expectations models, to three different versions of the neoclassical exogenous growth economy. In essence, all methods are very similar in spirit, all of them searching for the same stable manifold of the linearized/log-linearized system of equations to which they are applied, although important differences arise from the way they are usually implemented. We have also looked at the Parameterized Expectations solution, as a way to compare "almost" linear methods with a nonlinear method. It is a merit of this paper to present an homogeneous evaluation and comparison of the properties of the different solution methods, using a common realization of the shock/s in the economy.

The relative performance of the numerical solutions did not worsen when departing from the basic growth model. This may be due to the fact that endogenous dynamics are weak in all the model economies considered, and the shape of the time series generated by different methods inherit the pattern of the common exogenous shocks.

For the economies considered, refining a linear solution by mixing structure of the original non-linear problem with some structure of the system approximated in levels of the variables around steady-state, as in Sims (2002)-Novales *et al.* (1999), eliminates strange non-monotonicity properties that tend to appear with the standard linear quadratic approximation. As regards solutions in logs, the more refined approach in Sims (2002)-Novales *et al.* (1999) appears to be slightly more robust to extreme parameterizations than the solution in Uhlig (1999).

Also, the experiments we have carried out suggest that these two approaches, in logged variables, are as accurate as Parameterized Expectations in all the analyzed dimensions (expectations error rationality, mean and standard deviation of decision variables, cross correlations with output, induced decision rules) and for all the model economies considered. In particular, these two methods do not present any evidence on violation of rationality. On the contrary, approximations in levels tend to produce some deviations from rationality in the expectations error. There is some evidence suggesting that these approaches fail to rightly capture the dynamics embedded in the conditional expectations, in the sense of imposing more inertia in the conditional expectations than there actually is in the model. That leaves some spurious autocorrelation in the expectation error that leads to failure of the rationality tests. We have also shown that, as pointed out in Taylor and Uhlig (1990), numerical solutions that differ in their behavior with respect to the den Haan-Marcet rationality test, may also show significant differences in terms of other statistics: means, standard deviations or the sign of the relationships involved in the decision rules.

When working in logs, an increase in the variance of exogenous shock/s does not deteriorate the solution, possibly due to the homoskedasticity effect induced by the log-transformation. Working in logs seems to be very advisable when solving non-linear rational expectations models, specially in view of the rationality properties of the solutions implied in both cases. In fact, with independence of the solution approach, writing the model in logs of the variables seems to be more important than preserving more of the non-linear structure of the original model.

These results contrast with Dotsey and Mao (1992), where log-linear methods did not dominate linear methods, and where *more refined* linear or log-linear methods did not dominate *less refined* ones. Although the model they consider departs from the neoclassical growth model more than those we have analyzed, we presume that the different results may arise from their use of a five-state Markov chain for the only source of exogenous dynamics, a process for tax rates. Discussing solutions to the basic neoclassical growth model in section 5.1.2, we have also seen how the UHL and SIL log-linearizations seem to perform better than the log-linearization of the same model in Christiano (1990) where the log-linear decision rule presents the non-monotonic property for high variance technology shocks. These comparisons suggest that, contrary to some conventional wisdom, the choice of using discrete versus continuous probability distributions for exogenous shocks is fully relevant.

Several interesting questions have left aside in this paper and are important in the context of solution methods evaluation: considering models with more state variables, economies in transition to steady state after having experienced some perturbation, economies with a richer endogenous dynamics, or economies with heterogeneous agents. These are some of the interesting extensions of this work.

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Appendix

A Decision Rules

For an even more detailed exposition on how to obtain the results in this appendix, there is a Technical Appendix, available from the authors on request.

A.1 Basic Neoclassical Growth Model

For the LQA solution, we have $s_t = [k_{t-1}]$, $\mathbf{z}_t = [\log(z_t)]$ and $d_t = [x_t]$. For all the parametric cases considered, the coefficients in the decision rule $d_t = H[1, \mathbf{z}_t, s_t]^T$ are,

CASE	H
1,4,7	[1.9190, 3.2243, -0.0255]
2,5,8	[1.0512, 2.7668, -0.0027]
3,6,9	[0.7015, 2.7244, 0.0065]

changing only for different degrees of risk aversion. From the resource constraint and the production function, we can write consumption as a function of last period capital and the contemporaneous technology shock, $c_t = z_t k_{t-1}^{\alpha} - H[1, \log(z_t), k_{t-1}]^T$.

To solve with the UHL method, we choose: $s_t = [\tilde{k}_t]$, $v_t = [\tilde{c}_t,\ \tilde{R}_t,\ \tilde{y}_t]^T$, $\mathbf{z}_t = [\tilde{z}_t]$, where, along this Appendix, ˜ denotes log-deviations from steady state. Then, for the analyzed cases, the matrices in (9) become,

CASE	Ξ_1	Ξ_2	Ξ_3^T	Ξ_4^T
1,4,7	0.9495	0.0849	[0.8361, 0.1742, -0.0222]	[0.0348, 0.3600, 1.000]
2,5,8	0.9723	0.0728	[0.5210, 0.3403, -0.0222]	[0.0348, 0.3600, 1.000]
3,6,9	0.9815	0.0717	[0.3940, 0.3557, -0.0222]	[0.0348, 0.3600, 1.000]

As regards SIM method, we have: $u_t = [c_t - c_{ss},\ k_t - k_{ss},\ W_t - W_{ss},\ \log(z_t)]^T$, $\varepsilon_t = [\epsilon_t]$ and $\zeta_t = [\xi_t]$. Numerical estimates for the stability condition P^s are in each case,

CASE	P^s
1,4,7	[0.0000, 0.0071, 1.0000, 0.0303]
2,5,8	[0.0000, 0.0047, 1.0000, 0.0999]
3,6,9	[0.0000, 0.0015, 1.0000, 0.0474]

Notice that, since k_t is a non-linear function of k_{t-1} and z_t, the stability condition can also be expressed as a non-linear implicit function: $\varphi(W_t, k_{t-1}, z_t) = 0$.

With the SIL method the process is similar. We have: $u_t = [\tilde{c}_t,\ \tilde{k}_t,\ \tilde{W}_t,\ \tilde{z}_t]^T$, $\varepsilon_t = [\epsilon_t]$ and $\zeta_t = [\xi_t]$. The stability condition is,

CASE	P^s
1,4,7	[0.0000, 0.4403, 1.0000, 0.0497]
2,5,8	[0.0000, 0.8037, 1.0000, 0.4519]
3,6,9	[0.0000, 1.2043, 1.0000, 0.9807]

Finally, concerning the PEA solution to this model, in all the considered parametric cases, a second order polynomial approximation proved to be useful:

$$\begin{aligned}\psi_t(q;k_{t-1},z_t) &= q_1\ \exp\left(q_2\log(k_{t-1})+q_3\log(z_t)+q_4(\log(k_{t-1}))^2\right)\\ &\times\ \exp\left(q_5\log(k_{t-1})\log(z_t)+q_6(\log(z_t))^2\right).\end{aligned}$$

And we can obtain each time-t consumption from (2) using $\psi_t(q;k_{t-1},z_t)=E_t[c_{t+1}^{-\eta}R_{t+1}]$. The fixed point for vector q was calculated in each case using a sample size of 25000 observations and a four-digit accuracy stopping criterion. We set λ_q equal to one except for the cases when $\eta=0.5$, that we chose $\lambda_q=0.5$. We changed the polynomial until the solution passed the den Haan-Marcet test. Estimated parameter values were,

CASE	q_1	q_2	q_3	q_4	q_5	q_6
1	2.3473	-0.3253	-0.2258	-0.0126	0.0382	-0.0055
2	1.6293	-0.3156	-2.2440	-0.0642	0.4766	-0.4221
3	0.1162	-0.7187	-4.8308	-0.2635	1.0314	-0.5244
4	2.7395	-0.4170	-0.1762	-0.0008	0.0245	-0.0214
5	0.7466	0.1009	-1.0839	-0.1195	0.1561	-0.0971
6	1.6741	-0.7658	-3.5681	-0.0567	0.6828	-0.1533
7	2.4171	-0.3407	-0.2021	-0.0106	0.0315	-0.0207
8	3.1017	-0.6436	-0.9073	-0.0220	0.1080	-0.0861
9	2.8286	-1.0233	-2.3828	-0.0218	0.3569	-0.2553

A.2 Hansen (1985) Model

For the LQA solution, we have $s_t=[k_{t-1}]$, $\mathbf{z}_t=[\log(z_t)]$ and $d_t=[x_t,N_t]^T$. For the different parameter vectors considered, the decision rules $d_t=H[1,\mathbf{z}_t,s_t]^T$ are,

CASE	H		
1,4,7	0.7368	2.6129	-0.0332
	0.3801	0.7383	-0.0037
2,5,8	0.7368	1.7499	-0.0332
	0.5459	0.3718	-0.0168
3,6,9	0.7368	1.5342	-0.0332
	0.6127	0.2242	-0.0221

For the UHL method: $s_t=[\tilde{k}_t]$, $v_t=[\tilde{c}_t,\ \tilde{y}_t,\ \tilde{N}_t,\ \tilde{R}_t,\ \tilde{x}_t]^T$, $\mathbf{z}_t=[\tilde{z}_t]$. Then, we have,

CASE	Ξ_1	Ξ_2	Ξ_3^T	Ξ_4^T
1,4,7	0.9418	0.2063	[0.8210, 0.2702, -0.1403, -0.0254, -1.3273]	[0.4052, 2.4176, 2.2150, 0.0840, 8.2537]
2,5,8	0.9418	0.1382	[0.3930, -0.0481, -0.6376, -0.0364, -1.3273]	[0.3989, 1.7139, 1.1155, 0.0596, 5.5276]
3,6,9	0.9418	0.1212	[0.2206, -0.1763, -0.8380, -0.0409, -1.3273]	[0.2526, 1.4304, 0.6725, 0.0497, 4.8461]

Concerning SIM, we have: $u_t=[c_t-c_{ss},\ N_t-N_{ss},\ k_t-k_{ss},\ W_t-W_{ss},\ \log(z_t)]^T$, $\varepsilon_t=[\epsilon_t]$ and $\zeta_t=[\xi_t]$. Then,

CASE	P^s
1,4,7	[0.0000, 0.0000, 0.0363, 1.0000, 0.1188]
2,5,8	[0.0000, 0.0000, 0.0568, 1.0000, 0.5878]
3,6,9	[0.0000, 0.0000, 0.0724, 1.0000, 0.8781]

and for the SIL method, we have: $u_t=[\tilde{c}_t,\ \tilde{N}_t,\ \tilde{k}_t,\ \tilde{W}_t,\ \tilde{z}_t]^T$, $\varepsilon_t=[\epsilon_t]$ and $\zeta_t=[\xi_t]$. So

CASE	P^s
1,4,7	[0.0000, 0.0000, 0.4359, 1.0000, 0.1127]
2,5,8	[0.0000, 0.0000, 0.6260, 1.0000, 0.5119]
3,6,9	[0.0000, 0.0000, 0.7026, 1.0000, 0.6728]

Finally, for the PEA solution approach to this model, in all the considered parametric cases, a second order polynomial approximation proved again to be useful:

$$\begin{aligned}\psi_t(q;k_{t-1},z_t) &= q_1 \exp\left(q_2\log(k_{t-1})+q_3\log(z_t)+q_4(\log(k_{t-1}))^2\right)\\ &\times \exp\left(q_5\log(k_{t-1})\log(z_t)+q_6(\log(z_t))^2\right).\end{aligned}$$

And we can obtain each time-t consumption from (2) and $\psi_t(q;k_{t-1},z_t)=E_t[c_{t+1}^{-\eta}R_{t+1}]$. The fixed point for q was calculated in each case using again a sample size of 25000 observations and a four-digit accuracy criterion. We changed the polynomial until the solution passed the den Haan-Marcet test. We set λ_q equal to one except for the cases when $\eta=0.5$ that we set $\lambda_q=0.5$. For the different parameter values, estimated coefficients in the parameterized expectation were,

CASE	q_1	q_2	q_3	q_4	q_5	q_6
1	2.9009	-0.3869	-0.3265	-0.0047	0.0490	-0.0611
2	2.1570	0.0846	-0.7523	-0.1310	0.0612	-0.1099
3	2.2404	0.2335	-1.7301	-0.1757	0.3847	-0.2542
4	2.8956	-0.3866	-0.3594	-0.0046	0.0620	-0.0978
5	3.9471	-0.3810	-1.1585	-0.0415	0.2274	-0.1638
6	3.9666	-0.2217	-1.1962	-0.0848	0.1683	-0.0769
7	2.8364	-0.3693	-0.3593	-0.0082	0.0614	-0.1045
8	3.8988	-0.3776	-1.0778	-0.0405	0.1882	-0.1368
9	3.5446	-0.1183	-1.2966	-0.1065	0.2031	-0.0884

A.3 Cooley and Hansen (1989) Model

The equilibrium conditions for the Cooley and Hansen's problem are (5) and (6) together with $A_N=\lambda_t(1-\alpha)\frac{y_t}{N_t}$, the resource constraint and a last condition associated with the cash-in-advance constraint: $\hat{p}_t=\frac{1}{c_t}$, where $\hat{p}_t$ denotes the inverse of real money balances.

To solve this model using the LQA approach we simply took the H matrix reported by Cooley and Hansen in their paper. Now $[\hat{p}_t,N_t]^T=H\ [1,\log(z_t),\log(g_t),k_{t-1}]^T$, where

CASE	H			
1,3,5	1.88633	-0.58175	0.55474	-0.05898
	0.64419	1.73073	0.30219	-0.03318
2,4,6	2.07319	-0.66585	0.63537	-0.07726
	0.52716	1.51216	0.26423	-0.03318

Concerning the undetermined coefficients method, UHL, we have: $s_t=[\tilde{k}_t]$, $v_t=[\tilde{c}_t,\,\tilde{y}_t,\,\tilde{N}_t,\,\tilde{x}_t,\,\tilde{\hat{p}}_t,\,\tilde{\lambda}_t,\,\tilde{R}_t]^T$, $\mathbf{z}_t=[\tilde{z}_t,\,\tilde{g}_t]$. Then, for all the analyzed cases, we have $\Xi_1=[0.9418]$, $\Xi_2=[0.1552,\ 0.0271]$ and

$$\Xi_3=[0.5316,\ 0.0550,\ -0.4766,\ -1.3273,\ -0.5316,\ -0.5316,\ -0.0328]^T,$$

$$\Xi_4=\begin{bmatrix}0.4703 & 1.9417 & 1.4715 & 6.2091 & -0.4703 & -0.4703 & 0.0675\\ -0.4488 & -0.5555 & -0.0867 & 1.0850 & 0.4488 & -0.0312 & -0.0019\end{bmatrix}^T.$$

To implement the SIM solution, we have: $u_t = [c_t - c_{ss},\, N_t - N_{ss},\, k_t - k_{ss},\, W_t - W_{ss},\, \log(z_t),\, \log(g_t) - \log(g_{ss})]^T$, $\varepsilon_t = [\epsilon_{z_t},\, \epsilon_{g_t}]$ and $\zeta_t = [\xi_t]$. Then, the single stability condition is,

CASE	P^s
1,3,5	[0.0000, 0.0000, 0.0617, 1.0000, 0.4663, 0.0194]
2,4,6	[0.0000, 0.0000, 0.0699, 1.0000, 0.4663, 0.0194]

For the SIL method, we have: $u_t = [\tilde{c}_t,\, ,\, \tilde{N}_t,\, \tilde{k}_t,\, \tilde{W}_t,\, \tilde{z}_t,\, \tilde{g}_t]^T$, $\varepsilon_t = [\epsilon_{z_t},\, \epsilon_{g_t}]$ and $\zeta_t = [\xi_t]$. The single stability condition in each parametric case is

CASE	P^s
1,3,5	[0.0000, 0.0000, 0.5644, 1.0000, 0.3827, 0.0159]
2,4,6	[0.0000, 0.0000, 0.5644, 1.0000, 0.3827, 0.0159]

Finally, as regards the PEA solution for this model, to have an appropriate approximation we needed to use in this model a third order polynomial in all the considered parametric cases,

$$\begin{aligned}\psi_t(q; k_{t-1}, z_t) &= q_1 \exp\left(q_2 \log(k_{t-1}) + q_3 \log(z_t) + q_4 \log(g_t) + q_5 (\log(k_{t-1})^2\right) \\ &\times \exp\left(q_6 \log(k_{t-1}) \log(z_t) + q_7 (\log(z_t))^2 + q_8 (\log(z_t))^3\right)\end{aligned}$$

From (5) we have $\lambda_t = \beta \psi_t(q; k_{t-1}, z_t)$, and then consumption can be obtained from (6). The fixed point for q was calculated in each case as in the previous models. We iterated on the polynomial until the solution passed the den Haan-Marcet test. We set λ_q equal to one in all the cases. Estimated coefficients for the different parameter vectors were,

CASE	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
1	3.0710	-0.2498	-0.8460	-0.0362	-0.0569	0.1543	-0.1351	0.2371
2	3.6111	-0.4189	-0.7175	-0.0288	-0.0257	0.1111	-0.3122	-0.9004
3	3.8793	-0.4407	-0.9017	-0.0297	-0.0179	0.1790	-0.1316	0.1953
4	3.0250	-0.2686	-0.8818	-0.0223	-0.0576	0.1831	-0.1849	-0.6025
5	3.9664	-0.4567	-0.7432	-0.0582	-0.0154	0.1125	-0.1128	-0.0125
6	3.5363	-0.4038	-0.7614	-0.0191	-0.0291	0.1286	-0.1064	-0.0294

	Preparing the model		Stability analysis	Simulation
LQA	Computation of First Order Conditions for the optimization problem in order to compute the steady state	Linear approximation to the value function	Ricatti-equation iteration	Recursive solution of the original nonlinear problem plus the approximated policy function
SIM	Computation of First Order Conditions of the problem	Linear approximation to the system of First Order Conditions and constraints	Eigenvalue/eigenvector analysis	Solution of a nonlinear system of equations
SIL	Computation of First Order Conditions of the problem	Linear approximation in logged variables to the system of First Order Conditions and constraints	Eigenvalue/eigenvector analysis	Solution of a nonlinear system of equations
UHL	Computation of First Order Conditions of the problem	Linear approximation in logged variables to the system of First Order Conditions and constraints	Undetermined coefficients solution	Linear recursive system
PEA	Computation of First Order Conditions of the problem	Specification of an initial approximating function and computation of initial conditions	Fixed point iteration	Recursive solution of the original nonlinear problem plus the approximated policy function

Figure 1: Steps involved in the implementation of the selected solution methods

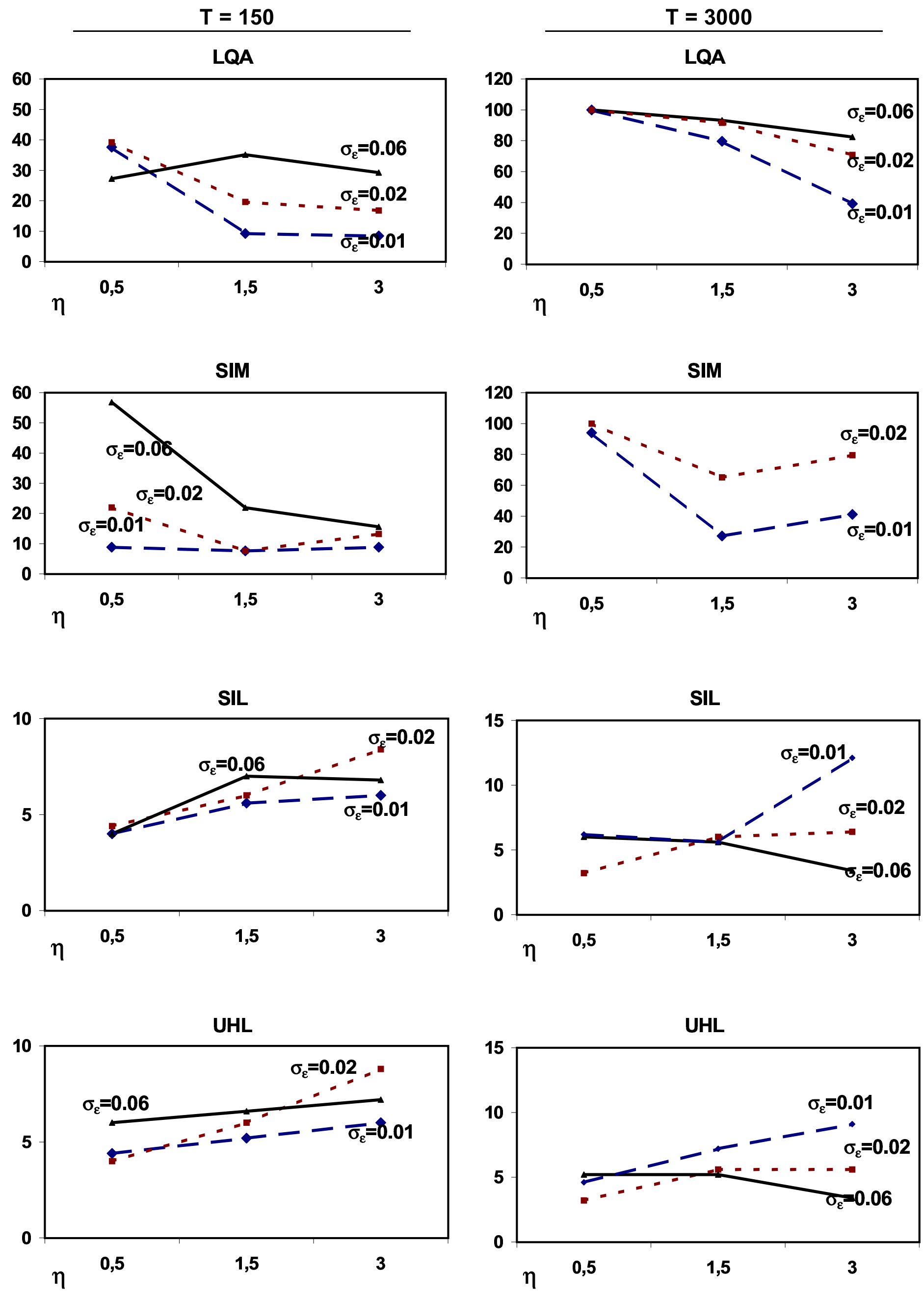

Figure 2: den Haan and Marcet (1994) test: Basic Neoclassical Growth Model. "Almost" linear methods. Percentage of realizations of the statistic in the 5% rejection region for the null hypothesis: $H_0 : E_t(\xi_{t+1}) = 0$. Instruments used: I_t = [constant, k_t, k_{t-1}, k_{t-2}, $\log(z_t)$,$\log(z_{t-1})$, $\log(z_{t-2})$].

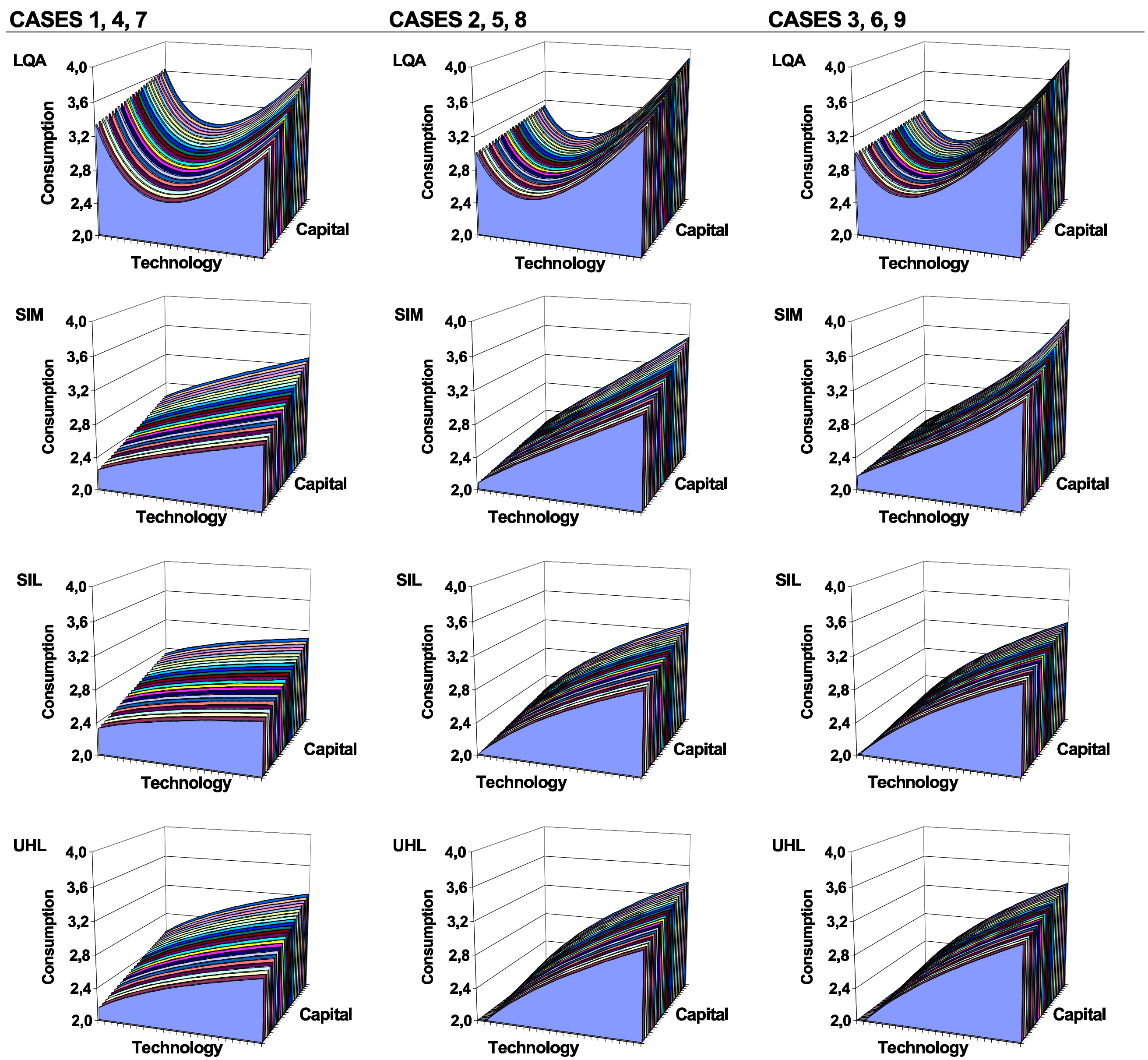

Figure 3: Decision rule for consumption. Basic Neoclassical Growth Model. “Almost” linear methods.

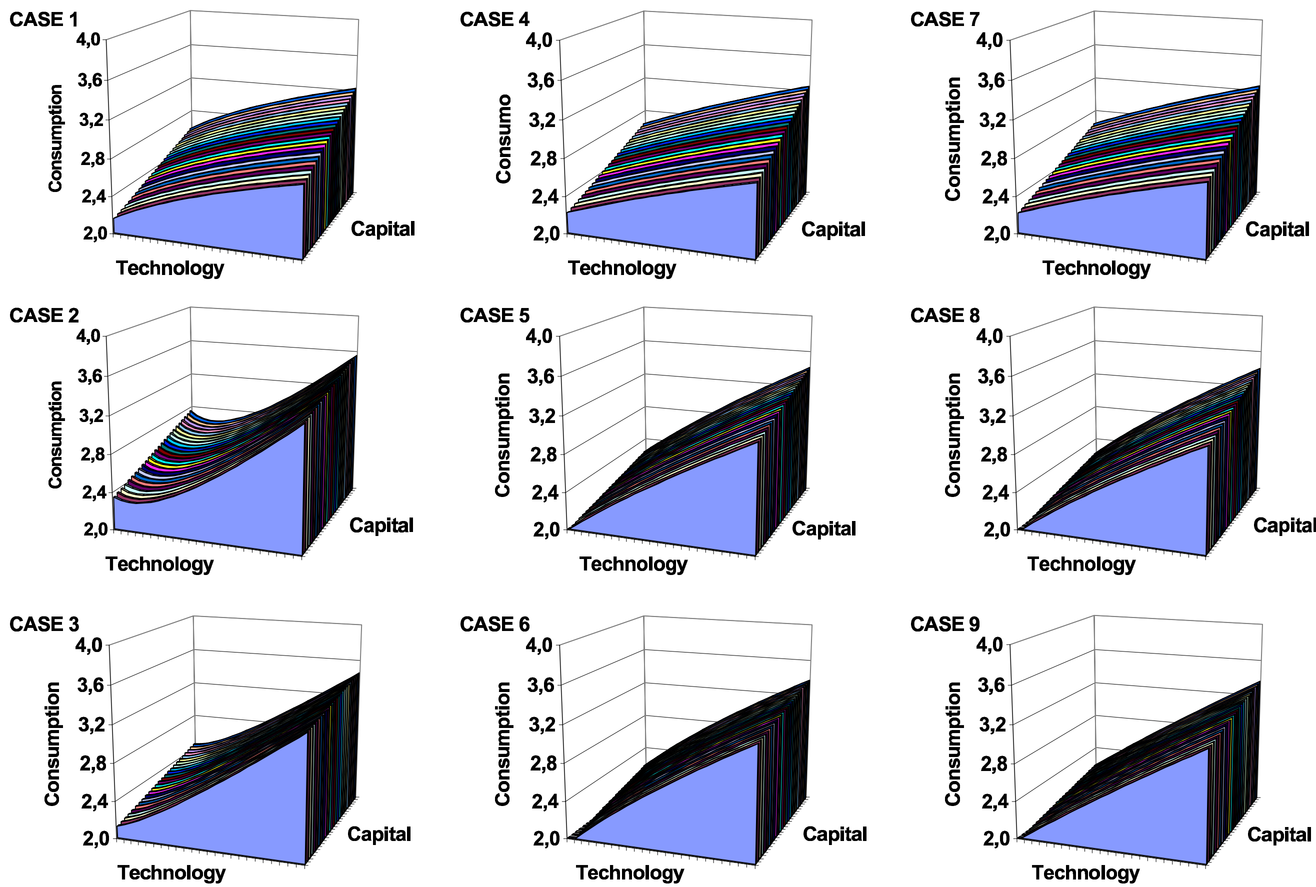

Figure 4: Decision rule for consumption. Basic Neoclassical Growth Model. Parameterized Expectations.

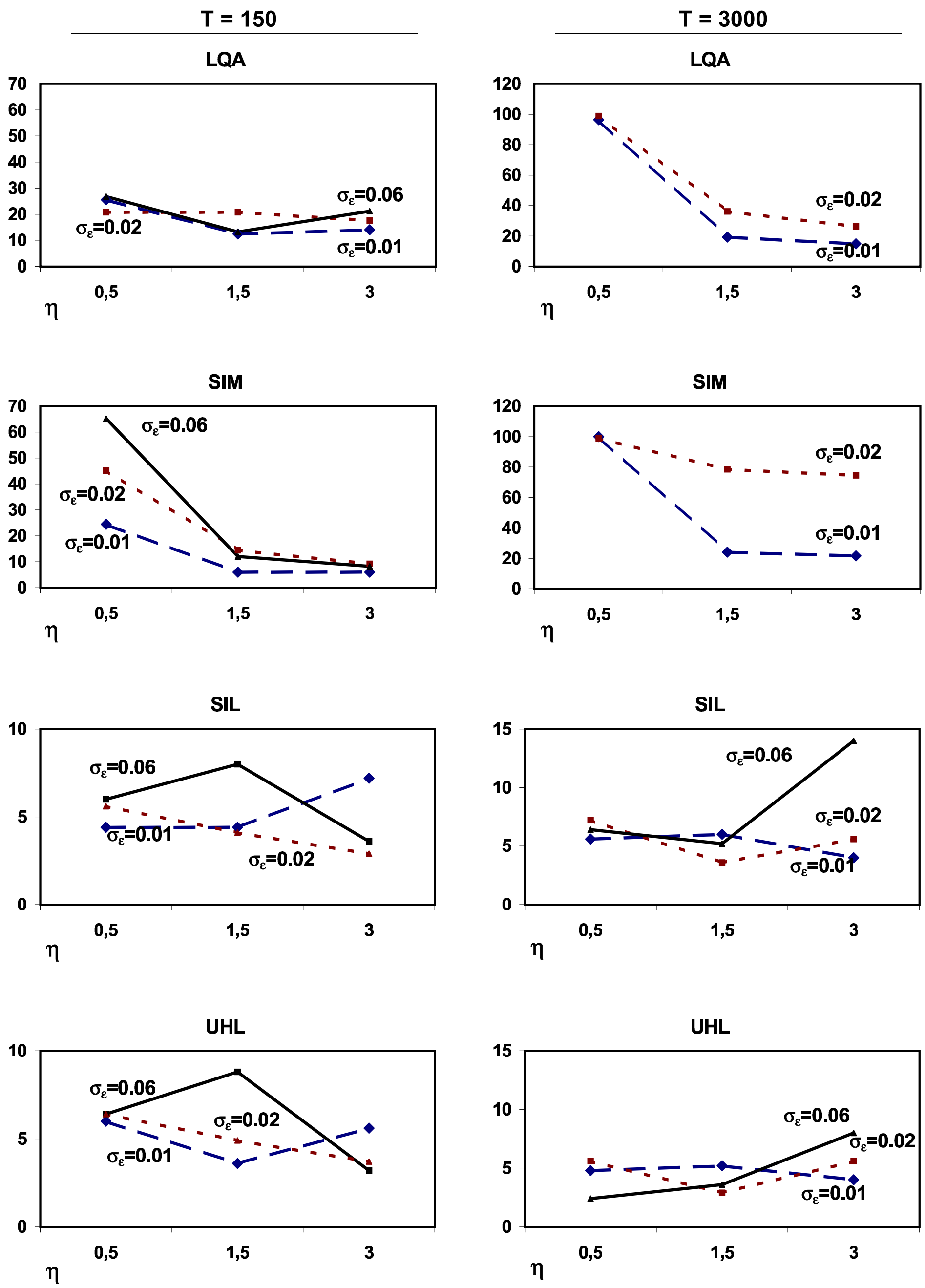

Figure 5: den Haan and Marcet (1994) test: Hansen (1985) Model. "Almost" linear methods. Percentage of realizations of the statistic in the 5% rejection region for the null hypothesis: $H_0 : E_t(\xi_{t+1}) = 0$. Instruments used: $I_t =$ [constant, k_t, k_{t-1}, k_{t-2}, $\log(z_t)$,$\log(z_{t-1})$, $\log(z_{t-2})$].

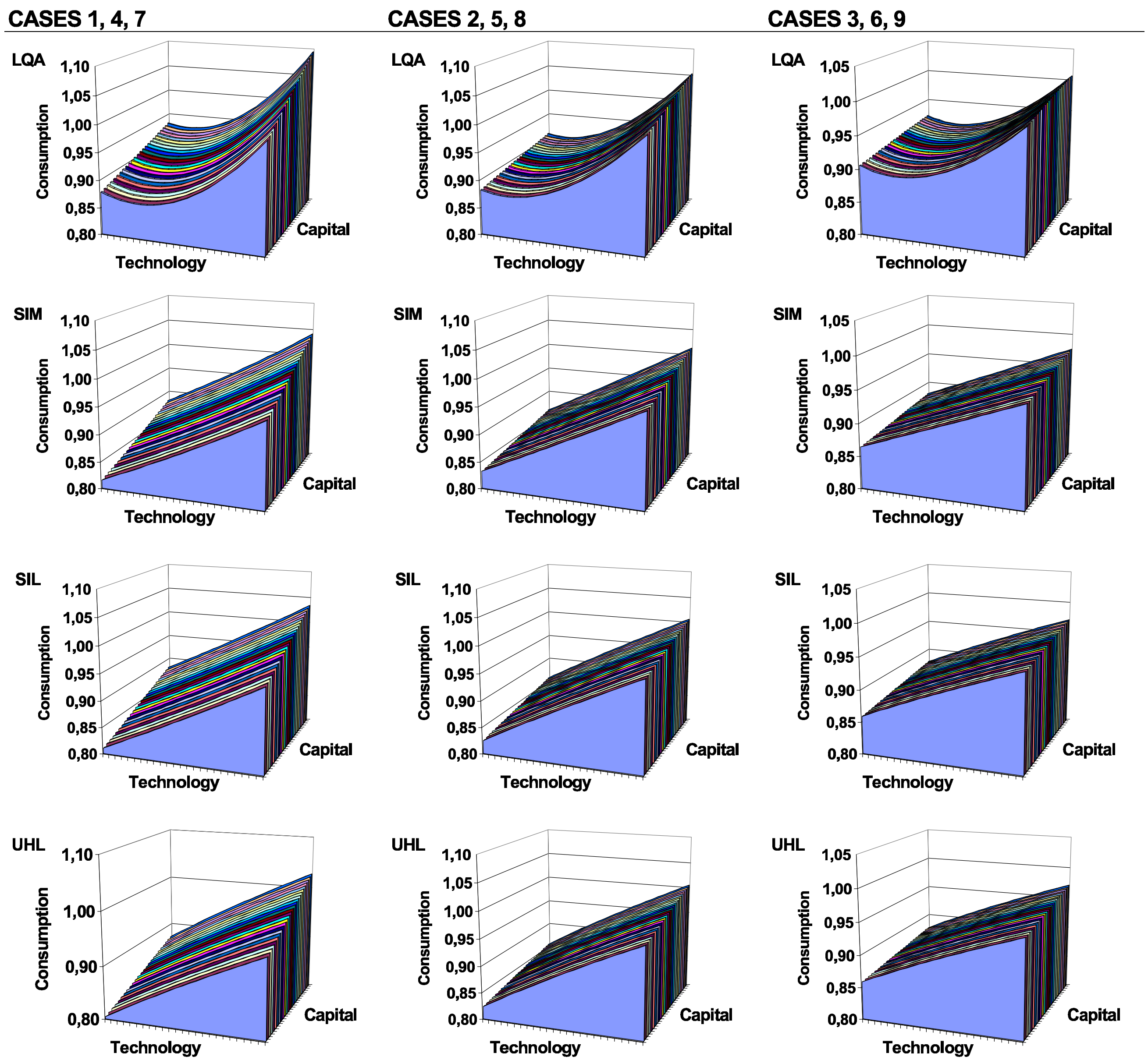

Figure 6: Decision rule for consumption: Hansen (1985) Model. “Almost” linear methods.

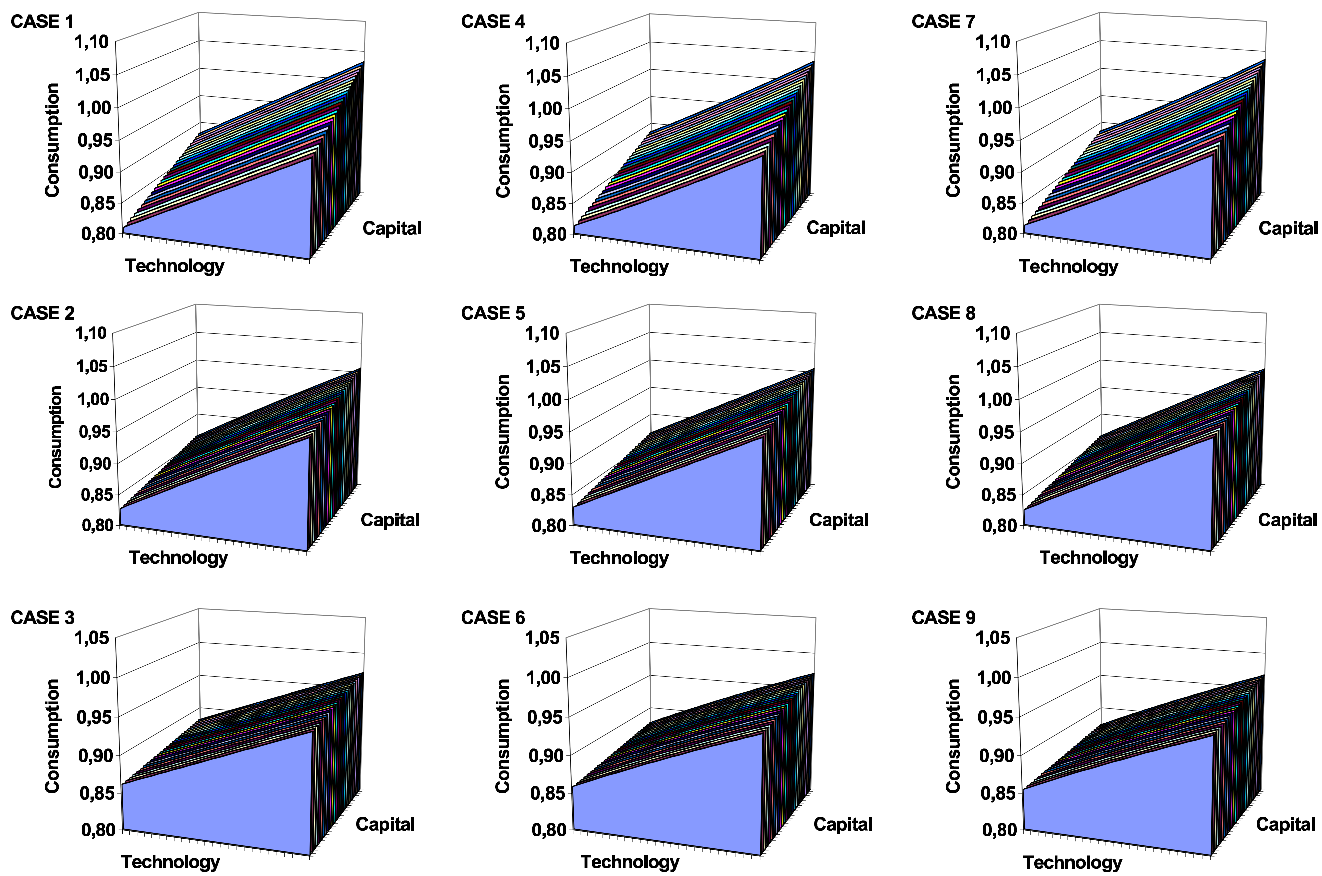

Figure 7: Decision rule for consumption: Hansen (1985) Model. Parameterized Expectations.

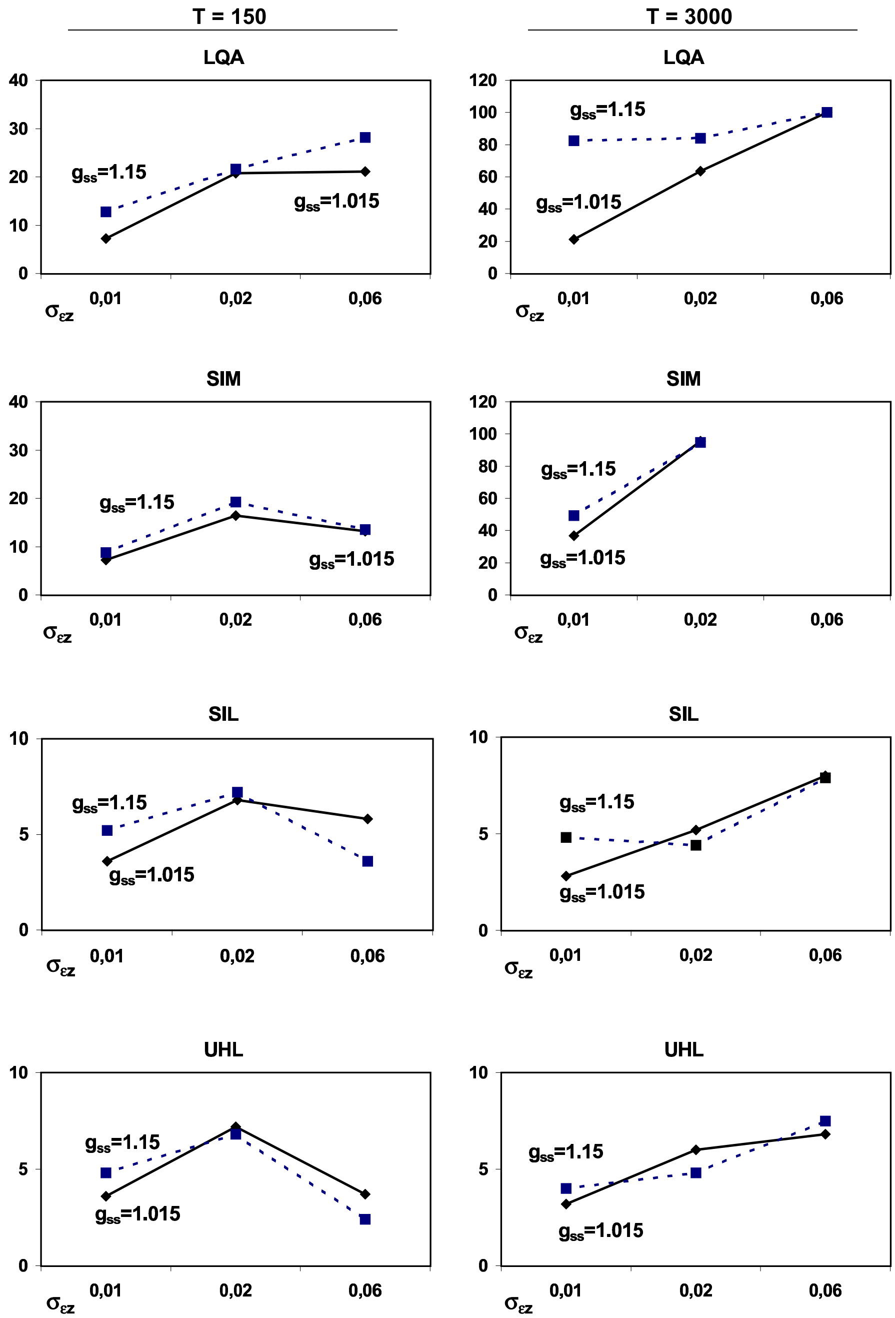

Figure 8: den Haan and Marcet (1994) test: Cooley and Hansen (1989) Model. “Almost” linear methods. Percentage of realizations of the statistic in the 5% rejection region for the null hypothesis: $H_0 : E_t(\xi_{t+1}) = 0$. Instruments used: I_t = [constant, k_t, k_{t-1}, k_{t-2}, $\log(z_t)$,$\log(z_{t-1})$, $\log(z_{t-2})$].

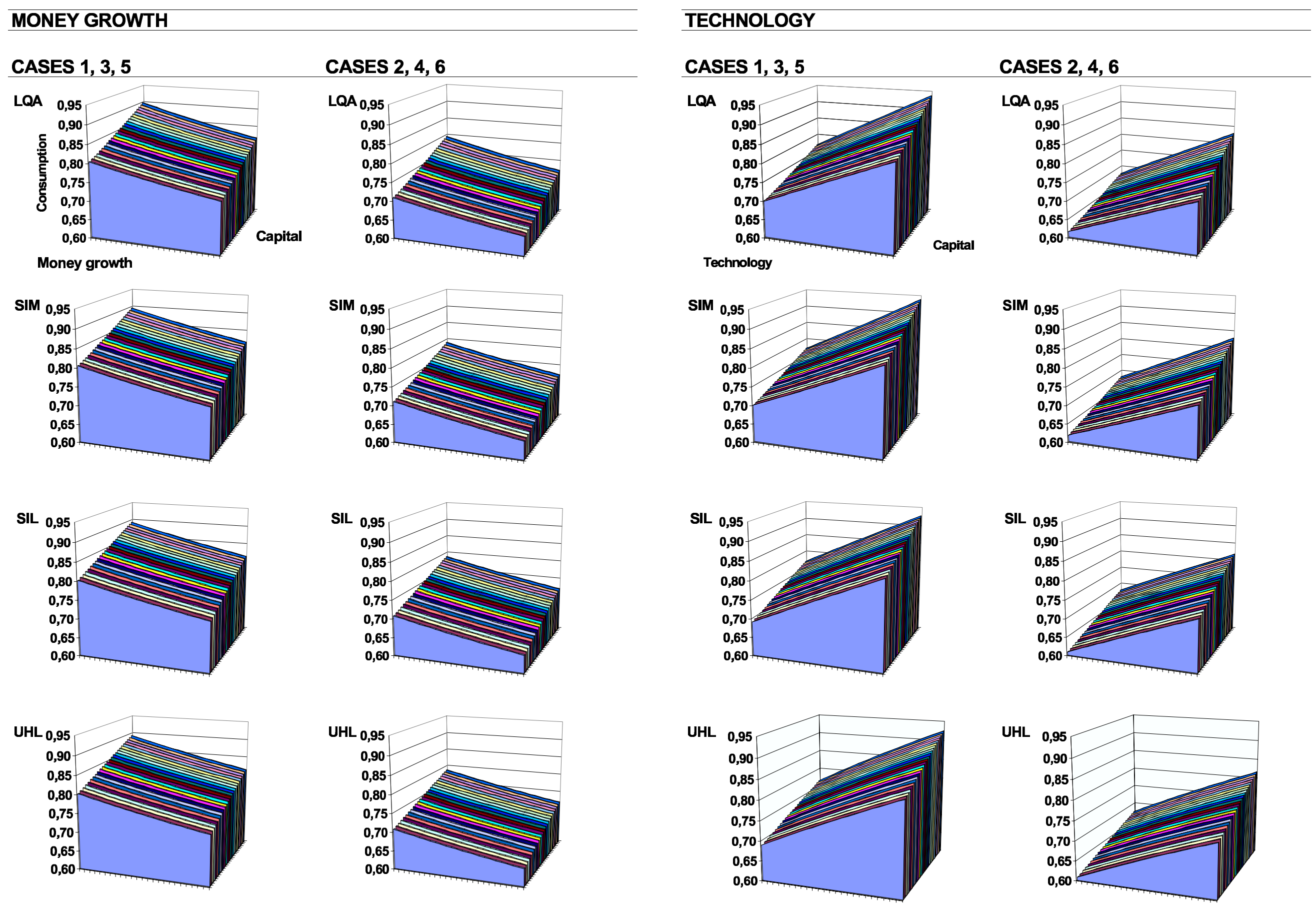

Figure 9: Decision rule for consumption: Cooley and Hansen (1989) Model. "Almost" linear methods.

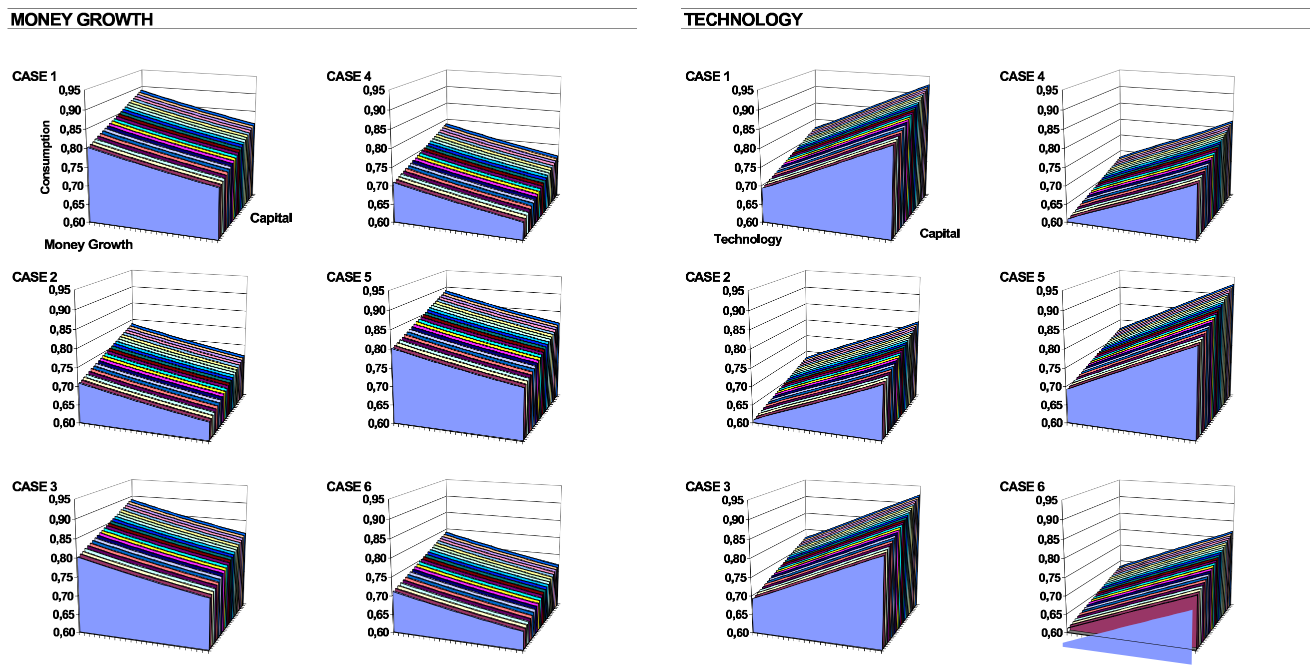

Figure 10: Decision rule for consumption: Cooley and Hansen (1989) Model. Parameterized Expectations.

Table 1: Basic Neoclassical Growth Model. Estimates of AR(1) for the expectations error. Percentage of realizations of the t-statistic in the 5% rejection region for the null hypothesis: $H_0 : \mu = 0$ (upper row), and $H_0 : \rho = 0$ (lower row).

T=150		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	8.8	5.2	3.2	3.2	3.2
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	5.2	4.4	4.8	4.8	4.8
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	4.8	4.0	4.4	4.4	4.4
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	12.4	14.8	2.4	2.4	2.4
Case 5	μ	0.0	0.0	0.0	0.0	0.0
	ρ	5.6	4.8	4.8	4.8	4.4
Case 6	μ	0.0	0.0	0.0	0.0	0.0
	ρ	6.0	4.0	2.8	2.0	2.0
Case 7	μ	0.0	0.0	0.0	0.0	0.0
	ρ	10.0	53.6	4.4	5.2	4.8
Case 8	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.6	8.8	6.8	6.4	7.6
Case 9	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.2	3.2	4.4	5.2	4.8

T=3000		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	9.6	16.0	3.6	3.6	3.2
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	8.0	6.4	8.4	8.4	8.0
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	6.0	5.6	5.6	5.6	5.6
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	19.2	75.6	4.4	4.4	2.8
Case 5	μ	0.0	0.0	0.0	0.0	0.0
	ρ	13.6	14.8	9.2	9.2	6.4
Case 6	μ	0.0	0.0	0.0	0.0	0.0
	ρ	18.7	24.9	6.2	6.2	5.3
Case 7	μ	0.0	0.0	0.0	0.0	0.0
	ρ	45.6	100	4.2	4.6	4.6
Case 8	μ	0.0	—	0.0	0.0	0.0
	ρ	83.8	—	7.4	7.9	9.6
Case 9	μ	4.4	—	0.0	0.0	0.0
	ρ	76.9	—	15.0	13.1	14.3

Table 2: Basic Neoclassical Growth Model, case 9. Test statistic for differences between cross correlations, means and standard deviations. In each panel, the upper corner corresponds to T=3000, the lower corner to T=150. Critical values at 95% and 90% significance levels are 1.6449 and 1.2816.

$\rho(y_t, y_{t-1})$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, y_{t-2})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.1357	0.0376	0.2143	LQA	—	—	0.1877	0.0466	0.2939
SIM	0.0971	—	—	—	—	SIM	0.1329	—	—	—	—
SIL	0.0305	0.1272	—	0.0974	0.0794	SIL	0.0396	0.1721	—	0.1403	0.1072
UHL	0.0469	0.1437	0.0163	—	0.1759	UHL	0.0663	0.1990	0.0267	—	0.2463
PEA	0.0286	0.1251	0.0018	0.0181	—	PEA	0.0377	0.1702	0.0018	0.0284	—

mean(c_t)	LQA	SIM	SIL	UHL	PEA	σ_{c_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	1.4382	0.0108	2.1787	LQA	—	—	2.3481	2.0376	2.9012
SIM	0.0772	—	—	—	—	SIM	1.0565	—	—	—	—
SIL	0.1065	0.1837	—	1.4624	0.7434	SIL	0.9891	1.9992	—	0.2453	0.5521
UHL	0.5667	0.6409	0.4648	—	2.2100	UHL	0.9398	1.9478	0.0429	—	0.7808
PEA	0.1916	0.2688	0.0852	0.3829	—	PEA	1.0951	2.0984	0.1077	0.1498	—

$\rho(y_t, c_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, c_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.3374	0.1749	0.9169	LQA	—	—	0.1705	0.0125	0.6916
SIM	0.8374	—	—	—	—	SIM	0.5649	—	—	—	—
SIL	0.3526	0.6739	—	0.1925	0.7164	SIL	0.3835	0.2072	—	0.1962	0.5677
UHL	0.3210	0.7206	0.0438	—	0.8910	UHL	0.3752	0.2175	0.0100	—	0.7545
PEA	0.1744	0.8778	0.2281	0.1864	—	PEA	0.2491	0.3615	0.1535	0.1438	—

Table 3: Hansen (1985) Model. Estimates of AR(1) for the expectations error. Percentage of realizations of the t-statistic in the 5% rejection region for the null hypothesis: $H_0 : \mu = 0$ (upper row), and $H_0 : \rho = 0$ (lower row).

T=150		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	10.8	13.6	3.2	3.6	3.6
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	7.6	4.0	4.4	4.4	4.4
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	7.6	4.4	5.6	5.2	4.8
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.2	35.2	4.4	4.4	4.4
Case 5	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.2	6.0	4.0	3.6	4.4
Case 6	μ	0.0	0.0	0.0	0.0	0.0
	ρ	10.8	6.0	6.0	5.6	5.6
Case 7	μ	0.0	0.0	0.0	0.0	0.0
	ρ	7.6	47.6	1.0	1.6	1.0
Case 8	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.6	8.8	6.8	6.4	7.6
Case 9	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.2	3.2	4.4	5.2	4.8

T=3000		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	25.6	90.8	4.4	4.4	4.8
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	6.8	4.4	2.4	3.2	1.6
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	8.0	6.0	6.0	5.6	5.6
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	16.8	38.8	0.8	1.2	1.2
Case 5	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.2	24.8	4.0	4.0	6.4
Case 6	μ	0.0	0.0	0.0	0.0	0.0
	ρ	17.1	26.2	6.7	6.7	3.7
Case 7	μ	—	—	0.0	0.0	0.0
	ρ	—	—	4.4	5.2	6.4
Case 8	μ	—	—	0.0	0.0	0.0
	ρ	—	—	8.4	8.4	8.4
Case 9	μ	—	—	0.0	0.0	0.0
	ρ	—	—	14.4	8.8	5.6

Table 4: Hansen (1985) Model, case 6. Test statistic for differences between cross correlations, means and standard deviations. In each panel, the upper corner corresponds to T=3000, the lower corner to T=150. Critical values at 95% and 90% significance levels are 1.6449 and 1.2816.

$\rho(y_t, y_{t-1})$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, y_{t-2})$	LQA	SIM	SIL	UHL	PEA
LQA	—	0.9259	0.0610	0.0460	0.2419	LQA	—	1.2902	0.0874	0.0670	0.3335
SIM	0.1170	—	0.8633	0.8779	0.6965	SIM	0.1744	—	1.2005	1.2203	0.9741
SIL	0.0028	0.1144	—	0.0149	0.1795	SIL	0.0019	0.1726	—	0.0203	0.2441
UHL	0.0004	0.1175	0.0032	—	0.1946	UHL	0.0029	0.1774	0.0048	—	0.2646
PEA	0.0708	0.0473	0.0681	0.0713	—	PEA	0.0958	0.0800	0.0939	0.0988	—
mean(N_t)	LQA	SIM	SIL	UHL	PEA	σ_{N_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	0.0237	0.0364	0.1250	0.0446	LQA	—	0.1789	0.0152	0.0021	0.1217
SIM	0.1125	—	0.0599	0.1004	0.0681	SIM	0.0360	—	0.1946	0.1818	0.2977
SIL	0.0102	0.1023	—	0.1615	0.0082	SIL	0.0002	0.0356	—	0.0131	0.1072
UHL	0.0560	0.1689	0.0663	—	0.1698	UHL	0.0036	0.0323	0.0033	—	0.1202
PEA	0.0097	0.1226	0.0200	0.0464	—	PEA	0.0604	0.0948	0.0604	0.0638	—
$\rho(y_t, N_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, N_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	1.2653	0.0897	0.1118	0.6331	LQA	—	1.2092	0.0887	0.1098	0.5991
SIM	0.5085	—	1.3504	1.3701	1.8822	SIM	0.5064	—	1.2936	1.3124	1.7936
SIL	0.0048	0.5126	—	0.0223	0.5422	SIL	0.0044	0.5100	—	0.0212	0.5093
UHL	0.0002	0.5086	0.0050	—	0.5190	UHL	0.0005	0.5060	0.0049	—	0.4874
PEA	0.2183	0.7214	0.2133	0.2186	—	PEA	0.1934	0.6947	0.1888	0.1940	—

Table 5: Hansen (1985) Model, case 8. Test statistic for differences between cross correlations, means and standard deviations. In each panel, the upper corner corresponds to T=3000, the lower corner to T=150. Critical values at 95% and 90% significance levels are 1.6449 and 1.2816.

$\rho(y_t, y_{t-1})$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, y_{t-2})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	—	—	—	LQA	—	—	—	—	—
SIM	0.1331	—	—	—	—	SIM	0.1534	—	—	—	—
SIL	0.0943	0.2253	—	0.0743	0.1434	SIL	0.1438	0.2940	—	0.0936	0.1832
UHL	0.0562	0.1878	0.0380	—	0.2176	UHL	0.0970	0.2481	0.0467	—	0.2766
PEA	0.1611	0.2910	0.0664	0.1044	—	PEA	0.2274	0.3762	0.0833	0.1300	—
mean(N_t)	LQA	SIM	SIL	UHL	PEA	σ_{N_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	—	—	—	LQA	—	—	—	—	—
SIM	1.2136	—	—	—	—	SIM	0.5834	—	—	—	—
SIL	0.3164	0.9159	—	1.6474	0.0786	SIL	0.0978	0.4969	—	0.3882	0.1349
UHL	0.4625	1.6724	0.7857	—	1.5759	UHL	0.0388	0.5486	0.0589	—	0.2527
PEA	0.2139	1.0172	0.1039	0.6831	—	PEA	0.0484	0.6241	0.1457	0.0870	—
$\rho(y_t, N_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, N_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	—	—	—	LQA	—	—	—	—	—
SIM	3.2537	—	—	—	—	SIM	3.0337	—	—	—	—
SIL	0.2161	3.0502	—	0.2296	0.7624	SIL	0.1705	2.8699	—	0.2420	0.7494
UHL	0.4761	2.8184	0.2586	—	0.9874	UHL	0.4570	2.6265	0.2836	—	0.9867
PEA	0.1757	3.3882	0.3885	0.6460	—	PEA	0.2255	3.2013	0.3915	0.6747	—

Table 6: Cooley and Hansen (1989) Model. Estimates of AR(1) for the expectations error. Percentage of realizations of the t-statistic in the 5% rejection region for the null hypothesis: $H_0 : \mu = 0$ (upper row), and $H_0 : \rho = 0$ (lower row).

T=150		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	5.6	6.4	5.2	5.2	4.4
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	6.0	4.0	3.6	3.6	3.6
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	10.4	8.0	4.4	4.0	4.4
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	7.2	7.6	2.0	2.0	2.0
Case 5	μ	0.0	0.0	0.0	0.0	0.0
	ρ	9.0	9.5	4.0	5.0	3.5
Case 6	μ	0.0	0.0	0.0	0.0	0.0
	ρ	10.5	11.0	6.8	6.8	7.2

T=3000		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	7.6	11.6	5.6	5.6	2.8
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	9.2	10.8	4.0	3.6	3.6
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	22.5	56.1	5.8	5.8	4.0
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	21.2	54.5	3.6	3.6	4.2
Case 5	μ	0.0	—	0.0	0.0	0.0
	ρ	90.8	—	2.2	2.9	3.8
Case 6	μ	3.1	—	0.0	0.0	0.0
	ρ	93.5	—	3.8	3.8	5.7

Table 7: Cooley and Hansen (1989) Model, case 6. Test statistic for differences between cross correlations, means and standard deviations. In each panel, the upper corner corresponds to T=3000, the lower corner to T=150. Critical values at 95% and 90% significance levels are 1.6449 and 1.2816.

$\rho(y_t, y_{t-1})$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, y_{t-2})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.3708	0.4047	0.4108	LQA	—	—	0.7159	0.7629	0.7831
SIM	1.2309	—	—	—	—	SIM	1.5889	—	—	—	—
SIL	0.8021	0.4411	—	0.0743	0.0877	SIL	1.0430	0.5613	—	0.0878	0.1254
UHL	0.8446	0.3971	0.0439	—	0.0134	UHL	1.0924	0.5107	0.0507	—	0.0376
PEA	0.8396	0.4027	0.0385	0.0054	—	PEA	1.0928	0.5109	0.0507	0.0000	—

mean(N_t)	LQA	SIM	SIL	UHL	PEA	σ_{N_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	1.0810	0.6934	1.1183	LQA	—	—	2.1126	2.0987	2.1147
SIM	1.5367	—	—	—	—	SIM	1.1693	—	—	—	—
SIL	0.7345	1.2395	—	2.1197	0.2086	SIL	0.5762	0.9115	—	0.3825	0.0614
UHL	0.3216	1.8793	0.6334	—	2.3468	UHL	0.5005	1.0072	0.1108	—	0.4417
PEA	0.8139	1.1240	0.1202	0.7564	—	PEA	0.5704	0.9180	0.0083	0.1024	—

$\rho(y_t, N_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, N_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	2.3463	2.2124	2.2568	LQA	—	—	2.4838	2.3217	2.3763
SIM	4.3355	—	—	—	—	SIM	4.2143	—	—	—	—
SIL	2.0852	2.7667	—	0.2451	0.1773	SIL	2.0822	2.7167	—	0.2882	0.2149
UHL	2.0817	2.7355	0.0181	—	0.0704	UHL	2.1199	2.6337	0.0743	—	0.0785
PEA	2.1431	2.7288	0.0546	0.0359	—	PEA	2.1623	2.6615	0.0801	0.0042	—

mean(π_t)	LQA	SIM	SIL	UHL	PEA	σ_{π_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.1258	0.1201	0.1262	LQA	—	—	0.7278	0.7159	0.7340
SIM	0.1008	—	—	—	—	SIM	0.3894	—	—	—	—
SIL	0.0268	0.1343	—	0.0073	0.0005	SIL	0.8684	0.5184	—	0.0793	0.0428
UHL	0.0205	0.0839	0.0496	—	0.0078	UHL	0.8446	0.4924	0.0262	—	0.1214
PEA	0.0185	0.1255	0.0087	0.0409	—	PEA	0.9050	0.5589	0.0420	0.0681	—

$\rho(y_t, \pi_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, \pi_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	2.9119	3.0739	3.1353	LQA	—	—	3.6807	3.7865	3.8903
SIM	0.5384	—	—	—	—	SIM	1.1232	—	—	—	—
SIL	1.2243	1.7588	—	0.3015	0.4137	SIL	1.8476	0.7659	—	0.1512	0.2989
UHL	1.5472	2.0784	0.3286	—	0.1118	UHL	2.0158	0.9392	0.1694	—	0.1477
PEA	1.7787	2.3061	0.5673	0.2396	—	PEA	2.2728	1.2274	0.4713	0.3066	—