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RESUMEN
En este artículo se introduce un nuevo concepto de solidaridad en términos de reciprocidad y se caracteriza el conjunto de funciones de elección social que son recíprocas (en un sentido fuerte y en otro más débil), anónimas y eficientes en un modelo estándar de provisión de bienes públicos en el que los agentes tienen preferencias unimodales sobre la cantidad del bien que se provee. Los procedimientos que obtenemos son los bien conocidos métodos del ganador de Condorcet generalizado, y por tanto proveemos una caracterización alternativa de esa clase de funciones de elección social basada en nuevas propiedades éticas referidas a la solidaridad.

Palabras clave: preferencias unimodales, solidaridad, dominancia en bienestar bajo reposicionamiento de preferencias.

ABSTRACT
In this paper we introduce a new concept of solidarity in terms of reciprocity and characterize the set of social choice functions that are reciprocate (in both a strong and a weak sense), anonymous and efficient in a standard public good provision model when the agents have single-peaked preferences on the amount of the good provided. The resulting procedures are the well-known Generalized Condorcet Winner Solutions, and therefore, we provide an alternative characterization of that class of social choice functions based in new ethical properties regarding solidarity.

Keywords: Single-peaked preferences, solidarity, welfare domination under preference replacement.
JEL classification: D71.

* This paper is part of my Ph.D. dissertation at the Departament d'Economía i d'Història Económica de la Universitat Autónoma de Barcelona, UAB. I wish to acknowledge to my thesis supervisor, Prof. Salvador Barberà for his guidance and support and to the Fundación Centro de Estudios Andaluces, CentrA.

Solidarity in Terms of Reciprocity*

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Abstract

In this paper we introduce a new concept of solidarity in terms of reciprocity and characterize the set of social choice functions that are reciprocate (in both a strong and a weak sense), anonymous and efficient in a standard public good provision model when the agents have single-peaked preferences on the amount of the good provided. The resulting procedures are the well-known Generalized Condorcet Winner Solutions, and therefore, we provide an alternative characterization of that class of social choice functions based in new ethical properties regarding solidarity.

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Resumen

En este artículo se introduce un nuevo concepto de solidaridad en términos de reciprocidad y se caracteriza el conjunto de funciones de elección social que son recíprocas (en un sentido fuerte y en otro más débil), anónimas y eficientes en un modelo estándar de provisión de bienes públicos en el que los agentes tienen preferencias unimodales sobre la cantidad del bien que se provee. Los procedimientos que obtenemos son los bien conocidos métodos del ganador de Condorcet generalizado, y por tanto proveemos una caracterización alternativa de esa clase de funciones de elección social basada en nuevas propiedades éticas referidas a la solidaridad.

1 Introduction

A solidarity principle applying to the fair allocation problem was introduced by Thomson [12] under the name of *replacement principle*. The idea is the following: every allocation problem can be described by some parameters or *data*, such as the set of agents involved in the decision, the description of their preferences or the possible amount of resources and their distribution among the agents. The replacement principle imposes solidarity among agents in the following sense. If some component of the *data* changes its value within the admissible domain, every agent should be affected in the same direction: either all of them improve their position or all of them lose. It is argued that *fair* and acceptable social choice rules should fulfill this equal treatment property when facing exogenously given shocks. The replacement principle has been widely explored in the literature in different contexts. When we consider the population as the *relevant* variable parameter, Thomson's [15], [16] concept of *population monotonicity* is the accurate translation of the replacement principle: every agent should lose when we add new agents to those initially present, since the growth of the population can be seen as a restriction of the opportunities available to society. This property was investigated by Moulin [7] in connection with strategy-proofness and by Thomson [11], [15], [16] and Ching and Thomson [4] in the context of single-peaked preferences. If we focus on a change on the amount of available resources, then, the replacement principle takes the form of the *resource monotonicity*, a property analyzed by Thomson [14].

The specific version of the replacement principle we are going to discuss here applies when the preferences of some individuals change. It was first defined by Moulin [8] under the name of *replacement domination* and later by many authors with the names replacement monotonicity or *welfare-domination under preference-replacement* ($WDUPR$). It requires that if somebody changes his preferences, and this shifts the social decision affecting the remaining agents, then, these should all move in the same direction: they should either all gain or all lose after the change. This property has been analyzed in the two contexts of private and public goods economies in Thomson [17], [18] and [13] respectively -see Thomson [19] for a comprehensive survey-.

We will consider here the provision of a public good, where there are a continuum of alternatives described by an interval of the real line. It has been shown by Thomson [13] that with single-peaked preferences, the only replacement monotonic and efficient social choice functions are those functions that choose a fixed given point in the interval if it is Pareto optimal and choose the nearest efficient point to this one if it is not.

This class of social choice functions constitutes a subclass of the family of *Generalized Condorcet winner solutions* defined by Moulin [6], and we feel that they are in fact very far from desirable. They are quite trivial decision rules, weighting excessively an arbitrary *status quo*, so they are very insensitive to changes in individual preferences.

This paper starts from this criticism and aims to provide a reasonable alternative to Thomson's principle of $WDUPR$. In order to enlarge the class of procedures, a new and intuitive concept of solidarity among agents -we call it *reciprocity*- is introduced in both a strong and a weak version. The social choice functions and voting schemes that preserve both reciprocity and anonymity are fully characterized.

The alternative property proposed here tries to embody part of the substance of the original idea of $WDUPR$, but differs from this in the sense that reciprocity can be considered as a somehow introspective conception of solidarity. Let us think of the society just before deciding what social choice function -from now on SCF- is going to be used when choosing the level of some public good. People is likely to accept a procedure that embodies some idea of solidarity in the sense that this rule provides some form of protection for every individual against the possible shifts in choice caused by the changes of preferences of others.

Thomson's requirement of $WDUPR$ can be reinterpreted in this context. The ex-ante social contract in the SCF guarantees that if any individual changes his preferences, the new value of the function is such that everybody moves in the same direction -all of them gain or all of them lose with the change-.

Our reciprocity condition can be seen as another type of social insurance: agents are no longer treated equally than the rest of individuals who maintain their original preferences, but equally than the agent who changed his own one. The idea is as follows: people may now gain or lose when somebody changes, but if I lose, I want to be sure that the agent who has caused my loss would be in the same situation than me if I had changed likewise and shifted the social decision. He would have been moved by my change in the same direction than I was moved by him.

By considering such contracts before deciding the optimal rule for society, people might be ready to accept this weaker and introspective concept of solidarity. Moreover, its philosophy is very intuitive and can be heard in the real world -people usually are much more permissive with the impositions of others when they dislike them if they know that they would be treated equally under similar situations-. The proposed property -in its weak version- allow for a larger and more flexible class of functions than those allowed by Thomson's $WDUPR$, although Thomson's class of efficient, replacement monotonic SCFs satisfies reciprocity.

The structure of this chapter is as follows. We first introduce the model in *Section 2*. In *Section 3*, the reciprocity properties are proposed and results are finally presented. It is shown that Thomson's class is a narrow subset of our class. We close with some comments and conclusions.

2 The model

Consider a society defined by a set of *agents or individuals*: $N = \{1, ..., n\}$, indexed by i and sometimes by j, h and l. Society must make decisions from some predetermined set of mutually exclusive *alternatives,* represented by A, whose elements will be denoted by $x, y, ... \in A$. The set of alternatives will sometimes be finite -representing discrete levels of the provision of some public good- and sometimes a closed interval of the real line, normalized for simplicity to the interval $[0, M]$ and standing for the continuous amount of the public good or the location of some public utility.

Every individual $i \in N$ is endowed with a complete *preference relation* over the set of alternatives denoted as R_i from some set of possible preferences $\Re$. We denote by P_i and I_i the asymmetric and symmetric part of R_i. The set of all possible strict orderings on the finite set of alternatives A is denoted by $\wp$.

When we consider the set of alternatives $[0, M]$, we also assume that the preference relations are single-peaked. A preference relation R_i on $[0, M]$ is *single-peaked* if and only if there exits a unique number $p(R_i) \in [0, M]$ such that $\forall x, y \in [0, M]$, if $y < x \leq p(R_i)$ or $p(R_i) \leq x < y$, then, xP_iy. The number $p(R_i)$ is the *peak* of agent $i's$ preference relation and is the most preferred alternative of agent $i \in N$.

We also assume for simplicity that single-peaked preferences are continuous. We say that preferences $R_i \in \Re$ are *continuous* if and only if for every alternative, both the upper and the lower contour sets are closed, i.e., $\forall x \in [0, M] = A$, $\{y \in A \mid yR_ix\}$ and $\{y \in A \mid xR_iy\}$ are closed. This is a natural assumption when dealing with infinite sets and it is sufficient to guarantee that for every closed interval contained in $[0, M]$, there exists a most-preferred alternative. Let $\Re^{SP}$ be set of all continuous and single-peaked preference relations on $A = [0, M]$.

An ordered list of preference relations for all the individuals is called a *preference profile* and denoted by $\mathbf{R} = (R_i)_{i \in N} = (R_1, ..., R_n)$. We often use the following notation: given a fixed preference profile $\mathbf{R} = (R_1, ..., R_n)$, $(R', \mathbf{R}_{-i})$ is the profile in which individual i takes preferences R' and any other agent $j \neq i$ remains with the same preferences he had in profile $\mathbf{R}$, i.e., R_j. $(R', R'', \mathbf{R}_{-i-j})$ is the profile such that the preference relations of agents i and j in profile $\mathbf{R}$, have been replaced by preference relations R' and R'' respectively and the other agents' preferences are the same than those they had in profile $\mathbf{R}$. Then, whatever preference relation is placed in the first component of some partitioned profile $(.,., \mathbf{R}_{-i-j})$ stands for the preference relation of agent i in that profile. Hence, the profile $(R', R'', \mathbf{R}_{-j-i})$ is intended to be the profile $\mathbf{R}$ when agent j has preferences R' and agent i is endowed with preferences R''. Moreover, our particular notation admits that some agent's new preference relation is the same preference relation of that of some other agent in the original profile $\mathbf{R}$, in which case we are allowed to refer to that preference relation with its former subscript in order to avoid notation; but notice that the subscript accompanying some individual preference relation in our partitioned notation is *not* related with the agent owning that preference relation in the actual profile, but with the agent that had it in the original -or *reference*- profile. Let us illustrate this important point with an example: The profile $(R_j, R', \mathbf{R}_{-i-j})$ should be read in the following way: "individual i has the same preference relation that individual j had in profile $\mathbf{R}$ (R_j), agent j possesses the preference relation R' and the remaining agents are endowed with the same preference relations they had in the reference profile $\mathbf{R}$".

When preferences are single-peaked, the associated vector of peaks will be: $p(\mathbf{R}) = (p(R_i))_{i \in N} \in [0, M]^n$.

Now, we model social objectives. A *social choice function* (SCF) is a function which associates a chosen alternative to every preference profile and will be denoted by $f : \Re^n \longrightarrow A$.

When we work with the set of alternatives $[0, M]$ and single-peaked preferences, we will be interested in a special kind of SCFs called *voting schemes*. Voting schemes only use information about the agents' peaks, so we can define a voting scheme Π as a SCF in which the following holds:

$$\forall \mathbf{R}, \mathbf{R}' \in \Re^n \; s.t. \; p(\mathbf{R}) = p(\mathbf{R}') \Longrightarrow \Pi(\mathbf{R}) = \Pi(\mathbf{R}').$$

Now we define the properties we shall deal with:

Definition 1 *For each given* $\mathbf{R} \in \Re^n$, x *is an* ***efficient alternative*** *if* $x \in A$ *and there is no* $x' \in A$ *with* $x' R_i x \; \forall i \in N$ *and* $x' P_i x$ *for some* $i \in N$. *The set of efficient alternatives associated to profile* $\mathbf{R}$ *will be denoted by* $x \in P(\mathbf{R})$

A SCF f *is efficient if and only if it selects efficient alternatives for each preference profile, i.e.,* $\forall \mathbf{R} \in \Re^n$, $f(\mathbf{R}) \in P(\mathbf{R})$.

In the case of R_i single- peaked for all $i \in N$ and $A = [0, M]$, it is easy to prove that f is efficient whenever $\forall \mathbf{R} \in \Re^n$,

$f(\mathbf{R}) \in P(\mathbf{R}) = [\min \{p(R_i) \ \mid i \in N\}, \ \max \{p(R_i) \ \mid i \in N\}]$.

Definition 2 *A SCF f is* ***manipulable*** *by agent $i \in N$ at profile $\mathbf{R} \in \Re^n$ via $R_i' \in \Re^n$ if and only if $f(R', \mathbf{R}_{-i}) P_i f(\mathbf{R})$. Whenever a SCF is manipulable by some agent at some profile via a preference relation we say that the SCF is manipulable.*

Definition 3 *A SCF f is* ***strategy-proof*** *if and only if it is not manipulable.*

This property constitutes a strong incentive compatibility requirement, meaning that agents' lies about their true preferences cannot be in any case profitable -whatever the declared preferences of others may be-. Strategy-proofness may therefore be interpreted as requiring that revealing actual preferences be a dominant strategy for all agents if the SCF is used to choose alternatives based on the agents' reported preferences.

Definition 4 *A SCF f is* ***anonymous*** *if any permutation of the different values of its arguments yields the same alternative -, i.e., for all one-to-one mappings $\sigma : N \to N$ and all $\mathbf{R} \in \Re^n$, $f(R_1, ..., R_n) = f(R_{\sigma(1)}, ..., R_{\sigma(n)})$.*

This property guarantees that no information about the individuals' names is used in the decision rule.

Definition 5 *A SCF f satisfies the property of* ***Welfare-domination under preference-replacement*** *(WDUPR)*[1] *if $\forall i \in N$, $\forall \mathbf{R} \in \Re^n$, $\forall R' \in \Re$, either $f(\mathbf{R}) R_j f(R', \mathbf{R}_{-i})$ $\forall j \in N \setminus \{i\}$ or $f(R', \mathbf{R}_{-i}) R_j f(\mathbf{R})$ $\forall j \in N \setminus \{i\}$.*

The change in the preferences of any individual makes that all the remaining agents *move* in the same welfare direction: either all of them gain or all of them lose -in the weak sense-.

We now introduce two versions of the main condition in this paper. The motivation for both versions is the same and they only differ in what they require when agents are left indifferent when facing somebody's change in preferences. Although both versions are quite similar, the possibilities of finding social choice functions are very different when we require each version to hold, so the apparently slight difference is proved to be crucial in allowing for positive results.

[1]This property has also been called "Replacement monotonicity" and "Replacement domination".

Definition 6 *A SCF f satisfies the property of* ***strong reciprocity*** *if* $\forall i \in N,\ \forall \mathbf{R} \in \Re^n,\ \forall R' \in \Re,\ \forall j \in N \setminus \{i\},\quad f(R', \mathbf{R}_{-i}) R_j f(\mathbf{R}) \Rightarrow f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$ *and* $f(\mathbf{R}) R_j f(R', \mathbf{R}_{-i}) \Rightarrow f(\mathbf{R}) R_i f(R', \mathbf{R}_{-j})$.

When agent i changes his preferences from R_i to R' and does not affect negatively to individual j, we require that if j were the one who changed his preferences from the initial profile to the same agent $i's$ new preferences -from R_j to R'-, and individual i would remain unchanged -with R_i-, i would not lose with $j's$ change either -so $f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$ holds-. Symmetrically, if such a change by individual i makes agent j be worse off -interchanging the roles of R_i and R' above-, the reasoning is the same, but individual i with initial preferences R' should now weakly lose -so that $f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$ holds in this case too-.

Definition 7 *A SCF f satisfies the property of* ***weak reciprocity*** *if* $\forall i \in N,\ \forall \mathbf{R} \in \Re^n,\ \forall R' \in \Re,\ \forall j \in N \setminus \{i\},\ f(R', \mathbf{R}_{-i}) R_j f(\mathbf{R}) \Rightarrow f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$.

In words, if agent i does not make me be (strictly) worse off by changing his preference to R', I should not be able to (strictly) damage him if I would be the agent who changes to R' and i will remain unchanged. Weak reciprocity imposes a ban on perverse asymmetric feelings.

Notice that weak reciprocity implies the following statements: if somebody makes me gain, I can either improve or not affect at all his position (a). If the changing agent is damaging me, again I can either cause him a loss or leaving him unaffected (b). Finally, if I am indifferent with i's change -he has not made me be (strictly) worse off-, the definition applies and I should not damage him: in my (reciprocate) turn, I should be able either to make him gain or break even (c).

(a). $f(R', \mathbf{R}_{-i}) P_j f(\mathbf{R}) \Rightarrow f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$.

(b). $f(\mathbf{R}) P_j f(R', \mathbf{R}_{-i}) \Rightarrow f(\mathbf{R}) R_i f(R', \mathbf{R}_{-j})$.

(c). $f(R', \mathbf{R}_{-i}) I_j f(\mathbf{R}) \Rightarrow f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$ &
$f(R', R_i, \mathbf{R}_{-i-j}) R' f(R', \mathbf{R}_{-i})$.

Weak reciprocity relaxes strong reciprocity in just one sense. We must take a short detour in order to explain the difference. It is not difficult to check that strong reciprocity implies the following: $\forall i \in N,\ \forall \mathbf{R} \in \Re^n,\ \forall R' \in \Re,\ \forall j \in N \setminus \{i\},$

$$f(R', \mathbf{R}_{-i}) I_j f(\mathbf{R}) \Rightarrow f(R', \mathbf{R}_{-j}) I_i f(\mathbf{R}) \ \&\ \ f(R', \mathbf{R}_{-i}) I' f(R', R_i, \mathbf{R}_{-i-j})$$

In order to prove this, just note that the indifference on the left hand side in the former statement is: $f(R', \mathbf{R}_{-i}) R_j f(\mathbf{R})$ and $f(\mathbf{R}) R_j\ f(R', \mathbf{R}_{-i})$;

by applying strong reciprocity to both expression, we get: $\forall i \in N$, $\forall \mathbf{R} \in \Re^n$, $\forall R' \in \Re$, $\forall j \in N \setminus \{i\}$,

$f(R', \mathbf{R}_{-i}) R_j f(\mathbf{R}) \Rightarrow$

$\Rightarrow f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$ (1) & $f(R', \mathbf{R}_{-i}) R' f(R', R_i, \mathbf{R}_{-i-j})$ (2).

$f(\mathbf{R}) R_j\, f(R', \mathbf{R}_{-i}) \Rightarrow$

$\Rightarrow f(R', R_i, \mathbf{R}_{-i-j}) R' f(R', \mathbf{R}_{-i})$ (3) & $f(\mathbf{R}) R_i f(R_i, R', \mathbf{R}_{-i-j})$ (4).

And since $f(R_i, R', \mathbf{R}_{-i-j}) = f(R', \mathbf{R}_{-j})$ by definition, (1) and (4) imply $f(R', \mathbf{R}_{-j}) I_i f(\mathbf{R})$

and (2) and (3) imply $f(R', \mathbf{R}_{-i}) I' f(R', R_i, \mathbf{R}_{-i-j})$, which means that whenever a change from some i leaves an agent j indifferent -for instance, when $i's$ change cannot shift the initial social choice-, the same change from j should not affect $i's$ preferences. This is a stronger requirement than desired, since there might be no reasons for forbidding i to strictly gain, while there may be reasons for i to lose when j change -the reciprocate symmetry might forbid "perverse" hypothetical effects, but there does not seem to be a strong reason to maintain such a strong implication . Weak reciprocity eliminates this requirement by allowing unaffected agents to improve $i's$ position, while not letting him become worse off.[2]

Notice that strong reciprocity always implies weak reciprocity but the converse is not true -(2) and (4) cannot be derived from weak reciprocity-.

Definition 8 *A SCF f is* ***dictatorial*** *if and only if $\exists i \in N$ such that $\forall R_i \in \Re$, $\forall \mathbf{R}_{-i} \in \Re_{-i}$, $f(R_i, \mathbf{R}_{-i}) \in \{a \in A \mid aR_i b\ \forall b \in A\}$.*

Dictatorial SCFs always select a given individual's first choice regardless of the preferences of the others. We will need this class of undesirable SCFs in some proofs.

Definition 9 *A SCF f is* ***constant*** *if and only if $\exists a \in A$ such that $\forall \mathbf{R} \in \Re^n$, $f(\mathbf{R}) = a$.*

Definition 10 *A SCF f is a* ***Generalized Condorcet winner solution*** *($GCWS(n+1)$) if $\exists \alpha = (\alpha_1, \alpha_2, ..., \alpha_{n+1}) \in [0, M]^{n+1}$, called phantom voters or fixed ballots, such that*

$$f(\mathbf{R}) = m\left(p(R_1), p(R_2), ..., p(R_n), \alpha_1, \alpha_2, ..., \alpha_{n+1}\right).$$

where m stands for the median. Notice that $GCWS(n+1)$ are voting schemes.

[2] There is still other possibility of defining an even weaker concept of reciprocity, consisting on not imposing any constraint at all on the behavior of the rule in indifference situations -and allowing for the "perverse" effect in indifference situations-. The author have explored this possibility but characterizations become much more complicated, although our intuition is that the results obtained with our version would not change very much.

Moulin (1980) showed that when preferences are single-peaked on the interval $[0, M]$, the only anonymous and strategy-proof voting schemes on $[0, M]$ are those belonging to the family of $GCWS(n+1)$. If efficiency is additionally imposed, the resulting class is also the median, but with only $n-1$ phantom voters. We will refer to this family as $GCWS(n-1)$.[3]

Definition 11 *A SCF f is* ***adjusted constant to*** *a ($a \in [0, M]$) if for all $R \in \Re^n$,*

$$f^a(R) = \begin{cases} a & if\ a \in P(\mathbf{R}) \\ \min\{p(R_i) \mid i \in N\} & if\ a < \min\{p(R_i) \mid i \in N\} \\ \max\{p(R_i) \mid i \in N\} & if\ a > \max\{p(R_i) \mid i \in N\} \end{cases}$$

Denote by Φ the family of adjusted-constant SCFs f, namely $\Phi = \{f^a \mid a \in [0, M]\ \text{ and } f^a \text{ is adjusted-constant to } a\}$.

Thomson [13] proved that class $\Phi \subset GCWS(n-1)$ contains the only efficient SCFs such that $WDUPR$ holds when preferences are single-peaked on $[0, M]$. Notice that all the SCFs within class Φ are anonymous, but not trivial and it is a subclass of the family $GCWS(n-1)$ where we have the $n-1$ phantom voters allocated to the same point.

3 Results

We will study the behavior of the reciprocity property under two different domain assumptions. First, we characterize the anonymous and strong reciprocate SCFs in the *unrestricted domain* of every preference relation when the set of alternatives is finite and we will obtain a result that establishes a close relationship between strategy-proofness and both reciprocity and anonymity. We will benefit from this property to prove the characterization result by means of the well-known Gibbard-Satterthwaite Theorem. The negative result shows the impossibility of finding strong reciprocate and anonymous SCFs in this domain.

Secondly, we investigate the existence of anonymous, and strong/weak reciprocate SCFs in contexts where preferences are restricted to satisfy single-peakedness on the closed interval of the real line. We can use some theorems related with strategy-proof SCFs with single-peaked preferences: Moulin's [6] characterization of strategy-proof voting schemes and the extensions of this result to general SCFs: Barberà & Jackson [3], Barberà, Sonnenschein &

[3]The median for the case of $n+1$ phantom voters is defined as:
$m\left(p(R_1), p(R_2), ..., p(R_n), \alpha_1, \alpha_2, ..., \alpha_{n+1}\right) \Leftrightarrow$
$\#\{i \mid p(R_i) \leq m\} + \#\{i \mid \alpha_i \leq m\} \geq n$ and
$\#\{i \mid p(R_i) \geq m\} + \#\{i \mid \alpha_i \geq m\} \geq n.$

Zhou [2], that will allow us to use the relation between strategy-proofness and reciprocity. The characterization theorems in this case results in an impossibility for anonymous, efficient and strong reciprocate SCFs and the $GCWS$ voting schemes are shown to be the only anonymous, efficient and weak reciprocate SCFs. Before establishing the main results in this section, we need to prove two useful lemmata. *Lemma 1*, is interesting on its own, since it shows that for any domain of preferences, weak reciprocity and anonymity together imply that if someone's change makes me strictly gain, I cannot make him be strictly better in my turn, but if someone strictly worsens my position, I can be sure that I would make him lose if I were the one who changed. The second lemma, *Lemma 2*, has not an easy interpretation and is intended to simplify some proofs.

Lemma 1 *Assume f is a weak reciprocate and anonymous SCF. Then,* $\forall i \in N$, $\forall \mathbf{R} \in \Re^n$, $\forall R' \in \Re$, $\forall j \in N \backslash \{i\}$,

$$f(R', \mathbf{R}_{-i}) P_j f(\mathbf{R}) \Rightarrow f(R', \mathbf{R}_{-j}) I_i f(\mathbf{R}) \;\&\; f(R', \mathbf{R}_{-i}) P' f(R', R_i, \mathbf{R}_{-i-j}).$$

Proof. Take any $i \in N$, $\mathbf{R} \in \Re^n$, $R' \in \Re$ and $j \in N \backslash \{i\}$, and suppose that $f(R', \mathbf{R}_{-i}) P_j f(\mathbf{R})$ (1). Since f is weak reciprocate, consider first the profile $\mathbf{R} = (R_i, R_j, \mathbf{R}_{-i-j})$ and suppose that individual i changes his preferences from R_i to R', reaching the profile $(R', R_j, \mathbf{R}_{-i-j})$. By (1), individual j strictly gains with $i's$ change, so by weak reciprocity, if it was individual j who was suffering the same change instead of i, the latter individual would not lose, so it holds that $f(R', \mathbf{R}_{-j}) R_i f(\mathbf{R})$ (2). Now, consider that the initial profile is $(R', \mathbf{R}_{-i})$ and agent i changes his preferences to R_i - the converse of the former shift -. By condition (1), individual j worsens his position, so by weak reciprocity, if j were the agent who changed to preference R_i - abstracting from the subscript - and i would remain unchanged, he would be weakly worsened likewise, so it also holds that $f(R', \mathbf{R}_{-i}) R' f(R', R_i, \mathbf{R}_{-i-j})$ (3). We know till now that (2) and (3) hold, but there may be two possibilities in each of those conditions: Each can be satisfied with strict preference or with indifference. We denote every possibility as: $(2P), (2I), (3P)$ and $(3I)$, i.e.,

$f(R', \mathbf{R}_{-j}) P_i f(\mathbf{R})$ $(2P)$ $\quad$ $f(R', \mathbf{R}_{-i}) P' f(R', R_i, \mathbf{R}_{-i-j})$ $(3P)$

$f(R', \mathbf{R}_{-j}) I_i f(\mathbf{R})$ $(2I)$ $\quad$ $f(R', \mathbf{R}_{-i}) I' f(R', R_i, \mathbf{R}_{-i-j})$ $(3I)$

Now, we check all the combined possibilities:

1- $(2P)$ and $(3P)$:

Let us consider $(2P)$ and focus on the profile $(R', \mathbf{R}_{-j}) = (R_i, R', \mathbf{R}_{-i-j})$. Now, imagine that individual j with preferences R' changes to preferences R_j, so that the final profile will be $(R_i, R_j, \mathbf{R}_{-i-j}) = \mathbf{R}$. Since we are assuming that $(2P)$ holds, agent i would be worse off, so by weak reciprocity, if i would had changed from preferences R_i in profile $(R_i, R', \mathbf{R}_{-i-j})$ to R_j, i remaining unchanged, he could not improve agent $i's$ situation. Hence,

$f(R_i, R', \mathbf{R}_{-i-j})R'f(R_j, R', \mathbf{R}_{-i-j})$ (4). Now, by anonymity, agents' names do not matter, so, we have: $f(R_i, R', \mathbf{R}_{-i-j}) = f(R', R_i, \mathbf{R}_{-i-j})$ and

$f(R_j, R', \mathbf{R}_{-i-j}) = f(R', \mathbf{R}_{-i})$, and we can write (4) as:

$f(R', R_i, \mathbf{R}_{-i-j})R'f(R', \mathbf{R}_{-i})$. Notice that this last expression directly contradicts $(3P)$, so the present possibility cannot appear.

2- $(2P)$ and $(3I)$:

Let us focus on $(3I)$ and profile $(R', \mathbf{R}_{-i}) = (R', R_j, \mathbf{R}_{-i-j})$: Suppose that individual j changes his preferences to R_i -agent $i's$ preferences in profile $\mathbf{R}$- so that the final situation is $(R', R_i, \mathbf{R}_{-i-j})$. $(3I)$ implies, then, that individual i with preferences R' is indifferent about the shift, so by weak reciprocity, if he were the one who changed to preferences R_i, he could not have worsened individual $j's$ position with preferences R_j, which can be written as: $f(R_i, R_j, \mathbf{R}_{-i-j}) = f(\mathbf{R})R_jf(R', \mathbf{R}_{-i}) = f(R', R_j, \mathbf{R}_{-i-j})$ (5). Notice that this statement contradicts directly the assumption (1), so this case is impossible.

3- $(2I)$ and $(3I)$:

This case cannot occur either, since it is identical to case 2 in the sense that only $(3I)$, when present, causes the contradiction with (1) whether the case is $(2P)$ or $(2I)$.

4- $(2I)$ and $(3P)$:

This turns out to be the only possibility allowed by both weak reciprocity and anonymity, so the lemma is proved. ■

Lemma 2 *Assume f is a strong reciprocate and anonymous SCF. Then,* $\forall i \in N$, $\forall \mathbf{R} \in \Re^n$, $\forall R' \in \Re$ *such that* $f(R', \mathbf{R}_{-i}) \neq f(\mathbf{R}) \Rightarrow$
$f(R', \mathbf{R}_{-i})I'f(R', \mathbf{R}_{-j})\ \ \forall j \in N \backslash \{i\}$.

Proof. Suppose any $i \in N$, $\mathbf{R} \in \Re^n$, $R' \in \Re$ such that $f(R', \mathbf{R}_{-i}) \neq f(\mathbf{R})$. Then, let us take some individual other than the one who shifted the decision (i), for example, agent j and find out in what direction he was affected by ith's shift from R_i to R'. there are two possibilities: either $f(R', \mathbf{R}_{-i})R_jf(\mathbf{R})$ or $f(\mathbf{R})R_jf(R', \mathbf{R}_{-i})$. We will distinguish both cases:

Case 1: $f(R', \mathbf{R}_{-i})R_jf(\mathbf{R})$ (1). Consider now the change of agent i from preferences R_i to R' : by assumption, j does not loose. We can use strong reciprocity with respect to agent i and obtain: $f(R', \mathbf{R}_{-j})R_if(\mathbf{R})$ (2). Consider now the profile $(R', \mathbf{R}_{-j})$ and suppose that agent j with preferences R' changes to his original one (R_j). We come back to the profile

$\mathbf{R}$. By expression (2), agent i weakly looses, and by strong reciprocity $f(R_i, R', \mathbf{R}_{-i-j})R'f(R_jR', \mathbf{R}_{-i-j})$ (3). But, by anonymity, any permutation of the arguments of the SCF cannot modify its value, and the following will hold: $f(R', R_j, \mathbf{R}_{-i-j}) = f(R_j, R', \mathbf{R}_{-i-j})$. We can, then, rewrite expression (3) in this way:

$f(R_i, R', \mathbf{R}_{-i-j})R'f(R', R_j, \mathbf{R}_{-i-j})$ (3′).

Let us focus now on profile $(R', R_j, \mathbf{R}_{-i-j})$ and imagine that agent i changes preferences R' to R_i -the converse of the initial change-. By assumption (1), j should be in a worse position, so by strong reciprocity, individual i should move in the same direction if j were the one who changed. In other words, the following holds true: $f(R', R_j, \mathbf{R}_{-i-j})R'f(R', R_i, \mathbf{R}_{-i-j})$ (4). Again by anonymity, permuting preferences of agents yields the same social choice, and this holds: $f(R', R_i, \mathbf{R}_{-i-j}) = f(R_i, R', \mathbf{R}_{-i-j})$. Expression (4) can be expressed this way: $f(R', R_j, \mathbf{R}_{-i-j})R'f(R_i, R', \mathbf{R}_{-i-j})$ (4′). Statements (3′) and (4′) are obtained from the assumptions, and both should be simultaneously true, so in this case we conclude: $f(R', R_j, \mathbf{R}_{-i-j}) = f(R', \mathbf{R}_{-i})R'f(R', \mathbf{R}_{-j}) = f(R_i, R', \mathbf{R}_{-i-j})$.

Case 2: $f(\mathbf{R})R_jf(R', \mathbf{R}_{-i})$. We can follow the same steps as in case 1. The only difference is that every preference relation is inverted, and we obviously reach the same conclusion as in case 1.

Corollary 1 *Let A be a finite set of alternatives and $\Re = \wp$. If SCF f is strong reciprocate and anonymous, then, f is strategy-proof.*

Proof. We prove it by contradiction: we suppose that f is not strategy-proof but it is both strong reciprocate and anonymous, and we will find a contradiction. If f is not strategy-proof, then, there exist: $\exists i \in N$, $\exists \mathbf{R} \in \Re^n$, $\exists R' \in \Re$, such that $f(R', \mathbf{R}_{-i})P_if(\mathbf{R})$. This obviously implies that $f(R', \mathbf{R}_{-i}) \neq f(\mathbf{R})$, so we can directly apply *Lemma 2* and obtain

$f(R', \mathbf{R}_{-i})I'f(R', \mathbf{R}_{-j}) \quad \forall j \in N\backslash\{i\}$ (1). Since we are working with strict orderings, it implies that

$f(R', \mathbf{R}_{-i}) = f(R', \mathbf{R}_{-j}) \ \ \forall j \in N\backslash\{i\}$. Consider now the change of individual j with preferences R_i in profile $(R', \mathbf{R}_{-j}) = (R', R_i, \mathbf{R}_{-i-j})$ to his original preferences R_j, reaching profile $(R', R_j, \mathbf{R}_{-i-j})$. From (1), individual i with preferences R' remains indifferent with the change, so by strong reciprocity and anonymity, we get $f(R', \mathbf{R}_{-i}) = f(\mathbf{R})$, contradicting our initial assumption. ■

Corollary 2 *Let A be a finite set of alternatives and $\Re = \wp$. There do not exist anonymous and strong reciprocate SCFs such that $\#(range(f)) \geq 3$.*

The proof is obvious by using *Corollary 1* and the Gibbard-Satterthwaite Theorem -Gibbard [5], Satterthwaite [9]-.

The former negative result leads us either to consider more restricted domains of preferences or to focus on weak reciprocity. If we relax the reciprocity condition to its weaker version, we can see that there exist efficient, anonymous and weak reciprocate SCFs, even for quite rich domains, like the one of *strict orderings* over alternatives. Let us consider $n = 3,\ A = \{a, b, c\}$ and the following class of SCFs.: $\forall \mathbf{R}^3 \in \Re^3 = \wp^3$,

$$f^a(R) = \begin{cases} a & if \quad a \in P(\mathbf{R}) \\ b & if \quad a \notin P(\mathbf{R}) \ \&\ D(b, c, \mathbf{R}) > D(c, b, \mathbf{R}) \\ c & if \quad a \notin P(\mathbf{R}) \ \&\ D(b, c, \mathbf{R}) < D(c, b, \mathbf{R}) \end{cases}$$

where $D(x, y, \mathbf{R}) = \# \{i \in N \ s.t. \ xR_iy\} \ \forall x, y \in A,\ x \neq y,\ \forall \mathbf{R} \in \Re^3$. It is not difficult to prove that this SCF and its analogous are efficient, anonymous, weak reciprocate and satisfy $WDUPR$. Unfortunately, they weight excessively an arbitrary status quo, they are not strategy-proof and it is not clear how they can be generalized to more than three alternatives or to domains admitting indifference sets. Therefore, We will now consider other domains. In order to compare both versions of reciprocity with $WDUPR$, we focus on the restricted domain of continuous, single-peaked preference relations on $A = [0, M]$. From now on, we should distinguish between both kinds of reciprocity, which will be separately explored. We start with our main results concerning strong reciprocity.

Theorem 1 *Let $\Re = \Re^{SP}$ and $n = 2$. Then, there do not exist efficient, strong reciprocate and anonymous SCFs.*

Proof. Let us consider any profile $R = (R_1, R_2)$ such that $P(R_1) = 0$ and $P(R_2) = M$ (see *Figure 1*). Suppose w.l.g. that $f(R_1, R_2) \in [0, M)$ -otherwise, just permute the names of the agents and the reasoning will be analogous-. Now, consider any profile $\widehat{R}_1$ such that $P(\widehat{R}_1) \in (f(R_1, R_2), M]$ and such that $\forall x \geq P(\widehat{R}_1),\ x\widehat{R}_1y \ \forall y < P(\widehat{R}_1)$. -Notice that there always exist admissible single-peaked preferences for which that condition holds-. Now, suppose that individual 1 in profile R changes his initial preferences R_1 to preferences $\widehat{R}_1$, such that the new profile will be $(\widehat{R}_1, R_2)$. Since there are just two agents, efficiency requires that $f(\widehat{R}_1, R_2) \in \left[P(\widehat{R}_1), P(R_2)\right]$, so $f(\widehat{R}_1, R_2) > f(R_1, R_2)$ and single-peaked preferences makes agent 2 in profile R with preferences R_2 be strictly better off with $i's$ change, since

$f(\widehat{R}_1, R_2)P_2\ f(R_1, R_2)$, so by strong reciprocity, if agent 2 were the one who changed to preferences $\widehat{R}_1 = \widehat{R}_2$ while agent 1 would remain unchanged with R_1, $f(R_1, \widehat{R}_2)R_1\ f(R_1, R_2)$. Hence, since $P(R_1) = 0$, it must be that $f(R_1, \widehat{R}_2) \leq f(R_1, R_2)$.

Now, let us consider profile $(R_1, \widehat{R}_2)$ and suppose that agent 2 with preferences $\widehat{R}_2$ changes to new preferences $\widehat{\widehat{R}}_2 = R_2$, the new profile being (R_1, R_2). since we know from above that $f(R_1, \widehat{R}_2) \leq f(R_1, R_2) = f(R_1, \widehat{\widehat{R}}_2)$, individual 1 with preferences R_1 can either be indifferent with the change whenever $f(R_1, \widehat{R}_2) = f(R_1, R_2)$ or strictly loose if the case is that of $f(R_1, \widehat{R}_2) < f(R_1, \widehat{\widehat{R}}_2) = f(R_1, R_2)$. Suppose first that $f(R_1, \widehat{R}_2) < f(R_1, R_2)$: this implies that 1 loses with the change, and strong reciprocity requires that, if he were the one who changed from R_1 to $\widehat{\widehat{R}}_1 = \widehat{\widehat{R}}_2 = R_2$, agent 2 with initial preferences $\widehat{R}_2$ in profile $(R_1, \widehat{R}_2)$ could never gain with the change. Therefore,

$$f(R_1, \widehat{R}_2)\widehat{R}_2 f(\widehat{\widehat{R}}_1, \widehat{R}_2) = f(R_2, \widehat{R}_2). \tag{1}$$

Since, by anonymity, $f(R_2, \widehat{R}_2) = f(\widehat{R}_2, R_2) = f(\widehat{R}_1, R_2)$, expression (1) can be rewritten as $f(\widehat{R}_1, R_2)\widehat{R}_2 f(\widehat{R}_1, R_2) \in \left[P(\widehat{R}_1), P(R_2)\right]$. But by definition of $\widehat{R}_1$,

$\forall x \in \left[P(\widehat{R}_1), P(R_2)\right]$, $x\widehat{P}_1 y\ \forall y \in \left[0, P(\widehat{R}_1)\right)$, so since $\widehat{R}_2 = \widehat{R}_1$ and $f(R_1, \widehat{R}_2) \leq f(R_1, R_2) < P(\widehat{R}_1) \leq P(R_2) = M$, $f(\widehat{R}_1, R_2)\widehat{P}_1 f(\widehat{R}_1, R_2)$, a contradiction.

It remains to check the case in which $f(R_1, \widehat{R}_2) = f(R_1, R_2)$ and agent 1 is indifferent with $2's$ change from $\widehat{R}_2$ to $\widehat{\widehat{R}}_2 = R_2$. By strong reciprocity, 1 should leave agent 2 indifferent if he were the one who changed preferences, so again by anonymity, it should hold that $f(\widehat{R}_1, R_2) = f(R_1, \widehat{R}_2) < P(\widehat{R}_1) \leq P(R_2)$, contradicting efficiency of f at $f(\widehat{\widehat{R}}_1, \widehat{R}_2) = f(R_2, \widehat{R}_2) = f(\widehat{R}_1, R_2)$. ■

[Insert *Figure 1* about here]

Theorem 2 *Let* $\Re = \Re^{SP}$ *and* $n \geq 3$*. Then, the only strong reciprocate and anonymous SCFs are constant.*

Proof. We have to prove both implications:

Step 1:($\Rightarrow$) $\Re$ single-peaked, f is a strong reciprocate and anonymous SCF with $\#N \geq 3 \Rightarrow f$ is constant.

We will first demonstrate that under the single-peakedness assumption and $\#N \geq 3$, every strong reciprocate and anonymous SCF has to be strategy- proof. It will be proved by contradiction: we first suppose that f is anonymous, manipulable and strong reciprocate and we will find a contradiction.

Let us consider only three ordered individuals to simplify the notation of the proof, and let us call them 1, 2 and 3. The profile that is supposed to be manipulated will be now the following: (R_1, R_2, R_3) and let agent 1 -without loss of generality- be the manipulator, changing to preferences R'.

We can now apply *Lemma 2* for $j = 2, 3$ with the above change, so the following statements are true: $f(R_1, R', R_3)I'f(R', R_2, R_3)$ (1) and

$f(R_1, R_2, R')I'f(R', R_2, R_3)$ (1′).

Consider now the change consisting of changing agent 1's preferences from R_1 to R_3 in the profile (R_1, R_2, R'), the final preference profile being: (R_3, R_2, R'). By anonymity, this profile has the same value that: (R', R_2, R_3). By expression (1′) both values are considered indifferent with preferences R'. Hence, by strong reciprocity in the two directions with respect to agent 1, the following holds true: $f(R_1, R_2, R_3)I_1f(R_1, R_2, R')$ (2).

Now, let us remember that, by the manipulability assumption at the original profile, it is true that: $f(R', R_2, R_3)P_1f(R_1, R_2, R_3)$ (3). Notice that we assumed that we have at least three individuals, so (1) and (1′) can be written:

$$f(R_1, R_2, R')I'f(R_1, R', R_3)I'f(R', R_2, R_3).$$

But notice that single- peaked preferences only allow for at most two distinct indifference points, so only two possibilities can occur:

1- $f(R', R_2, R_3) = f(R_1, R', R_3)$, -or $f(R', R_2, R_3)P_1f(R_1, R_2, R')$- in which case, using the analogous to expression (2) corresponding to the change from R_1 to R_2 in the profile (R_1, R', R_3), the final preference profile being: (R_2, R', R_3) : $f(R_1, R_2, R_3)I_1f(R_1, R', R_3)$ (4), and the following expression will hold:

$f(R_1, R_2, R_3)I_1f(R_1, R', R_3) = f(R', R_2, R_3)$. This contradicts directly the manipulability hypothesis -expression (1)-.

2- either $f(R_1, R', R_3) < f(R', R_2, R_3) < f(R_1, R_2, R_3)$ or

$f(R_1, R_2, R_3) < f(R', R_2, R_3) < f(R_1, R', R_3)$ and always: $f(R_1, R_2, R') = f(R_1, R', R_3)$. Because if $f(R', R_2, R_3) = f(R_1, R_2, R_3)$, there is a contradiction with the manipulability of the original profile, and it is the only possibility for (4) to hold true due to the single-peakedness of preferences. Notice that in this case we can consider profile: (R_1, R', R_3) and suppose that agent 1 changes his preferences from R_1 to R_2, reaching the profile of preferences

(R_2, R', R_3). Let us examine the effect of the change on agent 2 -with preferences R'-. As expression (1) holds and, by anonymity, $f(R_2, R', R_3) = f(R', R_2, R_3)$ - permuting agents' 1 and 2 preferences -, by reciprocity with respect to 2 the following relation should be true:

$f(R_1, R', R_3) I_1 f(R_2, R_1, R_3) = f(R_1, R_2, R_3)$ - by anonymity -.

Consider now the profile (R_1, R_2, R_3) and suppose that agent 3 with preferences R_3 moves to preferences R'. The final profile will be (R_1, R_2, R'), and by (2), the effect on agent 1 will be: $f(R_1, R_2, R_3) I_1 f(R_1, R_2, R')$. Using strong reciprocity and anonymity we have:

$f(R', R_2, R_3) = f(R', R_2, R_3) I_3 \ f(R_1, R_2, R_3)$ (5). But $f(R_1, R', R_3)$ is strictly on the right or strictly on the left of $f(R', R_2, R_3)$ and $f(R_1, R_2, R_3)$ and the peak of R_3 is such that: $p(R_3) \in [f(R_1, R_2, R_3), \ f(R', R_2, R_3)]$, so single-peakedness will imply:

$f(R', R_2, R_3) P_3 \ f(R_1, R', R_3) = f(R_1, R_2, R')$ (6) and

$f(R_1, R_2, R_3) P_3 \ f(R_1, R', R_3) = f(R_1, R_2, R')$ (7).

We can construct the symmetric change (R_2 moves to preferences R') to check the relation:

$f(R', R_2, R_3) P_2 \ f(R_1, R_2, R') = f(R_1, R', R_3)$ ($6'$) and

$f(R_1, R_2, R_3) P_2 \ f(R_1, R_2, R') = f(R_1, R', R_3)$ ($7'$).

Let us remember that we are in the only case allowed by the single-peakedness assumption in which: $f(R_1, R_2, R') = f(R_1, R', R_3)$. As both profiles achieve the same social choice, everybody will feel indifferent between them, and in particular, agents with preferences $I' : f(R_1, R_2, R') I' \ f(R_1, R', R_3)$. This can be written, by anonymity, in this way:

$f(R_1, R_2, R') I' \ f(R_1, R_3, R')$. Let us consider the first profile in the relation and suppose that agent 2 changes his preferences from R_2 to R_3, obtaining $f(R_1, R_3, R')$. By strong reciprocity with respect to agent 3, the following will be true:

$f(R_1, R_2, R_3) I_2 \ f(R_1, R', R_3) = f(R_1, R_3, R')$. But recalling expression ($6'$) and relation (5) for the symmetric case when agent 2 changes from preferences R_2 to $R' : f(R', R_2, R_3) I_2 \ f(R_1, R_2, R_3)$ (6) above. From ($6'$) and ($7'$), it should be true that:

$f(R_1, R', R_3) I_2 \ f(R_1, R_2, R_3)$. But we have seen that the following is true:

$f(R_1, R_2, R_3) P_2 \ f(R_1, R', R_3)$, and this is the contradiction we were looking for.

We have proved till now that under our assumptions, every strong reciprocate and anonymous SCF has to be strategy-proof. Using now Moulin's [6] characterization of anonymous, strategy-proof voting schemes and the results related for SCFs in the right direction, we obtain that such SCFs. should belong to the class of voting schemes defined by Moulin as *Generalized Condorcet winner solutions* $(n+1)$. Now, it suffices to prove that the only voting schemes belonging to the class of $GCWS(n+1)$ that are strong reciprocate are those that allocate all the phantom voters to the same point, i.e.,

$$\{\Pi : \Re^n \to A \mid \Pi(\mathbf{R}) = m\left(p(R_1), p(R_2), ..., p(R_n), a, a, ..., a_{n+1}\right)\}$$

Suppose that we face a voting scheme from the $GCWS(n+1)$ family such that there exist at least two phantom voters allocated in different points in the interval: $\exists \alpha_h, \alpha_l$ with $\alpha_h \neq \alpha_l,\ \alpha_h < \alpha_l$. Take, then any piece of the interval $[\alpha_h, \alpha_l]$ with no phantom voters in it and fix any profile with all the people's peaks inside that interval. It is not difficult to check that the social choice will coincide some of the agents' peaks, say individual i ($p(R_i) = m(x, \alpha)$). Consider that the agent which peak is closer to that of i -let's call him j- changes his preferences to any other with peak in the open interval between the initial peaks of i and j. It is straightforward that the median cannot change, so everybody feels indifferent with both profiles. By strong reciprocity, if agent i would change to $j's$ new peak and j would be the initial 1, i should be indifferent with both profiles. But this is impossible, since the new social choices change and cannot jump over anybody's peaks, so j would strictly gain and the voting scheme is not strong reciprocate.

The only voting schemes allowed are, then those with all the $n+1$ phantom voters located at the same point; but this is another expression for the constant function.

Step 2. ($\Leftarrow$) Any constant SCF is strong reciprocate. This part is obvious and follows directly from the definition of strong reciprocity. ∎

This result turns out to be even worse than expected, since constant SCFs are far more undesirable that Thomson's family Φ, which are at least efficient, so we can fear about the possibilities of introspective solidarity against $WDUPR$. Notice, however, that the apparently narrow behavior of strong reciprocity is extremely sensitive to the unnecessary and strong requirement that we have already seen behind the definition related to the responsiveness of strong reciprocity when facing indifference situations. In this line, we hope that weak reciprocity will yield better results than its stronger version. The problem is that the last proof cannot be applied to the weak reciprocity case because we have used the indifference features that are not shared by weak reciprocity. The analysis can however be simplified when we impose the additional property of efficiency. In exchange, we can forget about the minimal 3-agents size of society of the former result.

Theorem 3 *Let $\Re = \Re^{SP}$. The only weak reciprocate, anonymous and efficient SCFs are those belonging to the class $GCWS(n-1)$.*

Proof. As this is a characterization theorem, we must prove both directions:

$\Rightarrow$) First, we will prove that if f is a weak reciprocate, anonymous and efficient SCF, it has to be strategy-proof. We proceed by contradiction. Suppose that f is not strategy-proof, but it is weak reciprocate, anonymous and efficient and we will find a contradiction.

If f is not strategy-proof, we know that there exist: $\exists l \in N$, $\exists \mathbf{R} \in \Re^n$, $\exists R' \in \Re$, such that $f(R', \mathbf{R}_{-l}) P_l f(\mathbf{R})$. Since we suppose f to be anonymous, we can rename the individuals and the new SCF will be invariant, so consider the following permutation of agents such that all are reordered according to the following rule: $\forall j, h \in N$,

$if\ f(R', \mathbf{R}_{-i}) < f(\mathbf{R}),\ \ \sigma(j) < \sigma(h) \Leftrightarrow f(R', \mathbf{R}_{-i}) - p(R_j) > f(R', \mathbf{R}_{-i}) - p(R_h)$.

$if\ f(R', \mathbf{R}_{-i}) > f(\mathbf{R}),\ \ \sigma(j) > \sigma(h) \Leftrightarrow f(R', \mathbf{R}_{-i}) - p(R_j) < f(R', \mathbf{R}_{-i}) - p(R_h)$.

We can always construct the above permutation, which simply consists in ordering the individuals in direct relation with the distance from his peak to the extreme defined by the direction of the shift in the value of f due to the considered manipulation. Hence, call $i = \sigma(l)$ -the new name of the agent manipulating the rule-, and suppose without loss of generality that $f(R', \mathbf{R}_{-i}) < f(\mathbf{R})$ -all the argument can be easily replicated to the other case-. Now, by efficiency, somebody in the manipulable profile $\mathbf{R}$ should loose with the shift, and moreover, $\exists h > i$ such that $p(R_h) \geq f(\mathbf{R})$, since if not, $f(\mathbf{R})$ would not be an efficient alternative for $\mathbf{R}$ -everybody's peaks would be strictly on the left of $f(\mathbf{R})$-, so take the agent with the highest peak in profile $\mathbf{R}$ -if there are more than one, take any of them- and let us call him j, so it holds that $p(R_j) = \max_{h \in N} p(R_h)$. It holds for this individual that $f(\mathbf{R}) P_j f(R', \mathbf{R}_{-i})$ (1). Since f is weak reciprocate and anonymous, applying *Lemma 1* to (1),-while inverting roles of R_i and R'- we know that the following statements are true: $f(R', \mathbf{R}_{-i}) I' f(R', R_i, \mathbf{R}_{-i-j})$ (2) and $f(\mathbf{R}) P_i f(R_i, R', \mathbf{R}_{-i-j})$ (3). By linking the manipulability hypothesis with (3), we get: $f(R', \mathbf{R}_{-i}) P_i f(\mathbf{R}) P_i f(R_i, R', \mathbf{R}_{-i-j})$ $(3')$, so the former three profiles yield different outcomes and, by the single-peakedness assumption, only two possibilities can occur:

(i). $f(R_i, R', \mathbf{R}_{-i-j}) = f(R', \mathbf{R}_{-j}) < f(R', \mathbf{R}_{-i}) < f(\mathbf{R})$.

Notice that in this case, it must be by (2) and $(3')$ that

$p(R') \in (f(R', \mathbf{R}_{-j}),\ f(R', \mathbf{R}_{-i}))$, $p(R_i) \in (f(R', \mathbf{R}_{-j}),\ f(\mathbf{R}))$and

$p(R_j) \geq f(\mathbf{R})$. By efficiency of $f(R', \mathbf{R}_{-j})$, there exists some other individual h with preferences in $\mathbf{R}$ such that $p(R_h) \leq f(R', \mathbf{R}_{-j})$, so by single-peakedness, $f(R', \mathbf{R}_{-j}) P_h f(R', \mathbf{R}_{-i})$ $(1')$. Now, consider the change of individual i from preferences R_i in profile $(R_i, R', \mathbf{R}_{-i-j}) = (R', \mathbf{R}_{-j})$ to preferences R_j, such that the final profile will be $(R_j, R', \mathbf{R}_{-i-j})$. By anonymity, $f(R_j, R', \mathbf{R}_{-i-j}) = f(R', \mathbf{R}_{-i})$, and individual i gains with the change by $(3')$, the first profile being manipulable by i. Let us check the effect of the shift on individual h : by $(1')$, agent h strictly loses and, by reciprocity with respect to h, it holds that $f(R_i, R', R_j, \mathbf{R}_{-i-j-h}) = f(R', \mathbf{R}_{-h}) I_j f(R', \mathbf{R}_{-i})$, $(2')$, and $f(R_i, R', \mathbf{R}_{-i-j}) = f(R', \mathbf{R}_{-j}) P_i f(R', \mathbf{R}_{-h})$. Since $f(R', \mathbf{R}_{-h}) \neq f(R', \mathbf{R}_{-i})$ by the assumption of manipulability and single-peakedness, it is true that $f(R', \mathbf{R}_{-h}) > f(R', \mathbf{R}_{-i})$, and by $(2')$ and the above restrictions on the peaks, we know that $p(R_j) \in (f(R', \mathbf{R}_{-i}),\ f(R', \mathbf{R}_{-h}))$ and hence,

$p(R_i), p(R_j), p(R') <\ f(R', \mathbf{R}_{-h})$. But, notice that by construction,

$p(R_j) = \max_{g \in N} p(R_g)$, so it must be that

$f(R', \mathbf{R}_{-h}) > \max \left\{ \max_{g \in N} p(R_g),\ p(R') \right\}$, so $f(R', \mathbf{R}_{-h})$ cannot be an efficient alternative in profile $(R', \mathbf{R}_{-h})$, a contradiction.

(ii). $f(R', \mathbf{R}_{-i}) < f(\mathbf{R}) < f(R_i, R', \mathbf{R}_{-i-j}) = f(R', \mathbf{R}_{-j})$, so if both extreme profiles are considered indifferent by preferences R' by expression (2), since single-peaked indifferent sets have at most two points, it must be that $f(\mathbf{R}) P' f(R', \mathbf{R}_{-j})$ (4) and $f(\mathbf{R}) P' f(R', \mathbf{R}_{-i})$ $(4')$. Now, let us consider the profile $(R_i, R', \mathbf{R}_{-i-j})$; Notice that (4) and (3) respectively imply: $p(R') < f(R', \mathbf{R}_{-j})$ and $p(R_i) < f(R', \mathbf{R}_{-j})$, so by efficiency of $f(R', \mathbf{R}_{-j})$, $\exists j' \in N,\ j' \neq j$, such that $p(R_{j'}) \geq f(R', \mathbf{R}_{-j})$. Now, let us consider that individual j with preferences R' in profile $(R_i, R', \mathbf{R}_{-i-j})$ changes to preferences R_j -his initial ones-, reaching the profile R. By expression (4), j with preferences R' strictly gains by declaring R_j, so we have found another manipulable profile. Moreover, agent j' strictly looses with the change, so we also know that: $f(R_i, R', \mathbf{R}_{-i-j}) P_{j'} f(\mathbf{R})$. Now, we are in the conditions of applying *Lemma 12* and repeating all the former steps again, where only the case *(ii)* is to be considered, but now the role of preferences R_i is performed by R', the role of R' is carried out by R_j and the one of R_j is for $R_{j'}$, so we can always construct a sequence of profiles of the form:

$$
\begin{array}{l}
\mathbf{R}^{(1)} = (R_1, R', R_3, R_4, R_5, ..., R_n) \\
\mathbf{R}^{(2)} = (R_1, R_2, R', R_4, R_5, ..., R_n) \\
\mathbf{R}^{(3)} = (R_1, R_2, R_3, R', R_5, ..., R_n) \\
\mathbf{R}^{(4)} = (R_1, R_2, R_3, R_4, R', ..., R_n) \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{R}^{(n-1)} = (R_1, R_2, R_3, R_4, R_5, ..., R')
\end{array}
$$

in which some agent can manipulate the rule by changing preferences to another initially present in profile R and such that:

$f(\mathbf{R}^{(1)}) < f(\mathbf{R}^{(2)}) < f(\mathbf{R}^{(3)}) < f(\mathbf{R}^{(4)}) < ... < f(\mathbf{R}^{(n-1)})$ and
$\forall h \in \{1, ..., n-1\}$, $\forall l \in \{1, ..., h\}$, $f(\mathbf{R}^{(h)}) > p(R') \geq \max_l p(R_l)$.

Therefore, profile $\mathbf{R}^{(n-1)}$ cannot be efficient, since there are no more individuals with preferences in $\mathbf{R}$ with peaks on the right of $f(\mathbf{R}^{(n-1)})$ and every peak is strictly on the left of $f(\mathbf{R}^{(n-1)})$. This is a contradiction and f has to be strategy-proof. Now, we can apply Barberà & Jackson [3] result: the only strategy-proof SCFs must be voting schemes, and Moulin's [6] result, which states that every anonymous, efficient and strategy-proof voting scheme should belong to the family of $GCWS(n-1)$.

$\Leftarrow$) The implication: $\Pi \in GCWS(n-1) \Rightarrow \Pi$ is anonymous and efficient is easy and is already proved in Moulin [6]. So, it is sufficient to prove that every voting scheme in Moulin's class is weak reciprocate, and the characterization will be complete.

Let us take any voting scheme $\Pi \in GCWS(n-1)$; That is, we fix an arbitrary distribution of phantom voters $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{n-1})$. We will prove that the median of the peaks and phantoms gives us a voting scheme that preserves the weak reciprocity property. The median is defined as follows: $m\left(p(R_1), p(R_2), ..., p(R_n), \alpha_1, ..., \alpha_{n-1}\right) \quad \Leftrightarrow \quad \#\left\{i \mid p(R_i) \leq m\right\} + \#\left\{i \mid \alpha_i \leq m\right\} \geq n$ and
$\#\left\{i \mid p(R_i) \geq m\right\} + \#\left\{i \mid \alpha_i \geq m\right\} \geq n$.

Suppose any fixed distribution of peaks $p(\mathbf{R}) = (p(R_1), p(R_2), ..., p(R_n))$, so that the social decision is $m = m(p(\mathbf{R}), \alpha)$ and that somebody -let us call him i- changes his peak -without loss of generality one whose peak is on the left of the median $(p(R_i) \leq m)$- to $p(R') \neq p(R_i)$. The shift of the new choice will depend on the allocation of agent $i's$ new peak. There are two possibilities:

1- $p(R') \leq m(p(\mathbf{R}), \alpha)$:

We need to know how this change will affect the remaining agents, in order to check the reciprocity of the voting scheme. Notice that in this case the cardinality of the set of agents at both sides of the initial median will not vary and the distribution of phantoms is always the same. By the definition of median, the new choice will be the same:

$$m(p(R', \mathbf{R}_{-i}), \alpha) = m' = m = m(p(\mathbf{R}), \alpha).$$

Every individual will be then indifferent with both distributions of peaks and, by weak reciprocity we should check that every agent with peak $p(R')$ should either not affect i or improve $i's$ position. Let us consider any j such that $p(R_j) \leq m$. If j changes to $p(R')$, the total number of peaks on the left of the median will remain unchanged, so the median cannot vary: $m(p(R', \mathbf{R}_{-j}), \alpha) = m'' = m = m(p(\mathbf{R}), \alpha)$, so i does not loose. Now, let us fix any individual h with initial peak on the right of m : If he changes his preferences to x'_i, the left hand side of the median increases its weight relative to the right side -which lose $j's$ vote- so, in the case of shifting the choice, it has to be to the left of the initial median, so it is true that: $m(p(R', \mathbf{R}_{-j}), \alpha) = m'' \leq m = m(p(\mathbf{R}), \alpha)$. But the change of just one individual cannot make the median jump over anybody's peak, so everybody with peaks strictly on the left of the initial median -including agent i- should gain with the change. The only remaining possibility is that of $p(R_i) = m$, but in this case, whenever $p(R') \leq m$, if the initial change changes the rule's choice, we are in case 2, and if it does not, nobody can individually make the decision shift to the left, so reciprocity holds in this case.

2- $p(R') \geq m$:

In this case, it is easy to prove that $m' \geq m$ and every $p(R_j) > m$ implies $p(R_j) \geq m'$. By single-peakedness, $\forall j \in N$ such that $p(R_j) \geq m$, $m'R_jm$, and weak reciprocity: $m(p(R', \mathbf{R}_{-j}), \alpha)R_im$ should hold. Notice that $m(p(R', \mathbf{R}_{-j}), \alpha) \leq m(p(R', \mathbf{R}_{-i}), \alpha)$, and by single-peakedness again it will always be true that

$m(p(R', \mathbf{R}_{-j}), \alpha)R_im$. Let us see what happens with people on the left of the initial mean: For $p(R_j) \leq m$, everybody will be equal or worse off than before: $mR_jm(p(R', \mathbf{R}_{-i}), \alpha)$; so, by weak reciprocity we will expect i to weakly lose if some j such that $p(R_j) \leq m$ moves to $p(R')$. As $p(R') \geq m$, the following medians will coincide: $m(p(R', \mathbf{R}_{-j}), \alpha) = m(p(R', \mathbf{R}_{-i}), \alpha)$, and by single-peakedness -or simply looking at the definition of median above- i will not improve his position and this holds: $mR_im(p(R', \mathbf{R}_{-j}), \alpha)$. ■

The last result establishes the characterization of the large set of SCFs which are anonymous, efficient and weak reciprocate SCFs. As we said above, introspective solidarity interpreted as weak reciprocity allows for a larger set of procedures for making public decisions that the *effective* solidarity requirement represented by $WDUPR$. The important role given to the status quo when requiring the latter property along with efficiency disappears when we require weak reciprocity, so the SCF can be made much more sensitive and responsive to changes in the individuals' tastes.

Finally, it may be useful to comment the price we have to pay for this result with respect to that of strong reciprocity. We have yet argued that weak reciprocity is a weaker concept of solidarity than strong reciprocity, but it makes more sense, so that strong reciprocity is undoubtedly too stringent at a minimal conceptual cost. More interesting is the following question: since the efficiency and anonymity requirements are both needed for the last theorem to hold, we may wonder about what kind of weak reciprocate SCFs are we eliminating by imposing anonymity and efficiency together. Since anonymity is implied by replacement monotonicity combined with efficiency, the efficiency property is the crucial assumption in order to compare both solidarity principles. We should then, expect both weak reciprocate and replacement monotonic SCFs to exist outside the efficiency environment. The problem is that they may not be voting schemes and strategy-proof, so that the whole preference relations of the agents may be relevant to determine the outcome. This fact makes them too complex objects and difficult to implement. We can only provide the reader with two families of SCFs. of this kind that lay outside our analysis and they are anonymous, weak reciprocate and replacement monotonic, but they lack efficiency -in fact, they hardly select efficient alternatives and are manipulable-. The first class contains no voting scheme: Assume $\Re = \Re^{SP}$ and let us consider the family $\Psi = \{f^a \mid a \in [0, M]\}$. Given $a \in [0, M]$, let $f^a(\mathbf{R})$ be defined as: $\forall \mathbf{R} \in \Re,\ f^a(\mathbf{R}) =$

$$= \begin{cases} \arg\max\limits_{x} \bigcup\limits_{i \in N} \{x \in [0, M] \mid xI_i a\} & iff\ \ \forall i \in N,\ \#\{x \in [0, M] \mid xI_i a\} > 1 \\ M & \text{otherwise} \end{cases}$$

This function is not difficult to understand: it simply finds the largest point that is indifferent with the fixed one a -or M if there does not exist another one -for every individual- and then, selects the largest -the closest to M-. Notice that this SCF makes broad use of the information outside the agents' peaks. Let us define

$$b_i(\mathbf{R}) = \begin{cases} \max\{\{x \mid xI_i a\}, a\} & iff\ \exists x \neq a\ s.t.\ xI_i a. \\ M & \text{otherwise} \end{cases} \quad \forall \mathbf{R} \in \Re.$$

Notice that any function in the class Ψ is replacement monotonic since any shift in the function cannot jump over anybody's $b_i(\mathbf{R})$ so that either everybody gains or everybody loses. It is not strong reciprocate because whenever $f^a(\mathbf{R}) = b_j$, if agent i with $b_i(\mathbf{R}) < b_j(\mathbf{R})$ changes to preferences such that $b_i'(\mathbf{R}) = a$, since $b_i'(\mathbf{R}) < b_i(\mathbf{R}) < b_j(\mathbf{R})$, the social choice does not change, and leave all the others indifferent, but whenever agent j moves to a, the social choice shifts and everybody gains, so i will not be indifferent. Notice that weak reciprocity holds in any case.

The second class are voting schemes, and they are anonymous, weak reciprocate and replacement monotonic, but they are not efficient, strategy-proof and strong reciprocate. Consider the class $\Sigma = \{f^a \mid a \in [0, M]\}$. Given $a \in [0, M]$, let $f^a(\mathbf{R})$ be defined as: $\forall \mathbf{R} \in \Re$,

$$f^a(\mathbf{R}) = \begin{cases} p(R_1) & iff\ p(R_1) = p(R_2) = ... = p(R_n). \\ a & \text{otherwise} \end{cases}$$

4 Conclusions

We have investigated in this work the introspective solidarity principles of reciprocity in public goods environments when monetary compensations are not possible.

In a first step, we try to calibrate the power of the reciprocity property combined with anonymity in a general context with a finite set of alternatives, without imposing any domain restriction on the preference space. *Corollary 1* offers us a negative result. It is shown that we cannot find any anonymous and reciprocate SCF within this unrestricted domain. We are, then, compelled to impose some kind of structure on the space of preferences to obtain a positive result. In order to compare reciprocity with welfare-domination under preference-replacement, we move to the public good context with infinite alternatives defined into a closed interval on the real line, where the single-peakedness restriction is quite a natural assumption.

Theorem 3 provides the answer within this new context and proves that there exist efficient, anonymous and weak reciprocate SCFs. Moreover, all of them are fully characterized and the class of functions that preserve both properties turns out to coincide with Moulin's class of *Generalized Condorcet winner solutions*. This result can be considered in two different ways. First, it is clear that we have achieved our goal of enlarging the small class of replacement monotonic SCFs by allowing for procedures that are more sensitive to individual preferences. Secondly and in a strategic context, we can consider the result as some kind of reinforcement of the class of strategy-proof SCFs within the restricted domain of single-peakedness, since we show that they also satisfy some introspective solidarity principle.

Theorems 1 and *2* explore the strong version of reciprocity in the public good context and conclude that there do not exist minimally responsive strong reciprocate and anonymous SCFs. and they are not compatible with efficiency either. The reason why strong reciprocity is so much demanding than its weak version lies essentially on the treatment of changes in preferences that do not alter the social decision. Strong reciprocity is clearly overdemanding when it requires that when somebody changes and the social decision does not move, nobody else can make it shift with the other's preferences.

Whatever interpretation of the result we may like best, it may be worthwhile to point out the close relations between strategy-proofness and the reciprocity-anonymity condition in some restricted domains -not only that of single-peakedness, but that of strict orderings too-. When talking about reciprocate SCFs, we are imposing a fairness principle of equal treatment among individuals when someone suffers a preference mutation. The fairness principle in some of its usual forms may not make sense when people can *lie* about their real preferences. But notice that reciprocity is consistent even in this uncertain context of private information, and this conceptual consistency is obtained free from the implied strategy-proofness of the anonymous-reciprocate SCF.

References

[1] Barberà, S. & B. Peleg (1990): "Strategy- Proof Voting Schemes with Continuous Preferences". *Soc. Choice Welfare* 7, 31-38.

[2] Barberà, S., H. Sonnenschein & Zhou (1991): "Generalized Median Voter Schemes and Commitees". *J. Econ. Theory*

[3] Barberà, S., & M. Jackson (1994): "A characterization of Strategy- Proof Social Choice Functions for Economies with Pure Public Goods". *Soc. Choice Welfare* 11, 241-252.

[4] Ching, S., & W. Thomson (1993): "Population Monotonic Solutions in Public Good Economies with Single- peakedness. University of Rochester mimeo.

[5] Gibbard, A. (1973): "Manipulation of Voting Schemes: A General Result". *Econometrica* 41, 587-601.

[6] Moulin, H. (1980): "On Strategy- proofness and Single- peakedness". *Public Choice* 35, 437-455.

[7] Moulin, H. (1984): "Generalized Condorcet -winners for Single- peaked Preferences and Single- plateau Preferences". *Soc. Choice Welfare* 1, 127-147.

[8] Moulin, H. (1987): "The Pure Compensation Problem: Egalitarian versus laissez- fairism". *Quart. J. Econ.* 102, 769-783.

[9] Satterthwaite, M.A. (1975): "Strategy- proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions". *J. Econ. Theory* 10, 187-217.

[10] Schmeidler, D. & H. Sonnenschein (1978): "Two Proofs of the Gibbard-Satterthwaite Theorem on the Possibility of a Strategy- proof Social Choice Function". in Proceedings of a conference on decision theory and social ethics at Schloss Reisensburg, edited by H. Gottinger & W. Ensler. Dordrecht, Holland: Reidel.

[11] Thomson, W. (1983): "The Fair Division of a Fixed Supply among a Growing Population". *Mathematics of Operations Research* 8, 319-326.

[12] Thomson, W. (1990): "A Replacement Principle". University of Rochester mimeo.

[13] Thomson, W. (1993): "The Replacement Principle in Public Good Economies with Single- peaked Preferences". *Econ. Letters* 42, 31-36.

[14] Thomson, W. (1994): "Resource- Monotonic Solutions to the Problem of Fair Division when Preferences are Single- peaked". *Soc. Choice Welfare* 11, 205-223.

[15] Thomson, W. (1995a): "Population- Monotonic Solutions to the Problem of Fair Division when Preferences are Single- peaked". *Econ. Theory* 5, 229-246.

[16] Thomson, W. (1995b): "Population- Monotonic Allocation Rules", in "Social Choice, Welfare and Ethics" (W.A. Barnett, H. Moulin, M. Salles and N. Schofield, Eds.), Chap. 4, pp. 79-124, Cambridge Univ. Press, Cambridge, UK, 1995.

[17] Thomson, W. (1997): "The Replacement Principle in Economies with Single- peaked Preferences". *J. Econ. Theory* 76, 145-168.

[18] Thomson, W. (1998): "The Replacement Principle in Economies with Indivisible Goods". *Soc. Choice Welfare* 15, 57-67.

[19] Thomson, W. (1999): "Welfare-Domination under Preference-Replacement; A Survey and Open Questions". *Soc. Choice Welfare*, 16, 373-394.

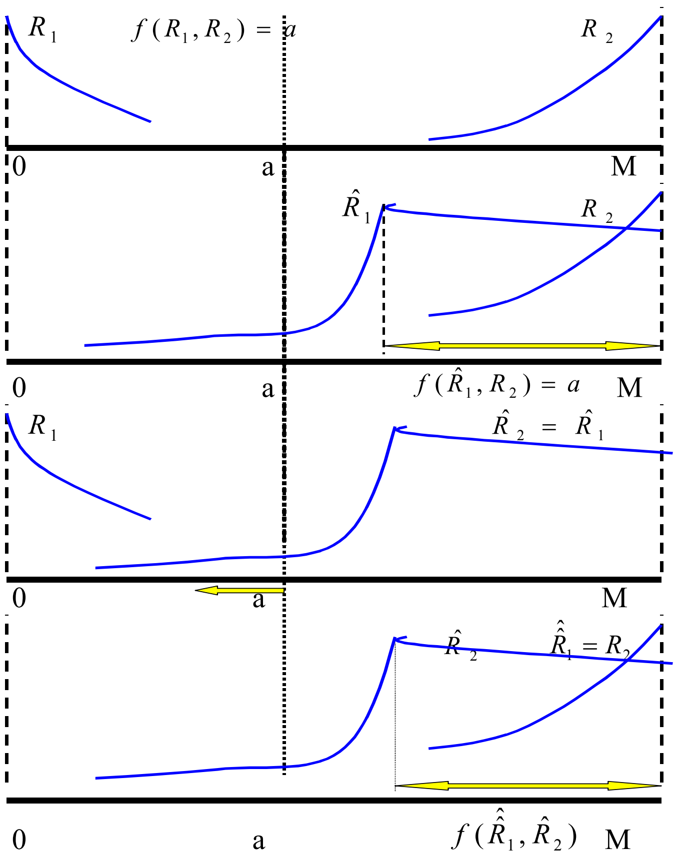
R_1
$f(R_1, R_2) = a$
R_2
0
a
M
$\hat{R}_1$
R_2
0
a
$f(\hat{R}_1, R_2) = a$
M
R_1
$\hat{R}_2 = \hat{R}_1$
0
a
M
$\hat{R}_2$
$\hat{\hat{R}}_1 = R_2$
0
a
$f(\hat{\hat{R}}_1, \hat{R}_2)$
M

Figure 1: