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Nash Implementation and Uncertain Renegotiation

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RESUMEN
Estudiamos el problema de implementación en equilibrios de Nash cuando el resultado del mecanismo puede ser renegociado entre los agentes, pero el planificador desconoce la función de renegociación que utilizarán. Caracterizamos los objetivos sociales que pueden implementarse en equilibrios de Nash cuando el mismo mecanismo debe funcionar para cualquier función de renegociación admisible. La correspondencia Walrasiana restringida, la correspondencia del núcleo, y la correspondencia Pareto-eficiente y libre de envidia satisfacen las condiciones necesarias y suficientes para esta forma de implementación si y sólo si se permite desperdiciar recursos. La regla uniforme, por otro lado, no es implementable en equilibrios de Nash para algunas funciones de renegociación admisibles.

Palabras clave: Teoría de la implementación, equilibrio de Nash, función de renegociación.

ABSTRACT
This paper studies Nash implementation when the outcomes of the mechanism can be renegotiated among the agents but the planner does not know the renegotiation function that they will use. We characterize the social objectives that can be implemented in Nash equilibrium when the same mechanism must work for every admissible renegotiation function. The constrained Walrasian correspondence, the core correspondence, and the Pareto-efficient and envy-free correspondence satisfy the necessary and sufficient conditions for this form of implementation if and only if free-disposal of the commodities is allowed. The uniform rule, on the other hand, is not Nash implementable for some admissible renegotiations functions.

Keywords: Implementation theory, Nash equilibrium, renegotiation function.
JEL classification: C 70, D 78.

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1 Introduction

Implementation theory concerns the problem of designing mechanisms (or game forms) whose equilibrium outcomes are desirable according to the objectives of a planner. Most papers related with implementation assume implicitly that the mechanisms are fully enforceable so that the agents are obliged to accept the outcomes that the mechanisms select, even if they are bad from the society's point of view. Although such bad outcomes will not generally occur in equilibrium, they are often incorporated in mechanisms off the equilibrium path as punishments for deviations. There are some studies which have addressed the ability of the planner to enforce out-of-equilibrium outcomes that are known to be undesirable (see, for example, Baliga et al., 1997, and Chakravorti et al., 2003).

Nevertheless, there exist numerous situations in which the agents are not bound to the mechanism.[1] In particular, Maskin and Moore (1999) argued that if the outcome of the mechanism is not Pareto-efficient from the agents' perspective, they may decide to renegotiate it. This can be problematic, to the extent that the good behavior of a mechanism may depend on Pareto-inefficient outcomes being enforced. Maskin and Moore considered implementation where any Pareto-inefficient outcome suggested by the mechanism is replaced by a Pareto-superior outcome according to an exogenous renegotiation function. In the same spirit, Jackson and Palfrey (2001) analyzed implementation where a general state-contingent function converts the outcomes of the mechanism. This function allows them to deal with different problems of enforcement other than renegotiation (for instance, the state-contingent function could also model individual rationality constraints).[2] In this paper, however, we will focus on its interpretation as a renegotiation function.

Given any such renegotiation function, Maskin and Moore (1999) and Jackson and Palfrey (2001) obtain characterizations of Nash implementation that have intuitive relationships to the standard results (see, e.g., Maskin, 1999). It must be stressed, however, that these characterizations depend

[1] See Hurwicz (1994) for a discussion of enforceability in mechanism design.

[2] Ma et al. (1988) were the first to point out that individual rationality constraints must be imposed both in and out of equilibrium. Jackson and Palfrey (2001) proposed to model these constraints by means of a function that reverts any non-individually rational outcome of the mechanism to the status quo.

on the exogenous specification of the renegotiation function.[3] This poses a new problem since, in many settings, the planner may not know the specific renegotiation function when the mechanism is designed (although the planner knows that any Pareto-inefficient outcome will be renegotiated, he may not know the particular bargaining strengths of each agent when he designs the mechanism).

This paper aims to study the Nash implementation problem in this sort of situations. For that, we assume that there exists a set of admissible renegotiation functions, G, so that, although the planner knows that the true renegotiation function must be in that set, the precise function is unknown to him. Specifically, we make two reasonable assumptions about admissible renegotiation functions: (1) renegotiated outcomes are always Pareto-efficient and, (2) no agent ends up worse off after renegotiating.

In the spirit of the Nash equilibrium concept, we assume that the agents know the true renegotiation function in G (i.e., they know each other's bargaining strenghts when playing the mechanism), but it is unknown to the planner. In this framework, we propose a new form of implementation where the same mechanism must work for every admissible renegotiation function in G (which we call Nash implementation in G). The characterizations of Nash implementation when the renegotiation function is fixed are extended to this setting. Not surprisingly, the fact that the planner ignores the renegotiation function acts as a constraint on what can be implemented.

Next, we examine some applications within the context of fair division problems. In this setting, free disposal of the commodities seems to be the key condition for Nash implementability in G. Thus, we show that the constrained Walrasian correspondence, the core correspondence, and the Pareto-efficient and envy-free correspondence are Nash implementable in G if and only if the mechanisms can throw away resources. That free disposal is a fundamental requirement is corroborated by the fact the uniform rule is not Nash implementable for some admissible renegotiation functions, despite being implementable in dominant strategies when no renegotiation is considered (remember that the uniform rule is a social choice rule defined within the context of a single-peaked preferences model where free disposal of the commodities is never allowed).

[3] In the face of individual rationality constraints, Jackson and Palfrey (2001) endogenized the state-contingent function in the context of a dynamic model where agents can force the mechanism to be replayed. In a previous paper, Jackson and Palfrey (1998) examined this form of implementation in a bargaining model.

The rest of the paper is organized as follows. Section 2 presents the general setting and explains how the implementation model is extended. Section 3 establishes necessary and sufficient conditions for Nash implementation in G. Section 4 deals with the Nash implementability in G of the constrained Walrasian correspondence, the core correspondence, the Pareto-efficient and envy-free correspondence, and the uniform rule. Finally, Section 5 makes some concluding remarks.

2 Definitions

Consider an environment with a known set A of feasible **alternatives or outcomes**, a set $I = \{1, 2, ..., n\}$ of **agents**, and a set S of possible **states**.

Agents' **preferences** over alternatives depend on the state: for each state $s \in S$, each agent $i \in I$ has a preference ordering $R_i(s)$ on the set A. Let $P_i(s)$ denote the strict part of $R_i(s)$.

Let 2^A denote the set of all subsets of A. A **social choice rule** (SCR) is a correspondence $F : S \to 2^A$, which associates each state s with a subset of alternatives $F(s) \subseteq A$.

A SCR is supposed to represent the objectives of a social planner (sometimes a real person, sometimes a surrogate of the agents). The implementation problem arises when the planner cannot achieve directly the outcomes recommended by the SCR. To obtain the alternatives prescribed by the SCR in a decentralized way, the planner must design a mechanism which specifies the "game rules". A **mechanism** is a pair $\Gamma = (M, h)$, where $M = \times_{i=1}^{n} M_i$, M_i is the set of possible messages for agent i, and $h : M \to A$ is the outcome function.

Most of the literature on implementation takes the alternatives selected by the mechanism as immutable: if the agents report $m \in M$, the alternative that they get is $h(m)$, and that would be that.

Following to Maskin and Moore (1999) and Jackson and Palfrey (2001), in this paper we study a more general setting in which the alternatives suggested by the mechanism may be altered in a state dependent way via some function $g : A \times S \to A$, which we call **renegotiation function**.

The renegotiation function reflects a renegotiation process. Suppose that the mechanism is $\Gamma = (M, h)$ and the agents report $m \in M$ at state $s \in S$. If $h(m)$ is inefficient from the agents' perspective at state s, they might decide to renegotiate the outcome to something which Pareto-dominates it. Although the issue of renegotiation is more apparent in a contractual context where the agents choose the mechanism themselves, it can well arise in other settings in which the planner is a real person who cannot prevent the agents from renegotiating (for example, Amorós and Moreno, 2001, studied the problem of implementation with renegotiation in a principal-agent model). Clearly, the renegotiation process depends on the default outcome and on the agents' preferences over alternatives (and it does not necessarily involve all agents).

Taking into account that the final outcome may not come directly from the mechanism but instead from the renegotiation function, we have the following natural extensions of the standard notions of Nash equilibrium and Nash implementation:

Definition 1 *The message profile $m \in M$ is a* **g-*Nash equilibrium*** *of mechanism $\Gamma = (M, h)$ at state $s \in S$ when $g(h(m), s)R_i(s)g(h(\hat{m}_i, m_{-i}), s)$ for all $i \in I$ and $\hat{m}_i \in M_i$. Let $N_g(\Gamma, s)$ denote the set of g-Nash equilibria of Γ at s.*

Definition 2 *The mechanism $\Gamma = (M, h)$* **g-*Nash implements*** *the SCR F when, for all $s \in S$:*

(1) For each $a \in F(s)$ there exists $m \in N_g(\Gamma, s)$ such that $g(h(m), s) = a$.

(2) If $m \in M$ is such that $g(h(m), s) \notin F(s)$, then $m \notin N_g(\Gamma, s)$.

If such a mechanism exists then F is g-Nash implementable.

Jackson and Palfrey (2001) showed that if a SCR is g-Nash implementable, then it satisfies the following condition:

Definition 3 *A SCR F is* **g-*monotonic*** *when, for all $s \in S$ and $a \in F(s)$, there exists $z \in A$ such that:*

(1) $g(z, s) = a$.

(2) For all $s' \in S$ such that $g(z, s') \notin F(s')$, there exists $y \in A$ and $i \in I$ such that $g(z, s)R_i(s)g(y, s)$ and $g(y, s')P_i(s')g(z, s')$.

Furthermore, Jackson and Palfrey showed that if there are at least three agents, g-monotonicity is not only a necessary condition for g-Nash implementability, but it is sufficient when combined with g-no veto power:

Definition 4 *A SCR F satisfies* **g-*no veto power*** *(**g-NVP**) when, for all $s \in S$, $z \in A$, and $i \in I$, the following is true: if $g(z, s)R_j(s)g(y, s)$ for all $y \in A$ and all $j \neq i$, then $g(z, s) \in F(s)$.*

It must be stressed that in the g-Nash implementation approach the renegotiation function is taken as fixed (i.e., g is part of the data of the problem). Then, the choice of the mechanism depends on the given g. On numerous occasions, however, the true renegotiation function that the agents will use is unknown to the planner when the mechanism is designed. This poses a new problem since the fact that a mechanism g-Nash implements F does not necessarily imply that the same mechanism $\tilde{g}$-Nash implements F for some other $\tilde{g} \neq g$. The following example may clarify this point.

Example 1 *Let $A = \{a, b, c, d, e\}$, $I = \{1, 2, 3\}$, and $S = \{s, s'\}$. Preferences of the three agents at the two states are described below (higher alternatives in the table are strictly preferred to lower alternatives).*

s			s'		
1	2	3	1	2	3
c	d	e	b	e	c
a	a	d	a	b	d
d	c	c	c	a	a
b	b	b	d	d	b
e	e	a	e	c	e

Note that alternative b is Pareto-dominated by alternatives c and d at state s. Consider the two following renegotiation functions, g and $\tilde{g}$, where alternative b is renegotiated at state s to alternatives c and d, respectively.

g		$\tilde{g}$	
s	s'	s	s'
$g(a, s) = a$	$g(a, s') = a$	$\tilde{g}(a, s) = a$	$\tilde{g}(a, s') = a$
$g(b, s) = c$	$g(b, s') = b$	$\tilde{g}(b, s) = d$	$\tilde{g}(b, s') = b$
$g(c, s) = c$	$g(c, s') = c$	$\tilde{g}(c, s) = c$	$\tilde{g}(c, s') = c$
$g(d, s) = d$	$g(d, s') = d$	$\tilde{g}(d, s) = d$	$\tilde{g}(d, s') = d$
$g(e, s) = e$	$g(e, s') = e$	$\tilde{g}(e, s) = e$	$\tilde{g}(e, s') = e$

Let $F(s) = a$ and $F(s') = b$. It is easy to see that F is g-Nash implementable via the simple mechanism where agent 2 chooses between alternatives a and b (even though b is renegotiated to c at state s, agent 2 prefers a rather than c at that state). Similarly, F is $\tilde{g}$-Nash implementable via the simple mechanism where agent 1 chooses between a and b. However, neither the former mechanism $\tilde{g}$-Nash implements F (agent 2 prefers to report b at

s, since $\tilde{g}(b,s)R_2(s)\tilde{g}(a,s)$) nor the later mechanism g-Nash implements F (agent 1 prefers to report b at s, since $g(b,s)R_1(s)g(a,s)$).

We will study the Nash implementation problem in this sort of situations. For that, we assume that there is a set G of **admissible renegotiation functions** such that, although the planner knows that the true renegotiation function must be in that set, the precise function is unknown to him.[4]

We will always assume that any renegotiation function in G satisfies the following two properties:

Pareto-efficiency. For all $a \in A$ and $s \in S$, there is no $b \in A$ such that $bR_i(s)g(a,s)$ for all $i \in I$, with strict preference for some $i \in I$.

Individual rationality. For all $a \in A$, $s \in S$, and $i \in I$, $g(a,s)R_i(s)a$.

In the spirit of the Nash equilibrium concept, we assume that the agents know the true renegotiation function $g \in G$ that they will use (suppose for example that they know each other's bargaining strenghts when they play the mechanism). In this case, when designing the mechanism, the planner knows that the agents will play any mechanism according to the g-Nash equilibrium concept for some $g \in G$, but he is unaware of the precise renegotiation function. Therefore, if the planner wants to be sure of implementing the SCR F, the same mechanism should g-Nash implement F for all $g \in G$. This is what we call Nash implementation in G.

Definition 5 *A SCR F is **Nash implementable in** **G** if and only if there exists a single mechanism which g-Nash implements F for all $g \in G$.*

Consider again Example 1 analyzed in the previous section. It is clear that none of the mechanisms proposed in that example Nash implemented F in $G = \{g, \tilde{g}\}$. Instead, the planner could use the following mechanism (where only agents 1 and 2 are strategically active):

[4] Alternatively, this could be interpreted as an enlargement of the set of possible states. As Maskin and Moore (1999) argue, two states s and s' might be identical in preferences and differ only in terms of how renegotiation would proceed. We prefer to model the set of admissible renegotiation functions separately in order to illustrate its effect on the set of implementable social choice rules.

		Agent 2	
		m_{21}	m_{22}
Agent 1	m_{11}	a	b
	m_{12}	e	a

Note that (m_{11}, m_{21}) is the only g-Nash equilibrium at s, (m_{12}, m_{22}) is the only $\tilde{g}$-Nash equilibrium at s, and (m_{11}, m_{22}) is the only g-Nash equilibrium and the only $\tilde{g}$-Nash equilibrium at s'. Therefore, this mechanism g-Nash implements and $\tilde{g}$-Nash implements F.

3 Necessary and sufficient conditions for Nash implementation in G

In this section, we will study necessary and sufficient conditions for Nash implementation in G. The key condition is what we call monotonicity in G. This is a generalization of the g-monotonicity condition that takes into account that, for any given alternative and state, the final outcome can be different depending on the true renegotiation function.

Definition 6 *A SCR F is* ***monotonic in*** **G** *when, for all $g \in G$, $s \in S$, and $a \in F(s)$, there exists $z \in A$ such that:*

(1) $g(z, s) = a$.

(2) For all $\tilde{g} \in G$ and $s' \in S$ such that $\tilde{g}(z, s') \notin F(s')$, there exists $y \in A$ and $i \in I$ such that $g(z, s) R_i(s) g(y, s)$ and $\tilde{g}(y, s') P_i(s') \tilde{g}(z, s')$.

The intuition behind this condition is that, if $a \in F(s)$, then Nash implementability in G implies the existence of a single mechanism where, for each $g \in G$, there exists a g-Nash equilibrium at s yielding a as final outcome. Moreover, if for some other $\tilde{g} \in G$ the final outcome associated with this equilibrium is not F-optimal in other state s', then it cannot be a $\tilde{g}$-Nash equilibrium at s'.

Theorem 1 *If the SCR F is Nash implementable in G, then F is monotonic in G.*

Proof. Let $\Gamma = (M, h)$ be a mechanism that Nash implements in G the SCR F. Let $g \in G$, $s \in S$, and $a \in F(s)$. Then, there exists $m \in N_g(\Gamma, s)$ with $g(h(m), s) = a$. Let $\tilde{g} \in G$ and $s' \in S$ be such that $\tilde{g}(h(m), s') \notin$

$F(s')$. Then $m \notin N_{\tilde{g}}(\Gamma, s')$ and so, there exists $i \in I$ and $\hat{m}_i \in M_i$ such that, $\tilde{g}(h(\hat{m}_i, m_{-i}), s')P_i(s')\tilde{g}(h(m), s')$. Since $m \in N_g(\Gamma, s)$, $g(h(m), s)R_i(s)$ $g(h(\hat{m}_i, m_{-i}), s)$. Let $z \equiv h(m)$ and $y \equiv h(\hat{m}_i, m_{-i})$ to satisfy the definition of monotonicity in G. ■

Although monotonicity in G alone is not a sufficient condition for Nash implementation in G, it is sufficient when combined with an adequately modified version of the g-NVP requirement defined in the previous section. The generalization of the no veto power condition to the case of Nash implementation in G only requires F to satisfy g-NVP for all $g \in G$.

Theorem 2 *If $n \geq 3$ and the SCR F is monotonic in G and satisfies g-NVP for all $g \in G$, then F is Nash implementable in G.*

The proof of this theorem follows from the logic of the proofs of Nash implementability and is provided in the Appendix.

While it is clear that if a SCR is Nash implementable in G then it is g-Nash implementable for all $g \in G$, the converse implication is not necessarily true (the fact that for each $g \in G$ there exists a mechanism g-Nash implementing F does not guarantee that there exists a single mechanism g-Nash implementing F for all $g \in G$). The following example illustrates this point.

Example 2 *Let $A = \{a, b, c, d, e, f\}$, $I = \{1, 2, 3\}$, and $S = \{s, s'\}$. Preferences are described by:*

s			s'		
1	2	3	1	2	3
d	b	e	c	e	b
c	e	c	d	a	e
a	a	b	a	d	d
b	c	f	e	f	f
f	f	d	f	b	c
e	d	a	b	c	a

Note that alternative f is Pareto-dominated by alternatives b and c at state s, and by alternatives d and e at state s'. Consider the two following renegotiation functions:

g		$\tilde{g}$	
s	s'	s	s'
$g(a,s)=a$	$g(a,s')=a$	$\tilde{g}(a,s)=a$	$\tilde{g}(a,s')=a$
$g(b,s)=b$	$g(b,s')=b$	$\tilde{g}(b,s)=b$	$\tilde{g}(b,s')=b$
$g(c,s)=c$	$g(c,s')=c$	$\tilde{g}(c,s)=c$	$\tilde{g}(c,s')=c$
$g(d,s)=d$	$g(d,s')=d$	$\tilde{g}(d,s)=d$	$\tilde{g}(d,s')=d$
$g(e,s)=e$	$g(e,s')=e$	$\tilde{g}(e,s)=e$	$\tilde{g}(e,s')=e$
$g(f,s)=b$	$g(f,s')=d$	$\tilde{g}(f,s)=c$	$\tilde{g}(f,s')=e$

Let $F(s)=a$ *and* $F(s')=b$. F *is* g*-monotonic, since: (1)* $g(a,s)=a$, $g(a,s)R_1(s)g(f,s)$, *and* $g(f,s')P_1(s')g(a,s')$, *and (2)* $g(b,s')=b$, $g(b,s')R_3(s')$ $g(e,s')$, *and* $g(e,s)P_3(s)g(b,s)$. *Similarly,* F *is* $\tilde{g}$*-monotonic, since: (1)* $\tilde{g}(a,s)=a$, $\tilde{g}(a,s)R_2(s)\tilde{g}(f,s)$, *and* $\tilde{g}(f,s')$ $P_2(s')$ $\tilde{g}(a,s')$, *and (2)* $\tilde{g}(b,s')=b$, $\tilde{g}(b,s')R_3(s')\tilde{g}(e,s')$, *and* $\tilde{g}(e,s)P_3(s)\tilde{g}(b,s)$. *Moreover,* F *trivially satisfies the* g*-NVP and* $\tilde{g}$*-NVP conditions. Therefore* F *is* g*-Nash implementable and* $\tilde{g}$*-Nash implementable. Nevertheless,* F *does not satisfy monotonicity in* $G=\{g,\tilde{g}\}$. *To see this, note that the only* $z\in A$ *such that* $g(z,s)=a$ *is* $z=a$. *However* $\tilde{g}(a,s')\notin F(s')$ *and, for all* $i\in I$ *and all* $y\in A$, *if* $g(a,s)R_i(s)g(y,s)$ *then* $\tilde{g}(a,s')R_i(s)\tilde{g}(y,s')$.[5]

Finally, note that a SCR may be Nash implementable in G but fail to be Nash implementable when no renegotiation is considered.[6] To see this, consider the following example:

Example 3 *Let* $A=\{a,b,c,d,e\}$, $I=\{1,2,3\}$, *and* $S=\{s,s'\}$. *Preferences are described by:*

[5] Note that in Example 2 we have not considered all possible Pareto-efficient and individually rational renegotiation functions. Indeed, if G is the set of all Pareto-efficient and individually rational renegotiation functions, then a SCR is Nash implementable in G if and only if it is g-Nash implementable for all $g\in G$.

[6] The fact that no enforcement of the mechanism can ease implementation has been already noted by Arya et al. (1997), and Jackson and Palfrey (2001). Nevertheless, their examples are in the context of individual rationality constraints, and the function that converts the outcomes of the mechanisms is taken as fixed.

s			s'		
1	2	3	1	2	3
e	c	b	e	a	b
d	a	e	d	b	e
a	b	d	a	d	d
c	d	c	c	e	c
b	e	a	b	c	a

Note that alternative c is Pareto-dominated by alternatives d and e at state s'. Then, the two only Pareto-efficient and individually rational renegotiation functions are the following:

g		$\tilde{g}$	
s	s'	s	s'
$g(a, s) = a$	$g(a, s') = a$	$\tilde{g}(a, s) = a$	$\tilde{g}(a, s') = a$
$g(b, s) = b$	$g(b, s') = b$	$\tilde{g}(b, s) = b$	$\tilde{g}(b, s') = b$
$g(c, s) = c$	$g(c, s') = d$	$\tilde{g}(c, s) = c$	$\tilde{g}(c, s') = e$
$g(d, s) = d$	$g(d, s') = d$	$\tilde{g}(d, s) = d$	$\tilde{g}(d, s') = d$
$g(e, s) = e$	$g(e, s') = e$	$\tilde{g}(e, s) = e$	$\tilde{g}(e, s') = e$

Let $F(s) = a$ and $F(s') = b$. Note that F is Nash implementable in $G = \{g, \tilde{g}\}$ via the following mechanism:

		Agent 2	
		m_{21}	m_{22}
Agent 1	m_{11}	a	b
	m_{12}	c	b

Nevertheless, F is not Nash implementable since it does not satisfy the standard condition of monotonicity (see Maskin, 1999)[7].

Not surprisingly, renegotiation may also act as a constraint on what SCRs can be Nash implemented. In fact, in the next section we will see that even if a SCR is implementable in dominant strategies when no renegotiation is considered, it might fail to be g-Nash implementable for some $g \in G$.

[7]Note that $F(s) = a$ and, for all $i \in I$ and $z \in A$, $[aR_i(s)z] \Rightarrow [aR_i(s')z]$. However, $a \notin F(s')$.

4 Applications

In this section we study the Nash implementability in G of four important social choice rules: the constrained Walrasian correspondence, the core correspondence, the Pareto-efficient and envy-free correspondence, and the uniform rule. Free disposal of the commodities seems to be the key condition to assure this form of implementation.

4.1 The constrained Walrasian correspondence

Consider the following setting. Each agent $i \in I$ owns a bundle $\omega_i \in \mathbb{R}^l_+$ of l goods which is fixed and known. For each state $s \in S$, each agent $i \in I$ has a preference ordering $R_i(s)$ over $\mathbb{R}^l_+$ which is continuous, strictly convex, and strictly monotone. Let $A = \{a \in \mathbb{R}^{l \times n}_+ : \sum a_i \leq \sum \omega_i\}$ be the set of feasible allocations, and let $A^* = \{a \in \mathbb{R}^{l \times n}_+ : \sum a_i = \sum \omega_i\}$ be the set of feasible allocations in which no resource is ever thrown away (where a_i denotes the i-th entry of a). Denote the vector of market prices for goods by $p \in \mathbb{R}^l$. The allocation $a \in A$ and the price vector $p \in \mathbb{R}^l$ constitute a **constrained Walrasian equilibrium** at $s \in S$ if, for each agent $i \in I$, a_i maximizes $R_i(s)$ over the set $\{b_i \in \mathbb{R}^l_+ : b_i \leq \sum \omega_i \text{ and } pb_i \leq p\omega_i\}$. The **constrained Walrasian correspondence**, $W : S \to 2^A$, selects for each state s the set of feasible allocations that can be supported as a constrained Walrasian equilibrium for some $p \in \mathbb{R}^l$.

In this context, we say that a renegotiation function is **feasible** when, for all $a \in A$ and $s \in S$, $\sum g_i(a, s) \leq \sum a_i$ (where $g_i(a, s)$ denotes the i-th entry of $g(a, s)$). Let G^* be the set of all feasible renegotiation functions satisfying Pareto-efficiency and individual rationality.

First we will show that if free disposal is not allowed (i.e., if no resource is ever thrown away) then W is not Nash implementable in G^*.

Proposition 1 *If free disposal is not allowed, then the constrained Walrasian correspondence does not satisfy g-monotonicity for some $g \in G^*$.*

Proof. Note that if free-disposal is not allowed then any implementing mechanism $\Gamma = (M, h)$ must me such that $h(m) \in A^*$ for all $m \in M$. Consider the two-person, two-good example represented in Figure 1. There are two states, s and s'. In state s, the agents have preferences represented by the indifference curves I^s_1 and I^s_2, respectively. In state s', the indifference curves of agent

1 are represented by the dotted curves $I_1^{s'}$, while the preferences of agent 2 do not change (i.e., $I_2^{s'} = I_2^s$). In this example, allocation a can be supported as a constrained Walrasian equilibrium at state s for the price vector p (i.e., $a \in W(s)$). Let $g \in G^*$ be a renegotiation function such that any possible gains from renegotiation are given to agent 1.[8] Let $z \in A^*$ be such that $g(z, s) = a$. Notice that: (1) z must be in the indifference curve $I_2^s(a)$ of agent 2, and (2) $g(z, s') = g(a, s')$. Moreover, $g(a, s')$ cannot be supported as a constrained Walrasian equilibrium at s', and then $g(z, s') \notin W(s')$. It is easy to see, however, that any $y \in A^*$ such that $g(y, s')P_1(s')g(z, s')$ must be in a indifference curve of agent 2 below $I_2^s(a)$, and therefore $g(y, s)P_1(s)g(z, s)$. Similarly, any $y \in A^*$ such that $g(y, s')P_2(s')g(z, s')$ must be in a indifference curve of agent 2 above $I_2^s(a)$, and therefore $g(y, s)P_2(s)g(z, s)$. Hence, F does not satisfy g-monotonicity. ■

[FIGURE 1 HERE]

The former impossibility result can be avoided if we allow free disposal. Let us reconsider the example analyzed in Proposition 1. Let $y = (y_1, 0) \in A$, where bundle y_1 is as represented in Figure 1. Notice that $aR_1(s)y$, $yP_1(s')g(a, s')$, and $g(y, s) = g(y, s') = y$ (since $y_2 = 0$, no renegotiation is possible). Then we have $g(a, s) = a$, $g(a, s)R_1(s)g(y, s)$, and $g(y, s')P_1(s')$ $g(a, s')$, so that the g-monotonicity requirement holds. In fact, if free-disposal is allowed, then W is Nash implementable in G^*.[9]

Proposition 2 *If $n \geq 3$ and free disposal is allowed, then the constrained Walrasian correspondence is Nash implementable in G^*.*

Proof. First note that, as soon as there are at least three agents, g-NVP is trivially satisfied for all $g \in G^*$, since its hypothesis is never met (given that preferences are strictly monotone).

[8] We have chosen this renegotiation function to ease the exposition. One can find similar examples with symmetric renegotiation functions.

[9] One could view free disposal as a problematic assumption, since it allows inefficient outcomes to stand. Thus, in a contractual context where there is no planner and the mechanism is a sort of constitution, the agents could rescind their mechanism to exploit any ex-post gains. We must stress, however, that in many situations the planner is a real person who can through away resources (but cannot avoid agents' renegotiation when they have enough commodities).

Next we will show that W satisfies monotonicity in G^*. Let $g \in G^*$, $s \in S$, and $a \in W(s)$. By the first fundamental theorem of welfare economics we know that a is Pareto-efficient at s, and then $g(a,s) = a$. Let $\tilde{g} \in G^*$ and $s' \in S$ be such that $\tilde{g}(a,s') \notin W(s')$.

Claim 1. There exists $b \in A$ and $i \in I$ such that $aR_i(s)b$ and $bP_i(s')\tilde{g}(a,s')$.

We prove this claim in three steps.

Step 1.1. If Claim 1 is false then $\tilde{g}(a,s') \neq a$.

Suppose that $\tilde{g}(a,s') = a$. Let $p \in \mathbb{R}^l$ be a price vector that supports a as a constrained Walrasian equilibrium at s. Since $a \notin W(s')$, there is some $i \in I$ and $b_i \neq a_i$ such that $b_i \leq \sum \omega_i$, $pb_i \leq p\omega_i$, and $b_i P_i(s') a_i$. Moreover, since (a,p) constitute a constrained Walrasian equilibrium at s, we have $a_i R_i(s) b_i$. Let $b \in A$ be a feasible allocation where agent i gets b_i. Then $aR_i(s)b$ and $bP_i(s')\tilde{g}(a,s')$. Therefore Claim 1 is not false.

Step 1.2. If Claim 1 is false then $\tilde{g}(a,s')R_i(s)a$ for all $i \in I$.

Suppose that $a_i P_i(s) \tilde{g}_i(a,s')$ for some $i \in I$. Then, since preferences are strictly monotone, $\tilde{g}_i(a,s') \neq \sum \omega_i$. By continuity of preferences, there is an $\epsilon > 0$ such that, for all $b_i \in \mathbb{R}^l_+$ with $\|b_i - \tilde{g}_i(a,s')\| < \epsilon$, then $a_i P_i(s) b_i$. Moreover, since preferences are strictly monotone, there is some $b_i \leq \sum \omega_i$ such that $\|b_i - \tilde{g}_i(a,s')\| < \epsilon$ and $b_i P_i(s') \tilde{g}_i(a,s')$. Therefore, Claim 1 is not false.

Step 1.3. If Claim 1 is false then a cannot be Pareto efficient at s.

Suppose that Claim 1 was false. Then, by Steps 1.1 and 1.2, and given that preferences are strictly convex, we have $[\lambda\tilde{g}(a,s') + (1-\lambda)a]P_i(s)a$ for all $i \in I$ and all $\lambda \in (0,1)$, which contradicts that a is Pareto-efficient at s.

Claim 2. There exists $y \in A$ and $i \in I$ such that $g(a,s)R_i(s)g(y,s)$ and $\tilde{g}(y,s')P_i(s')\tilde{g}(a,s')$.

Let $b \in A$ and $i \in I$ be as defined in Claim 1, and let $y = (0,...,b_i,...,0)$. Since only one agent has a positive amount of at least one good, no renegotiation is possible, and then $g(y,s) = \tilde{g}(y,s') = y$. Then, since $g(a,s) = a$, by Claim 1 we have $g(a,s)R_i(s)g(y,s)$ and $\tilde{g}(y,s')P_i(s')\tilde{g}(a,s')$. ■

4.2 The core

Consider the setting defined in the previous subsection. A **coalition** is a non-empty subset of agents $I' \subseteq I$. A coalition $I' \subseteq I$ **blocks** a feasible allocation $a \in A$ at state $s \in S$ if there exists $b \in A$ such that: (1) $\sum_{i\in I'} b_i \leq \sum_{i \in I'} \omega_i$ and, (2) $bR_i(s)a$ for all $i \in I'$, with strict preference for some $i \in I'$. The

core correspondence, $C : S \to 2^A$, selects for each state s the set of feasible allocations that cannot be blocked by any coalition.

Like the constrained Walrasian correspondence, the core correspondence is Nash implementable in G^* if and only if free disposal of the commodities is allowed.

Proposition 3 *If free disposal is not allowed, then the core correspondence does not satisfy g-monotonicity for some* $g \in G^*$.

Proof. It is analogous to the proof of Proposition 1 (note that in the example illustrated in Figure 1 $a \in C(s)$ and, since $\omega P_1(s')g(a, s')$, $g(a, s') \notin C(s')$). ■

Proposition 4 *If* $n \geq 3$ *and free disposal is allowed, then the core correspondence is Nash implementable in* G^*.

Proof. As in Proposition 2, g-NVP is trivially satisfied for all $g \in G^*$. Next we will show that C satisfies monotonicity in G^*. Let $g \in G^*$, $s \in S$, and $a \in C(s)$. Then a is Pareto-efficient at s, and so $g(a, s) = a$. Let $\tilde{g} \in G^*$ and $s' \in S$ be such that $\tilde{g}(a, s') \notin C(s')$. Then, since preferences are continuous and strictly monotone, there exists $I' \subseteq I$ and $b \in A$ such that: $\sum_{i \in I'} b_i \leq \sum_{i \in I'} \omega_i$ and, (2) $bP_i(s')\tilde{g}(a, s')$ for all $i \in I'$. Moreover, since $a \in C(s)$, there is some $i \in I'$ such that $aR_i(s)b$. Let $y = (0, ..., b_i, ..., 0)$. Since only one agent has a positive amount of at least one good, no renegotiation is possible, and then $g_i(y, s) = \tilde{g}_i(y, s') = b_i$. Then, we have $g(a, s)R_i(s)g(y, s)$ and $\tilde{g}(y, s')P_i(s')\tilde{g}(a, s')$. ■

4.3 The Pareto-efficient and envy-free correspondence

Consider the same setting of the previous subsections and suppose that, instead of individual endowments, there is a fixed bundle of goods, $\Omega \in \mathbb{R}^l_{++}$, to be divided among the agents (suppose that all agents have equal claims on Ω). Let $A = \{a \in \mathbb{R}^{l \times n}_+ : \sum a_i \leq \Omega\}$ and $A^* = \{a \in \mathbb{R}^{l \times n}_+ : \sum a_i = \Omega\}$. The **Pareto-efficient and envy-free correspondence**, $E : S \to 2^A$, selects for each state s the set of feasible allocations $a \in A$ such that: (1) there is no $b \in A$ such that $b_i R_i(s) a_i$ for all $i \in I$, with strict preference for some $i \in I$ (i.e., a is Pareto-efficient at s) and, (2) $a_i R_i(s) a_j$ for all $i, j \in I$ (i.e., a is envy-free at s). Once again, free-disposal of the commodities is the necessary and sufficient condition for Nash implementation in G^*.

Proposition 5 *If free disposal is not allowed, then the Pareto-efficient and envy-free correspondence does not satisfy g-monotonicity for some $g \in G^*$.*

Proof. The proof is analogous to the proof of Proposition 1 (consider the example illustrated in Figure 1 and suppose that $\omega_1 = \omega_2$ and $\Omega = \sum \omega_i$; note that $a \in E(s)$ and, since $g_2(a,s')P_1(s')g_1(a,s')$, $g(a,s') \notin E(s')$). ■

Proposition 6 *If $n \geq 3$ and free disposal is allowed, then the Pareto-efficient and envy-free correspondence is Nash implementable in G^*.*

Proof. Since preferences are strictly monotone and $n \geq 3$, g-NVP is trivially satisfied for all $g \in G^*$. Next we will show that E satisfies monotonicity in G^*. Let $g \in G^*$, $s \in S$, and $a \in E(s)$. Since a is Pareto-efficient at s, $g(a,s) = a$. Let $\tilde{g} \in G^*$ and $s' \in S$ be such that $\tilde{g}(a,s') \notin E(s')$. Suppose that $\tilde{g}(a,s') = a$. Then a is Pareto-efficient at s'. Since $\tilde{g}(a,s') \notin E(s')$, then a is not envy-free at s'. Therefore, $a_j P_i(s') a_i$ for some $i,j \in I$. Let $\pi(a) \in A$ be a permutation of a such that i gets a_j. Then $\pi(a) P_i(s') \tilde{g}(a,s')$. Moreover, since a is envy-free at s, then $a R_i(s) \pi(a)$. Therefore, there exists $b \in A$ and $i \in I$ such that $a R_i(s) b$ and $b P_i(s') \tilde{g}(a,s')$. The rest of the proof is analogous to the proof of Proposition 2. ■

4.4 The uniform rule

Consider now the following single-peaked preferences model. A fixed amount $\Omega \in \mathbb{R}_{++}$ of some commodity has to be allocated among a set $I = \{1,...n\}$ of agents whose preferences are single peaked: for each state $s \in S$ and each agent i there is a number $p_i(s) \in [0,\Omega]$ (called agent i's **peak**) such that, for all $x_i, x_i' \in [0,\Omega]$, if $p_i(s) \leq x_i < x_i'$, or if $x_i' < x_i \leq p_i(s)$, then $x_i P_i(s) x_i'$. The set of feasible allocations is $A = \{x \equiv (x_i)_{i\in I} \in \mathbb{R}^n_+ : \sum x_i = \Omega\}$ (free disposal of the commodity is not allowed). The following social choice rule has been characterized on the basis of a number of implementability conditions (see, for example, Sprumont, 1991).

Uniform rule (U). For all $s \in S$, $U(s) \in A$ is such that: (1) if $\sum p_i(s) \geq \Omega$, then $U_i(s) = \min\{p_i(s), \lambda\}$ for all $i \in I$, where λ solves $\sum \min\{p_i(s), \lambda\} = \Omega$, and (2) if $\sum p_i(s) \leq \Omega$, then $U_i(s) = \max\{p_i(s), \lambda\}$ for all $i \in I$, where λ solves $\sum \max\{p_i(s), \lambda\} = \Omega$.

Let G^* be the set of all renegotiation rules satisfying Pareto-efficiency and individual rationality. When no renegotiation function is considered, the uniform rule is implementable in dominant strategies. However, as we show in the next proposition, the uniform rule is not g-Nash implementable for some $g \in G^*$, and then it is not Nash implementable in G^*. The reason for this result seems to be that, in the single-peaked preferences model, free disposal is never allowed.

Proposition 7 *The uniform rule is not g-Nash implementable for some $g \in G^*$.*

Proof. Let $\Omega = 9$ and $I = \{1, 2, 3\}$. Suppose that the agents' preferences can be represented by an utility function of the form $u_i^s(x) = -|x_i - p_i(s)|$. Let $s, s' \in S$ be two states in which the agents have the following peaks:

s	s'
$p_1(s) = 7$	$p_1(s') = 1$
$p_2(s) = 5$	$p_2(s') = 5$
$p_3(s) = 8$	$p_3(s') = 8$

Note that $U(s) = (3, 3, 3)$, and $U(s') = (1, 4, 4)$. Let $g \in G^*$ be such that any possible gains from renegotiation are first given to agents 1 and 2. Note that (1) the only $z \in A$ such that $g(z, s) = U(s)$ is $z \equiv U(s)$, and (2) $g(U(s), s') = (1, 5, 3) \neq U(s')$. Since agents 1 and 2 are receiving their peaks at s' in $g(U(s), s')$, it is clear that there is no $y \in A$ such that $g(y, s')P_1(s')g(U(s), s')$ or $g(y, s')P_2(s')g(U(s), s')$. Moreover, any $y \in A$ such that $g(U(s), s)R_3(s)g(y, s)$ must be such that $y_3 \leq 3$. From the assumption made on the renegotiation function, this implies that in $g(y, s')$ agent 3 will not receive more than three units of commodity, and then $g(U(s), s')R_3(s')g(y, s')$. Therefore, the uniform rule does not satisfy g-monotonicity. ■

5 Concluding remarks

In this paper, we have characterized Nash implementation when the outcomes of the mechanism can be renegotiated but the planner does not know the nature of the renegotiation process. We call this Nash implementation in G. We have shown that the constrained Walrasian correspondence, the core correspondence, and the Pareto-efficient correspondence are Nash implementable in G if and only if free disposal of the commodities is allowed.

The uniform rule, however, is not Nash implementable for some admissible renegotiations functions.

We see some scope for further development and extension of the model studied in this paper. One line of research could involve to study the case in which the true renegotiation function is unknown not only to the planner, but also to the agents (this situation can arise when the agents do not know each other's bargaining strenghts when they play the mechanism). Another line of research could involve to extend our analysis to the case in which the no enforcement of the mechanism is due to individual rationality constraints.

Appendix

Proof of Theorem 2. For all $g \in G$, $s \in S$, and $a \in F(s)$, let $A_g^{s,a}$ be the set of alternatives satisfying points (i) and (ii) of the definition of monotonicity in G.

Let $\Gamma = (M, h)$ be the following mechanism. For all $i \in I$, the message space is $M_i = G \times S \times A \times \{0, 1, 2, ...\}$. The outcome function $h : M \to A$ is defined as follows:

Rule 1. If there is $g \in G$, $s \in S$, $a \in F(s)$, and $z \in A_g^{s,a}$ such that $m_i = (g, s, z, 0)$ for all $i \in I$, then $h(m) = z$.

Rule 2. Suppose there exists $g \in G$, $s \in S$, $a \in F(s)$, $z \in A_g^{s,a}$, and $j \in I$ such that $m_i = (g, s, z, 0)$ for all $i \neq j$, but $m_j = (\tilde{g}, s', y, k) \neq (g, s, z, 0)$. Then

$$h(m) = \begin{cases} z; \text{ if } g(y,s)P_j(s)g(z,s) \\ y; \text{ if } g(z,s)R_j(s)g(y,s) \end{cases}$$

Rule 3. In all other cases, let $h(m)$ be the alternative announced by the agent who announced the highest integer (possible ties are broken by choosing the agent with the lowest index).

Step 1: For all $g \in G$, $s \in S$ and $a \in F(s)$, there is $m \in N_g(\Gamma, s)$ such that $g(h(m), s) = a$.

Let $g \in G$, $s \in S$ and $a \in F(s)$. Since F is monotonic in G, there exists $z \in A_g^{s,a}$. Let $m \in M$ be such that $m_i = (g, s, z, 0)$ for all $i \in I$. Then Rule 1 applies to m and $h(m) = z$. Since $z \in A_g^{s,a}$, $g(h(m), s) = a$. Moreover, $m \in N_g(\Gamma, m)$. To see this consider any unilateral deviation by some agent i to $\hat{m}_i = (\tilde{g}, s', y, k)$. Then Rule 2 comes into effect, and therefore $g(h(m), s)R_i(s)g(h(\hat{m}_i, m_{-i}), s)$.

Step 2: For all $g \in G$, $s \in S$ and $m \in N_g(\Gamma, s)$, $g(h(m), s) \in F(s)$.

Let $g \in G$, $s \in S$ and $m \in N_g(\Gamma, s)$. Suppose first that Rule 1 applies to m. Then, there is $\tilde{g} \in G$, $s' \in S$, $a' \in F(s')$, and $z' \in A_{\tilde{g}}^{s',a'}$ such that $m_i = (\tilde{g}, s', z', 0)$ for all $i \in I$. Therefore $h(m) = z'$ and $\tilde{g}(z', s') = a'$. Suppose by contradiction that $g(z', s) \notin F(s)$. By monotonicity in G, there exists $y \in A$ and $i \in I$ such that $\tilde{g}(z', s')R_i(s')\tilde{g}(y, s')$ and $g(y, s)P_i(s)g(z', s)$. Consider a unilateral deviation by agent i to $\hat{m}_i = (g, s, y, 1)$. By Rule 2 we have $h(\hat{m}_i, m_{-i}) = y$, which contradicts that $m \in N_g(\Gamma, s)$.

Suppose now that either Rule 2 or Rule 3 applies to m. Then there is $j \in I$ such that, by making a unilateral deviation, any agent $i \neq j$ can make the mechanism to select any alternative $y \in A$ via Rule 3. Therefore, since $m \in N_g(\Gamma, s)$, for all $y \in A$ and all $i \neq j$, $g(h(m), s) R_i(s) g(y, s)$. Then, since F satisfies g-NVP, $g(h(m), s) \in F(s)$. ■

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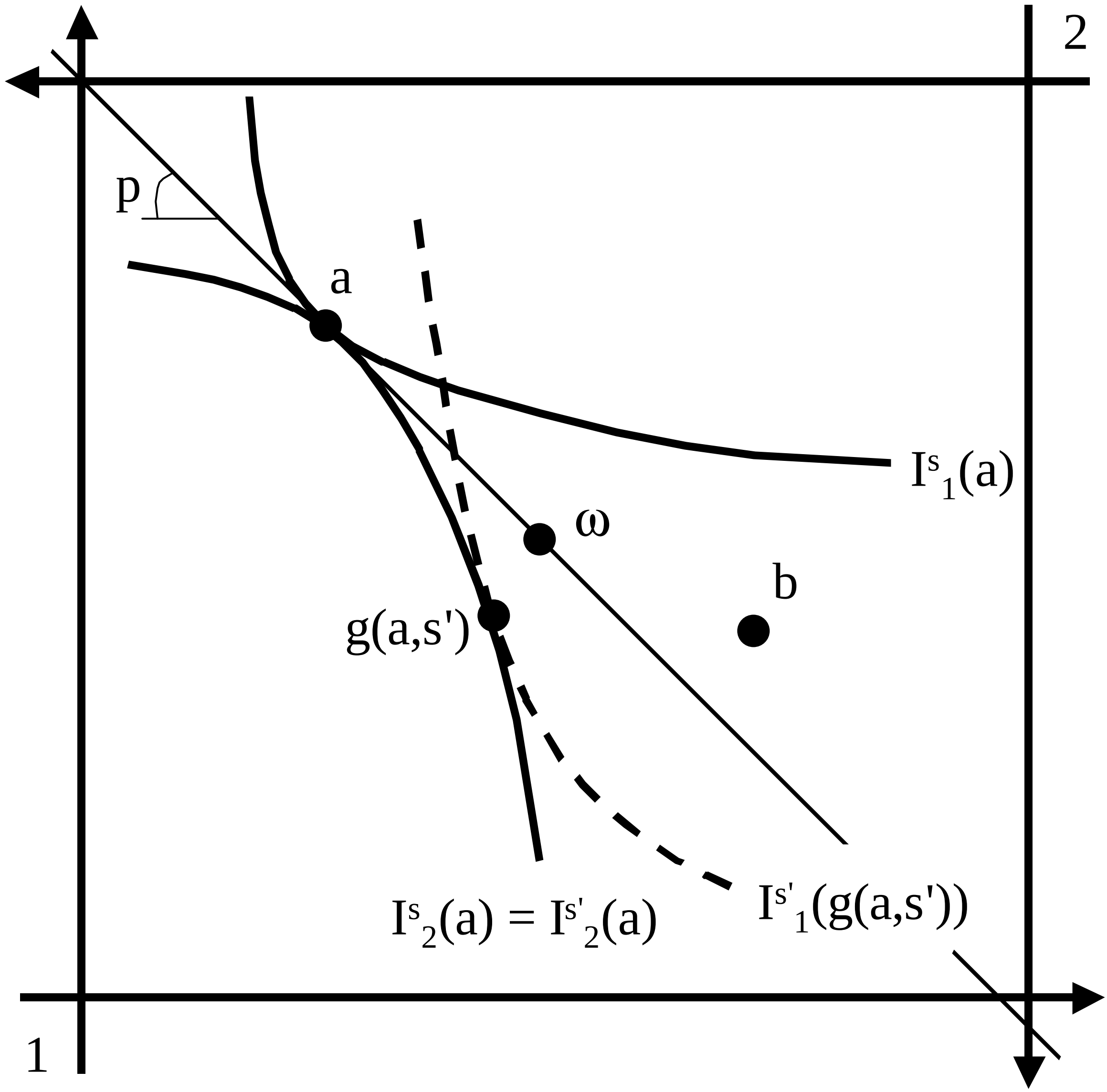

Figure 1: Illustration of Propositions 1, 3 and 5