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" Measuring polarization, inequality, welfare and poverty"

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RESUMEN
Este artículo analiza la relación que existe entre las medidas de polarización y desigualdad, bienestar y pobreza. En primer lugar, se generaliza el índice de polarización de Wolfson, en términos de los componentes intergrupos e intragrupos del índice de Gini, a grupos de renta separados por cualquier valor z. En segundo lugar, se demuestra que la medida propuesta de polarización es la diferencia entre el bienestar de los ricos y el bienestar de los pobres cuando los sentimientos de identificación se basan en sus funciones de utilidad. En tercer lugar, el índice Generalizado de Wolfson es una función del índice de pobreza de Sen, de su extensión debida a Shorrocks (1995) y del índice de déficit de pobreza normalizado cuando el valor z adoptado coincide con la línea de pobreza. Además, estos resultados son puestos en relación con los índices de polarización de Esteban y Ray (1994) y Esteban et al. (1999).

Palabras clave: polarización, desigualdad, bienestar, pobreza.

ABSTRACT
This paper analyzes the relationship between polarization and inequality, welfare and poverty measures. First, the Wolfson polarization measure is generalized in terms of the between-groups and within-groups Gini components for income groups separated by any z income value. Second, it is shown that polarization is the difference between the welfare levels of rich and poor income groups when feelings of identification between individuals are based on their utility functions. Third, the proposed polarization measure is a function of the Sen poverty index, its extension due to Shorrocks (1995) and the normalized poverty deficit index when the z income value represents the poverty line. In addition, these results are linked to the Esteban and Ray (1994) and Esteban et al. (1999) polarization measures.

Keywords: polarization, inequality, welfare, poverty.

JEL classification: D39, D63, H30.

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1 Introduction

Polarization measures have recently been proposed as useful characterizations of income distributions.[1] Polarization is widely accepted as being distinct from inequality. Polarization concentrates the income distribution on several focal or polar modes, whereas inequality relates to the overall dispersion of the distribution. A more polarized income distribution is one that is more spread out from the middle, so there are fewer individuals or families with mid-level incomes (Wolfson, 1994). Therefore, polarization measures can be used to complement the analysis of an income distribution. To make income distribution comparisons, it is useful to study not only inequality, poverty and welfare, but also polarization.

Relationships between inequality, poverty and welfare measures have been the focus of a large body of research on distribution (see, for instance, Lambert, 2001 and the references therein). However, relationships between these concepts and income polarization have hardly been analyzed. The similarities and differences between welfare, inequality and poverty measures are well known, but we know little about the meaning of income polarization in terms of welfare, poverty and inequality. This is the main disadvantage of using polarization measures as complementary tools analyzing income distribution.

In this paper, the measurement of polarization is linked to the other primary features of an income distribution: inequality, welfare and poverty.

First, a general relationship between the Wolfson polarization index and the Gini-based inequality measurement is formally established. The Wolfson polarization measure in terms of the between-groups and within-groups Gini components for income groups separated by any z income value is obtained. Then, polarization measures for the median and mean income values found in the literature (see Rodríguez and Salas, 2003 and Prieto *et al.*, 2004a) are represented as special cases. Polarization (for any z income value) and inequality are viewed within the same framework, with subtraction and addition of the within-groups dispersion corresponding to polarization and inequality, respectively. In addition, it is shown that the *Generalized Wolfson polarization measure*

[1] See, among others, Foster and Wolfson (1992), Esteban and Ray (1994 and 1999), Wolfson (1994 and 1997), Esteban *et al.* (1999), Tsui and Wang (2000), Gradín (2000), Zhang and Kanbur (2001), D'Ambrosio and Wolff (2001), Chakravarty and Majumder (2001), Rodríguez and Salas (2003), Prieto *et al.* (2004a and 2004b) and Duclos *et al.* (2004).

is a function of the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization measures. The proposed polarization measure is a function of the Esteban and Ray (1994) polarization index when only two groups are considered. The Generalized Wolfson polarization measure is also a function of the Esteban *et al.* (1999) polarization index when two income groups are considered and the measurement error weight β is equal to unity. Therefore, the relationships between the Generalized Wolfson bipolarization index and the welfare and poverty indexes (developed below) can be linked to the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization measures.

Second, a relationship between polarization and welfare measures arises when envy between people is considered in the utility function. It is shown that polarization is the difference between the richer income group's welfare and the poorer income group's welfare when individuals' utilities depend not only on their own income but also on their group incomes. Consequently, polarization increases when the welfare of the richer income group rises or the welfare of the poorer income group falls. In addition, the feelings of identification between individuals are based on their utility functions in the framework used for analysis. This seems closer to the original motivation of the identification–alienation framework (see Esteban and Ray, 1994 and Duclos *et al.*, 2004) than just relying the identification term on the density function value.

Third, polarization and poverty measures are closely related when the z income value used to separate income groups represents the poverty line. In that case, polarization between the poor and those elsewhere in the income distribution explicitly considers the value of a poverty index. In particular, the Generalized Wolfson polarization measure can be expressed as a function of Sen's poverty index (Sen, 1976), its extension due to Shorrocks (1995) and one of the Foster–Greer–Thorbecke family of poverty measures, the so-called normalized poverty deficit (Foster *et al.*, 1984). Moreover, the proposed polarization measure is a function of wealth, measured by the *Normalized wealth surplus index*. However, it is shown that the Generalized Wolfson polarization measure is not an increasing function of these four measures. Only if poverty increases due to the income gap ratio and/or wealth increases due to the *income overabundance gap ratio* greater polarization is assured. This is a relevant point as a greater proportion of poor people in society does not imply necessarily more social conflict measured by a polarization index.

The paper is organized as follows. The Generalized Wolfson polarization index for any z income value is obtained in section 2. In section 3, the relationship between polarization and welfare is analyzed. Poverty and polarization measures are linked in section 4. Section 5 concludes the paper.

2 Polarization and inequality: the Generalized Wolfson polarization index

Let $F \in \Re^n$ be an income distribution of n individuals, families or households, with a mean income value μ and a median income value m.

Wolfson's index of bipolarization (see Wolfson, 1994) was originally proposed for a population divided into two groups by the median value:

$$P_m^W(F) = 4\frac{\mu}{m}P_1 = 4\frac{\mu}{m}\left[T_m - \frac{G(F)}{2}\right] = 2\frac{\mu}{m}\left[2(0.5 - L(0.5)) - G(F)\right] \tag{1}$$

where P_1 is the lightly shaded area in Figure 1, $G(F)$ is the Gini coefficient of the income distribution, F, and T_m is the trapezoid area delimitated by the diagonal line and the tangent to the Lorenz curve (L) at the 50th population percentile. This trapezoid area is equal to the vertical distance between the 45-degree line and the Lorenz curve at the median percentile, $L(0.5)$. The larger the shaded area P_1, the fewer individuals or households there are with mid-level incomes, and hence, the greater is polarization.

If we divide the population into two groups based on the mean income value (instead of the median), the average difference in income pairs within both groups—that is, the dispersion within each group, measured by the Gini coefficient—is minimized (see Aghevli and Merhan, 1981 and Davies and Shorrocks, 1989). In this case, expression (1) can be rewritten as:

$$P_\mu^W(F) = 2\left[2(q_\mu - L(q_\mu)) - G(F)\right] \tag{2}$$

where q_μ is the population percentile at the mean income value and $L(q_\mu)$ is the value of the Lorenz curve evaluated at q_μ. Note that the trapezoid area is easy to determine when the income groups are separated by the median or mean incomes. However, difficulties arise when different income values are considered (see theorem 1 below).

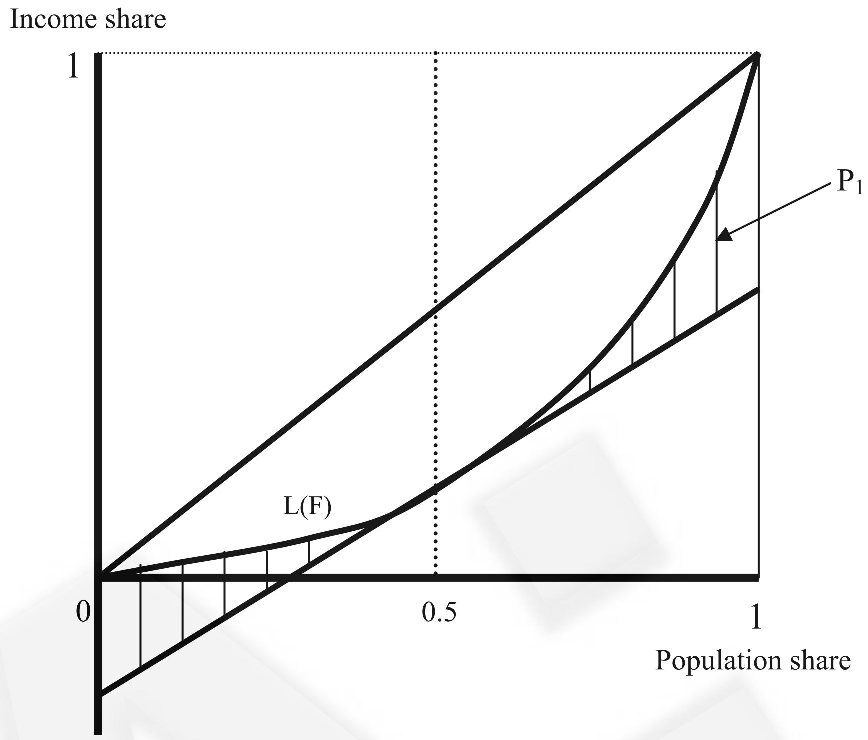

Figure 1. The Wolfson bipolarization Measure

The Wolfson index of polarization has been reformulated in terms of Gini components. The additive decomposition of the Gini coefficient by groups of the population (see, for instance, Bhattacharya and Mahalanobis, 1967, Pyatt, 1976 or Lambert and Aronson, 1993), when incomes groups do not overlap, is:

$$G(F) = G^B(F) + G^W(F) = G^B(F) + 2\left[\int_0^1 L_B(q)dq - \int_0^1 L(q)dq\right] = G^B(F) + \sum_k t_k r_k G_k \qquad (3)$$

where $G^B(F)$ is the between-groups Gini coefficient, $G^W(F)$ is the within-groups Gini coefficient, L_B is the between-groups Lorenz curve, t_k is the proportion of the population in group k, r_k is group k's share of total income and G_k is the Gini coefficient of group k.

The Wolfson index of polarization has been reformulated in terms of the between-groups Gini coefficient and the within-groups Gini coefficient as follows (see Rodríguez and Salas, 2003 and Prieto *et al.*, 2004a):

$$P_m^W(F) = 2\frac{\mu}{m}\left[G_m^B(F) - G_m^W(F)\right] \qquad (4)$$

$$P_{\mu}^{W}(F) = 2\left[G_{\mu}^{B}(F) - G_{\mu}^{W}(F)\right] \tag{5}$$

where the income groups are separated by the median and the mean income values, respectively. Therefore, polarization and inequality are explicitly represented within the same framework, with subtraction and addition of the within-groups dispersion corresponding to polarization and inequality, respectively. In other words, progressive income transfers between groups reduce inequality and polarization, while progressive income transfers within groups reduce inequality but increase polarization.

Another advantage of these reformulations is that a link is established between the Wolfson concept of polarization and the polarization model of Esteban and Ray (1994) and Duclos *et al.* (2004). The polarization measures in Esteban and Ray (1994) and Duclos *et al.* (2004) rely almost exclusively on the identification–alienation framework. Alienation relates to the accentuation of polarization by intergroup heterogeneity, while identification relates to the accentuation of polarization by intragroup homogeneity. Hence, in our framework, $G^B(F)$ represents feelings of alienation between dissimilar individuals and $G^W(F)$ represents feelings of identification between similar individuals. A different interpretation of this identification-alienation framework in terms of individual utility functions and the difference between mean income values is proposed in section 3 below.

Now we generalize the Wolfson bipolarization index in terms of the between-groups and within-groups Gini components for any z income value.

Theorem 1 (*the Generalized Wolfson polarization index*): Let $F \in \Re^n$ be an income distribution separated into two groups by any income value z. Then, the Generalized Wolfson polarization index ($GP_z(F)$ henceforth) in terms of the between-groups and within-groups Gini components is:

$$GP_z(F) = 2\frac{\mu}{z}\left[G_z^B(F) - G_z^W(F)\right] + \frac{2}{z}(1-2q_z)(\mu - z). \tag{6}$$

Proof: In what follows, consider a z income value below the median (without loss of generality). We obtain the following expression for bipolarization when the Wolfson methodology (*mutatis mutandis*) is applied:

$$GP_z(F) = 2\frac{\mu}{z}\left[2T_z - G(F)\right] \tag{7}$$

where T_z is the trapezoid area delimited by the 45-degree line and the tangent to the Lorenz curve at the z population percentile. This area is equal to the vertical distance between the 45-degree line and the tangent value at the median population percentile (see Figure 2).

The vertical distance between the Lorenz curve value at the z population percentile, $L(q_z)$, and the 45-degree line, is equal to the between-groups Gini coefficient by construction (see Figure 2):

$$B = q_z - L(q_z) = G_z^B(F)\,. \tag{8}$$

Therefore, we must obtain T_z as a function of B to generalize the Wolfson bipolarization index in terms of the between-groups and within-groups Gini components for any z income value.

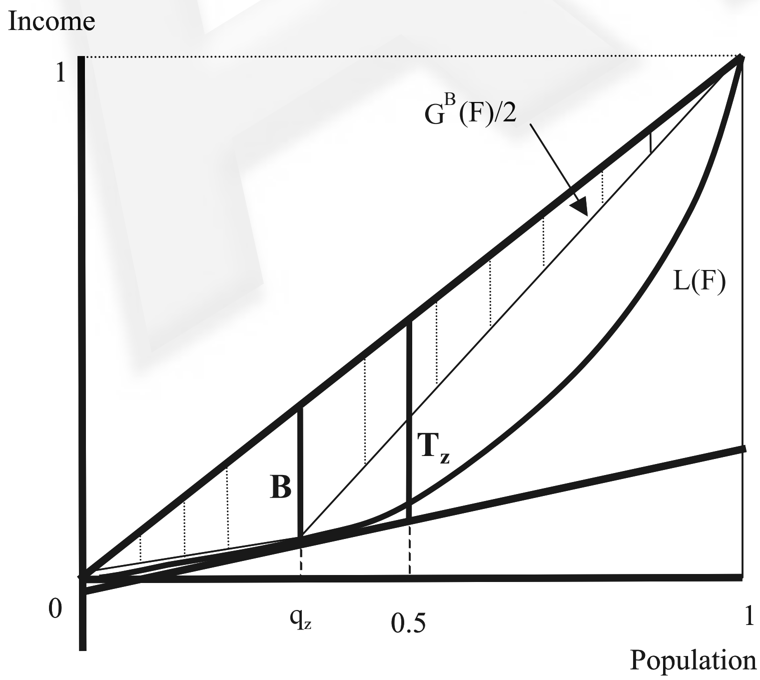

Figure 2. Bipolarization according to a *z* income value

Consider the trapezoid delimited by the diagonal line, with a slope of unity, the tangent to the Lorenz curve at q_z, with a slope of z/μ, and the vertical distances B and T_z in Figure 2. We change the coordinates (see Figure 3) and apply some geometric results.

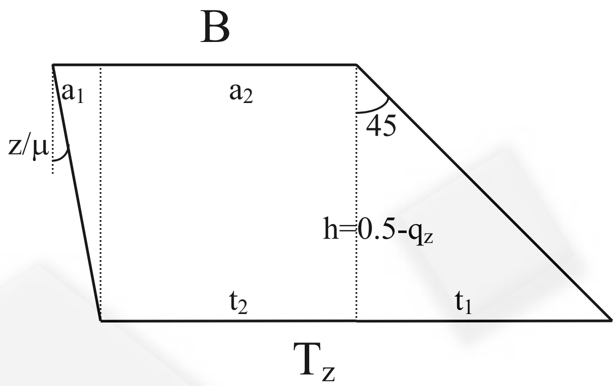

Figure 3. The B-T_z trapezoid area

Since the slope of the diagonal line is unity, the height of the segment t_1 is 0.5-q_z. If we apply the *straight-line equation* [2] it is easy to show that segment a_1 is equal to $(z/\mu)\cdot(0.5-q_z)$. Thus, $B = (z/\mu)\cdot(0.5 - q_z) + a_2$ and $T_z = (0.5 - q_z) + t_2$. Since $a_2 = t_2$, we have:

$$T_z = (0.5 - q_z) + B - \frac{z}{\mu}(0.5 - q_z) = B + (0.5 - q_z)\left(1 - \frac{z}{\mu}\right). \tag{9}$$

Substituting (8) and (9) into equation (7) yields:

$$GP_z(F) = 2\frac{\mu}{z}\left[2G_z^B(F) + 2(0.5 - q_z)\left(1 - \frac{z}{\mu}\right) - G(F)\right]. \tag{10}$$

Hence, expression (6) for the Generalized Wolfson index of polarization is obtained. The proof is similar if we consider a z income above the median value and expression (6) does not change. The following two corollaries are obtained.

[2] Recall that the point-slope form of the straight-line equation is: $(y_1-y_0)=\delta\cdot(x_1-x_0)$, where δ is the slope.

Corollary 1: Let $F \in \Re^{n}$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. If $z = m$, then $GP_m(F) = 2\frac{\mu}{m}\left[G_m^B(F) - G_m^W(F)\right]$.

Corollary 2: Let $F \in \Re^{n}$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. If $z = \mu$, then $GP_\mu(F) = 2\left[G_\mu^B(F) - G_\mu^W(F)\right]$.

The polarization expressions for the median and the mean income values found in the literature (see expressions (4) and (5)) are represented as special cases of the Generalized Wolfson polarization measure.

To complete this section, it is shown that the Generalized Wolfson bipolarization measure is a function of the Esteban *et al.* (1999) polarization index when two income groups are considered and the measurement error weight β is equal to unity. When only two groups are considered, the Generalized Wolfson bipolarization is a function of the Esteban and Ray (1994) polarization measure. These relationships are used later to generalize some of the results obtained for the Esteban *et al.* (1999) and Esteban and Ray (1994) polarization measures.

Theorem 2 (*the Generalized Wolfson polarization measure as a function of the Esteban et al. (1999) polarization index*): Let $F \in \Re^{n}$ be an income distribution and $GP_z(F)$ be the Generalized Wolfson polarization measure. Then, it follows that:

$$GP_z(F) = \frac{2\mu}{zT} P_z^{EGR}(F;\alpha;1) + \frac{2}{z}\left[(1-2q_z)(\mu - z) - \mu G_z^W(F)\left(1 - \frac{1}{T}\right)\right] \qquad (11)$$

where $P_z^{EGR}(F;\alpha;\beta)$ is the Esteban *et al.* (1999) polarization index for the income distribution F separated into two groups by the z income value, α is the identification sensitivity parameter, β is the measurement error weight and T is $q_z^\alpha + (1-q_z)^\alpha$.

Proof: The Esteban *et al.* (1999) polarization index is:

$$P_z^{EGR}(F;\alpha;\beta) = P_z^{ER}(F;\alpha) - \beta\varepsilon(F;\ell) \qquad (12)$$

where $P_z^{ER}(F;\alpha)$ is the Esteban and Ray (1994) polarization index for two income groups separated by the z income value and $\varepsilon(F;\ell)$ is the measurement error that occurs when ℓ (which requires agrupated data) is considered as the relevant income distribution instead of F.

The Esteban and Ray (1994) polarization index is:

$$P^{ER}(F;\alpha) = \sum_i \sum_j q_i^{1+\alpha} q_j \left|\mu_i - \mu_j\right| \tag{13}$$

where q_i and μ_i are the population quintile and the mean income value of income group i, respectively. Therefore, when we consider two income groups:

$$P_z^{EGR}(F;\alpha;\beta) = \left[q_z^{1+\alpha}(1-q_z) + (1-q_z)^{1+\alpha} q_z\right](\mu_2 - \mu_1) - \beta\left[G(F) - G(\ell)\right]. \tag{14}$$

The mean income values are:

$$\mu_1 = \frac{L(q_z)}{q_z} \quad \text{and} \quad \mu_2 = \frac{1 - L(q_z)}{1 - q_z}\text{, so} \tag{15}$$

$$P_z^{EGR}(F;\alpha;\beta) = \left[q_z^{\alpha} + (1-q_z)^{\alpha}\right] G_z^B(F) - \beta\, G_z^W(F). \tag{16}$$

Expression (16) with $\beta = 1$, together with (6), proves theorem 2.

Corollary 3: Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the z income value. When the identification sensitivity parameter α and the measurement error weight β are equal to unity, it follows that:

$$GP_z(F) = \frac{2\mu}{z} P_z^{EGR}(F;1;1) + \frac{2}{z}(1 - 2q_z)(\mu - z). \tag{17}$$

Corollary 4: Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the m income value. When the identification sensitivity parameter α and the measurement error weight β are equal to unity, it follows that:

$$GP_m(F) = \frac{2\mu}{m} P_m^{EGR}(F;1;1). \tag{18}$$

Corollary 5: Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the μ income value. When the identification sensitivity parameter α and the measurement error weight β are equal to unity, it follows that:

$$GP_\mu(F) = 2P_\mu^{EGR}(F;1;1). \tag{19}$$

Theorem 3 (*the Generalized Wolfson polarization measure as a function of the Esteban and Ray (1994) polarization index*): Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the z income value. Then, it follows that:

$$GP_z(F) = \frac{2\mu}{zT} P_z^{ER}(F;\alpha) + \frac{2}{z}\left[(1-2q_z)(\mu - z) - \mu G_z^W(F)\right] \tag{20}$$

where α is the identification sensitivity parameter and T is $q_z^\alpha + (1-q_z)^\alpha$.

Proof: When we consider the following expression in (6), the result above is obtained.

$$P_z^{ER}(F;\alpha) = \left[q_z^\alpha + (1-q_z)^\alpha\right] G_z^B(F). \tag{21}$$

Corollary 6: Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $P_m^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the m income value. It follows that:

$$GP_m(F) = \frac{2\mu}{mT} P_m^{ER}(F;\alpha) - \frac{2\mu}{m} G_m^W(F). \tag{22}$$

Corollary 7: Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $P_\mu^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the μ income value. It follows that:

$$GP_\mu(F) = \frac{2}{T} P_\mu^{ER}(F;\alpha) - 2G_\mu^W(F). \tag{23}$$

However, note that the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes can be applied to any number of income groups. By contrast, the Generalized Wolfson polarization measure can only be applied to two income groups.

In the next section, we use abbreviated welfare functions containing the Gini coefficient to interpret polarization in terms of welfare.

3 Polarization and welfare

An interesting relationship between polarization and welfare measures arises when envy between people is incorporated into their utility functions. We show that polarization increases when the welfare of the richer income group increases or the welfare of the poorer income group falls.

It is well known that the rankings induced on any two income distributions with the same mean income value by a symmetric, increasing and individualistic abbreviated welfare function W, and by $-G$, are not necessarily the same (Newbery, 1970). Nevertheless, the use of an abbreviated welfare function containing the Gini coefficient can be justified when W is non-individualistic (see, for example, Sheshinski, 1972, Kakwani, 1980 and 1986).

Let $D(x;y)$ be the relative deprivation felt by an individual with income x in relation to an individual with income y, where:

$$D(x;y) = y - x \qquad \text{if } x \le y$$

$$D(x;y) = 0 \qquad \text{if } x \ge y \tag{24}$$

(see Runciman, 1966). Then, the overall deprivation felt by an individual with income x is:

$$D_F(x) = \int D(x;y) f(y) dy\,. \tag{25}$$

Now let $U^D(x, F)$ be the utility function of an individual with income x, where:

$$U^D(x,F) = ax - bD_F(x) \qquad a,\ b > 0. \tag{26}$$

The individual cares not only about his or her own income but also about the distribution to which he or she belongs. In particular, the higher the deprivation felt by the individual, the lower his or her utility.

The following result justifies the use of an abbreviated welfare function containing the Gini coefficient when W is non-individualistic.

Result 1 (Lambert, 2001, pp. 123-124):[3] when $U^D(x,F) = ax - bD_F(x)$, $W_F^D = \int U^D(x,F) f(x) dx = \mu_F(a - bG_F)$ for every income distribution F.

We use this result (for $a = b = 1$) later to link overall polarization in society to the welfare level of the rich income group.

A parallel result arises when the new concept of *relative abundance* is introduced.

Let $A(x;y)$ be the relative abundance felt by an individual with income x in relation to an individual with income y, where:

$$A(x;y) = x - y \qquad \text{if } x \ge y$$

[3] A similar result, $W_F = \mu_F\left[a - 0.5b(1 + G_F)\right]$, is obtained when the altruistic utility function $U(x,F) = x\left[a - bF(x)\right]$ $a,\ b > 0$ is used, in which the arguments are the individual's own income level and the proportion of people who are worse off than that individual.

$$A(x;y)=0 \qquad \text{if } x \le y. \tag{27}$$

The overall abundance felt by an individual with income x is:

$$A_F(x)=\int A(x;y)f(y)dy\,. \tag{28}$$

Now let $U^A(x, F)$ be the utility function of the individual with income x, where:

$$U^A(x,F)=ax+bA_F(x) \quad a,\ b>0. \tag{29}$$

In this case, envy is different: an individual with income x is better off when more people have less income than he or she has. People care for status. Consequently, the more relative abundance felt by an individual with income x, the higher his or her utility.

The following result allows the use of an abbreviated welfare function (containing the Gini coefficient) when W is non-individualistic in a different way than in the context of result 1.

Theorem 4 (*a welfare function based on the relative abundance concept*):[4] let $F \in \Re^n$ be an income distribution, let $A_F(x)$ be the relative abundance function and let $U^A(x,F)=ax+bA_F(x)$ for $a,\ b>0$; then $W_F^A=\int U^A(x,F)f(x)dx=\mu_F(a+bG_F)$.

Proof: Substituting equations (27), (28) and (29) into the welfare function yields:

$$W_F^A=\int_0^\infty U^A(x,F)f(x)dx=a\mu+b\int_0^\infty\left[\int_0^x (x-y)f(y)dy\right]f(x)dx\,. \tag{30}$$

Since $q = F(x)$ and $L(q)=\dfrac{1}{\mu}\displaystyle\int_0^x yf(y)dy$, we have:

[4] A similar result, $W_F=\mu_F\left[a+0.5b(1+G_F)\right]$, is obtained when the utility function $U(x,F)=x\left[a+bF(x)\right]$ $a,\ b>0$ is used, in which the arguments are the individual's own income level and the proportion of people who are worse off than that individual.

$$\mu L(q)=\int_0^x y f(y)dy \tag{31}$$

$$L'(q)=\frac{dL(q)}{dx}\frac{dx}{dq}=\frac{xf(x)}{\mu}\frac{1}{f(x)}=\frac{1}{\mu}x\,. \tag{32}$$

Substituting (31) and (32) into (30) yields:

$$W_F^A=\mu\left[a+b\int_0^1\left[qL'(q)-L(q)\right]dq\right]. \tag{33}$$

The definition of the Gini inequality index implies that:

$$\int_0^1 L(q)dq=\frac{1-G(F)}{2}. \tag{34}$$

Integrating (34) by parts yields:

$$\int_0^1 qL'(q)dq=\frac{G(F)+1}{2}. \tag{35}$$

Substituting expressions (34) and (35) into equation (33) completes the proof.

We use this theorem (for a = b = 1) later to link overall polarization in society to the welfare level of the poor income group. However, we must prove the following lemma before linking abbreviated welfare functions and economic polarization.

In the following lemma, we decompose the Generalized Wolfson polarization measure into two terms, which correspond to the two transformed areas (below and above $L(q_z)$) that define polarization (see Figure 1).

Lemma 1 (*the Generalized Wolfson polarization measure decomposition*): Let $F \in \Re^n$ be an income distribution and let $GP_z(F)$ be the Generalized Wolfson polarization measure. Then, it follows that:

$$GP_z(F)=4\frac{\mu}{z}\left[\frac{1}{2}\left(q_z^2\left(1+\frac{z}{\mu}\right)-2q_zL(q_z)\right)-\int_0^{q_z}(q-L(q_z))dq\right]+$$

$$+4\frac{\mu}{z}\left[\frac{1}{2}\left(1+q_z-2L(q_z)-\frac{z}{\mu}(1-q_z)\right)(1-q_z)-\int_{q_z}^{1}(q-L(q_z))dq\right] \quad (36)$$

where each term on the right-hand side of equation (36) corresponds to the two transformed areas (below and above $L(q_z)$) that define polarization (represented by the shaded areas in Figure 1).

Proof: Theorem 1 implies that:

$$GP_z(F)=2\frac{\mu}{z}\left[G_z^B(F)-G_z^W(F)\right]+\frac{2}{z}(1-2q_z)(\mu-z)=$$

$$=4\frac{\mu}{z}\left[G_z^B(F)-\frac{G(F)}{2}\right]+4\frac{\mu}{z}(\frac{1}{2}-q_z)\left(1-\frac{z}{\mu}\right). \quad (37)$$

Substituting expression (8) into equation (37) yields:

$$GP_z(F)=4\frac{\mu}{z}\left[\frac{1}{2}-L(q_z)-\frac{z}{2\mu}+\frac{z}{\mu}q_z-\frac{G(F)}{2}\right]=$$

$$=4\frac{\mu}{z}\left[\left(q_z^2\left(1+\frac{z}{\mu}\right)-2q_zL(q_z)+(1-q_z)+q_z(1-q_z)-2(1-q_z)L(q_z)-\frac{z}{\mu}(1-q_z)^2\right)\frac{1}{2}\right]$$

$$-4\frac{\mu}{z}\left[\int_0^{q_z}(q-L(q))dq+\int_{q_z}^{1}(q-L(q))dq\right]. \quad (38)$$

Reordering terms in (38) yields expression (36).

Note that the two terms on the right-hand side of equation (36) are the two areas below and above $L(q_z)$, which define polarization. The following term corresponds to the trapezoid area below q_z:

$$\frac{1}{2}\left(q_z^2\left(1+\frac{z}{\mu}\right)-2q_zL(q_z)\right)=\frac{1}{2}\left[\left[q_z-L(q_z)\right]+\left[0-\left(L(q_z)-\frac{z}{\mu}q_z\right)\right]\right]q_z. \quad (39)$$

The term $\left(L(q_z)-\frac{z}{\mu}q_z\right)$ is the negative vertex of the trapezoid and is obtained by applying the point-slope form of the straight-line equation (see footnote 2).

The following term corresponds to the trapezoid area above q_z:

$$\frac{1}{2}\left(1+q_z-2L(q_z)-\frac{z}{\mu}(1-q_z)\right)(1-q_z)=\frac{1}{2}\left[[q_z-L(q_z)]+\left[1-L(q_z)-\frac{z}{\mu}(1-q_z)\right]\right](1-q_z) \tag{40}$$

The vertex $\left[L(q_z)-\frac{z}{\mu}(1-q_z)\right]$ is also obtained by applying the point-slope form of the straight-line equation (see footnote 2).

Having proved Lemma 1, consideration of the welfare functions discussed above (see result 1 and theorem 4) leads to the following result.

Theorem 5 (*the Generalized Wolfson polarization measure as a function of the welfare levels of the income groups*): Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure, let W_P^A be the welfare level of the poor income group and let W_R^D be the welfare level of the rich income group. Then, it follows that:

$$GP_z(F)=2q_z^2\left[1-\frac{W_P^A}{z}\right]+2(1-q_z)^2\left[\frac{W_R^D}{z}-1\right]. \tag{41}$$

Polarization increases when the welfare of the rich income group rises or the welfare of the poor income group falls, and *vice versa*.

Proof: Let μ_P be the mean income value of the poor income group (the one below the z income value) and let μ_R be the mean income value of the rich income group (above the z income value). Then, it follows that:

$$L_B(q)=\frac{1}{\mu}\int_0^q \mu_P dF = q\frac{\mu_P}{\mu} \qquad \forall\ q \in [0, q_z] \tag{42}$$

and

$$L_B(q)=\frac{1}{\mu}\int_0^{q_z} \mu_P dF+\frac{1}{\mu}\int_{q_z}^{q} \mu_R dF = q_z\frac{\mu_P}{\mu}+(q-q_z)\frac{\mu_R}{\mu} \qquad \forall\ q \in (q_z, 1]. \tag{43}$$

We derive from (3):

$$\int_0^{q_z} L_B(q)dq - \int_0^{q_z} L(q)dq = \frac{1}{2}q_z\left(q_z\frac{\mu_P}{\mu}\right)G_P \tag{44}$$

$$\int_{q_z}^{1} L_B(q)dq - \int_{q_z}^{1} L(q)dq = \frac{1}{2}(1-q_z)\left((1-q_z)\frac{\mu_R}{\mu}\right)G_R\,. \tag{45}$$

Therefore, given expressions (42), (43), (44) and (45), we have:

$$\int_0^{q_z}(q-L(q_z))dq = \left[\int_0^{q_z} qdq - \int_0^{q_z} L_B(q)dq\right] + \left[\int_0^{q_z} L_B(q)dq - \int_0^{q_z} L(q)dq\right] = \frac{1}{2}q_z^2\left(1-\frac{\mu_P}{\mu}\right) + \frac{1}{2}q_z^2\frac{\mu_P}{\mu}G_P \tag{46}$$

$$\int_{q_z}^{1}(q-L(q))dq = \left[\int_{q_z}^{1} qdq - \int_{q_z}^{1} L_B(q)dq\right] + \left[\int_{q_z}^{1} L_B(q)dq - \int_{q_z}^{1} L(q)dq\right] =$$

$$= \frac{1}{2}\left(1-q_z^2\right) - \frac{1}{2}\left(1-q_z^2\right)\frac{\mu_P}{\mu} + q_z(1-q_z)\frac{\mu_R-\mu_P}{\mu} + \frac{1}{2}(1-q_z)^2\frac{\mu_R}{\mu}G_R \tag{47}$$

Substituting expressions (46) and (47) into equation (36), given that $L(q_z) = L_B(q_z) = q_z\frac{\mu_P}{\mu}$, yields:

$$GP_z(F) = 4\frac{\mu}{z}\left[\frac{1}{2}q_z^2\frac{z}{\mu} - \frac{1}{2}q_z^2\frac{\mu_P}{\mu} - \frac{1}{2}q_z^2\frac{\mu_P}{\mu}G_P\right] +$$

$$+4\frac{\mu}{z}\left[q_z\frac{z}{\mu} - \frac{z}{2\mu} - \frac{1}{2}q_z^2\frac{z}{\mu} + \frac{1}{2}\left(1+q_z^2\right)\frac{\mu_R}{\mu} - q_z\frac{\mu_R}{\mu} - \frac{1}{2}(1-q_z)^2\frac{\mu_R}{\mu}G_R\right] \quad \Leftrightarrow$$

$$GP_z(F) = \left[2q_z^2 - 2q_z^2\frac{\mu_P}{z} - 2q_z^2\frac{\mu_P}{z}G_P\right] +$$

$$+\left[4q_z - 2 - 2q_z^2 + 2(1+q_z^2)\frac{\mu_R}{z} - 4q_z\frac{\mu_R}{z} - 2(1-q_z)^2\frac{\mu_R}{z}G_R\right] \quad \Leftrightarrow$$

$$GP_z(F) = 2q_z^2\left[1-\frac{\mu_P}{z}(1+G_P)\right] + 2(1-q_z)^2\left[\frac{\mu_R}{z}(1-G_R)-1\right]. \tag{48}$$

We need only consider result 1 and theorem 2 for $a = b = 1$ to obtain expression (41) in Theorem 5.

Polarization is viewed as a function of individuals' welfare levels that depends not only on individuals' own incomes but also on their feelings of envy of others in their own income groups. In particular, people in the rich income group feel envy (relative deprivation) of individuals with higher incomes, whereas people in the poor income group feel envy (relative abundance) of individuals with lower incomes.

On the one hand, income polarization increases when the mean income value of the rich income group increases (that is, when μ_R moves away from z), whereas polarization decreases when the mean income value of the poor income group increases (that is, when μ_P moves closer to z). On the other hand, income polarization increases when relative deprivation in the rich income group or relative abundance in the poor income group decreases. This polarization result has the following clear interpretation in the context of the identification–alienation framework.

(a) When μ_R moves away from z, alienation (between the income groups) increases; when μ_P moves closer to z, alienation decreases.

(b) When relative deprivation in the rich income group decreases, identification (within the rich income group) increases. When relative abundance in the poor income group decreases, identification (within the poor income group) increases.

Alienation is determined by the difference between μ_R and μ_P. Identification depends negatively on the levels of envy, relative deprivation and relative abundance felt by individuals.

A relevant question arises. The polarization models of Esteban and Ray (1994) and Duclos *et al.* (2004) treated the identification term as the value of the density function. However, there is no reason to believe that the grouping of income distribution data conveniently conforms to the psychological demands of group identification, as the authors acknowledged. In this respect, our framework for analysis seems closer to their original motivation for the identification–alienation framework. In fact, feelings of identification are based on individuals' utility functions, in which not only are individuals' own incomes important, but also their feelings of envy matter.

Note that the welfare of the rich income group is generally, but not necessarily, higher than the welfare of the poor income group. It is possible that the level of relative abundance or feelings of deprivation experienced by people in the poor and rich income groups, respectively, more than offset the difference between the mean values of the rich and poor income groups.

A straightforward result derived from theorem 5 is stated in the following corollary.

Corollary 8: Let $F \in \Re^n$ be an income distribution and let $GP_z(F)$ be the Generalized Wolfson polarization measure. Then, if $z = m$, it follows that:

$$GP_m(F) = \frac{1}{2m}\left[W_R^D - W_P^A\right]. \tag{49}$$

Polarization is half the difference between the normalized (by the median income) welfare levels of the rich and poor income groups when income groups are separated by the median income value.

In this case, the welfare of the rich income group is unambiguously higher than the welfare level of the poor income group because polarization cannot be negative. Hence, polarization decreases when the welfare level of the poor income group approaches that of the rich income group.

The following theorem generalizes the relationship between polarization and welfare to the Esteban *et al.* (1999) polarization index.

Theorem 6 (*the Esteban et al. (1999) polarization index as a function of the welfare levels of the income groups*): Let $F \in \Re^n$ be an income distribution, let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization measure for two income groups separated by the z income value, let $W_P{}^A$ be the welfare level of the poor income group and let $W_R{}^D$ be the welfare level of the rich income group. Then, it follows that:

$$P_z^{EGR}(F;1;1) = \frac{z}{\mu}\left[q_z^2\left(1-\frac{W_P^A}{z}\right)+(1-q_z)^2\left(\frac{W_R^D}{z}-1\right)\right]-\frac{1}{\mu}(1-2q_z)(\mu-z). \tag{50}$$

According to the Esteban *et al.* (1999) polarization index, polarization is a function of individuals' welfare levels when the identification sensitivity parameter α and the parameter β are equal to unity and there are only two income groups.

Proof: Given expressions (17) and (41), the proof of this result is straightforward.

Corollary 9: Let $F \in \Re^n$ be an income distribution, let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization measure for two income groups separated by the *m* income value, let W_P^A be the welfare level of the poor income group and let W_R^D be the welfare level of the rich income group. Then, when α and β are equal to unity:

$$P_m^{EGR}(F;1;1) = \frac{1}{4\mu}\left[W_R^D - W_P^A\right]. \tag{51}$$

Corollary 10: Let $F \in \Re^n$ be an income distribution, let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization measure for two income groups separated by the μ income value, let W_P^A be the welfare level of the poor income group and let W_R^D be the welfare level of the rich income group. Then, when α and β are equal to unity:

$$P_\mu^{EGR}(F;1;1) = q_\mu^2\left(1 - \frac{W_P^A}{\mu}\right) + (1-q_\mu)^2\left(\frac{W_R^D}{\mu} - 1\right). \tag{52}$$

4 Polarization and poverty

Polarization and poverty measures can be related when the *z* income value that separates the income groups represents the poverty line. In this case, polarization between poor people and those elsewhere in the income distribution explicitly considers the value of a poverty index. In the next three results, the Generalized Wolfson polarization measure is expressed as a function of the Sen poverty index (Sen, 1976), its extension due to Shorrocks (1995) and as a function of one of the Foster–Greer–Thorbecke family of poverty measures, the normalized poverty deficit (see Foster *et al.*, 1984). It is shown that more poverty, due to an increase in the income gap ratio, implies greater income polarization in society. Furthermore, more wealth, due to an increase in the *income overabundance gap ratio*, also implies greater polarization.

First, recall some concepts. The Sen poverty index is:

$$S_z^S(F) = H_z(F)\left[I_z(F) + (1 - I_z(F))G_P\right] \tag{53}$$

where z is the poverty line, $H_z(F) = q_z$ is the headcount ratio or the proportion of the population who are poor in F, and $I_z(F) = 1 - \frac{\mu_P}{z}$ is the income gap ratio (Sen, 1976).[5]

Shorrocks (1995) proposed the following generalization of the Sen poverty index:

$$S_z^{SH}(F) = (2 - H_z(F))H_z(F)I_z(F) + H_z(F)(1 - I_z(F))G_P. \tag{54}$$

This poverty index is not only replication invariant but also continuous and consistent with the progressive transfer axiom.

The family of poverty indices introduced by Foster *et al.* (1984) is:

$$S_z^{FGT}(F;\gamma) = \int_0^z \Gamma(x)^\gamma f(x)dx \tag{55}$$

where $\Gamma(x) = \max\left\{\frac{z-x}{z}, 0\right\}$ and $\gamma \geq 0$. Note that since $D_z(F) = \int_0^z (z-x)f(x)dx$ is the poverty deficit index, the Foster–Greer–Thorbecke family of poverty measures is the normalized poverty deficit index or the product of the headcount and income gap ratios, $D_z(F)/z = H_z(F)I_z(F)$, when $\gamma = 1$.

Analogously to the normalized poverty deficit index, the *normalized wealth surplus index* is defined as:

$$R_w(F) = \overline{H}_w(F)O_w(F) = (1 - q_w)\left[\frac{\mu_R}{w} - 1\right] \tag{56}$$

where w is the wealth line, that is, the income value above which anyone is considered rich. This index is the product of the proportion of the population who are rich in F,

[5] This is the *orthodox* replication invariant version of the original Sen poverty index $S_z^S(F) = H_Z(F)I_Z(F) + [r/r+1](1 - I_Z(F))G_P$, where r is the number of poor persons.

$\overline{H}_w(F) = (1 - q_w)$, and the *income overabundance gap ratio*, $O_w(F)$, which is the average wealth gap, μ_R-w, normalized by the wealth threshold.[6]

The wealth line w and the poverty line z are the same in a bipolarized society where there are only rich and poor people.

***Theorem** 7* (*the Generalized Wolfson polarization measure as a function of the Sen poverty index*): Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $S_z^S(F)$ be the Sen poverty measure. Then, it follows that:

$$GP_z(F) = 2\left[q_z \left(S_z^S(F) - q_z \frac{\mu_P}{z} G_P \right) + (1 - q_z) R_z(F) - \frac{\mu}{z} G_z^W(F) \right] \tag{57}$$

where G_P is the Gini coefficient for the poor income group, $R_z(F)$ is the normalized wealth surplus index and $G_z^W(F)$ is the within-groups Gini coefficient.

Proof: Consider equation (48) and expression (53) together:

$$GP_z(F) = 2q_z^2 [1 - (1 - I_z(F))(1 + G_P)] - 2(1 - q_z)^2 \frac{\mu_R}{z} G_R + 2(1 - q_z)^2 \left[\frac{\mu_R}{z} - 1 \right]. \tag{58}$$

That is:

$$GP_z(F) = 2q_z^2 [I_z(F) - (1 - I_z(F))G_P] - 2(1 - q_z)^2 \frac{\mu_R}{z} G_R + 2(1 - q_z) R_z(F)$$

$$= 2q_z \left[S_z^S(F) - 2q_z \frac{\mu_P}{z} G_P \right] - 2(1 - q_z)^2 \frac{\mu_R}{z} G_R + 2(1 - q_z) R_z(F). \tag{59}$$

Given the within-groups Gini coefficient, the result is proven:

$$G_z^W(F) = q_z^2 \frac{\mu_P}{\mu} G_P + (1 - q_z)^2 \frac{\mu_R}{\mu} G_R. \tag{60}$$

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[6] The wealth surplus index is $M_w(F) = \int_w^\infty (x - w)\, dF(x)$ and therefore $R_w(F)=M_w(F)/w$.

Corollary 11 (*the Generalized Wolfson polarization measure as a function of the Shorrocks poverty index*): Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $S_z^{SH}(F)$ be the Shorrocks poverty measure. Then, it follows that:

$$GP_z(F) = 2\left[\frac{q_z}{2-q_z}\left(S_z^{SH}(F) - q_z^2\frac{\mu_P}{z}G_P\right) + (1-q_z)R_z(F) - \frac{\mu}{z}G_z^W(F)\right] \tag{61}$$

where $R_z(F)$ is the normalized wealth surplus index. This corollary is straightforward given the proof of theorem 7 and expression (54) (*mutatis mutandis*).

Corollary 12 (*the Generalized Wolfson polarization measure as a function of the normalized poverty deficit index*):[7] Let $F \in \Re^n$ be an income distribution, let $GP_z(F)$ be the Generalized Wolfson polarization measure and let $S_z^{FGT}(F;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures. Then

$$GP_z(F) = 2\left[q_z S_z^{FGT}(F;1) + (1-q_z)R_z(F) - \frac{\mu}{z}G_z^W(F)\right] \tag{62}$$

where $S_z^{FGT}(F;1)$ is the normalized poverty deficit index and $R_z(F)$ is the normalized wealth surplus index.

Bipolarization between poor people and those elsewhere in the income distribution explicitly considers the value of a poverty index: the Sen poverty index, its extension due to Shorrocks (1995) or the normalized poverty deficit index. Moreover, polarization is a function of wealth according to the normalized wealth surplus index. However, the proposed polarization measure is not an increasing function of these four measures.

On one hand, polarization depends negatively on the dispersion within the income groups according to the Gini coefficient. As shown in section 2, progressive transfers within groups increase polarization. As a result, when the Gini coefficient for the poor income group changes, polarization and poverty (measured by the Sen poverty index or

[7] It can be shown that the area below the first polarization curve (see Wolfson 1994, 1997) for incomes below z is equal to the normalized poverty deficit index.

the Shorrocks poverty index) vary in opposite directions.[8] On the other hand, the effect of a change in the proportion of poor or rich people on polarization is ambiguous.[9] When the proportion of poor or rich people changes, polarization, poverty and wealth can vary in the same direction but also in the opposite one. As a result, a greater proportion of poor people in society does not imply necessarily more social conflict measured by a polarization index.

Only changes in the income gap ratio guarantee that polarization and poverty vary in the same direction (more poverty implies more polarization); only changes in the income overabundance gap ratio guarantee that polarization and wealth vary in the same direction (more wealth implies more polarization).

In what follows, we generalize the relationship between poverty and polarization to the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes.

Theorem 8 (*the Esteban and Ray (1994) polarization measure as a function of the Sen poverty index*): Let $F \in \Re^n$ be an income distribution, let $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the *z* income value and let $S_z^S(F)$ be the Sen poverty measure. Then, it follows that:

$$P_z^{ER}(F;\alpha) = \frac{zT}{\mu}\left[q_z\left(S_z^S(F) - q_z\frac{\mu_P}{z}G_P\right) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) \qquad (63)$$

where $R_z(F)$ is the normalized wealth surplus index. Note that the negative second term on the right-hand side of equation (63) is zero when *z* is equal to *m* or μ.

Proof: Given expressions (20) and (57) and the proof of theorem 8, this proof is straightforward.

[8] Note that when only the Gini coefficient for the poor income group changes, $dGP_z(F) = -2q_z^2\frac{\mu_P}{z}dG_P$.

[9] This can be checked by deriving $GP_z(F)$ with respect to q_z.

Corollary 13 (*the Esteban and Ray (1994) polarization measure as a function of the Shorrocks poverty index*): Let $F \in \Re^n$ be an income distribution, let $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the *z* income value and let $S_z^{SH}(F)$ be the Shorrocks poverty measure. Then, it follows that:

$$P_z^{ER}(F;\alpha) = \frac{zT}{\mu}\left[\frac{q_z}{2-q_z}\left(S_z^{SH}(F) - q_z^2\frac{\mu_P}{z}G_P\right) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) \tag{64}$$

where $R_z(F)$ is the normalized wealth surplus index. Note that the negative second term on the right-hand side of equation (64) is zero when *z* is equal to *m* or μ.

Corollary 14 (*the Esteban and Ray (1994) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^n$ be an income distribution, let $P_z^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the *z* income value and let $S_z^{FGT}(F;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures. Then, it follows that:

$$P_z^{ER}(F;\alpha) = \frac{zT}{\mu}\left[q_z S_z^{FGT}(F;1) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) \tag{65}$$

where $R_z(F)$ is the normalized wealth surplus index.

Corollary 15 (*the Esteban and Ray (1994) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^n$ be an income distribution, let $P_m^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the *m* income value and let $S_z^{FGT}(F;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures. Then, it follows that:

$$P_m^{ER}(F;\alpha) = \frac{mT}{2\mu}\left[S_m^{FGT}(F;1) + R_m(F)\right] \tag{66}$$

where $R_z(F)$ is the normalized richness surplus index.

Corollary 16 (*the Esteban and Ray (1994) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^n$ be an income distribution, let $P_\mu^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for two income groups separated by the μ income value and let $S_z^{FGT}(F;\gamma)$ be the Foster-Greer-Thorbecke family of poverty measures. Then, it follows that:

$$P_\mu^{ER}(F;\alpha) = T\left[q_\mu S_\mu^{FGT}(F;1) + (1-q_\mu)R_\mu(F)\right]. \tag{67}$$

In the last two results, polarization between poor people and those elsewhere in the income distribution is simply a function of poverty, according to the normalized poverty deficit index, and wealth, according to the normalized wealth surplus index.

Theorem 9 (*the Esteban et al. (1999) polarization measure as a function of the Sen poverty index*): Let $F \in \Re^n$ be an income distribution, let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the *z* income value and let $S_z^S(F)$ be the Sen poverty measure. Then

$$P_z^{EGR}(F;\alpha;1) = \frac{T}{\mu}\left[zq_z\left(S_z^S(F) - q_z\frac{\mu_P}{z}G_P\right) + z(1-q_z)R_z(F) - \frac{\mu}{T}G_z^W(F) - (1-2q_z)(\mu - z)\right] \tag{68}$$

where $R_z(F)$ is the normalized wealth surplus index.

Proof: Given expressions (11) and (57) and the proof of theorem 9, this proof is straightforward.

Corollary 17 (*the Esteban et al. (1999) polarization measure as a function of the Shorrocks poverty index*): Let $F \in \Re^{n}$ be an income distribution, let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the *z* income value and let $S_z^{SH}(F)$ be the Shorrocks poverty measure. Then, it follows that:

$$P_z^{EGR}(F;\alpha;1) = \frac{T}{\mu}\left[\frac{zq_z}{2-q_z}\left(S_z^{SH}(F) - q_z^2\frac{\mu_P}{z}G_P\right) + z(1-q_z)R_z(F)\right]$$

$$-G_z^W(F) - \frac{T}{\mu}(1-2q_z)(\mu - z) \qquad (69)$$

where $R_z(F)$ is the normalized wealth surplus index.

Corollary 18 (*the Esteban et al. (1999) polarization index as a function of the normalized poverty deficit index*): Let $F \in \Re^{n}$ be an income distribution, let $P_z^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for two income groups separated by the *z* income value and let $S_z^{FGT}(F;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures. Then

$$P_z^{EGR}(F;\alpha;1) = \frac{zT}{\mu}\left[q_z S_z^{FGT}(F;1) + (1-q_z)R_z(F)\right] - \frac{T}{\mu}(1-2q_z)(\mu - z) - \frac{1}{T}G_z^W(F) \qquad (70)$$

where $R_z(F)$ is the normalized wealth surplus index.

The Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes explicitly consider the value of a poverty index (the Sen poverty index, its extension due to Shorrocks (1995) and the normalized poverty deficit index) and the value of a wealth index (the normalized wealth surplus index). However, it is guaranteed that polarization, poverty and wealth vary in the same direction if only the income gap ratio and/or the income overabundance gap ratio change. Again, a greater proportion of poor people does not imply necessarily more social conflict measured by a polarization index.

All the proposed results in this section are related to the measurement of bipolarization. That is, the results only apply to income distributions that are divided into two groups. It may be necessary to generalize to more than two income groups, for example, to analyze a society in which there are poor people, middle-income people and rich people. In this case the following two results arise.

Theorem 10 (*the Esteban and Ray (1994) polarization measure as a function of the normalized poverty deficit index when there are three income groups*): Let $F \in \Re^n$ be an income distribution, let $P_{z,w}^{ER}(F;\alpha)$ be the Esteban and Ray (1994) polarization index for three income groups separated by the *z* and *w* income values and let $S_z^{FGT}(F;\gamma)$ be the Foster-Greer-Thorbecke family of poverty measures. Then, it follows that:

$$P_{z,w}^{ER}(F;\alpha) = Az\left(S_z^{FGT}(F;1) - q_z\right) + B\mu_2 + Cw\left(R_w(F) + (1 - q_w)\right) \tag{71}$$

where $R_w(F)$ is the normalized wealth surplus index and

A $= q_z^{\alpha}(1-q_z) + (q_w - q_z)^{1+\alpha} + (1-q_w)^{1+\alpha}$

B = $\left(q_w - q_z\right)\left(q_z^{1+\alpha} + (q_w - q_z)^{\alpha}(q_w + q_z - 1) - (1-q_w)^{1+\alpha}\right)$ and

C = $(q_w - q_z)^{1+\alpha} + (1-q_w)^{\alpha} q_w + q_z^{1+\alpha}$.

Proof: Given expression (13) for three income groups separated by the poverty line *z* and the wealth line *w* the result is obtained after a few transformations.

Corollary 19 (*the Esteban et al. (1999) polarization index as a function of the normalized poverty deficit index when there are three income groups*): Let $F \in \Re^n$ be an income distribution, let $P_{z,w}^{EGR}(F;\alpha;\beta)$ be the Esteban *et al.* (1999) polarization index for three income groups separated by the *z* and *w* income values and let $S_z^{FGT}(F;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures. Then

$$P_{z,w}^{EGR}(F;\alpha;\beta) = Az\left(S_z^{FGT}(F;1) - q_z\right) + B\mu_2 + Cw\left(R_w(F) + (1 - q_w)\right) - \beta G^{W}(F) \tag{72}$$

where $R_w(F)$ is the normalized wealth surplus index and A, B and C are as in theorem 10.

In a similar way, *mutatis mutandis*, the relationship between polarization (measured by the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes), poverty (measured by the normalized poverty deficit index) and wealth (measured by the normalized wealth surplus index) can be obtained when the income range is divided in four or more income groups. Again, the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes explicitly consider the value of the normalized poverty deficit index and the value of the normalized wealth surplus index. As a result, more poverty and wealth in terms of the income gap ratio and/or the income overabundance gap ratio imply greater polarization independently of the number of income groups that we consider.

5 Concluding remarks

Several links between polarization measures and inequality, welfare and poverty measures have been established in this paper. The Wolfson polarization measure is generalized in terms of the between-groups and within-groups Gini components for income groups separated by any z income value. In addition, links between the Generalized Wolfson polarization measure and the Esteban and Ray (1994) and Esteban *et al.* (1999) polarization indexes have been proposed. It has also been shown that polarization, according to the Generalized Wolfson polarization index and the Esteban *et al.* (1999) polarization measure, is the difference between the welfare levels of rich and poor income groups when individuals' feelings of identification with others are based on their utility functions. Furthermore, the proposed polarization measure, the Esteban and Ray (1994) and the Esteban *et al.* (1999) polarization measures are a function of the Sen poverty index, its extension due to Shorrocks (1995) and the normalized poverty deficit index when the z income value represents the poverty line.

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