A NON-POSSIBILITY THEOREM FOR JOINT-STABILITY IN INTERINDUSTRY MODELS

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ABSTRACT: Joint-stability in interindustry models relates to the mutual simultaneous consistency of the demand-driven and supply-driven models of Leontief and Ghosh, respectively. Previous work has claimed joint-stability to be an acceptable assumption from the empirical viewpoint, provided only small changes in exogenous variables are considered. We show in this note, however, that the issue has deeper theoretical roots and offer an analytical demonstration that shows the impossibility of consistency between demand-driven and supply-driven models.

KEYWORDS: Interindustry modeling, joint-stability, demand-driven, supply-driven.

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1. Introduction

Interindustry models are essentially of two types. The demand-driven model of Leontief (1936) takes final demand as external with gross output and primary factors use responding to accommodate demand while keeping the demand-supply balance. In contrast, Ghosh (1958) takes primary factors as external whereas gross inputs and final demand adjust to the availability in production of primary factors. Clearly, these two models emphasize different driving forces while attempting to determine total activity levels in both cases. In the Leontief version, activity levels refer to total or gross outputs, from the use perspective. In Ghosh, they refer to total inputs, or resource perspective. But we know from basic National Accounting rules that gross outputs and gross inputs will necessarily coincide in equilibrium. It is from this accounting connection that the two competing models naturally arise. It is also known that the behavioral information of both models is not independent. Equilibrium changes in the demand-driven model will modify the underlying coefficient matrix in the supply-driven version, and vice versa. Thus joint-stability, or mutually compatible coefficient matrices, would guarantee quantitative applications with a sound theoretical basis and a common accounting platform.

Previous research by Chen and Rose (1986, 1991), Bon (1986), and Rose and Allison (1989), among others, has correctly identified a version of the joint-stability condition but has not taken it to its theoretical limit. In fact, it has been argued the condition to be sufficiently acceptable for empirical quantitative work as long as only small changes in external parameters are considered. We consider this conclusion to be faulty and prove so by formally showing its theoretical unsuitability. In Section 2 we introduce the elements of the discussion and redefine the joint-stability condition. Section 3 explores the implications of the condition and presents the main theoretical result showing the condition is contradictory with basic axioms. Section 4 summarizes.

2. Preliminaries

An n-sector interindustry economy $\mathcal{E}(n)$ is characterized by a $n\times n$ matrix of bilateral aggregate flows $\mathbf{Z}$, a (column) n vector of final demands f and a (row) n vector of primary factors use v', or value-added. In compact expression we may represent this economy by $\mathcal{E}(n) = (\mathbf{Z}, f, v')$. Matrix $\mathbf{Z} = (z_{ij})$ contains interindustry exchange between sectors i and j. The balance national accounting identities ensure the following relationship for all i:

$$\sum_{j=1}^{n} z_{ij} + f_i = \sum_{j=1}^{n} z_{ji} + v_i = x_i \tag{1}$$

where x_i stand for benchmark gross output (left-hand side) *and* gross input (rigth-hand side) level for sector *i*. We introduce now behavioral assumptions. From the perspective of production (i.e. inputs) we define a $n \times n$ technical coefficient matrix a_{ij} = $[\mathbf{A}]_{ij}$ by setting $a_{ij} = z_{ij} / x_j$. Similarly, from the perspective of distribution (i.e. outputs), we introduce a $n \times n$ allocation matrix $b_{ij}=[\mathbf{B}]_{ij}$ by taking $b_{ij} = z_{ij} / x_i$. Introducing these definitions in expression (1) we transform it into two behavioral equations:

$$\sum_{j=1}^{n} a_{ij} \cdot x_j + f_i = x_i \tag{2}$$

$$\sum_{j=1}^{n} b_{ij} \cdot x_i + v_i = x_i \tag{3}$$

In compact matrix notation we can write (and solve) them as

$$x = \mathbf{A} \cdot x + f = (\mathbf{I} - \mathbf{A})^{-1} \cdot x \tag{2a}$$

$$x' = \mathbf{B} \cdot x' + v' = v' \cdot (\mathbf{I} - \mathbf{B})^{-1} \tag{3a}$$

The first of these two expressions is the basic Leontief quantity model and the second one corresponds to the quantity model of Ghosh. In Leontief's model total production is determined as a result of demand-driven (i.e. *f*) inputs adjustments whereas in Ghosh is a consequence of supply-driven (i.e. v') allocation adjustments in output. If we use $\hat{\mathbf{X}}$ as the diagonal matrix version of a vector *x*, we can easily check from the definitions of **A** and **B** that $\mathbf{A} = \mathbf{Z} \cdot \hat{\mathbf{X}}^{-1}$ and $\mathbf{B} = \hat{\mathbf{X}}^{-1} \cdot \mathbf{Z}$. It is therefore immediate that **A** and **B** are similar matrices through change of basis matrices $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}^{-1}$:

$$\mathbf{B} = \hat{\mathbf{X}}^{-1} \cdot \mathbf{A} \cdot \hat{\mathbf{X}} \tag{4}$$

It can quickly be seen that for the same bases, similarity of matrices **B** and **A** implies similarity of (**I-A**) and (**I-B**) and, provided **B** and **A** are invertible, similarity of $\mathbf{B}^{-1}$ and $\mathbf{A}^{-1}$ as well. As a combined result, similarity on the Leontief and Ghosh inverses, if they exist, also follows for the same bases:

$$(\mathbf{I} - \mathbf{B})^{-1} = \hat{\mathbf{X}}^{-1} \cdot (\mathbf{I} - \mathbf{A})^{-1} \cdot \hat{\mathbf{X}} \tag{5}$$

Take now expression (5) and substitute into equation (3a):

$$x' = v' \cdot (\mathbf{I} - \mathbf{B})^{-1} = v' \cdot \left(\hat{\mathbf{X}}^{-1} \cdot (\mathbf{I} - \mathbf{A})^{-1} \cdot \hat{\mathbf{X}} \right) \tag{6a}$$

For simplicity, let now $\boldsymbol{\alpha}$ denote the coefficients of the Leontief inverse $\boldsymbol{\alpha} = (\mathbf{I} - \mathbf{A})^{-1}$ so that (6a) becomes:

$$x' = v' \cdot (\hat{\mathbf{X}}^{-1} \cdot \boldsymbol{\alpha} \cdot \hat{\mathbf{X}}) \tag{6b}$$

Similarly $\boldsymbol{\beta} = (\mathbf{I} - \mathbf{B})^{-1}$ stands now for the Ghosh inverse matrix. If we now expand (6b) and write the equivalent algebraic relationship, we obtain

$$x_j = \sum_{i=1}^{n} v_i \cdot x_j \cdot \alpha_{ij} \cdot \frac{1}{x_i} = \sum_{i=1}^{n} v_i \cdot \alpha_{ij} \cdot \left(\frac{x_j}{x_i} \right) = \sum_{i=1}^{n} v_i \cdot \alpha_{ij} \cdot \gamma_i^j \tag{7}$$

where γ_i^j represent the output ratio between output in sector j and output in sector i , i.e. $\gamma_i^j = x_j / x_i$. Take partial derivatives in equivalent expressions (3a) and (7) to obtain:

$$\frac{\partial x_j}{\partial v_i} = \beta_{ij} = \alpha_{ij} \cdot \gamma_i^j \tag{8}$$

Both interindustry models will appear to be simultaneously equivalent in their partial effects provided the output ratios (x_j / x_i) remain unaltered after a change in sector *i*'s value-added. This is only possible if in the new equilibrium quantities do not change or changes are proportional everywhere, i.e. a balanced growth situation. This is the same conclusion reached by Chen and Rose (1986, 1991), Bon (1986), and Rose and Allison (1989) but we have used the inverse matrix coefficients instead of the direct input and allocation coefficients in matrices **A** and **B**. This novel presentation will allows us to further explore the implications of the constancy of output ratios.

3. Main Result

We first state and prove the following:

Lemma: *If output ratios are constant, the Leontief inverse* $\boldsymbol{\alpha}$ *is a singular matrix.*

Proof: We have seen in (8) that $\partial x_j / \partial v_i = \alpha_{ij} \cdot \gamma_i^j$. The additive effects on all sectors *j* of a change in value-added in sector *i* will therefore be:

$$x_j' = x_j + \frac{\partial x_j}{\partial v_i} = x_j + \alpha_{ij} \cdot \gamma_i^j \quad (j \neq i)$$
$$x_i' = x_i + \frac{\partial x_i}{\partial v_i} = x_i + \alpha_{ii} \tag{9}$$

Constancy of output ratios between equilibria implies:

$$\frac{x_j'}{x_i'} = \frac{x_j}{x_i} = \gamma_i^j \quad (\text{all } j \text{ and } i) \tag{10}$$

Substitute (9) into expression (10) to obtain:

$$\frac{x_j + \alpha_{ij} \cdot \gamma_i^j}{x_i + \alpha_{ii}} = \gamma_i^j \tag{11}$$

Solve now (11) for x_j:

$$x_j = \gamma_i^j \cdot (x_i + \alpha_{ii} - \alpha_{ij}) \tag{12}$$

A simple trick to reintroduce the output ratio on the left hand side yields:

$$\frac{x_j}{x_i} = \gamma_i^j \cdot \frac{x_i + \alpha_{ii} - \alpha_{ij}}{x_i} \tag{13}$$

And from here it follows immediately that:

$$\alpha_{ij} = \alpha_{ii} \quad (\text{all } j) \tag{14}$$

In other words, the *j*-th row of matrix $\boldsymbol{\alpha}$ has the same coefficients. Being this true for all *j*, the matrix $\boldsymbol{\alpha}$ has determinant equal to zero and hence turns out to be singular. *QED*.

This has severe implications for joint stability of the Leontief and Ghosh interindustry models. Simply stated, it is theoretically impossible for the property to hold since it would violate basic productivity assumptions of matrix **A** as well as the Perron-Frobenius theorem. We can state the following:

Proposition: *Let* **A** *and* **B** *be respectively the Leontief and Ghosh non-negative coefficient matrices of an interindustry economy* $\mathcal{E}(n)$ *. Assume matrix* **A** *is productive* *(maximal eigenvalue smaller than 1). Then (a) matrix* **B** *is productive as well, (b) matrix* $\mathbf{(I - A)}$ *is non-singular and in addition its inverse* $\mathbf{(I - A)^{-1}}$ *is non-negative.*

Proof: (a) follows from matrix similarity (Shores, 2007, chapter 5), (b) from the Perron-Frobenius theorem (Nikaido, 1972, chapter 3). *QED.*

The following corollary is now trivial:

Joint-stability non-possibility theorem. *Constant output ratios are incompatible with the basic interindustry axioms since they would violate productivity of* **A**.

4. Summary Remarks

The joint-stability property requires that matrices **A** and **B** be independent from the effects of changes in output. In general this is not the case. Simple examples show that when production adjusts to new final demand schedules, the derived Ghosh matrix **B** changes. Similarly, in the Ghosh model, when output adjusts to new value-added levels, the derived Leontief matrix **A** is affected. Joint-stability is the condition that guarantees mutual consistency of both models.

Empirical work has liberally used both versions and has even built and used mixed versions of both models (Davis & Salkin, 1984, Cronin, 1984). For joint-stability to hold, however, output ratios must be unaffected by exogenous demand or value-added changes. There are two possibilities here: (1) balanced growth and all sectors' output change in equilibrium at the same rate. Our result above shows this case is not possible since the Leontief inverse would be a degenerate matrix. The remaining possibility is (2) that physical output levels remain constant. Under this proviso, output changes in the Ghosh model can only be interpreted as changes in the value of output. But with constant production levels, this can only imply changes in prices. This observation reinforces the 'vindication' of the Ghosh model by Dietzenbacher (1997) as being exclusively a price model. In essence, the very same price model as the conventional Leontief interindustry price model.

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REFERENCES

Bon, R. (1986). “Comparative Stability Analysis of Demand-side and Supply-side Input-output Models”. *International Journal of Forecasting*, 2, 231-235.

Chen, C. J. and Rose, A. (1986). “The Joint Stability of Input-Output Production and Allocation Coefficients”. *Modeling and Simulation*, 17, 251-255.

Chen, C. J. and Rose, A. (1991). “The Absolute and Relative Joint Stability of Input-Output Production and Allocation Coefficients”. In: Peterson, W. (ed.), *Advances in Input-Output Analysis*, chapter 3, Oxford University Press, New York.

Cronin, F. (1984). “Analytical Assumptions and Causal Ordering in Interindustry Modeling”. *Southern Economic Journal*, 50, 531-529.

Davis, H.C. and Salkin, E.L. (1984). “Alternative Approaches to the Estimation of Economic Impacts Resulting from Supply Constraints”. *Annals of Regional Science*, 18, 25-34.

Dietzenbacher, E. (1997). “In Vindication of the Ghosh Model: a Reinterpretation as a Price Model”. *Journal of Regional Science*, 37, 629-651.

Ghosh, A. (1958). “Input-Output Approach in an Allocation System”. *Economica*, 25, 58-64.

Leontief, W. (1936). “Quantitative input and output relations in the economic system of the United States”. *Review of Economic and Statistics*, 28, 105-125.

Nikaido, H. (1972). *Introduction to Sets and Mappings in Modern Economics*. North-Holland, Amsterdam, the Netherlands.

Rose, A. and Allison, T. (1989). “On the Plausibility of the Supply-driven Input-output Model: Empirical Evidence on Joint Stability”. *Journal of Regional Science*, 29, 451-458.

Shores, T. (2007). *Applied Linear Algebra and Matrix Analysis*. Springer, New York.