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An Efficient Multi-Item Dynamic Auction with Budget Constrained Bidders

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Abstract: An auctioneer wishes to sell several heterogeneous indivisible items to a group of potential bidders. Each bidder has valuations over the items but faces a budget constraint and may therefore not be able to pay up to his valuations. In such markets, a competitive equilibrium typically fails to exist. We develop a dynamic auction and prove that the auction always finds a core allocation in finitely many rounds. The core allocation consists of an assignment of the items and its associated supporting price vector.

Keywords: Dynamic auction, budget constraint, core.

JEL classification: D44.

1 Introduction

Auction theory typically assumes that all bidders can pay up to their values on the goods for sale. However, in reality buyers often face budget or liquidity constraints and may therefore be unable to afford what the goods are worth to them. Financial constraints arise in a variety of situations, such as less developed countries, business downturns and financial crises, see Che and Gale (1998) and Maskin (2000). Financial constraints can pose a serious obstacle to the efficient allocation of the goods, thus resulting in the loss of market efficiency. It is known that even when a single item is sold, it is generally impossible to have a mechanism for achieving full market efficiency when bidders face budget constraints, because budget constraints can fail the existence of Walrasian equilibrium. Moreover, auctions that do very well without budget constraints, if implemented under budget constraints, often produce highly inefficient outcomes.

Although budget constraints make full market efficiency unattainable due to nonexistence of Walrasian equilibrium, it is natural to ask whether there exists any mechanism that can achieve the allocation of goods as efficient as possible. This paper aims to answer this question in the affirmative. We examine a general model in which n (indivisible)

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items are sold to m budget constrained bidders. Each bidder wants to consume at most one item. When bidders face no budget constraints, the model reduces to the well-known assignment model as studied by Shapley and Shubik (1972), Crawford and Knoer (1981), and Demange et al. (1986), among others. We propose a dynamic auction and prove that the auction always finds a core allocation of the goods among bidders in finitely many steps. The notion of core is more general than that of competitive equilibrium and is a widely used solution concept for general exchange economies and NTU games, see Scarf (1967).

In early literature, to cite but a few, Che and Gale (1998), Maskin (2000), Krishna (2002), and Zheng (2001) have analysed various auctions for selling a single item when bidders are financially constrained. Moreover, Benoît and Krishna (2001), Brusco and Lopomo (2008), and Pitchik (2009) have studied auctions for selling two items to budget constrained bidders. Recently, van der Laan and Yang (2008) have examined a similar model as the current one and developed an ascending auction that always finds a constrained equilibrium. The constrained equilibrium possesses several interesting properties but does not necessarily yield a core allocation.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 describes and analyzes the auction. Section 4 concludes.

2 The Model

Consider an auction model consisting of a seller (i.e., auctioneer) and m potentially bidders. The seller has n indivisible goods for sale. Let $N = \{1, \ldots, n\}$ denote the set of the items, and $M = \{1, 2, \cdots, m\}$ the set of bidders. Let 0 denote the *dummy good* which has no value and costs nothing for any agent. Every real item $j \in N$ is inherently indivisible and thus can be assigned to at most one bidder. The seller has for each real item $j \in N$ a reservation price $c(j) \in \mathbb{Z}_+$ below which the item will not be sold, where $\mathbb{Z}$ and $\mathbb{Z}_+$ are the set of integers and the set of nonnegative integers, respectively. The reservation price of the dummy good is known to be $c(0) = 0$. Every bidder $i \in M$ demands at most one item and has a (possibly negative) monetary valuation to each item in $N \cup \{0\}$ given by the function $V^i\colon N \cup \{0\} \to \mathbb{Z}$ with $V^i(0) = 0$, and is endowed with a budget $m^i \in \mathbb{Z}_+$ units of money. All values $V^i(j)$, $j \neq 0$, and m^i are private information and thus only bidder i knows his own values $V^i(j)$, $j \neq 0$ and m^i.

It is assumed that buying an item $j \in N \cup \{0\}$ against price $p(j)$ by bidder i yields him a utility U^i equal to

$$U^i = \begin{cases} V^i(j) + m^i - p(j) & \text{if } p(j) \leq m^i, \\ -\infty & \text{if } p(j) > m^i. \end{cases}$$

That is, bidders are not allowed to have a deficit of money. By this assumption, no bidder is willing to pay a price for any item above his budget m^i. We say that bidder i is *financially constrained* if $m^i < \max_{j\in N} V^i(j)$, i.e., the valuation of bidder i for some items exceeds what he can afford, and that bidder i faces *no financial constraint* otherwise. In this paper, we allow $m^i < \max_{j\in N} V^i(j)$ for some $i \in M$. Observe that when no bidder faces any financial constraint, this model becomes identical to the classical assignment market model as studied by Shapley and Shubik (1972), Crawford and Knoer (1981), and Demange et al. (1986) among others.

A price vector $p \in \mathbb{R}^{n+1}_+$ gives a price $p(j) \geq 0$ for each item $j \in N \cup \{0\}$. A price vector $p \in \mathbb{R}^{n+1}_+$ is *feasible* if $p(j) \geq c(j)$ for every $j \in N$ and $p(0) = 0$. The price of the dummy good is always zero. Let $N_0 = N \cup \{0\}$ denote the set of all items (including the dummy) in the market. Let $M_0 = M \cup \{0\}$ denote the set of all agents, where 0 stands for the seller. An assignment is a vector $\pi = (\pi(1), \cdots, \pi(m))$ of items among all bidders in M such that $\pi(i) = \pi(j)$ for $i \neq j$ implies $\pi(i) = 0$. At an assignment π, every bidder $i \in M$ gets one item $\pi(i)$ (which can be real or dummy). Notice that the dummy good can be assigned to several bidders while every real item is assigned to at most one bidder. With respect to π, let $N_\pi = \{k \in N \mid k \neq \pi(i),\ \forall i \in M\}$ denote the set of unsold items which will be kept by the seller. Let $\mathcal{A}$ denote the family of all assignments.

At a feasible price vector $p \in \mathbb{R}^{n+1}_+$, the demand set of bidder $i \in M$ is given by

$$D^i(p) = \{j \mid\ V^i(j) - p(j) = \max\{V^i(k) - p(k) \mid p(k) \leq m^i,\ k \in N_0\}\}.$$

Observe that for any feasible p, the demand set $D^i(p) \neq \emptyset$, because $p(0) = 0 \leq m^i$ and thus the dummy item is always in the budget set. This means that the bidder has always the possibility not to buy any real item.

A pair (p, π) of a feasible price vector p and an assignment $\pi \in \mathcal{A}$ is said to be *implementable* if $p(\pi(i)) \leq m^i$ for all $i \in M$ and $p(j) = c(j)$ for all $j \in N_\pi$, i.e., every bidder i can afford to buy the item $\pi(i)$ assigned to him, and the price of every unsold item equals its reservation price. A *Walrasian equilibrium* (**WE**) is an implementable pair (p^*, π^*) such that $\pi^*(i) \in D^i(p^*)$ for all $i \in M$. It is well known from Shapley and Shubik (1972) that when there is no financial constraint for any bidder, a Walrasian equilibrium always exists. Moreover, Crawford and Knoer (1981) and Demange et al. (1986) have developed auction mechanisms for finding Walrasian equilibria in such markets.

The following example shows that a Walrasian equilibrium may fail to exist if buyers face financial constraints.

Example 1. A seller wants to sell item 1 to buyers 1 and 2. The seller's reservation price is zero, $c(1) = 0$. The two buyers' valuations and budgets are $V^1(1) = 4$, $V^2(1) = 6$, and $m^1 = m^2 = 2$.

This market has no equilibrium, although both buyers have valuations and budgets above the seller's zero reservation price. To see this, we have to consider two cases for the price of the item. At $p(1) \leq 2$, the item is over-demanded, whereas at $p(1) > 2$ the item is over-supplied.

Although a Walrasian equilibrium fails to exist, it is interesting to observe that implementable pairs $(p, \pi^1) = ((0, 2), (1, 0))$ and $(p, \pi^2) = ((0, 2), (0, 1))$ are core allocations and Pareto efficient. To verify this, consider $(p, \pi^1) = ((0, 2), (1, 0))$. At (p, π^1), the seller gets utility of $p(1) = 2$, buyer 1 utility of $V(1) + m^1 - p(1) = 4$, and buyer 2 utility of $m^2 = 2$. This allocation is stable and thus in the core in the sense that the seller and buyer 2 cannot block it because it is impossible for the seller to achieve a utility of more than 2. The same argument applies to (p, π^2).

The above example shows that when bidders face financial constraints, the Walrasian equilibrium is not guaranteed to exist but the core may be still nonempty. It is well known from Scarf (1965) that an exchange economy can have a nonempty core under fairly general conditions. In the sequel, we shall prove the existence of a nonempty core for the auction model with budget constrained bidders. We establish this result by proposing a dynamic auction, which actually finds a core allocation in finitely many steps.

To introduce the notion of core, we first give several definitions. At an implementable pair (p, π), the utilities that the bidders $i \in M$ and the seller $i = 0$ can achieve are given and denoted by

$$U^i(p, \pi) = V^i(\pi(i)) + m^i - p(\pi(i)), \ i \in M,$$

and

$$U^0(p, \pi) = \sum_{i \in M} p(\pi(i)) + \sum_{j \in N_\pi} c(j),$$

respectively. An implementable pair (p, π) is *Pareto efficient* if there does not exist another implementable pair (p', π') such that $U^i(p', \pi') > U^i(p, \pi)$ for all $i \in M_0$. A nonempty subset of M_0 is called a coalition. Given a coalition $S \subseteq M_0$, a *feasible assignment* ρ^S is an assignment in $\mathcal{A}$ such that $\rho^S(i) = 0$ for every $i \in M \setminus S$. That is, at ρ^S, every bidder in S receives at most one real item but every bidder outside S receives a dummy item. A pair (q, ρ^S) of a feasible price vector $q \in \mathbb{R}^{n+1}_+$ and a feasible assignment ρ^S is *implementable* if $q(\rho^S(i)) \leq m^i$ for all $i \in S$ and if $q(j) = c(j)$ for every unsold item $j \in N_{\rho^S}$. An implementable pair (p, π) is a *core allocation* if there does not exist any implementable pair (q, ρ^S) with some coalition S such that $U^i(q, \rho^S) > U^i(p, \pi)$ for all $i \in S$. Clearly, a core allocation is Pareto efficient. Also a core allocation (p, π) ensures individual rationality, that is, $U^i(p, \pi) \geq m^i$ for every bidder $i \in M$ and $U^0(p, \pi) \geq \sum_{h \in N} c(h)$ for the seller. Observe that when we consider any coalition of two or more agents, we only need to concentrate on those coalitions that contain both the seller and at least one bidder. Any coalition consisting of only bidders can be excluded.

3 The dynamic auction

We now establish the existence of a core allocation for the auction model with budget constrained bidders. This result can be seen as a generalization of the classic existence theorem for the assignment markets without budget constraints to the more general case which permits budget constraints. While in the classical model (see Shapley and Shubik (1972)) the set of Walrasian equilibrium allocations coincides with the strong core, in the current model the core is shown to be non-empty but can be strictly larger than the set of Walrasian equilibrium allocations.

Theorem 3.1 *There exists at least one core allocation in the auction model with budget constrained bidders.*

We shall design a dynamic auction that can actually find in finitely many steps a core allocation, thus yielding a constructive proof for this theorem. Briefly speaking, the auction works as follows. Initially, every bidder submits a bid for every item that he is willing and able to pay. Subsequently, taking all the current bids and her own reservation prices into account the auctioneer chooses a profile of bids that can give her the highest revenue and offers a spot market price for each item. Then each bidder updates his bid for every item according to the spot market prices, his current bids and his budget and his valuations. Again, the auctioneer updates. Repeat this process until no bidder is willing to offer any new bid.

In each round $t \in \mathbb{Z}_+$ of the auction, every bidder $i \in M$ offers a (feasible) bidding function $p_t^i : N_0 \to \mathbb{Z}$ with $p_t^i(0) = 0$ and $p_t^i(k) \leq \min\{m^i, V^i(k)\}$ for every $k \in N$. That is, no bidder is willing to bid above his budget or his value for any item. Let $P_t = (p_t^1, \cdots, p_t^m)$ be the bidding system at round t. Since the auctioneer wishes to achieve the highest revenue, her choice set at round t, is determined by

$$S(P_t) = \{\pi \in \mathcal{A} \mid \sum_{k \in N_\pi} c(k) + \sum_{i \in M} p_t^i(\pi(i)) = \max_{\rho \in \mathcal{A}} (\sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i)))\}.$$

We can now give a detailed description of the dynamic auction.

The dynamic auction

Step 1: Every bidder $i \in M$ offers a bidding function p_0^i. Set $t = 0$ and go to Step 2.

Step 2: Based on the current bidding system P_t, the auctioneer announces an assignment $\pi_t \in S(P_t)$ and a spot market price vector $\bar{p}_t \in \mathbb{R}^{n+1}$ as follows. If $t = 0$ or $\pi_{t-1} \notin S(P_t)$, take π_t to be any element of $S(P_t)$, and set $\bar{p}_t(0) = 0$, $\bar{p}_t(\pi_t(i)) = p_t^i(\pi_t(i))$ when $\pi_t(i) \in N$ for some $i \in M$, and $\bar{p}_t(k) = c(k)$ when $k \in N_{\pi_t}$, and go to Step 3. If $t > 0$ and $\pi_{t-1} \in S(P_t)$, then set $\pi_t = \pi_{t-1}$, $\bar{p}_t(0) = 0$, for any $\pi_t(i) \in N$ for some $i \in M$, set $\bar{p}_t(\pi_t(i)) = \bar{p}_{t-1}(\pi_t(i)) + 1$ when, at round $t-1$, some bidder j increased his bid for the item $\pi_t(i)$ to $\bar{p}_{t-1}(\pi_t(i)) + 1$, and set $\bar{p}_t(k) = \bar{p}_{t-1}(k)$ otherwise, and go to Step 3.

Step 3: Each bidder $i \in M$ updates his bids by setting $\tilde{p}_t^i(k) = \min\{p_t^i(k), \bar{p}_t(k)\}$ for all $k \in N$. For any bidder $i \in M$, if there exists some item $k \in N$ such that $V^i(k) - \tilde{p}_t^i(k) > V^i(\pi_t(i)) - \tilde{p}_t^i(\pi_t(i))$ and $\tilde{p}_t^i(k) < m^i$, bidder i updates his bidding function by setting $p_{t+1}^i(k) = \tilde{p}_t^i(k) + 1$ for one such k, and setting $p_{t+1}^i(h) = \tilde{p}_t^i(h)$ for any other item $h \in N$. Every other bidder i sets $p_{t+1}^i = \tilde{p}_t^i$. When $p_{t+1}^i \neq \tilde{p}_t^i$ for some $i \in M$, then set $t = t + 1$ and go back to Step 2. Otherwise, the auction stops and the output is $(\tilde{P}_t, \pi_t)$.

This auction bears some similarity with pay-as-you-bid auctions used in internet or more traditionally by governments for selling treasury bills, but it differs from them in three crucial aspects: First, in Step 2, the auctioneer does not merely announce the spot price for each item but more importantly she adjusts the spot price upwards when she observes that some bidder increases his bid. Second, in Step 3, unlike the existing pay-as-you-bid auctions in which bidders are not allowed to decrease their bids, the current auction permits bidders to reduce their bids in order to avoid overbidding. Third, in Step 3, the current auction has a flexible rule for the bidders to adjust their bids for those items which they can afford and give them higher profits, whereas the existing auctions typically require the bidders to adjust their bids for those items which give the bidders the highest profits. See Ausubel and Milgrom (2002), Bernheim and Whinston (1986) for pay-as-you-bid auctions in detail.

Observe that when the auction stops with $(\tilde{P}_t, \pi_t)$, it is possible that some bidder i gets item $\pi_t(i)$ which is not contained in his demand set $D^i(\tilde{p}_t^i)$. In this case, we must have $\tilde{p}_t^i(k) = m^i$ for any $k \in N$ satisfying $V^i(k) - \tilde{p}_t^i(k) > V^i(\pi_t(i)) - \tilde{p}_t^i(\pi_t(i))$.

Proposition 3.2 *In every round t of the auction, we have $\pi_t \in S(\tilde{P}_t)$.*

Proof. In case of $\pi_{t-1} \notin S(P_t)$, we have

$$\sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i)) \geq \sum_{k \in N_\rho} c(k) + \sum_{i \in M} \tilde{p}_t^i(\rho(i))$$

for all $\rho \in \mathcal{A}$. Moreover, we have

$$\sum_{k\in N_{\pi_t}} c(k) + \sum_{i\in M} \tilde{p}_t^i(\pi_t(i)) = \sum_{k\in N_{\pi_t}} c(k) + \sum_{i\in M} p_t^i(\pi_t(i)),$$

$$\sum_{k\in N_{\pi_t}} c(k) + \sum_{i\in M} p_t^i(\pi_t(i)) \geq \sum_{k\in N_\rho} c(k) + \sum_{i\in M} p_t^i(\rho(i))$$

for all $\rho \in \mathcal{A}$. Therefore for all $\rho \in \mathcal{A}$, we get

$$\sum_{k\in N_{\pi_t}} c(k) + \sum_{i\in M} \tilde{p}_t^i(\pi_t(i)) \geq \sum_{k\in N_\rho} c(k) + \sum_{i\in M} \tilde{p}_t^i(\rho(i)).$$

Similarly we can show the case of $\pi_{t-1} \in S(P_t)$ in which $\pi_t = \pi_{t-1} \in S(\tilde{P}_t)$. □

The existing auctions are typically either ascending or decreasing. It is relatively easy to prove their finite convergence because of obvious monotonicity. However, this is not the case for the current auction, because in the current auction, the bidders may increase or decrease their bids, this feature makes it impossible to use the familar argument. Instead, we have to explore a new approach which makes use of the bids of all bidders and the revenues of the seller.

Lemma 3.3 *The auction terminates in finitely many rounds.*

Proof. Let $R(\pi, P) = \sum_{k\in N_\pi} c(k) + \sum_{i\in M} p^i(\pi(i))$ denote the revenue of the seller at (π, P), and let $R(P) = \max\{R(\pi, P) \mid \pi \in \mathcal{A}\}$ be the highest revenue at P. In each round t, we have $R(\pi_t, P_t) = R(P_t)$. We need to consider two cases. In case of $\pi_{t-1} \notin S(P_t)$, we have $R(P_t) = R(\pi_t, P_t) > R(\pi_{t-1}, P_t)$, by the auction rule we have $R(\pi_{t-1}, P_t) \geq R(\pi_{t-1}, \tilde{P}_{t-1})$, and from Proposition 3.2 it follows that $R(\pi_{t-1}, \tilde{P}_{t-1}) = R(\pi_{t-1}, P_{t-1}) = R(P_{t-1})$. This proves $R(P_t) > R(P_{t-1})$.

In case of $\pi_{t-1} \in S(P_t)$, we have $R(P_t) = R(P_{t-1})$, because $p_t^i(\pi_t(i)) = p_{t-1}^i(\pi_{t-1}(i))$ for all $i \in M$. In this case, at least one bidder increases his bid but no bidder decreases any of his bids.

The two arguments imply that in each round, either the revenue of the seller is strictly increasing, or the revenue of the seller remains constant and no bidder is bidding less but at least one bidder is bidding more. Therefore, because of finite values and budgets, the auction must stop in finitely many rounds. □

Let $\pi^* = \pi_t$ and $p^* = \bar{p}_t$ when the auction stops at round t. The following theorem shows that the outcome (p^*, π^*) is a core allocation.

Theorem 3.4 *The outcome (p^*, π^*) found by the auction is in the core and thus Pareto efficient.*

Proof. Suppose to the contrary that (p^*, π^*) is not in the core. Clearly, the pair (p^*, π^*) is individually rational. Then there exist a coalition S consisting of the seller and at least one bidder and an implementable pair (q, ρ^S) such that $U^i(q, \rho^S) > U^i(p^*, \pi^*)$ for all $i \in S$. So for the seller we have

$$\begin{array}{rcl} \sum_{j\in N} q(j) & = & \sum_{i\in S} q(\rho^S(i)) + \sum_{j\in N_{\rho^S}} c(j) \\ & = & U^0(q, \rho^S) \\ & > & U^0(p^*, \pi^*) \\ & = & \sum_{i\in M} p^*(\pi^*(i)) + \sum_{j\in N_{\pi^*}} c(j) \\ & = & \sum_{j\in N} p^*(j). \end{array} \tag{3.1}$$

It is clear that there exists some $j^* \in N$ such that $q(j^*) > p^*(j^*)$. This means that some bidder $i^* \in S$ is assigned item j^* at ρ^S, i.e., $\rho^S(i^*) = j^*$, because it holds $\min\{p^*(j), q(j)\} \geq c(j)$ for all $j \in N$ and $q(j) = c(j)$ for every unassigned item $j \in N_{\rho^S}$. Observe that since (q, ρ^S) is implementable and $U^{i^*}(q, \rho^S) > U^{i^*}(p^*, \pi^*)$, we have $V^{i^*}(j^*) - q(j^*) > V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*))$ and $q(j^*) \leq m^{i^*}$. It follows from $q(j^*) > p^*(j^*)$ that

$$\begin{array}{rcl} V^{i^*}(j^*) - p^*(j^*) & > & V^{i^*}(j^*) - q(j^*) \\ & > & V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*)) \\ & = & V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)), \end{array}$$

where t is the round when the auction stops. Because $\tilde{p}_t^i(k) \leq \bar{p}_t(k) = p^*(k)$ for any $i \in M$ and $k \in N$, we have

$$\begin{array}{rcl} V^{i^*}(j^*) - \tilde{p}_t^{i^*}(j^*) & \geq & V^{i^*}(j^*) - p^*(j^*) \\ & > & V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)). \end{array}$$

Moreover, $\tilde{p}_t^{i^*}(j^*) \leq \bar{p}_t(j^*) = p^*(j^*) < q(j^*) \leq m^{i^*}$. But then at prices $\tilde{p}_t^{i^*}$, bidder i^* should have rejected the item $\pi^*(i^*)$ and made a new offer, and therefore the auction could not have stopped at round t. □

4 Concluding remarks

In this article we have proposed a dynamic auction for finding a core allocation in a setting where bidders are budget constrained and each bidder demands at most one item. It is worth pointing out that the auctions developed by Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), Ausubel and Milgrom (2002), Perry and Reny (2005), Ausubel (2004, 2006), Sun and Yang (2009) allow bidders to demand multiple items, albeit in the absence of budget constraint. This more general but also more difficult case remains to be explored when bidders face budget constraints.

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