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Elastic Money, Inflation, and Interest Rate Policy

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Abstract

We study optimal monetary policy in an environment in which money plays a basic role in facilitating exchange, aggregate shocks affect households asymmetrically and exchange may be conducted using either bank deposits (inside money) or fiat currency (outside money). A central bank controls the stock of outside money in the long-run and responds to shocks in the short-run using an interest rate policy that manages private banks' creation of inside money and influences households' consumption. The zero bound on nominal interest rates prevents the central bank from achieving efficiency in all states. Long-run inflation can improve welfare by mitigating the effect of this bound.

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1 Introduction

In this paper, we study optimal monetary policy in an environment in which money is essential and aggregate shocks affect individual agents differentially. Exchange may be conducted using either bank deposits (inside money) or fiat currency (outside money). A central bank conducts a monetary policy with two components: It controls the issuance of inside money by private banks by managing short-run interest rates and sets the trend inflation rate by controlling the quantity of outside money. We show that both components of the central bank's policy are useful for maximizing welfare. Long-run inflation mitigates the effect of the zero bound and is necessary for the implementation of the central bank's interest rate policy.

In models in which money plays an explicit role as the medium of exchange, monetary policy is typically modeled as direct control of the supply of fiat money. In many settings this is natural as such a policy is equivalent to one based on the setting of nominal interest rates. This analysis is not well suited, however, for understanding the reasons why central banks use interest rates as their primary short-run policy tool (Alvarez, Lucas, and Weber (2001)). A separate literature explores interest rate policies in models in which money plays no explicit role and indeed may not even be present (Woodford (2003)). Here we focus on a class of policies in which the central bank varies the interest rate in response to shocks in the short-run and uses transfers only to maintain the long-run rate of inflation. Within this class we characterize an optimal monetary policy for an economy in which money is necessary for exchange.

Berentsen and Waller (2010) consider optimal monetary policies of a certain class using an extension of the environment of Lagos and Wright (2005) in which anonymous agents have access to a credit market. We extend their analysis by replacing the credit market with a large number of identical private institutions which can both accept deposits and make loans. These *banks* can create money through short-term loans in excess of their collected deposits. The creation of bank deposits through this channel makes the money supply elastic in the short-run even though the central bank is limited with regard to the frequency with which it can make transfers.

In our economy, aggregate shocks affect differently households who do and do not have access to banks and who have different incentives to save. Each period a fraction of households are excluded from interaction with banks and learn that they will exit the economy immediately after trading. Households' desire for insurance against finding themselves in this situation generates both an essential role for money and the possibility of welfare improving policy.

Under the optimal policy, the central bank pays interest on the reserves of private banks subject to two requirements. First, they must settle net interbank transactions in outside money. Second, at the end of each period, they must demonstrate solvency, by which we mean that any liabilities (deposits) must be matched by assets (loans and outside money). By setting the rate at which it pays interest on reserves the central bank can control the loan rate charged by private banks and thus the supply of inside money. Through this channel, the central bank can raise and lower output and consumption in response to fluctuations in consumers' marginal utilities. Interest rate movements thus redistribute wealth among households by changing the value of existing holdings of outside money.

The ability of the central bank to control output through its interest rate policy is limited, however, by the zero bound on nominal interest rates. The effects of this bound can be mitigated by maintaining sufficiently high inflation on average. Inflation, the costs of which are offset by paying interest on reserves, enables the central bank to engineer negative real interest rates when needed as described by Summers (1991) and others.

A large literature considers the issuance and acceptance of inside money in models in which money functions as a medium of exchange (see, for example, Bullard and Smith (2003), Cavalcanti, Erosa, and Temzelides (1999, 2005); Freeman (1996a), He, Huang, and Wright (2005); and Sun (2010, 2007)). This literature focuses to a large extent on the incentive problems associated with acceptance and creation of inside money, devoting significant attention to the possibility of oversupply. To focus on the short-run elasticity of the money supply and its role in monetary stabilization policy, we largely abstract from these issues. Our solvency requirement effectively eliminates the possibility of oversupply of inside money in equilibrium.

In some ways, our work is closely related to that of Champ, Smith, and Williamson (1996) and, especially, Freeman (1996b). We depart from Freeman in that we consider a setting in which the central bank is limited with regard to both the frequency with which it can make transfers and its ability to target them to particular households. These features of our economy account for our main results: An interest rate based policy is not equivalent to one employing transfers of outside money only; and an elastic money supply does not necessarily lead to an efficient outcome.

Our work also extends that of Berentsen and Monnet (2008), who study monetary policy implemented through a channel system in a similar framework, in that we consider the role of a private banking system in implementing monetary policy. Berentsen and Waller (2008) also consider optimal monetary policy in a model in which the central bank supplies an elastic currency in the presence of aggregate shocks. We, however, focus on the incentive of some households to overconsume rather than on frictions associated with firm entry. In our economy, unlike those studied in these other papers, the zero bound imposes a significant impediment to monetary policy—indeed it is the factor which prevents the central bank from attaining the efficient outcome.

The remainder of the paper is organized as follows. Section 2 describes the environment. Section 3 analyzes households' optimal choices. Section 4 defines a symmetric stationary monetary equilibrium and presents an example in which the central bank sets nominal interest rates to zero in all states and maintains a constant stock of outside money. Section 5 characterizes the optimal policy. The implications of this policy for long-run inflation are considered in Section 6. Section 7 concludes and describes future work. Longer proofs and derivations are included in the appendix.

2 The environment

Time is discrete and indexed by $t = 1, 2, \ldots$ etc.. Building on the environment introduced by Lagos and Wright (2005), each time period is divided into $n \geq 2$ distinct and consecutive sub-periods the first $n - 1$ of which are symmetric and different from the nth. In each sub-period of every period it is possible to produce a distinct good. All of these goods are non-storable both between sub-periods and periods of time. To begin with, we describe the case of $n = 3$, so that there are two initial sub-periods in which households are differentiated by type in addition to the final one in which they are symmetric. We will then describe the effects of eliminating one of the initial sub-periods. Adding more sub-periods (beyond three) makes little difference and so we comment on it only briefly at the end.

At the beginning of each period (comprised of three sub-periods) there exists a unit measure of identical households. These households are then differentiated by type according to a random process. Households are distinguished by type along three lines. First, each household is *active* in only one of the first two sub-periods. We assume that one-half of all households are active in each of these sub-periods. Second, during the sub-period in which it is active, a household is either a *producer* or a *consumer*. Finally, fraction λ of the households learn at the beginning of the period that after acting as a consumer in one of the first two sub-periods they will exit the economy. These households have no access to the banking system at any time during the current period. Let α denote the fraction of households that act as consumers in one of the first two sub-periods and do *not* exit the economy. The fraction of households that act as producers in one of the first two sub-periods is then given by $1 - \alpha - \lambda$.

Consumers active in sub-period $j = 1, 2$ have preferences given by

$$u(c_j) = A_j u(c_j) \tag{1}$$

where $u(\cdot)$ satisfies $u'(c) > 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$, and c_j denotes consumption of the sub-period j good in the current period. We will distinguish consumption by households that exit in the current period from that of those that *stay* by superscripts: c_j^e vs. c_j^s. A_j is a preference shock common to all consumers in a given sub-period of activity. This shock is realized at the beginning of the relevant sub-period and independent over time. For $j = 1, 2$, A_j is non-negative, has compact support, and is distributed according to the cumulative distribution function $F(A)$ with density $f(A)$.

Let y_j denote the output of an individual producer active in sub-period j. Production results in disutility $g(y)$ where $g'(y) > 0$ and $g''(y) > 0$. Producers derive no utility from consumption during either of the first two sub-periods. Unlike consumers, producers active in different sub-periods are entirely symmetric.

At the beginning of the final (*i.e.* third) sub-period, λ new households arrive to replace those who exited at the end of the previous two sub-periods. During this final sub-period all households can both consume and produce. Regardless of whether it was a consumer, producer, or was not yet present earlier in the period, the household's preferences are given by

$$U(x) - h, \tag{2}$$

where x is the quantity of sub-period three good consumed, with $U'(0) = \infty$, $U'(+\infty) = 0$ and $U''(x) \leq 0$. We assume that the sub-period 3 good can be produced at constant marginal disutility and interpret h as the quantity of "labour" used to produce one unit of the good. Linear disutility plays the same role here as in Lagos and Wright (2005).

In all sub-periods, exchange takes place in Walrasian markets. In the first two sub-periods households are anonymous to each other. As a result, cannot credibly commit to repay trade credit extended to them by either sellers or other buyers. Anonymity motivates the need for a medium of exchange in the first two sub-periods. In the final sub-period, households are not anonymous, trade credit is in principle feasible, and there is no need for a medium of exchange.[1]

In addition to households, there also exists in the economy a large fixed number, N, of *private* institutions which we will refer to as *banks.* Banks are owned by households and act so as to maximize dividends, which are paid during the final sub-period of each period. Private banks are able to recognize in the final sub-period households with whom they

[1] We will refer to the final sub-period as the *frictionless* market, and the preceding sub-periods as periods of *anonymous exchange*.

have contracted in either of the sub-periods of anonymous exchange. Similarly, households are able to find banks with which they have contracted earlier in the period. This, together with an assumption that contracts between banks and individual households can be perfectly enforced enables private banks feasibly to take deposits and make loans within a time period.[2]

The institutions that we refer to as banks function much as the credit market in Berentsen, Camera, and Waller (2007) and Berentsen and Waller (2010,b). The key difference here is that we allow these institutions to extend loans in excess of their deposits. They will do this by creating deposits which function effectively like checking accounts. We will refer to these deposits as *inside money*. We do not consider the possibility of private bank notes and exclude them by assumption.

There also exists in the economy an institution which we refer to as the *central bank*. Unlike the private banks described above, the central bank does not have the ability to identify and contract with households during the sub-periods of anonymous exchange. The central bank can, however, interact with private banks at any time and has the ability both to enforce agreements into which it has entered with these banks and to impose taxes upon both banks and households in the final sub-period.

The central bank maintains a stock of *fiat money* which can in principle serve as a medium of exchange in any sub-period. Let M_t denote the quantity of fiat money in existence at the beginning of period t. At the beginning of the initial period, all households are endowed with equal shares of this money. Central bank money will be referred to as *outside money* to distinguish it from the deposits created and maintained by private banks.

Timing

Figure 1 depicts the timing of events within period t under the assumption that there are two sub-periods of anonymous exchange. Agents enter the first sub-period owning shares in banks and holding any outside money acquired during period $t-1$. At the beginning of the period households are randomly divided by type as described above. Immediately thereafter, households who are not exiting the economy this period may access the banking system. That is, they may take out a loan from a bank, make deposits and/or shift deposits among banks. At this time banks may also interact with the central bank if they so choose. Exchange then takes place among those consumers and producers who are active in the first sub-period.

Buyers may purchase goods with either outside money or bank deposits. To the extent

[2]Because households which exit have no access to banks within their final period in the economy, there is no possibility for banks to offer insurance against the exit "shock". Below, we discuss the role of central bank solvency requirements which prevent banks from issuing liabilities in order to generate current profits.

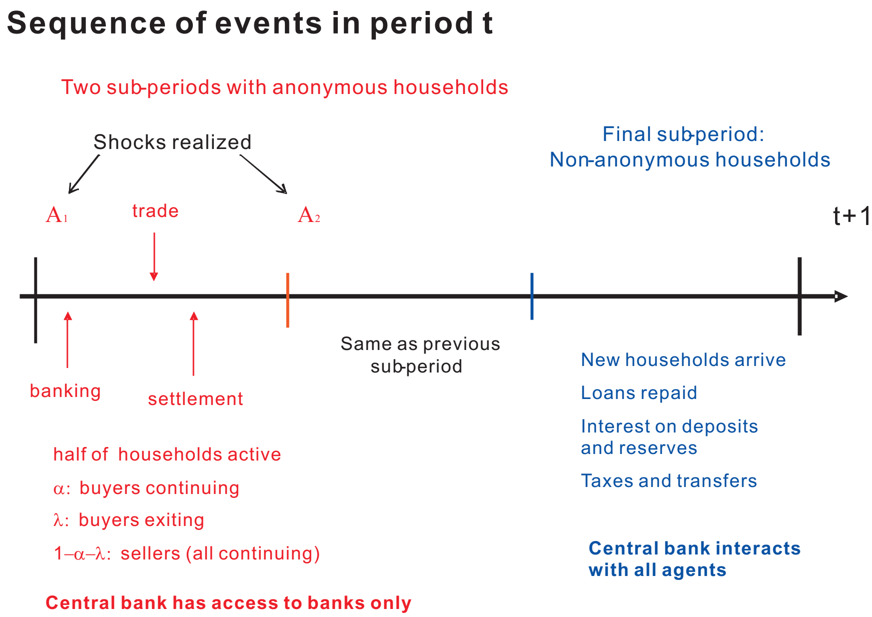


Figure 1: Timing

that deposits are used, the central bank organizes payments and requires banks to settle net balances using central bank money, which it may offer to lend to them if necessary. Settlement takes place immediately after exchange through a process that will be described below. After the settlement of interbank transactions, sub-period 1 ends and sub-period 2, which is identical, begins.

In sub-period 3 all households have identical preferences and productive capacities but differ with regard to their asset holdings as a result either of transactions earlier in the period or of having just arrived in the economy. In this sub-period the sequence of events does not matter. Banks collect interest on reserves from the central bank, pay interest to their depositors and pay dividends. The central bank requires repayment of loans to banks in outside money and requires banks to be *solvent* at the end of each period. By this we mean that before paying dividends, banks must demonstrate that any deposits (liabilities) are matched by assets (loans and outside money).[3] Borrowers re-pay their loans and all households exchange in a Walrasian market. Through exchange households acquire both goods for consumption and currency to carry into period $t+1$.

[3]This aspect of the central bank's policy prevents banks from overissuing inside money. Households will not have incentive to borrow from banks at the end of any period in exchange for deposits. First, if they must exit next period they will have no access to these deposits. Second, if they do not exit, then they have access to loans in the next period *after* observing their preference shock. Thus, at the end of each period, there will be no outstanding loans. The solvency requirement thus forces banks' deposits (if any) to be equal to their holdings of outside money.

Monetary policy

The central bank is able to commit fully to state-contingent policy actions. It organizes payments and conducts a policy with two components. It pays interest on reserves of outside money held by private banks and makes lump-sum transfers (or collects lump-sum taxes) in outside money. Interest is paid and taxes/transfers take place during the final sub-period only.

In each of the sub-periods of anonymous exchange, $j = 1, 2$, the central bank sets an interest rate, i^c_j, (possibly contingent on the realizations of the aggregate shocks A_1 and A_2) at which it is willing to accept deposits (in units of outside money) from or make loans to banks. As noted above, loans to banks are repaid in the final sub-period. At that time interest on reserves is paid to the bank holding the deposit at the end of the relevant sub-period of anonymous exchange.[4]

In the final sub-period, the central bank can adjust the supply of outside money by choosing the growth rate of the money stock from period t to $t + 1$:

$$M_{t+1} = \gamma_t M_t. \tag{3}$$

Like interest rates, γ_t can be contingent on the realized shocks. The central bank adjusts the money stock by conducting equal lump-sum transfers to all households, with the total transfer equal to $(\gamma_t - 1)M_t$ minus the total interest paid on reserve deposits to all banks plus the interest charged on central bank loans (if any). If reserve interest exceeds the desired increase in the aggregate supply of outside money, then the transfer is negative—a lump-sum tax.

Transactions, banking, and money flows

A detailed description of the gross monetary flows within each period is given in the appendix. Having learned their state, households which will continue in the economy may deposit their currency holdings in banks and/or take out loans. We denote initial deposits and beginning of period loans made by a representative bank D_0 and L_1, respectively. Let i^d_j denote the net interest paid (in the final sub-period) to households who hold deposits in a bank at the end of sub-period $j = 1, 2$. Similarly, let i^ℓ_j denote the net interest to be paid on a loan taken out in sub-period j. Households can choose among different banks and may move their deposits from one to another at any time. Interest on deposits is paid and loans are re-paid in the final sub-period.

[4] For example, a bank which accepts a deposit from a household at the beginning of sub-period 1 and then immediately transfers it to another bank through the settlement process following goods trading in sub-period 1 receives no interest on that deposit as the interest is paid instead to the bank of the seller. If the first bank transfers it to another bank following goods trade in sub-period 2, then it receives interest i^c_1 for sub-period 1, but nothing for sub-period 2, etc..

When a bank makes a loan, it increases the borrower's deposits resulting in an increase in the total quantity of deposits in the economy. There is no explicit limit on the quantity of credit that any bank can extend. At any time private banks may deposit their outside money reserves at the central bank and earn net interest i_j^c, which is paid in the final sub-period as described above on reserves held at the central bank through sub-period j.

We assume that all households with the opportunity deposit in banks any outside money they either carry into the period or receive in transactions. In this case we can further assume that exchange involving continuing households takes place using bank deposits only (for example, by means of checks). There are a large number of symmetric banks, and so we assume that deposits spent by buyers on goods flow to sellers that are equally distributed among all banks. Interbank settlement in outside money of net balances takes place immediately following exchange. This settlement process may be thought of as a component of monetary policy. If an individual bank requires more outside money for settlement than its collected initial deposits, it must borrow outside money from the central bank at rate i_j^c. After settlement, banks can deposit newly received reserves at the central bank.

The central bank pays interest on reserves held at the end of each sub-period. When a buyer spends deposits in exchange, reserves associated with these deposits will be transferred to the bank of seller who receives the deposits in payment. As a result, the seller's bank will end up receiving interest from the central bank on these reserves and will pay interest to the seller on the deposit. The buyer's bank will receive no interest on reserves and will pay none to the original depositor.

3 Optimal choices

We now consider households' optimal choices in a representative time period, t. Agents behave competitively, taking the central bank's monetary policy and all prices as given. To economize on notation, we will omit the subscript "t" throughout. We use "$t-1$" and "$t+1$" to denote the previous and next periods, respectively. Continuing with the case of $n = 3$, let p_1, p_2 and p_3 denote the nominal price level in sub-periods 1, 2 and 3, respectively and let $\phi = \frac{1}{p_3}$ denote the real value of money in the final sub-period.

Let $V(m)$ denote the expected value of a representative household at the beginning of the current period (before the realization of shocks) with m units of outside money. We restrict attention to situations in which all non-exiting households deposit their money holdings in banks. Let d_0 represent initial nominal wealth. We will construct an expression for $V(d_0)$ (and describe households' optimal choices) by working backward through period t.

3.1 The final sub-period (frictionless market)

Let $W(d_2, \ell_2)$ denote the value of a household entering the final sub-period with deposits d_2 and outstanding loan balance ℓ_2. We have

$$W(d_2, \ell_2) = \max_{x,h,d_{0,t+1}} [U(x) - h + \beta \mathrm{E_t} V_{t+1}(d_{0,t+1})] \tag{4}$$

$$\text{subject to}: \quad x + \phi d_{0,t+1} = h + \phi\tau + \phi(1 + i_2^d)d_2 + \phi\Pi - \phi\ell_2(1 + i_2^\ell) \tag{5}$$

where the budget, (5), is written in units of sub-period 3 consumption good. Here τ denotes the tax or transfer from the central bank and Π is bank profits, also distributed lump-sum. Using (5) to eliminate h in (4) we have

$$\begin{aligned} W(d_2, \ell_2) \quad &= \quad \phi \left[\tau + \Pi + (1 + i_2^d)d_2 - \ell_2(1 + i_2^\ell)\right] \\ &\quad + \max_{x,d_{0,t+1}} [U(x) - x - \phi d_{0,t+1} + \beta \mathrm{E_t} V_{t+1}(d_{0,t+1})]. \end{aligned} \tag{6}$$

The first-order conditions for optimal choice of x and $d_{0,t+1}$ are given by

$$U'(x) = 1 \tag{7}$$

$$\phi = \beta \mathrm{E_t} \left[V'_{t+1}(d_{0,t+1})\right] \tag{8}$$

where $\mathrm{E_t} V'_{t+1}(d_{0,t+1})$ is the expected marginal value of an additional unit of deposits carried into period $t+1$ (the expectation here is with respect to the realizations of A_{1t+1} and A_{2t+1}). The envelope conditions for d_2 and ℓ_2 are

$$W_d = \phi(1 + i_2^d) \tag{9}$$

$$W_\ell = -\phi(1 + i_2^\ell). \tag{10}$$

As in Lagos and Wright (2005) the optimal solution for x is the same for all households and the choice of $d_{0,t+1}$ is independent of the deposit and loan carried into sub-period 3.[5] As a result, all households choose to carry the same quantity of money into period $t + 1$ and thus have the same deposit balance, $d_{0,t+1}$, at the beginning of that period. Define the common real balance carried into the current period by

$$\omega \equiv \frac{d_0}{p_{3,t-1}} = d_0\phi_{t-1}. \tag{11}$$

[5] We can adjust $U(x)$ such that people always produce positive amount of goods in sub-period 3 so as to avoid the corner solution in which people select $h = 0$. Please see appendix B for details.

3.2 Sub-period 2 (anonymous exchange)

Let $V_2(d_1, \ell_1)$ denote the value of a household entering the second sub-period of anonymous exchange with deposits d_1 and outstanding loan ℓ_1. Households inactive in sub-period 2 were necessarily active in sub-period 1. Any of these households which will exit this period may be thought of as already gone at this point. Those that will remain in the economy will simply roll over their loans and deposits and wait for sub-period 3. For these households we may therefore write

$$V_2(d_1, \ell_1) = W(d_1(1 + i_1^d), \ell_1(1 + i_1^\ell)). \tag{12}$$

Households which are active in the second sub-period are divided into buyers (staying and exiting) and sellers. All of these households have either deposits (d_1) or money (m) equal to d_0 and no outstanding loans; $\ell_1 = 0$. Let $V_2^s(d_0)$, $V_2^e(d_0)$, and $V_2^y(d_0)$ denote the values of staying buyers, exiting buyers, and sellers active in the second sub-period respectively.

Sellers

Sellers do not borrow from banks as in any equilibrium the lending rate will be at least equal to the deposit rate (this is shown below). Thus, sellers' optimization problem may be represented by the following Bellman equation:

$$V_2^y(d_0) = \max_{y_2}[-g(y_2) + W(\underbrace{d_0(1 + i_1^d) + p_2 y_2}_{d_2}, 0)] \tag{13}$$

where y_2 is the quantity of goods they sell and $p_2 y_2$ is their monetary revenue deposited into their bank on settlement following goods trading. Optimization requires

$$-g'(y_2) + p_2 W_d = 0. \tag{14}$$

Using (9), we may write

$$g'(y_2) = \frac{p_2}{p_3}(1 + i_2^d). \tag{15}$$

Since the marginal cost of producing in sub-period 3 is 1, sellers choose y_2 such that the ratio of marginal costs across markets $g'(y_2)/1$ is equal to the relative nominal price $p_2\phi$, multiplied by the rate of return $1 + i_2^d$ on deposits held through sub-period 2 (after settlement). Thus, the price level in sub-period 2 satisfies

$$p_2 = \frac{g'(y_2)}{\phi(1 + i_2^d)}. \tag{16}$$

Buyers who will remain in the economy

All buyers receive preference shock A_2. For those that are not exiting this period there are two possibilities: Either they find d_0 sufficient for the purchases that they would like to make at the prevailing price level, in which case they do not borrow from banks, or their deposits are insufficient and they take out a loan. We consider the two cases separately.[6]

For continuing buyers who choose not to borrow we may write[7]

$$V_2^s(d_0) = \max_{c_2}[A_2 u(c_2) + W(\underbrace{d_0 - p_2 c_2 + d_0 i_1^d}_{d_2}, 0)]. \tag{17}$$

The first-order condition is

$$A_2 u'(c_2^s) = p_2 W_d. \tag{18}$$

Using (9) and (15), we get

$$A_2 u'(c_2^s) = p_2 \phi(1 + i_2^d) = g'(y_2). \tag{19}$$

If the household wants to take out a loan, the Bellman equation may be written:

$$V_2^s(d_0) = \max_{c_2}[A_2 u(c_2) + W(\underbrace{d_0 i_1^d}_{d_2}, \ell_2)] \tag{20}$$

$$\text{subject to : } \quad p_2 c_2 = d_0 + \ell_2 \tag{21}$$

The first order condition in this case is

$$A_2 u'(c_2^s) = -p_2 W_\ell \tag{22}$$

and using (10) we then have

$$A_2 u'(c_2^s) = p_2 \phi(1 + i_2^\ell). \tag{23}$$

Exiting buyers

Buyers who will exit at the end of this sub-period are unable to borrow and have no incentive to save. Optimization for them is trivial: They simply consume the value of their money holdings:

$$c_2^e = \frac{m}{p_2} \tag{24}$$

and their value is given by

$$V_2^e(m) = A_2 u\left(\frac{m}{p_2}\right). \tag{25}$$

[6]It can be easily shown that contingent on the loan rate, i_2^ℓ, there is a critical value of A_2 above which continuing buyers will choose to borrow. This of course may be equal to either the lower or upper support of the distribution.

[7]The transfer policy in the final sub-period is certain once the state in the second sub-period is known.

3.3 Sub-period 1 (anonymous exchange)

The first period of anonymous exchange is similar to the second except that no household enters with an outstanding loan. Households who are not active in sub-period 1 keep all their money in banks and the deposit balance that they carry into the second sub-period, d_1, is equal to d_0. These households' value after the realization of shocks in sub-period 1 is thus given by $\mathrm{E}_1[V_2(d_0, 0)]$, where the expectation is with respect to A_2.

Sellers

The value for a seller active in the first sub-period is

$$V_1^y(d_0) = \max_{y_1}[-g(y_1) + \mathrm{E}_1 W(\underbrace{(d_1 + p_1 y_1)(1 + i_1^d)}_{d_2}, 0)], \tag{26}$$

were we have taken into account that a seller active in the first sub-period is necessarily inactive in the second sub-period. The first order condition for y_1 is

$$-g'(y_1) + p_1(1 + i_1^d)\mathrm{E}_1 W_d = 0. \tag{27}$$

Using (9), we may write

$$g'(y_1) = p_1(1 + i_1^d)\mathrm{E}_1\left[\phi(1 + i_2^d)\right] \tag{28}$$

so that p_1 satisfies

$$p_1 = \frac{g'(y_1)}{(1 + i_1^d)\mathrm{E}_1\left[\phi(1 + i_2^d)\right]}. \tag{29}$$

Buyers who remain in the economy

When non-exiting buyer's own deposits are sufficient for their consumption purchases, we have

$$V_1(d_0) = \max_{c_1^s}[A_1 u(c_1^s) + \mathrm{E}_1 W(\underbrace{(d_0 - p_1 c_1^s)(1 + i_1^d)}_{d_2}, 0)]. \tag{30}$$

The first order condition for c_1^s is

$$A_1 u'(c_1^s) = p_1(1 + i_1^d)\mathrm{E}_1 W_d. \tag{31}$$

Using (9) and (28), we have

$$A_1 u'(c_1^s) = p_1(1 + i_1^d)\mathrm{E}_1[\phi(1 + i_2^d)] = g'(y_1). \tag{32}$$

For a buyer that wants to borrow, the Bellman equation may be written:

$$\max_{c_1^s}[A_1 u(c_1^s) + \mathrm{E}_1 W(0, \ell_1(1+i_1^\ell))] \tag{33}$$

$$\text{subject to}: \quad p_1 c_1^s = d_0 + \ell_1. \tag{34}$$

The first order condition in this case is

$$A_1 u'(c_1^s) = -p_1(1+i_1^\ell)\mathrm{E}_1 W_\ell \tag{35}$$

and using (10) and (28), we have

$$A_1 u'(c_1^s) = p_1(1+i_1^\ell)\mathrm{E}_1[\phi(1+i_2^\ell)] = g'(y_1)\frac{1+i_1^\ell}{1+i_1^d}\frac{\mathrm{E}_1[\phi(1+i_2^\ell)]}{\mathrm{E}_1[\phi(1+i_2^d)]}. \tag{36}$$

Exiting buyers

Again, exiting buyers simply spend all their money. Their consumption is

$$c_1^e = \frac{m}{p_1}, \tag{37}$$

and their value is given by

$$V_1^e(m) = A_1 u\left(\frac{m}{p_1}\right). \tag{38}$$

4 Equilibrium

We now define and characterize a stationary symmetric monetary equilibrium contingent on the central bank's monetary policy (*i.e.* for a fixed profile of central bank deposit rates, i_1^c, i_2^c, and money creation rates γ, all of which we take to be contingent on the realizations of A_1 and A_2). We will turn to the optimal selection of these policy variables later.

In a symmetric equilibrium all households of a particular type active in a particular sub-period make the same choices. Similarly, all banks set the same deposit and loan rates, take in the same quantity of deposits, make the same loans, receive the same payments, and as a result earn the same profit. All choices, including the central bank policy are contingent on the aggregate state variables, A_1 and A_2. We define a stationary symmetric monetary equilibrium as follows:

A *stationary monetary equilibrium* (SME) is a list of quantities, c_1^s, c_1^e, y_1, c_2^s, c_2^e, y_2 and x; work efforts in sub-period 3 by non-exiting buyers, sellers, and newly arrived households, h_1^s, h_1^y, h_2^s, h_2^y and h^n; nominal prices p_1, p_2 and p_3, interest rates, i_1^d, i_1^ℓ, i_2^d, and i_2^ℓ; and a central bank policy i_1^c and i_2^c and γ (all of which are contingent on A_1 and A_2) such that:

1. Taking the central bank policy and prices as given, households choose quantities to solve the optimization problems described in the previous section.

2. Taking the central bank policy as given, banks set i_j^d and i_j^ℓ in each sub-period to maximize profits with no bank wanting to deviate individually.

3. Goods markets clear:

$$\text{sub}-\text{period } 1: \quad \alpha c_1^s + \lambda c_1^e = (1-\alpha-\lambda)y_1 \tag{39}$$

$$\text{sub}-\text{period } 2: \quad \alpha c_2^s + \lambda c_2^e = (1-\alpha-\lambda)y_2 \tag{40}$$

$$\text{sub}-\text{period } 3: \quad \frac{\alpha}{2}(h_1^s + h_2^s) + \frac{1-\alpha-\lambda}{2}(h_1^y + h_2^y) + \lambda h^n = x \tag{41}$$

4. The market for money clears:

$$(1-\lambda)d_0 + \lambda m = M \tag{42}$$

5. Money has value; *i.e.* for all t, $\phi_t > 0$.

We begin our characterization of an equilibrium by deriving some characteristics of bank deposits and lending rates in any SME. We consider only the case in which banks set short-term rates in each sub-period. Deposits and loans carried over into sub-period 2 from sub-period 1 are rolled over at the short-term rates set in the second sub-period, contingent on the realization of A_2.[8] The following proposition establishes some properties of lending and deposit rates in equilibrium:

Proposition 1. *In any SME, in each sub-period* $j = 1, 2$*:*

$$i_j^d = i_j^\ell = i_j^c, \qquad \forall A_j. \tag{43}$$

Proof: See appendix.

Thus, in each sub-period the deposit rate and lending rate of private banks will be equal to the interest rate on reserves set by the central bank. The intuition for this result depends on two key characteristics of the economy: First, banks compete with each other for deposits by setting interest rates. Second, there are a large number of banks which must settle their net balances in outside money.

[8] We do not consider the case where the interest rates are fixed for two sub-periods. It can be shown, however, that this assumption does not affect our results. With linear utility of agents in the centralized market, there is no advantage for them to enter fixed interest rate contracts in the first sub-period.

Consider first the equality $i_j^d = i_j^c$ for $j = 1, 2$. When a bank receives a deposit of outside money from a household, it may deposit it at the central bank and earn i_j^c to be paid in the final sub-period. A profit maximizing bank will not offer a rate higher than i_j^c on deposits as this will result in a loss. Banks are willing to pay up to i_j^c on deposits of outside money and will be forced up to this rate by interest rate competition among banks.

The result that $i_j^\ell = i_j^c$ depends on the settlement requirement. When a bank makes a new loan in sub-period j, borrowers spend their deposits resulting in an outflow of reserves to other banks (those holding the accounts of the sellers from whom the borrower purchases). The marginal cost of the new loan is thus the central bank interest that could have been earned by holding onto these reserves, i_j^c. Banks will therefore not be willing to lend for less. For loans made in the previous sub-period, the opportunity cost for the bank to roll over the loan and continue to hold it is equal to the new central bank rate on reserves.

Using (8) lagged one period we have

$$\phi_{t-1} = \beta(\mathrm{E}_{t-1}V'(d_0)). \tag{44}$$

In the appendix we derive an expression for $\mathrm{E}_{t-1}\left[V'(d_0)\right]$ in the case in which lending and deposit rates are equal (as they must be in any SME). Using this and (8) we derive the following equation

$$\begin{aligned} \frac{1}{\beta} &= (1-\lambda)\int_{A_1}\left[(1+i_1^d)\mathrm{E}_1\left(\frac{\phi}{\phi_{t-1}}(1+i_2^d)\right)\right]dF(A_1) + \frac{\lambda}{2}\int_{A_1} A_1 u'\left(\frac{m}{p_1}\right)\frac{1}{p_1\phi_{t-1}}dF(A_1) \\ &+ \frac{\lambda}{2}\int_{A_1}\mathrm{E}_1\left[A_2 u'\left(\frac{m}{p_2}\right)\frac{1}{p_2\phi_{t-1}}\right]dF(A_1). \end{aligned} \tag{45}$$

This equation can be written in terms of ω, with the details depending on the form of the utility function. Given a monetary policy, a solution to (45) is an SME. Existence is standard in our economy and the details are essentially the same as those described in Berentsen, Camera, and Waller (2007). Since our focus in this paper is on optimal monetary policy, we do not conduct an analysis similar to theirs here. Rather, before turning to the analysis of optimal policy, we describe some key properties that an SME must have.

A formal proof of the following proposition is omitted as these results are intuitive and follow immediately from expressions in Section 3.

Proposition 2. *If $\lambda = 0$ (no households exit), then there exists a unique symmetric equilibrium for which*

1. *outside money is not essential.*

2. *the allocation is efficient*

With equal lending and deposit rates (from Proposition 1) equations (19), (23), (32), and (36) imply efficient consumption and production in sub-periods 1 and 2. From (7) it is clear that production and consumption are always efficient in the final sub-period. It is also clear that this allocation can be attained without outside money. Consumers borrow from producers in sub-periods 1 and 2 effectively using private banks as a record-keeping mechanism. In this case the economy may be viewed as a series of one-period economies and both existence and uniqueness of an equilibrium is straightforward.

Another straightforward result for which we omit a formal proof is the following:

Proposition 3. *If $\lambda > 0$ (a positive fraction of households exit), then outside money is essential.*

If $\lambda > 0$ but there is no outside money, then there will exist a unique symmetric equilibrium in which consumers that stay in the economy and producers consume and produce the same quantities as in the case of $\lambda = 0$. In this equilibrium, however, exiting households will consume nothing (*i.e.* $c_j^e = 0$ for j=1,2). Valued outside money can improve on this allocation by enabling these households to consume a positive amount.

While the introduction of outside money can improve welfare, neither its presence nor a choice of monetary policy can succeed in getting the economy to an efficient allocation in an SME:

Proposition 4. *An SME allocation is not Pareto efficient, regardless of monetary policy, except in the case of no aggregate uncertainty (i.e. when A is constant).*

Proof: In our economy, because all households are *ex ante* identical and all goods are non-storable, it is straightforward to show that efficiency requires that in every state, (A_1, A_2):

$$A_1 u'(c_1^s) = g'(y_1) \tag{46}$$

$$A_1 u'(c_1^e) = g'(y_1) \tag{47}$$

$$A_2 u'(c_2^s) = g'(y_2) \tag{48}$$

$$A_2 u'(c_2^e) = g'(y_2). \tag{49}$$

We will show that these four equations cannot hold simultaneously in all states in an SME. As shown in Section 3 above, irrespective of monetary policy, in any SME (46) and (48) hold in all states. In this sense, buyers that will continue in the economy always consume the "right" amount. Thus we focus here on a basic tension between (47) and (49).

Combining (16) and (24), and making use of the definition of ω we may write in any

SME:

$$\begin{aligned} c_2^e &= \frac{m\phi(1+i_2^c)}{g'(y_2)} \\ &= \frac{\omega}{g'(y_2)}\frac{\phi}{\phi_{t-1}}(1+i_2^c) \\ &= \frac{\omega}{g'(y_2)}r_2 \end{aligned} \tag{50}$$

where,

$$r_2 \equiv \frac{\phi}{\phi_{t-1}}(1+i_2^c) \tag{51}$$

is the real return associated with holding a unit of money through sub-period 2 and into the final sub-period. Rearranging (50) and making use of (49) we have

$$r_2 = \frac{A_2 u'(c_2^e)c_2^e}{\omega}, \tag{52}$$

an expression which must hold in any SME in which (46)—(49) are satisfied.

Similar calculations using (29) and (37) lead to the following expressions for sub-period 1:

$$\begin{aligned} c_1^e &= \frac{\omega}{g'(y_1)}(1+i_1^c)\mathrm{E}_1\left[\frac{\phi}{\phi_{t-1}}(1+i_2^c)\right] \\ &= \frac{\omega}{g'(y_1)}r_L \end{aligned} \tag{53}$$

where

$$r_L \equiv (1+i_1^c)\mathrm{E}_1\left[\frac{\phi}{\phi_{t-1}}(1+i_2^c)\right] \equiv (1+i_1^c)\mathrm{E}_1\left[r_2\right] \tag{54}$$

is the "long-run" gross real interest rate associated with holding money from sub-period 1. Rearranging (53) and making use of (47) we have a first sub-period counterpart to (52):

$$r_L = \frac{A_1 u'(c_1^e)c_1^e}{\omega}. \tag{55}$$

Since (46)—(49) are required to hold in all states, consider a state in which $A_1 = A_2 = A$. Given the restrictions we have imposed on $u(\cdot)$ and $g(\cdot)$, it is clear that in such a state (46)—(49) imply $c_1^s = c_1^e = c_2^s = c_2^e = c$. So, (52) and (55) may be written:

$$r_2 = \frac{Au'(c)c}{\omega} \tag{56}$$

$$r_L = \frac{Au'(c)c}{\omega} \tag{57}$$

or, $r_2 = r_L$. But, since $r_L \equiv (1+i_1^c)\mathrm{E}_1[r_2]$ and $i_1^c \geq 0$ we have

$$r_L \geq \mathrm{E}_1[r_2] \qquad \text{or} \qquad r_2 \geq \mathrm{E}_1[r_2] \tag{58}$$

which, of course, can hold only if r_2 is constant across states. Thus, for any monetary policy that does not maintain a constant r_2, (46)—(49) cannot be maintained in all states and an SME allocation is not Pareto efficient. At the same time, it is straightforward to show, given the properties of $u(\cdot)$ and $g(\cdot)$, that any monetary policy which maintains a constant r_2 across states is inconsistent with maintaining (46)—(49) in all states, except in the extreme case in which there are no aggregate shocks (A constant). Thus, irrespective of monetary policy, if A is not constant, then the allocation in any SME is not Pareto efficient. ■

Note that Proposition 4 relies on there being *at least* two sub-periods of anonymous exchange. If there is only one, then the central bank can always attain the first-best by choice of either $\gamma_t = \frac{\phi}{\phi_{t-1}}$ or i^c so that the analog of (52) holds in all states. Also, the key source of friction here is the incentive of exiting households to overconsume. In our economy, these households cannot be induced to save by *any* policy affecting interest rates. This assumption is admittedly extreme, and in our concluding section we discuss ways to relax it without changing either the basic result of Proposition 4 or the qualitative aspects the optimal policy derived in Section 5.

We now illustrate some properties of an SME, including its inefficiency, for an example in which monetary policy is entirely passive. The central bank maintains a constant money stock (*i.e.* $\gamma_t = 1$ for all t) and sets its short-term interest rates, i_1^c and i_2^c equal to zero regardless of the realizations of A_1 and A_2. Note that in this case r_2 is indeed a constant. We choose the following parameters and functional forms more or less arbitrarily since they do not matter much for the qualitative aspects of the equilibrium. We set the discount factor $\beta = 0.99$ and let utility be logarithmic: $u(c) = \ln c$. We set $g(y) = y + \frac{1}{2}y^2$. We let A be uniformly distributed on $[0.4, 1.1]$ in each sub-period. We set $\alpha = 0.6$ and $\lambda = 0.2$. The larger the share of buyers that do not exit the economy, α, the higher aggregate bank lending whenever A_j is sufficiently high that these buyers would like to borrow. The larger the share of buyers that exit, λ, the larger the aggregate welfare loss associated with their exclusion from the banking system and subsequent sub-optimal consumption.

With the chosen functional forms and an arbitrary monetary policy we may write (45) as

$$\frac{1}{\beta} = (1-\lambda)\int_{A_1} r_L dF(A_1) + \frac{\lambda}{2}\int_{A_1}\frac{A_1}{\omega}dF(A_1) + \frac{\lambda}{2}\int_{A_1}\mathrm{E}_1\left[\frac{A_2}{\omega}\right]dF(A_1) \tag{59}$$

$$= (1-\lambda)\mathrm{E}[r_L] + \frac{\lambda}{\omega}\mathrm{E}[A] \tag{60}$$

where $\mathrm{E}[A]$ is the time-invariant expected value of A in any sub-period. Since $\int_{A_1} \mathrm{E}_1\left[\frac{A_2}{\omega}\right] dF(A_1) = \mathrm{E}[A]/\omega$, we may write

$$\omega = \frac{\lambda \mathrm{E}[A]}{\frac{1}{\beta} - (1-\lambda)\mathrm{E}[r_L]}. \tag{61}$$

An SME exists for this example economy if the denominator of (61) is positive, *i.e.* if

$$\mathrm{E}[r_L] < \frac{1}{\beta(1-\lambda)}. \tag{62}$$

Since in this case $i_1^c = i_2^c = 0$ and $\gamma = 1$, we have $r_L = 1$, and so in equilibrium ω is given by

$$\omega = \frac{\lambda \mathrm{E}[A]}{\frac{1}{\beta} - (1-\lambda)}. \tag{63}$$

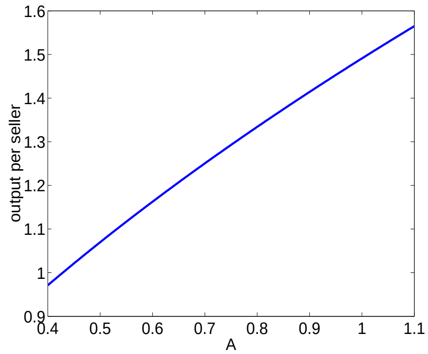
In the appendix we calculate equilibrium quantities and prices. Figure 2 illustrates these for the first sub-period as functions of A_1. The figure can summarized as follows:

1. Both production and the sub-period 1 real price are increasing in A_1.
2. The consumption of buyers that do not exit is increasing in A_1. In contrast, that of buyers who do exit decreases with A_1 as the increase in the price level erodes their real balances, given that their nominal balance is fixed at $m = d_0$.
3. When A_1 is sufficiently high, the loan balance is positive and the aggregate money supply exceeds the quantity of outside money.
4. For high values of A_1 the marginal utility of buyers who exit exceeds sellers' marginal cost. In this case, exiting buyers *under-consume*. Conversely, for low values of A_1, exiting buyers *over-consume* and their marginal utility is below marginal cost.

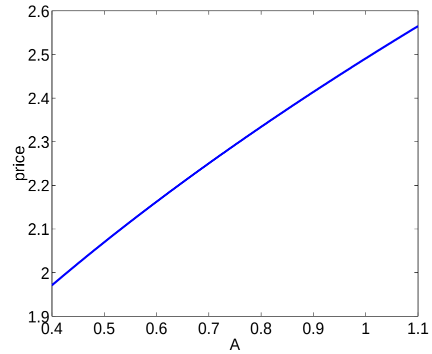
We do not describe sub-period 2 in detail because sub-periods of anonymous exchange are largely symmetric and differ significantly only with regard to the aggregate loan balance and total money supply. The total outstanding loan in sub-period 2 is the new loans plus those extended in the previous sub-period (and rolled over). Similarly, the total money supply in sub-period 2 is measured by total deposits some of which are created in each of the first two sub-periods.

To understand the relationships depicted in Figure 2, note first that with $i_j^c = 0$ in all states (29) may be written

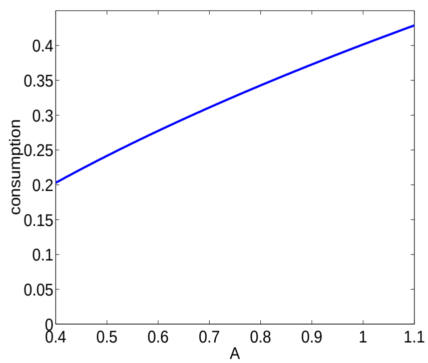
$$p_1 = \frac{g'(y_1)}{\phi}. \tag{64}$$



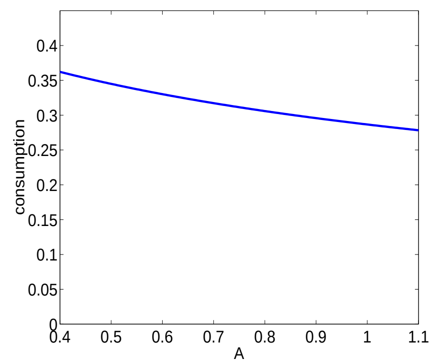
(a) Production of each seller



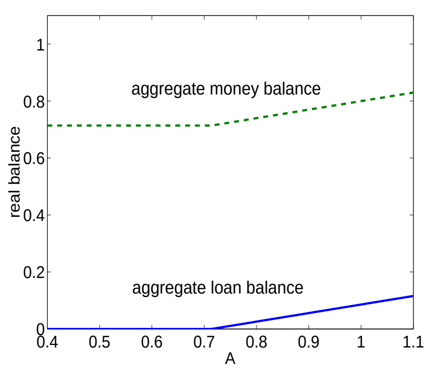
(b) Real Price $(p_1\phi_{t-1})$



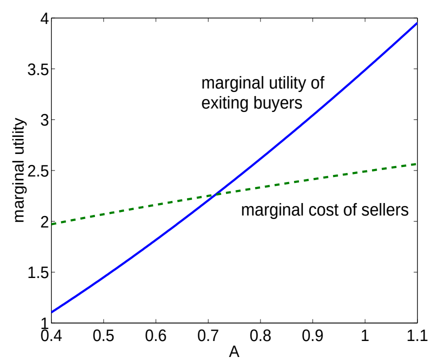
(c) Consumption by non-exiting buyers



(d) Consumption of exiting buyers



(e) Total Loans and Money



(f) Marginal utilities and costs

Figure 2: Sub-period 1 in an example with passive monetary policy.

With a constant money stock, $\phi = \phi_{t-1}$, and $p_1\phi_{t-1} = g'(y_1)$. When an increase in A_1 raises households' demand for goods, marginal cost increases and $p_1\phi_{t-1}$ must rise. An increase in both the quantity produced and the real price level is financed by newly created inside money. Since the total money supply is stochastic (it depends on A_1), p_1 is as well. An increase in p_1 reduces exiting buyers' real balances and so lowers their consumption and utility. Inside money creation therefore redistributes wealth from exiting buyers to those that remain in the economy.

Because the inflation rate is controlled by outside money growth γ, the increase in the nominal price, p_1 (and in p_2 as well) is only temporary and the price level must subsequently fall between sub-periods 1 and 3. Since bank loans are repaid in sub-period 3 and households carry only outside money into the next period, the creation of inside money affects the price level only in the short-run and does not contribute to long-run inflation.

5 Optimal monetary policy

We now consider the problem of a central bank which solves a "Ramsey" problem. That is, it chooses a policy to maximize social welfare subject to the constraint that (45) hold (*i.e.* that the resulting allocation be an SME allocation). We have already shown that given its instruments, the central bank cannot attain the first-best.

The welfare criterion

We assume that the central bank maximizes the expected utility of households present in the economy at the beginning of the current period (period t) plus the expected utility of all households that will enter in the future discounted by the factor β. At the beginning of the current period the expected lifetime utility of a representative household with money holdings $m = d_0$ is denoted $\mathrm{E}[V(d_0)]$. Similarly, the expected utility of a household that will arrive in the final sub-period of this period is given by

$$\mathcal{W}_n = \int_{A_1}\int_{A_2} [U(x) - h^n] dF(A_2) dF(A_1) + \beta \mathrm{E}_{\mathrm{t}} \left[V_{t+1}(d_{0,t+1})\right]. \tag{65}$$

Since the environment is stationary, the central bank maximizes

$$\mathrm{E}[V(d_0)] + \lambda \sum_{i=0}^{\infty} \beta^i \mathcal{W}_n = \mathrm{E}[V(d_0)] + \lambda \frac{\mathcal{W}_n}{1-\beta}. \tag{66}$$

Multiplying by $(1-\beta)$ define the central bank's objective by

$$\mathcal{W} \equiv (1-\beta)\mathrm{E}[V(d_0)] + \lambda \mathcal{W}_n. \tag{67}$$

Making use of the optimization problems in Section 3 we may expand $V(d_0)$ and combine the result with (65) and the goods market clearing conditions (39), (40), and (41) to get

$$\begin{aligned} \mathcal{W} &= \frac{1}{2}\int_{A_1}[\alpha A_1 u(c_1^s) + \lambda A_1 u(c_1^e) - (1-\alpha-\lambda)c\left(\frac{\alpha c_1^s + \lambda c_1^e}{1-\alpha-\lambda}\right)]dF(A_1) \qquad (68) \\ &+ \frac{1}{2}\int_{A_1}\int_{A_2}\left[\alpha A_2 u(c_2^s) + \lambda A_2 u(c_2^e) - (1-\alpha-\lambda)c\left(\frac{\alpha c_2^s + \lambda c_2^e}{1-\alpha-\lambda}\right)\right]dF(A_2)dF(A_1), \end{aligned}$$

where since x is a constant we can ignore $U(x)$ and x for policy analysis.

Define *realized* utility in sub-periods 1 and 2 as

$$\mathcal{W}_1 = \alpha A_1 u(c_1^s) + \lambda A_1 u(c_1^e) - (1-\alpha-\lambda)g(y_1) \qquad (69)$$

$$\mathcal{W}_2 = \alpha A_2 u(c_2^s) + \lambda A_2 u(c_2^e) - (1-\alpha-\lambda)g(y_2) \qquad (70)$$

We then have the following proposition:

Proposition 5. *A monetary policy: $i_1^c(A_1)$, $i_2^c(A_1, A_2)$, and $\gamma(A_1, A_2)$ maximizes $\mathcal{W}$ subject to (45) if it maximizes $\mathcal{W}_1 + \mathrm{E}_1\mathcal{W}_2$ subject to (45).*

Proof: See appendix.

The optimal policy

We first partially characterize analytically the policy which maximizes $\mathcal{W}_1 + \mathrm{E}_1[\mathcal{W}_2]$, and then turn to a computed example. To begin with, note that the central bank sets r_2^* and r_L^* using (52) and (55), respectively, whenever A_1 is such that the zero bound on i_1^{c*} is slack. This is consistent with many different choices of $i_2^{c*}(A_1, A_2)$ and $\gamma^*(A_1, A_2)$, including the possibility of setting $i_2^{c*} = 0$ in all of these states and varying only the inflation rate through taxes and transfers in sub-period 3. For each choice of i_2^{c*} and γ^*, however, the optimal policy requires a unique choice of $i_1^{c*}(A_1)$. Note that for each realization A_1 for which the zero bound is not hit, (46)—(49) hold regardless of the realization of A_2.

If A_1 is sufficiently low that continuing households do not wish to spend all of their money holdings in exchange, then the central bank will want to discourage over-consumption by exiting households that are active in the first period. Since it does not have the ability to tax these households directly, the central bank can lower their consumption only by raising the price level, p_1, to erode the value of their money holdings. Because of the zero bound, however, the only way to raise p_1 is to reduce r_L by lowering $\mathrm{E}_1 r_2^*$. The following proposition describes the central bank's optimal choice of r_2 across states:

Proposition 6. *Given $A_1 \in \mathcal{A}$, under the optimal policy $\frac{\partial \mathcal{W}_2(A_2)}{\partial r_2}$ is equal for all A_2 and*

$$\frac{\partial \mathcal{W}_1(A_1)}{\partial r_L} = -\frac{\partial \mathcal{W}_2(A_2)}{\partial r_2}, \qquad \forall A_2 \in \mathcal{A}. \qquad (71)$$

Proof: See appendix.

The intuition for this proposition is straightforward. Given $\mathrm{E}_1[r_2]$, the central bank will vary r_2^* across states to equate $\partial \mathcal{W}_2(A_2)/\partial r_2$. If it did not do so, then welfare could be improved by providing better insurance in sub-period 2 without affecting sub-period 1 in any way (*i.e.* without changing $\mathrm{E}_1[r_2]$). At the same time, the central bank must choose $\mathrm{E}_1[r_2]$ so as to equate the marginal gains from providing insurance in the first sub-period to the marginal loss associated with lower short-term real interest rates in sub-period 2, conditional on the realization of A_1.

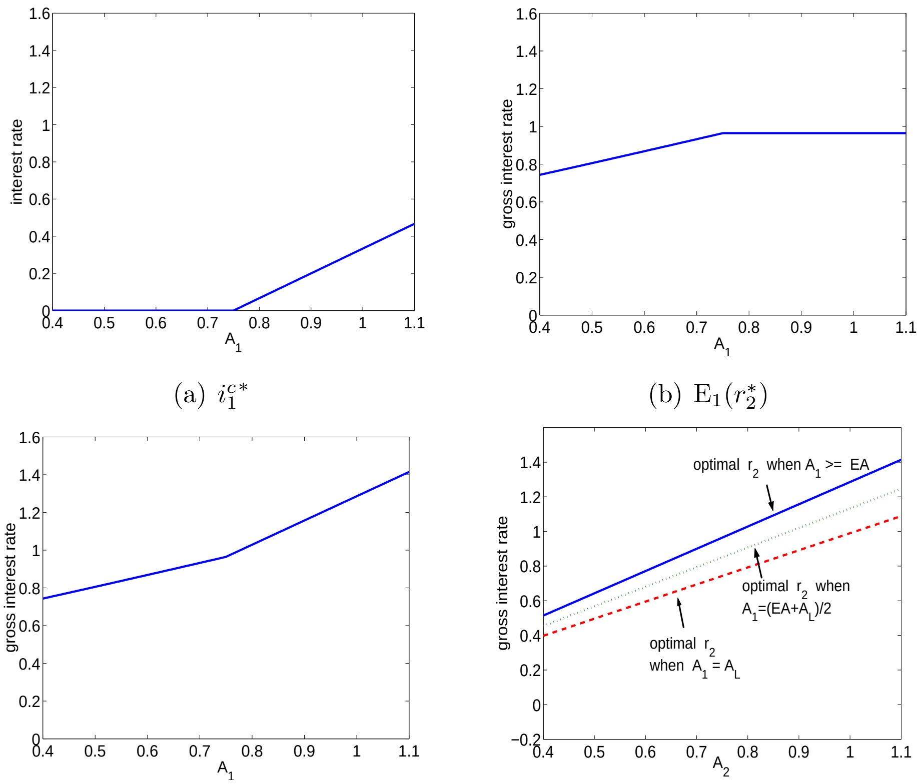
A numerical example

Figure 3 illustrates the key aspects of the optimal monetary policy for the example economy introduced in Section 4. Again we view this example as purely illustrative rather than quantitatively meaningful. Panels (a), (b), and (c) of the figure illustrate i_1^{c*}, $\mathrm{E}[r_2^*]$, and r_L^* as functions of A_1 under the optimal policy. Panel (d) depicts *realized* r_2^* as a function of A_2 conditional on some specific values of A_1.

As A_j rises in either sub-period, continuing households' demand increases. In sub-period 2, the central bank increases r_2^* with A_2 in all states to protect the exiting households active in that sub-period from a rising price level by limiting money creation and encouraging sellers to supply more at a given price. In sub-period 1, the central bank pursues the same goal by increasing r_L^* with A_1.

With log utility, for any state in which $A_1 \geq \mathrm{E}[A]$, continuing buyers want to borrow, and the central bank sets r_2^* according to the highest (solid line) schedule for r_2^* in panel (d). Thus $\mathrm{E}_1[r_2^*]$ is constant across all of theses states. Because $\mathrm{E}_1[r_2^*]$ is constant, the central bank can only vary r_L^* in these states by raising i_1^{c*} above zero and making it increasing in A_1. As noted earlier, in these states all buyers consume equally and efficiently. Thus, the combination of a constant $\mathrm{E}_1[r_2^*]$ and a positive i_1^{c*} effectively "cools down" an economy which would otherwise be "overheated" in the sense that high money creation to satisfy the demand of continuing buyers would raise the price level excessively and harm households who exit.

If $A_1 < \mathrm{E}[A]$ (again, for the case of log utility), the central bank discourages over-consumption by exiting households active in sub-period 1 by lowering r_L^*. It cannot, however, achieve this by further lowering i_1^{c*}, because of the zero bound. Rather, the central bank must reduce $\mathrm{E}_1[r_2^*]$. The entire schedule for r_2^* thus shifts down as A_1 falls. In panel (d) the dashed line depicts the schedule for the case in which A_1 is at its lower support. As shown above, this policy cannot attain the first-best. Because of the zero bound the central bank cannot generate enough demand from continuing households to erode the value of exiting



(a) i_1^{c*} (b) $\mathrm{E}_1(r_2^*)$

(c) r_L^* (d) r_2^*

Figure 3: Interest rates under the optimal policy.

household's money holdings without violating (71) and reducing welfare.

The value of i_1^{c*} in each state is unique under the optimal policy. The implied nominal central bank rate in the *second* sub-period (i_2^{c*}) is, however, indeterminate. An optimal nominal rate can, though, be calculated from (52) and (51) for a given choice of inflation target, $\frac{\phi}{\phi_{t-1}}$. Overall, it is clear from Figure 3 that it is not optimal for the central bank to maintain zero nominal interest rates in all states, regardless of its inflation policy. That is, the nominal interest rate is a necessary component of the optimal policy.

6 Implications of the optimal policy for long-run inflation

The optimal policy is characterized in the second sub-period by the real return r_2^*. This implies that the central bank can use many different combinations of i_2^{c*} and γ^* to carry out the required policy. For example, in any state the central bank can set $i_2^{c*} = 0$, provided that it sets γ^* according to

$$\gamma^* = \frac{\phi}{\phi_{t-1}} = r_2^* \qquad \text{or} \qquad \gamma^* = \frac{1}{r_2^*}. \tag{72}$$

Alternatively, the central bank can adopt a *fixed* inflation rate and rely on i_2^{c*} to reach the required r_2^* in each state.[9] In this case, however, the constant inflation rate must be high enough so that i_2^{c*} never hits the zero bound. Define the lowest *constant* money creation rate such that this is the case as γ_L. Clearly,

$$\gamma_L = \frac{1}{\underline{r_2^*}} \tag{73}$$

where $\underline{r_2^*}$ is the minimum of r_2 in any state under the optimal policy.

In general, the *trend* or *long-run* inflation rate under the optimal policy is given by its average across states:

$$\bar{\gamma} \equiv \int_{A_1} \int_{A_2} \gamma(A_1, A_2) dF(A_1) dF(A_2). \tag{74}$$

Clearly, maintaining a constant inflation rate over time requires a *higher* trend inflation rate than any optimal policy which makes use of a time-varying inflation rate. Trend inflation can be minimized by setting i_2^{c*} in all states so that

$$\bar{\gamma}_L \equiv \int_{A_1} \int_{A_2} \frac{1}{r_2^*(A_1, A_2)} dF(A_1) dF(A_2). \tag{75}$$

[9] We may think of a constant inflation policy in this economy as corresponding to the "price-level targeting" policy of Berentsen and Waller (2010). In this case, the price level follows a deterministic path.

The minimum trend inflation rate consistent with optimal policy depends on the parameters of the economy. Clearly, however, comparing (75) with (73) we have $\gamma_L > \bar{\gamma}_L$. The minimum average inflation rate under the optimal policy can be positive and in principle quite large.

The following table contains γ_L and $\bar{\gamma}_L$ for different distributions of shocks. Here we maintain the assumption of a uniform distribution and change the variance by changing the length of the interval $[A_L, A_H]$.

Table 1

$A_H - A_L$	Minimum constant inflation rate (γ_L)	Minimum stochastic inflation rate ($\bar{\gamma}_L$)
0	0.99	—
0.1	1.1045	1.0067
0.3	1.4012	1.0514
0.5	1.8359	1.1143
0.7	2.5171	1.2021

The last line of Table 1 contains γ_L and $\bar{\gamma}_L$ for our parametric example. Note that when A is constant (*i.e.* there are no shocks) the minimum constant inflation rate is equal to the discount factor, β, and there is no need for time varying inflation. In this case the optimal policy (constant deflation) attains the first-best.

7 Conclusion

We have characterized an optimal interest rate based monetary policy in an environment in which money plays an essential role as the medium of exchange and aggregate shocks affect households, some of which have no access to the banking system, asymmetrically. Using short-run interest rates the central bank can improve welfare by exploiting the effect of inside money creation on the distribution of wealth. The optimal monetary policy requires the central bank to set positive short-term nominal interest rates in some states. It does not attain the first-best and its implementation requires sufficient long-run inflation.

As noted above, the incentive of at least some households to overconsume in some states is central to our results. Here we have modeled this by assuming that a fraction of households is unwilling to save under any circumstance. Similar results can be achieved, however, without totally eliminating any household's incentive to save. Suppose that whenever any agent carries cash following anonymous exchange, they are robbed (or simply lose their money) with probability δ before entering the frictionless market. Then, suppose that as in the current model, fraction λ of households are buyers who have no access to the banking system in the current period. In this case, regardless of monetary policy, buyers excluded from the banking system have incentive to overconsume in states when A is sufficiently low. The result of Propositions 4 and 6 continue to hold, and the optimal policy is qualitatively similar to that present in Figure 3. Our current model may be thought of as corresponding to the case of $\delta = 1$.

We have abstracted from several phenomena which are clearly of importance to the role of the banking system in the implementation of monetary policy. For example, we ignore default risk associated with bank loans and we have assumed that the central bank has sufficient powers to prevent banks from becoming insolvent in any state. We are exploring the implications of relaxing these potentially restrictive assumptions in separate research.

8 Appendices

A Proofs and Derivations

A.1 Proof of proposition 1

We first consider the deposit rate i^d. In the first sub-period, if households deposit outside money, then the bank can deposit the outside money into the central bank and earn i_1^c. If households make deposits by transferring deposits from other banks, then the bank will receive an equal amount of outside money. So banks are willing to offer i_1^c to attract new deposits. This implies that banks must also offer i_1^c to existing deposits in order to prevent households from switching to other banks. Thus, $i_1^d = i_1^c$. In the second sub-period, the deposit rate is decided in the same way, and so $i_2^d = i_2^c$.

Next, we consider the lending rate. In the first sub-period, when a bank makes new loans L, an equivalent amount of deposits will be created. After the borrower spends the money, $\frac{N-1}{N}L$ will be paid to sellers in other banks, and $\frac{1}{N}L$ will be paid to sellers in the same bank. Due to the inter-bank settlement process, the final change is a net decrease in reserve by $\frac{N-1}{N}L$ and a net increase in sellers' deposit in the same bank by $\frac{1}{N}L$. The opportunity cost for reserve is i_1^c, and the cost for sellers' new deposit is $i_1^d = i_1^c$. Thus,

$$i_1^\ell = \frac{N-1}{N} i_1^c + \frac{1}{N} i_1^d = i_1^c \tag{76}$$

In the second sub-period, for new loans, the lending rate is decided in the same way as in the first sub-period. For existing loans, the opportunity cost is the return that could have been earned if the bank asked the borrower to repay the loan. If the borrower repays the loan using outside money, then the money can be used to earn i_2^c. It would the same if the loan is repaid using deposits from other banks because the bank will receive outside money after settlement. If the loan is repaid using the deposits of the same bank, then the opportunity cost is the deposit interest, which is also i_2^c. As a result, $i_2^\ell = i_2^c$. ■

A.2 Calculating equilibrium quantities

These quantities are useful for producing Figures 2 and 3.

The second sub-period

Equilibrium quantities can be computed using the following equations:

$$Au'(c_2^s) = g'(y_2) \tag{77}$$

$$(1-\alpha-\lambda)y_2 = \alpha c_2^s + \lambda\frac{d_0}{p_2} = \alpha c_2^s + \lambda\frac{\omega r_2}{g'(y_2)}. \tag{78}$$

The first equation is the first order condition of continuing buyers, the second equation is goods market clearing, where we use the result that $c_2^e = \frac{d_0}{p_2} = \frac{\omega r_2}{g'(y_2)}$.

In our example, we have $u'(c^s) = \frac{1}{c^s}$ and $g'(y) = 1 + y$, so we can write (77) and (78) as

$$\frac{A}{c_2^s} = 1 + y_2 \tag{79}$$

$$(1-\alpha-\lambda)y_2 = \alpha c_2^s + \lambda\frac{\omega r_2}{1+y_2} \tag{80}$$

Using (79) to replace y_2 in (80), we have

$$(1-\alpha-\lambda)\left(\frac{A}{c_2^s}-1\right) = \alpha c_2^s + \lambda\frac{\omega r_2 c_2^s}{A}$$

$$\Rightarrow \left(\alpha + \lambda\frac{\omega r_2}{A}\right)(c_2^s)^2 + (1-\alpha-\lambda)c_2^s - A(1-\alpha-\lambda) = 0 \tag{81}$$

The positive root of (81) is the equilibrium c_2^s.

The nominal loan level for each continuing buyer is $\ell = c_2^s p_2 - d_0$. Since $Au'(c_2^s) = g'(y_2)$, using (16), p_2 can be written as as $\frac{Au'(c_2^s)}{\phi(1+i_2^d)}$. Using $u'(c_2^s) = \frac{1}{c_2^s}$, we have

$$\ell = c_2^s \frac{1}{\phi}\frac{Au'(c_2^s)}{1+i_2^d} - d_0 = \frac{A}{\phi(1+i_2^d)} - d_0. \tag{82}$$

The aggregate quantity of new loans is $L = \frac{1}{2}\alpha\ell_2$ and the aggregate money supply is $M + L_1 + L_2$ where L_1 and L_2 are new deposits created through loans in the two sub-periods with anonymous exchange.

If we measure loans and deposits in real terms (using $p_{3,t-1}$) we have

$$\ell\phi_{t-1} = \frac{A}{(1+i_2^d)}\gamma - \omega = \frac{A}{r_2} - \omega. \tag{83}$$

Using (83), we then have that the lending takes place whenever

$$A_2 > \omega r_2. \tag{84}$$

The first sub-period

The equations are similar to those of the second sub-period

$$\begin{aligned} Au'(c_1^s) &= g'(y_1) && (85)\\ (1-\alpha-\lambda)y_1 &= \alpha c_1^s + \lambda\frac{d_0}{p_1} = \alpha c_1^s + \lambda\frac{\omega r_L}{g'(y_1)} && (86)\end{aligned}$$

where we use the result that the consumption of exiting buyer is $c_1^e = d_0/p_1 = \frac{\omega r_L}{g'(y_1)}$. Comparing (79) and (80) to (85) and (86), we can now solve for the first sub-period equilibrium quantities by replacing r_2 in (81) by r_L. Clearly, equilibrium output and consumption in the second sub-period depend only on ωr_2, and in the first sub-period they depend only on ωr_L. This is true not only for the example, but also when $u(\cdot)$ and $g(\cdot)$ take general forms. We make use of this result in the proofs which follow.

A.3 Proof of Proposition 5

Define $\mathcal{W}_1 + \mathrm{E}_1 W_2$ as $\mathcal{W}_s$, then (68) can be written as $\mathcal{W} = \int_{A_1} \mathcal{W}_s dF(A_1)$. Let $A_1(i)$ denote state i of A_1. The interest rate policy in $A_1(i)$ will affect the welfare in other states indirectly only through the level of ω. If we can show that under the optimal policy $\frac{\partial \mathcal{W}_s(j)}{\partial \omega} = 0$ in all states of A_1, then the policy that maximizes $\mathcal{W}_s$ will also maximize $\mathcal{W}$. We first derive several interim results.

Lemma 1. *Define*

$$\widetilde{\xi}_2 = \frac{\partial \mathcal{W}_2}{\partial \omega r_2} \tag{87}$$

then when the zero bound is binding, given A_1, $\widetilde{\xi}_2$ is the same in all states in the second sub-period.

Proof: From the equilibrium solution, we know that the equilibrium production and consumption in the second sub-period only depend on ωr_2, so in every state of A_2, we have

$$\frac{\partial \mathcal{W}_2}{\partial r_2} = \frac{\partial \mathcal{W}_2}{\partial \omega r_2}\frac{\partial \omega r_2}{\partial r_2} = \widetilde{\xi}_2 \omega \tag{88}$$

We will show in Proposition 6 that $\frac{\partial \mathcal{W}_2}{\partial r_2}$ is the same in every state of the second sub-period, so $\widetilde{\xi}_2$ is the also the same in every state.

Lemma 2. *Define*

$$\widetilde{\xi}_1 = \frac{\partial \mathcal{W}_1}{\partial (\omega r_L)} \tag{89}$$

we have

$$\widetilde{\xi}_1 = -\widetilde{\xi}_2 \tag{90}$$

Proof: Since the output and consumption in the first sub-period only depend on ωr_L, we can write

$$\frac{\partial \mathcal{W}_1}{\partial r_L} = \frac{\partial \mathcal{W}_1}{\partial (\omega r_L)} \frac{\partial (\omega r_L)}{\partial r_L} = \widetilde{\xi}_1 \omega \tag{91}$$

in Proposition 6, we show that $\frac{\partial \mathcal{W}_1}{\partial r_L} = -\frac{\partial \mathcal{W}_2}{\partial r_2}$. Using lemma 1, we get $\widetilde{\xi}_1 \omega = -\widetilde{\xi}_2 \omega$, thus (90). ■

Using (69) and (70), $\frac{\partial \mathcal{W}_s}{\partial \omega}$ can be written as

$$\frac{\partial \mathcal{W}_s}{\partial \omega} = (A_1 u'(c_1^e) - g'(y_1)) \frac{\partial c_1^e}{\partial \omega} + \mathrm{E}_1 \left[(A_2 u'(c_2^e) - g'(y_2)) \frac{\partial c_2^e}{\partial \omega} \right] \tag{92}$$

where we have omitted the terms associated with $Au'(c^s) - g'(y)$ because they are zero. In those states where the zero bound is not binding, (92) is zero because $Au'(c^e) - g'(y) = 0$ holds on both markets.

When the zero bound is binding, since the equilibrium output and consumption on the first sub-period only depend on $\omega r_L = \omega \mathrm{E}_1 r_2$, and the equilibrium quantities on the second sub-period only depend on ωr_2, we can write (92) as

$$\begin{aligned} & (A_1 u'(c_1^e) - g'(y_1)) \frac{\partial c_1^e}{\partial (\omega \mathrm{E}_1 r_2)} \frac{\partial (\omega \mathrm{E}_1 r_2)}{\partial \omega} + \mathrm{E}_1 \left[(A_2 u'(c_2^e) - g'(y_2)) \frac{\partial c_2^e}{\partial (\omega r_2)} \frac{\partial (\omega r_2)}{\partial \omega} \right] && (93) \\ = \; & \widetilde{\xi}_1 \mathrm{E}_1 r_2 + \mathrm{E}_1 \left[\widetilde{\xi}_2 r_2 \right] = \widetilde{\xi}_1 \mathrm{E}_1 r_2 + \widetilde{\xi}_2 \mathrm{E}_1 \left[r_2 \right] = 0 && (94) \end{aligned}$$

where we use the result of lemma 1 that $\widetilde{\xi}_2$ is the same in all states in the second sub-period and the result of lemma 2 that $\widetilde{\xi}_1 + \widetilde{\xi}_2 = 0$.

The result means that the marginal effects of ω on the first sub-period and the second sub-period will exactly offset each other under the optimal policy. ■

A.4 Proof of proposition 6

Proposition 6 says that given A_1, $\frac{\partial \mathcal{W}_2(A_2)}{\partial r_2(A_2)}$ is the same in every state of A_2. In addition,

$$\frac{\partial \mathcal{W}_1}{\partial r_L} = -\frac{\partial \mathcal{W}_2(A_2)}{\partial r_2(A_2)} \tag{95}$$

Proof: It is easy to see that the above results will hold when the zero bound is not binding because the derivatives are zero. When the zero bound is binding, the central bank would reduce the expected interest rate in the second sub-period in order to achieve a better outcome in the first sub-period. Given the optimal $\mathrm{E}_1 r_2$, the central bank would choose the distribution of r_2 such that the welfare can not be further improved. Let i and j denote any of the two states in the second sub-period, then it must be the case that if we change $r_2(i)$ and $r_2(j)$ marginally in such a way that $\mathrm{E}_1 r_2$ is not changed, then the marginal change in social welfare should be zero, otherwise, the social welfare can be further improved.

Suppose we change $r_2(i)$ by $\Delta r_2(i)$, then the change in $r_2(j)$ that maintains the same $\mathrm{E}_1 r_2$ should satisfy

$$\Delta r_2(i) f(A_i) + \Delta r_2(j) f(A_j) = 0 \tag{96}$$

The marginal change in the social welfare should be zero:

$$\frac{\partial \mathcal{W}_2(i)}{\partial r_2(i)} \Delta r_2(i) f(A_i) + \frac{\partial \mathcal{W}_2(j)}{\partial r_2(j)} \Delta r_2(j) f(A_j) = 0 \tag{97}$$

using (96), we get

$$\frac{\partial \mathcal{W}_2(i)}{\partial r_2(i)} = \frac{\partial \mathcal{W}_2(j)}{\partial r_2(j)} \tag{98}$$

which implies that $\frac{\partial \mathcal{W}_2}{\partial r_2}$ is the same in all states in the second sub-period.

The optimal r_L should satisfy

$$\frac{\partial \mathcal{W}_s}{\partial r_L} = \frac{\partial \mathcal{W}_1}{\partial r_L} + \mathrm{E}_1 \frac{\partial \mathcal{W}_2}{\partial r_L} = 0 \tag{99}$$

$$\Rightarrow \quad (A_1 u'(c_1^e) - g'(y_1)) \frac{\partial c_1^e}{\partial r_L} + \mathrm{E}_1 \left[(A_2 u'(c_2^e) - g'(y_2)) \frac{\partial c_2^e}{\partial r_2} \frac{\partial r_2}{\partial \mathrm{E}_1 r_2} \right] = 0 \tag{100}$$

$$\Rightarrow \quad (A_1 u'(c_1^e) - g'(y_1)) \frac{\partial c_1^e}{\partial r_L} + \xi \mathrm{E}_1 \left[\frac{\partial r_2}{\partial \mathrm{E}_1 r_2} \right] = 0 \tag{101}$$

where we use the result that $r_L = \mathrm{E}_1 r_2$ when the zero bound is binding. $\frac{\partial r_2}{\partial \mathrm{E}_1 r_2}$ is the optimal marginal change in r_2 in each state when we change $\mathrm{E}_1 r_2$. $(A_2 u'(c_2^e) - g'(y_2)) \frac{\partial c_2^e}{\partial r_2}$ is $\frac{\partial \mathcal{W}_2}{\partial r_2}$, and is the same in every state, which we denote as ξ. Note that

$$\mathrm{E}_1 \left[\frac{\partial r_2}{\partial \mathrm{E}_1 r_2} \right] = 1 \tag{102}$$

because the average changes in r_2 is the same as the change in $\mathrm{E}_1 r_2$. As a result, we get (95). ■

B Production and monetary flows in the equilibrium

In this appendix we show the details of production, consumption and money flows in equilibrium. In the exposition that follows, we assume that preference shocks are sufficiently high that bank loans are positive in both markets. Analysis of cases in which no borrowing takes place is similar, although slightly simpler, and is therefore omitted here.

B.1 Consumption and production

Let m denote the nominal money balance carried by every household into sub-period 1 in period t. We assume that all continuing households (consumers and sellers) deposit their money in banks. Exiting households have no access to banks and therefore hold on to their cash themselves. Thus we have $M = (1-\lambda)d_0 + \lambda m$, where $d_0 = m$.

In sub-period 1, each continuing buyer spends $d_0 + \ell_1$, each exiting buyer spends $m = d_0$, and each seller earns $\frac{\alpha(d_0+\ell_1)+\lambda d_0}{1-\alpha-\lambda}$. Consumption is $c_1^s = \frac{d_0+\ell_1}{p_1}$ for continuing buyers with $\ell_1 = c_1^s p_1 - d_0$. Exiting buyers consume $c_1^e = \frac{m}{p_1}$. The production of each seller is given by $y_1 = \frac{\alpha c_1^s + \lambda c_1^e}{1-\alpha-\lambda}$.

In the symmetric equilibrium, $i_1^d = i_1^\ell = i_1^c$. Bank reserves are initially equal to the beginning of period deposit, $(1-\lambda)d_0$. After exchange and settlement, reserves are equal to $(1-\lambda)d_0 + \frac{\lambda d_0}{2}$, where $\frac{\lambda d_0}{2}$ is the money spent by exiting buyers in sub-period 1. These reserves accrue interest for the first sub-period at rate i_1^c.

In sub-period 2, deposit and loan balances carried from sub-period 1 are rolled over at second sub-period interest rate. Bank reserves increase by $\frac{\lambda d_0}{2}$ following settlement as sellers deposit their income, including money spent by exiting buyers active in sub-period 2.

In sub-period 3, equation (7) gives $x^* = U'^{-1}(1)$. The transfer required to ensure that the money supply grows at gross rate γ is given by

$$\tau = (\gamma-1)d_0 - \left(1-\frac{\lambda}{2}\right)d_0[(1+i_1^c)(1+i_2^c)-1] - \frac{\lambda d_0}{2} i_2^c \tag{103}$$

$$= \gamma d_0 - d_0(1+i_1^c)(1+i_2^c) + \frac{\lambda d_0}{2} i_1^c (1+i_2^c). \tag{104}$$

where the net interest payment of the central bank on reserves carried from sub-period 1 is $\left(1-\frac{\lambda}{2}\right)d_0[(1+i_1^c)(1+i_2^c)-1]$ and the net interest on the increase in reserves due to the expenditure of exiting households active in sub-period 2 is $\frac{\lambda d_0}{2} i_2^c$.

In an SME, bank profits (Π) equal zero. As L_1 and L_2 denote new loans issued in the first and second sub-periods, respectively, we can also use them to denote the loans extended by each bank. We then have

$$\begin{aligned} \Pi &= \left[\left(1-\frac{\lambda}{2}\right) M(1+i_1^c)(1+i_2^c)+\frac{\lambda}{2}M(1+i_2^c)\right]+L_1(1+i_1^\ell)(1+i_2^\ell)+L_2(1+i_2^\ell) \\ &\quad -(1-\alpha-\lambda)M(1+i_1^d)(1+i_2^d)-\left[\frac{\alpha+\lambda}{2}M+L_1\right](1+i_1^d)(1+i_2^d) \\ &\quad -\left[\frac{\alpha+\lambda}{2}M+L_2\right](1+i_2^d)-\frac{\alpha}{2}Mi_1^d(1+i_2^d)=0 \end{aligned} \tag{105}$$

The first term is the gross return from depositing the banks' reserve with the central bank. The second and the third terms are associated with payment of bank loans. The fourth term is the payment for sellers' initial deposit. The fifth term is the payment for sellers' income earned in the first sub-period, where the spending of exiting buyers is $\frac{\lambda}{2}M$, and the spending of continuing buyers is $\frac{\alpha}{2}M+L_1$. Similarly, the sixth term is the payment for sellers' income from the second sub-period. The last term is the interest payment the accrued interest of buyers active in the second sub-period. Using the results of Proposition 1, (105) is equal to zero.

At the end of the final sub-period, all households carry the same nominal balance, $m_{t+1} = \gamma d_0$, into the next period. The production of continuing buyers who are active in the first sub-period is:

$$\begin{aligned} h_1^s &= x^*+\phi[d_0\gamma-\tau+(1+i_1^\ell)(1+i_2^\ell)\ell_1] \\ &= x^*+\phi\left[d_0\gamma-\left(d_0\gamma-d_0(1+i_1^c)(1+i_2^c)+\frac{\lambda d_0}{2}\lambda i_1^c(1+i_2^c)\right)+(1+i_1^\ell)(1+i_2^\ell)\ell_1\right] \\ &= x^*+\phi\left[d_0(1+i_1^c)(1+i_2^c)-\frac{\lambda d_0}{2}i_1^c(1+i_2^c)+\ell_1(1+i_1^\ell)(1+i_2^\ell)\right] \end{aligned} \tag{106}$$

where we use (104) to replace τ. For continuing buyers who are active in the second sub-period:

$$\begin{aligned} h_2^s &= x^*+\phi[d_0\gamma-\tau-d_0(i_1^d)(1+i_2^d)+(1+i_2^\ell)\ell_2] \\ &= x^*+\phi\left[d_0(1+i_2^d)-\frac{\lambda d_0}{2}i_1^c(1+i_2^c)+(1+i_2^\ell)\ell_2\right] \end{aligned} \tag{107}$$

For sellers active in the first sub-period, the initial deposit is d_0 and sales revenue is $\frac{\alpha(d_0+\ell_1)+\lambda d_0}{1-\alpha-\lambda}$. Production by these sellers in the final sub-period is given by

$$\begin{aligned} h_1^y &= x^*+\phi\left[d_0\gamma-\tau-d_0(1+i_1^d)(1+i_2^d)-\frac{\alpha(d_0+\ell_1)+\lambda d_0}{1-\alpha-\lambda}(1+i_1^d)(1+i_2^d)\right] \\ &= x^*-\phi\left[\frac{\lambda d_0}{2}i_1^c(1+i_2^c)+\frac{\alpha(d_0+\ell_1)+\lambda d_0}{1-\alpha-\lambda}(1+i_1^d)(1+i_2^d)\right] \end{aligned} \tag{108}$$

Similarly, for sellers active in the second sub-period we have

$$\begin{aligned} h_2^y &= x^* + \phi\left[d_0\gamma - \tau - d_0(1+i_1^d)(1+i_2^d) - \frac{\alpha(d_0+\ell_2)+\lambda d_0}{1-\alpha-\lambda}(1+i_2^d)\right] \\ &= x^* - \phi\left[\frac{\lambda}{2}d_0 i_1^c(1+i_2^c) + \frac{\alpha(d_0+\ell_2)+\lambda d_0}{1-\alpha-\lambda}(1+i_2^d)\right] \end{aligned} \tag{109}$$

Finally, the production of the newly arrived households is

$$h^n = x^* + \phi\left[d_0\gamma - \tau\right] = x^* + \phi\left[d_0(1+i_1^c)(1+i_2^c) - \frac{1}{2}\lambda d_0 i_1^c(1+i_2^c)\right]. \tag{110}$$

Combining the above expressions, we have

$$\frac{1}{2}\alpha(h_1^s+h_2^s) + \frac{1}{2}(1-\alpha-\lambda)(h_1^y+h_2^y) + \lambda h^n = x^* \tag{111}$$

From (110), $h_n > 0$. Since in most cases, sellers active in the first sub-period produce the smallest quantity of goods in the final sub-period, we show here that $h_1^y > 0$. A similar argument can be used to establish that $h_2^y > 0$.

Using (108) and making use of the result that $i_j^d = i_j^c$ in an SME, h_1^y can be written as

$$\begin{aligned} h_1^y &= x^* - \phi(d_0+\ell_1)(1+i_1^c)(1+i_2^c)\frac{\alpha+\lambda}{1-\alpha-\lambda} - \phi d_0 i_1^c(1+i_2^c)\frac{\lambda}{2} \\ &+ \phi(1+i_1^c)(1+i_2^c)\frac{\lambda\ell_1}{1-\alpha-\lambda} \end{aligned} \tag{112}$$

In our economy, equilibrium real balances, ω, output, and consumption in the sub-periods of anonymous exchange do not depend on $U(x)$. So, $U(x)$ can be chosen so that $x^* = U'^{-1}(1)$ is equal to any finite number. Thus, it is sufficient to show that the second and third terms on the RHS of (112) are bounded. If they are, then the model may be parameterized so as to ensure that $h^* > 0$ in an SME for all households.

Since in equilibrium $d_0 + \ell_1 = p_1 c_1^s$, and $p_1 = \frac{g'(y_1)}{(1+i_1^c)\mathrm{E}_1\left[\phi(1+i_2^c)\right]}$ we have

$$\phi(d_0+\ell_1)(1+i_1^c)(1+i_2^c) = \phi(1+i_2^c)\frac{g'(y_1)c_1^s}{\mathrm{E}_1\left[\phi(1+i_2^c)\right]} = g'(y_1)c_1^s\frac{r_2}{\mathrm{E}_1\left[r_2\right]}. \tag{113}$$

Boundedness of the second term of the RHS of (112) thus hinges on that of the RHS of (113). Since we focus on equilibrium under the optimal policy, $0 < \mathrm{E}_1[r_2] < \infty$ and that $r_2/\mathrm{E}_1[r_2]$ is bounded. $\mathrm{E}_1 r_2 = 0$ would imply $r_2 = 0$ for all A_2. Such a policy is clearly sub-optimal because it implies that money has no value in sub-period 2 and so forces $c_2^e = 0$. Similarly, $r_2 = \infty$ is not consistent with equilibrium because it implies a nominal prices level of zero in sub-period 2. Similarly, it is clear that $g'(y_1)c_1^s$ is also bounded as c_1^s satisfies $Au'(c_1^s) = g'(y_1)$. Overall, the expression defined by (113) is bounded and so the second term on the RHS of (112) is bounded as well. Moreover, since $\phi d_0 i_1^c(1+i_2^c) < \phi(d_0+\ell_1)(1+i_1^c)(1+i_2^c)$, the third term on the RHS of (112) is smaller than the second term, and we have that $h_1^y > 0$.

B.2 The creation and circulation of money

Table 2 illustrates the flows of money in an SME using the balance sheet of a representative bank. Since M is the stock of outside money, at the beginning of sub-period 1, reserves and deposits both equal $(1-\lambda)M$. In sub-period 1, bank loans result in an increase of both deposits (liabilities) and loans (assets) by L_1. When continuing buyers make purchases their deposits are transferred to sellers who also deposit any cash they receive from exiting buyers. The additional deposits raise bank reserves one-for-one, by $\frac{\lambda}{2}M$. This pattern repeats in the second sub-period.

In the final sub-period, the order in which the gross flows depicted in 2 take place does not matter. For illustrative purposes only, consider the following order: First, the central bank pays interest on reserves to banks using outside money. The central bank then makes the transfer, which changes households' deposits and bank reserves by the same amount. Banks pay interest to depositors by crediting their deposits. Next, bank loans are repaid. When continuing buyers from the first sub-period repay their loans, deposits equal to $L_1(1+i_1^\ell)(1+i_2^\ell)$ are destroyed. Similarly, when continuing buyers from the second sub-period repay their loans, bank deposits are reduced by $L_2(1+i_2^\ell)$. Finally, all households exchange their deposits for goods and outside money so as to consume x^* and carry the (common) optimal real balance, ω, into the next period.

To summarize, deposits are created when banks make loans, sellers deposit cash income, the central bank transfers, and when banks pay deposit interest. All of the deposits created here $\lambda M + M(\gamma-1) + L_1[(1+i_\ell)(1+i_2^\ell)] + L_2(1+i_2^\ell)$. $L_1[(1+i_\ell)(1+i_2^\ell)] + L_2(1+i_2^\ell)$ are eliminated when loans are repaid. The net change in bank deposits is $\lambda M + M(\gamma-1)$, where λM is the cash income earned by sellers from exiting buyers, and $M(\gamma-1)$ is equal to the change in outside money as a result of the central bank's policy.

C The derivation of $\mathrm{E_{t-1}}\left[V'(d_0)\right]$

Collecting expressions from Section 3, we have

$$\begin{aligned}
\mathrm{E}_{t-1}\left[V(d_0)\right] &= \frac{1}{2}\left\{(1-\alpha-\lambda)\int_A \left[-g(y_1) + \mathrm{E}_1 W((d_0+p_1y_1)(1+i_1^d),0)\right] dF(A)\right.\\
&+ \left.\alpha\int_A \left[A_1u(c_1^s) + \mathrm{E}_1 W(d_2,\ell_2)\right] dF(A) + \lambda\int_A A_1 u\left(\frac{m}{p_1}\right) dF(A)\right\}\\
&+ \frac{1}{2}\int_A\left\{(1-\alpha-\lambda)\mathrm{E}_1\left[-g(y_2) + W(d_0(1+i_1^d)+p_2y_2,0)\right]\right.\\
&+ \left.\alpha\mathrm{E}_1\left[A_2u(c_2^s) + W(d_2,\ell_2)\right] + \lambda\mathrm{E}_1\left[A_2u\left(\frac{m}{p_2}\right)\right]\right\}dF(A). \qquad (115)
\end{aligned}$$

Table 2: The creation, circulation and destruction of inside money

Commercial Bank Balance sheet (With positive loan)									
	Asset Side			Liability side					
	Reserve	Loan (continuing buyers 1)	Loan (continuing buyers 2)	Deposit (continuing buyers 1)	Deposit (sellers 1)	Deposit (continuing buyers 2)	Deposit (sellers 2)	Deposit (newly arrived)	Bank equity
Initial Balance									
Balance	$(1-\lambda)M$	0	0	$\frac{\alpha}{2}M$	$\frac{(1-\alpha-\lambda)}{2}M$	$\frac{\alpha}{2}M$	$\frac{(1-\alpha-\lambda)}{2}M$	0	0
Changes in Sub-period 1									
Bank makes loans		$+L_1$		$+L_1$					
Buyers consume	$+\frac{\lambda}{2}M$			$-L_1-\frac{\alpha}{2}M$	$+L_1+\frac{\alpha}{2}M$ $+\frac{\lambda}{2}M$				
Changes in Sub-period 2									
Bank makes loans			$+L_2$			$+L_2$			
Buyers consume	$+\frac{\lambda}{2}M$					$-L_2-\frac{\alpha}{2}M$	$+L_2+\frac{\alpha}{2}M$ $+\frac{\lambda}{2}M$		
Changes in Sub-period 3									
Central bank pays the interest	$+(1-\frac{\lambda}{2})M[(1+i_1^c)(1+i_2^c)-1]+\frac{\lambda}{2}Mi_2^c$								$+(1-\frac{\lambda}{2})M[(1+i_1^c)(1+i_2^c)-1]+\frac{\lambda}{2}Mi_2^c$
Central bank makes the transfer	$+\tau$			$+\frac{\alpha}{2}\tau$	$+\frac{(1-\alpha-\lambda)}{2}\tau$	$+\frac{\alpha}{2}\tau$	$+\frac{(1-\alpha-\lambda)}{2}\tau$	$+\lambda\tau$	
Bank pays the deposit interest					$+(L_1+\frac{M}{2})[(1+i_1^d)(1+i_2^d)-1]$	$\frac{\alpha}{2}Mi_1^d(1+i_2^d)$	$+\frac{(1-\alpha-\lambda)}{2}M[(1+i_1^d)(1+i_2^d)-1]$ $+(L_2+\frac{(\alpha+\lambda)}{2}M)i_2^d$		-sum1
Sellers spend the extra deposit				$+Income(1)$	$-SP_1$	$+Income(2)$	$-SP_2$	$+Income_n$	
Buyers repay the bank loan		$-L_1$		$-L_1(1+i_1^\ell)(1+i_2^\ell)$					$+L_1[(1+i_1^\ell)(1+i_2^\ell)-1]$
			$-L_2$			$-L_2(1+i_2^\ell)$			$+L_2i_2^\ell$
Final Balance									
Balance	$M\gamma$	0	0	$\frac{\alpha}{2}M\gamma$	$\frac{1-\alpha-\lambda}{2}M\gamma$	$\frac{\alpha}{2}M\gamma$	$\frac{1-\alpha-\lambda}{2}M\gamma$	$\lambda M\gamma$	0

1. "sum1" is the sum of the interest payment in the same row.

2. SP_1 and SP_2 are spending of sellers active in sub-period 1 and sub-period 2, respectively. We use $Income(1)$, $Income(2)$, and $Income_n$ to denote the income of continuing buyers active in sub-period 1 and 2, and the newly arrived households, respectively. Their levels can be computed using the final balance in the balance sheet. For example, for the newly arrived households, we have

$$0(\text{initial balance}) + \lambda\tau + Income_n = \lambda M\gamma(\text{final balance}) \Longrightarrow Income_n = \lambda(M\gamma - \tau) \quad (114)$$

Differentiating (115) with respect to d_0 we obtain[10]

$$
\begin{aligned}
\mathrm{E}_{t-1}[V'(d_0)] &= \frac{1}{2}\Big\{(1-\alpha-\lambda)\int_{A_1}\left[-g'(y_1)\frac{\partial y_1}{\partial d_0}+(1+i_1^d)\mathrm{E}_1 W_d(1+p_1\frac{\partial y_1}{\partial d_0})\right]dF(A_1)\\
&+ \alpha\int_{A_1\leq \mathcal{A}_{b1}}\left[A_1 u'(c_1^s)\frac{\partial c_1^s}{\partial d_0}+(1+i_1^d)\mathrm{E}_1 W_d(1-p_1\frac{\partial c_1^s}{\partial d_0})\right]dF(A_1)\\
&+ \alpha\int_{A_1> \mathcal{A}_{b1}}\left[A_1 u'(c_1^s)\frac{\partial c_1^s}{\partial d_0}+(1+i_1^\ell)\mathrm{E}_1 W_\ell(p_1\frac{\partial c_1^s}{\partial d_0}-1)\right]dF(A_1)\\
&+ \lambda\int_{A_1} A_1 u'\left(\frac{m}{p_1}\right)\frac{1}{p_1}dF(A_1)\Big\}\\
&+ \frac{1}{2}\int_{A_1}\Big\{(1-\alpha-\lambda)\mathrm{E}_1\left[-g'(y_2)\frac{\partial y_2}{\partial d_0}+W_d(1+i_1^d+p_2\frac{\partial y_2}{\partial d_0})\right]\\
&+ \alpha\mathrm{E}_{1(\mathrm{A}_2\leq\mathcal{A}_{\mathrm{b}2})}\left[A_2 u'(c_2^s)\frac{\partial c_2^s}{\partial d_0}+W_d((1+i_1^d)-p_2\frac{\partial c_2^s}{\partial d_0})\right]\\
&+ \alpha\mathrm{E}_{1(\mathrm{A}_2>\mathcal{A}_{\mathrm{b}2})}\left[A_2 u'(c_2^s)\frac{\partial c_2^s}{\partial d_0}+W_d i_1^d+W_\ell(p_2\frac{\partial c_2^s}{\partial d_0}-1)\right]\\
&+ \lambda\mathrm{E}_1\left[A_2 u'\left(\frac{m}{p_2}\right)\frac{1}{p_2}\right]\Big\}dF(A_1). \qquad (116)
\end{aligned}
$$

For sellers active in the first sub-period, $\frac{\partial y_1}{\partial d_0}=0$ because the choice of q_s is independent of d_0. For continuing buyers active in the first sub-period, when the deposit balance is not binding

$$
\begin{aligned}
& A_1 u'(c_1^s)\frac{\partial c_1^s}{\partial d_0}+(1+i_1^d)\mathrm{E}_1 W_d(1-p_1\frac{\partial c_1^s}{\partial d_0})\\
&= \frac{\partial c_1^s}{\partial d_0}\left[A_1 u'(c_1^s)-p_1(1+i_1^d)\mathrm{E}_1 W_d\right]+(1+i_1^d)\mathrm{E}_1 W_d=(1+i_1^d)\mathrm{E}_1 W_d \qquad (117)
\end{aligned}
$$

where we use (31). When the deposit balance is binding, we have

$$
\begin{aligned}
& A_1 u'(c_1^s)\frac{\partial c_1^s}{\partial d_0}+(1+i_1^\ell)\mathrm{E}_1 W_\ell(p_1\frac{\partial c_1^s}{\partial d_0}-1)\\
&= \frac{\partial c_1^s}{\partial d_0}\left[A_1 u'(c_1^s)+(1+i_1^\ell)\mathrm{E}_1 W_\ell\right]-(1+i_1^\ell)\mathrm{E}_1 W_\ell=-(1+i_1^\ell)\mathrm{E}_1 W_\ell \qquad (118)
\end{aligned}
$$

where we use (35).

Similarly, for sellers active in the second sub-period, $\frac{\partial y_2}{\partial d_0}=0$. For continuing buyers in the second sub-period, when the deposit balance is not binding, we have

$$
\begin{aligned}
& A_2 u'(c_2^s)\frac{\partial c_2^s}{\partial d_0}+W_d((1+i_1^d)-p_2\frac{\partial c_2^s}{\partial d_0})\\
&= \frac{\partial c_2^s}{\partial d_0}(A_2 u'(c_2^s)-p_2 W_d)+W_d(1+i_1^d)=W_d(1+i_1^d) \qquad (119)
\end{aligned}
$$

[10]Let A_{bj} denote the critical value of A_j at which continuing buyers wish to borrow.

where we use (18). When the deposit balance is binding, we have

$$\begin{aligned}
&A_2 u'(c_2^s)\frac{\partial c_2^s}{\partial d_0} + W_d i_1^d + W_\ell (p_2 \frac{\partial c_2^s}{\partial d_0} - 1) \\
&= \frac{\partial c_2^s}{\partial d_0}\left(A_2 u'(c_2^s) + W_\ell p_2\right) - W_\ell + W_d i_1^d = -W_\ell + W_d i_1^d
\end{aligned} \tag{120}$$

where we use (22).

Using $W_d = \phi(1+i_2^d)$ and $W_\ell = -\phi(1+i_2^\ell)$, and also the result that lending rate is equal to deposit rate, we get

$$\begin{aligned}
\mathrm{E_{t-1}}[V'(d_0)] &= \frac{1}{2}\Bigg\{(1-\alpha-\lambda)\int_{A_1}\left[(1+i_1^d)\mathrm{E}_1(\phi(1+i_2^d))\right]dF(A_1) \\
&+ \alpha\int_{A_1}\left[(1+i_1^d)\mathrm{E}_1(\phi(1+i_2^d))\right]dF(A_1) + \lambda\int_{A_1} A_1 u'\left(\frac{d_0}{p_1}\right)\frac{1}{p_1}dF(A_1)\Bigg\} \\
&+ \frac{1}{2}\int_{A_1}\big\{(1-\alpha-\lambda)(1+i_1^d)\mathrm{E}_1(\phi(1+i_2^d)) \\
&+ \alpha(1+i_1^d)\mathrm{E}_1(\phi(1+i_2^d)) + \lambda\mathrm{E}_1\left[A_2 u'\left(\frac{d_0}{p_2}\right)\frac{1}{p_2}\right]\bigg\}\,dF(A_1).
\end{aligned} \tag{121}$$

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