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ON A CLASS OF PRODUCTION FUNCTIONS

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DISCUSSION PAPER NO. 68

* I am indebted to Professor P.C. Thanh, who generously extended his time and advice, to Professor D. Usher for encouragement to take up this topic as well as to Mr. G. Warskett for numerous discussions. Each of these persons is responsible for a number of improvements and none for any of the remaining errors.

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I. Introduction

When we look at the production process at microeconomic level, it seems plausible to assume that at the time when a piece of equipment is installed in the firm, there is no or very little room for variations in the factor proportions. But as the time goes by, factor rewards may change, and for optimum output entrepreneur may wish to vary the factor's proportions. The shortcoming of the fixed coefficient type of production function is that it implies that the entrepreneur can never do that, and the shortcoming of a C.E.S. is that he can always do this smoothly.

It seems more reasonable to assume that the entrepreneur cannot vary or can vary little the factor proportions when the piece of equipment is installed. But if we allow a period of time to elapse then for technical reasons or otherwise, he may be able to change this factor proportions to a greater extent. In other words, at the microeconomic level, for any specific type of equipment

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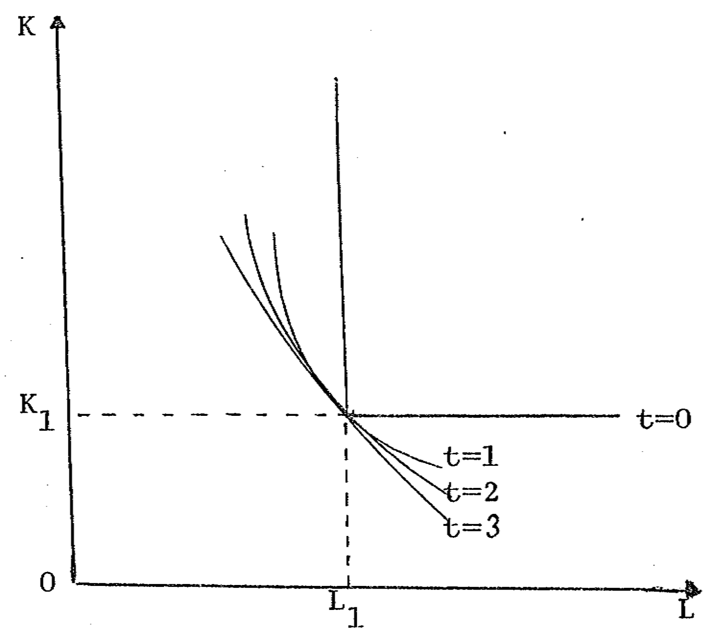
one may assume that the elasticity of substitution is zero or very near zero at the time when that equipment is installed, but that the elasticity of substitution would be greater the longer the period of time has elapsed. This is to say that it is more reasonable to assume that the elasticity of substitution is a variable over time.

As an illustration, in the diagram below the idea of the elasticity of substitution being a function of time is presented. Imagine that historically the entrepreneur found efficient to combine

Figure 1



OL_1 units of labour with OK_1 units of capital to produce one unit of output. This combination of factors of production corresponds to point A in Figure 1. The entrepreneur will remain at point A as long as the existing wage/rental ratio prevails. A change in the rewards of factors of production will induce the producer to choose a new combination of labour and capital. However, at different moments of time there exists different possible combinations of inputs which is illustrated by the fact that there is a number of unit isoquants passing through point A. Each of these isoquants is characterized by a constant elasticity of substitution as one moves along it, however the isoquants associated with later (greater) t are more elastic, i.e. are flatter. It should be noticed that at time $t = 0$ no substitution is possible and the producer operates under the Leontief production function.

2. Production Functions with Variable Elasticity of Substitution

In what follows we investigate a family of V.E.S. production functions in which the elasticity of substitution σ is a function of time, i.e. $\sigma = \varphi(t)$.* Here t is understood to be the period of

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* On V.E.S. production functions see: R. Sato:"Linear Elasticity of Substitution Production Functions", Metroeconomica, April, 1967; R. Sato and R.F. Hoffman: "Production Functions with Variable Elasticity of Factor Substitution: Some Analysis and Testing", The Review of Economics and Statistics, November, 1968; Yao-chi Lu and L.B. Fletcher: "A Generalization of the C.E.S. Production Function", The Review of Economics and Statistics, November, 1968; N.S. Revankar: "A Class of Variable Elasticity of Substitution Production Functions", Econometrica, January, 1971.
In these studies the elasticity of substitution is usually a function of capital/labour ratio. Only R. Sato and R.E. Hoffman make some attempts to construct a production function characterized by σ being a linear function of time.

time which elapses from the moment when a change in factors rewards ratio took place to the moment when it causes a corresponding change in the capital/labour ratio. This response delay implies φ is a monotonic increasing function of t :

$$\frac{d}{dt}\{\varphi(t)\} > 0 \qquad (1)$$

We also assume that at the instant when the wage/rental ratio changes we have:

$$\varphi(0) = \varepsilon > 0 \qquad (2)$$

where ε is a sufficiently small number.

Further,

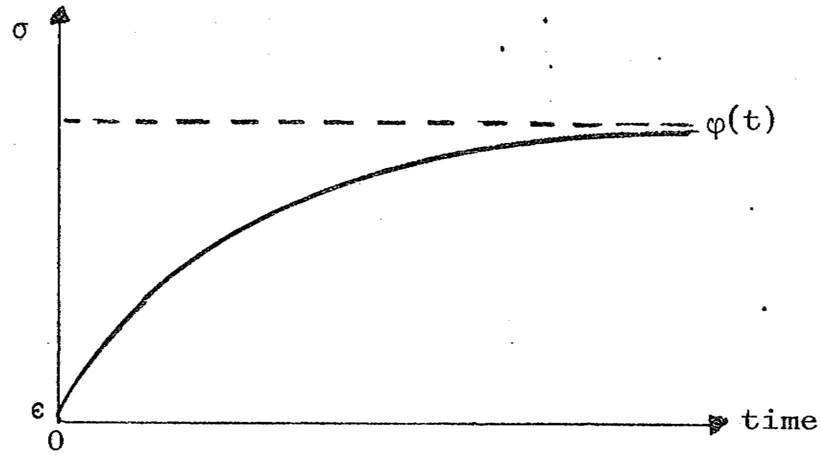
$$\frac{d^2}{dt^2}\{\varphi(t)\} < 0 \qquad (3)$$

Assume that there exists an upper bound for $\varphi(t)$ at a , so that

$$\lim_{t\to\infty} \varphi(t) = a \qquad (4)$$

These properties are illustrated in Figure 2.

Figure 2



In the following, we derive the production function corresponding to the class of elasticities defined by

$$\sigma = \varphi(t) = \frac{an^t - b}{n^t} \tag{5}$$

where $n > 1$, and $b = a - \varepsilon > 0$.
This family of functions possesses all the desired properties (1), (2), (3), (4).* The appeal of this form for σ is that the corresponding production function contains the Leontief, Cobb-Douglas and C.E.S.production functions as special cases.

As is usual, assume the production function to be homogeneous of degree one in capital (K) and labour (L) .

$$Q = F(K,L) = LF(\frac{K}{L},1) \tag{6}$$

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* In the appendix we derive the production function for the general case.

Or in intensive form

$$q = F(k,1) = f(k) \tag{7}$$

where $\frac{K}{L} = k$.

In terms of the production function write the elasticity of substitution as:

$$\sigma = - \frac{f'(k)[f(k)-kf'(k)]}{kf(k)f''(k)} \tag{8}$$

Our postulate implies that:

$$- \frac{f'(k)[f(k)-kf'(k)]}{kf(k)f''(k)} = \frac{an^t-b}{n^t} \tag{9}$$

Define the transformation*

$$f(k) = e^{u(k)} \tag{10}$$

where $\mu(k) > 0$.

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* In the derivation of the production function we follow C.E. Ferguson. See his "The Neoclassical Theory of Production and Distribution", Cambridge University Press, 1969. Ch. 5. For an alternative method see M. Brown: "On the Theory of Measurement of Technological Change", Cambridge University Press, 1966.

Applying this transformation to (9) we get:

$$kn^t u'^2(k) - n^t u'(k) = (an^t - b)ku''(k) + (an^t - b)ku'^2(k) \tag{11}$$

Let

$$\mu'(k) = \omega(k) \tag{12}$$

It follows that

$$\frac{\omega'(k)}{\omega^2(k)} + [1 - \frac{n^t}{an^t - b}] + \frac{1}{\omega(k)} \frac{1}{k} \frac{n^t}{at^n - b} = 0 \tag{13}$$

Introduce a new function $z(k)$ defined by

$$\frac{1}{\omega(k)} = z(k) \tag{14}$$

in order to reduce equation (13) to:

$$z'(k) - [1 - \frac{n^t}{an^t - b}] - z(k) \frac{1}{k} \frac{n^t}{an^t - b} = 0 \tag{15}$$

The above first order, linear differential equation can be made exact by applying the integrating factor

$k^{-\frac{n^t}{an^t - b}}$ giving:

$$k^{-\frac{n^t}{an^t-b}} dz - [z\frac{1}{k} \frac{n^t}{an^t-b} + (1 - \frac{n^t}{an^t-b})]k^{-\frac{n^t}{an^t-b}} dk = 0 \qquad (16)$$

To see this, write

$$M(k,z)\,dz + M(k,z)\,dk = 0 \qquad (17)$$

where

$$M(k,z) = k^{-\frac{n^t}{an^t-b}} \qquad (18)$$

and

$$N(k,z) = [-z\frac{1}{k} \frac{n^t}{an^t-b} - (1 - \frac{n^t}{an^t-b})]k^{-\frac{n^t}{an^t-b}} \qquad (19)$$

Differentiation of $M(k,z)$ with respect to k and of $N(k,z)$ with respect to z yields:

$$\frac{\partial M}{\partial z} = - \frac{n^t}{an^t-b} k^{-\frac{n^t}{an^t-b}-1} \qquad (20)$$

and

$$\frac{\partial N}{\partial z} = - \frac{n^t}{an^t-b} k^{-\frac{n^t}{an^t-b}-1} \qquad (21)$$

We see immediately that $\frac{\partial M}{\partial k} = \frac{\partial N}{\partial z}$ and hence by a well known theorem on line integrals the condition for exactness is fulfilled. By integrating equation (16) we obtain:

$$zk^{-\frac{n^t}{an^t-b}} - k^{-\frac{n^t}{an^t-b}+1} = \left(\frac{\alpha}{1+\alpha}\right)^{\frac{n^t-an^t+b}{an^t-b}} \tag{22}$$

where $\left(\frac{\alpha}{1-\alpha}\right)^{\frac{n^t-an^t+b}{an^t-b}}$ is the constant of integration.

Rearrange equation (22) to obtain

$$z = k\left[\left(\frac{\alpha}{1-\alpha}\right)^{\frac{n^t-an^t+b}{an^t-b}} k^{\frac{n^t}{an^t-b}-1} + 1\right] \tag{23}$$

Recall that $z = \frac{1}{\omega}$. Therefore

$$\omega = \frac{1}{k\left[\left(\frac{\alpha}{1-\alpha}\right)^{\frac{n^t-an^t+b}{an^t-b}} k^{\frac{n^t}{an^t-b}-1} + 1\right]} \tag{24}$$

Since $\mu'(k) = \omega(k)$ by (12), we get

$$\frac{du}{dk} = \frac{1}{k\left[\left(\frac{\alpha}{1-\alpha}\right)^{\frac{n^t-an^t+b}{an^t-b}} k^{\frac{n^t-an^t+b}{an^t-b}} + 1\right]} \tag{25}$$

Integrate (25):

$$\mu = \frac{an^t-b}{n^t-an^t-b} \ln \left[\frac{k^{\frac{n^t-an^t+b}{an^t-b}}}{1 + \left(\frac{\alpha}{1-\alpha}\right)^{\frac{n^t-an^t+b}{an^t-b}} k^{\frac{n^t-an^t+b}{an^t-b}}} \right]$$

$$+ \ln \left[\frac{\Upsilon}{1-\alpha} \left(\frac{1}{2}\right)^{\frac{an^t-b}{an^t-b-n^t}} \right] \qquad (26)$$

Where $\frac{\Upsilon}{1-\alpha} \left(\frac{1}{2}\right)^{\frac{an^t-b}{an^t-b-n^t}}$ is the constant of integration.

In view of (10) we may write:

$$f(k) = \frac{\Upsilon}{1-\alpha} \left(\frac{1}{2}\right)^{\frac{an^t-b}{an^t-b-n^t}} \left[k^{\frac{an^t-n^t-b}{an^t-b}} + \left(\frac{\alpha}{1-\alpha}\right)^{\frac{n^t-an^t+b}{an^t-b}} \right]^{\frac{an^t-b}{an^t-n^t-b}} \qquad (27)$$

Multiply both sides of the above equation by L to obtain $Q = L.f(k)$.

$$Q = \Upsilon \left(\frac{1}{2}\right)^{\frac{an^t-b}{an^t-b-n^t}} \left[\left(\frac{K}{1-\alpha}\right)^{\frac{an^t-n^t-b}{an^t-b}} + \left(\frac{L}{\alpha}\right)^{\frac{an^t-n^t-b}{an^t-b}} \right]^{\frac{an^t-b}{an^t-n^t-b}} \qquad (28)$$

Equation (28) is the desired production function corresponding to $\varphi(t)$ given by (5) .

3. Some Final Remarks on VES Production function.

Obviously the above VES production function is homogeneous of degree one. The marginal products are as follows:

(i) Marginal product of capital

$$\frac{\partial Q}{\partial K} = \frac{Q}{\left[1 + \left(\frac{1}{k}\right)^{\frac{an^t - n^t - b}{an^t - b}} \left(\frac{1-\alpha}{\alpha}\right)^{\frac{an^t - n^t - b}{an^t - b}}\right] K} > 0 \qquad (29)$$

(ii) Marginal product of labour

$$\frac{\partial Q}{\partial L} = \frac{Q}{\left[1 + k^{\frac{an^t - n^t - b}{an^t - b}} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{an^t - n^t - b}{an^t - b}}\right] L} > 0 \qquad (30)$$

These yield shares of capital and labour:

$$\frac{\partial Q}{\partial K}\frac{K}{Q} = \frac{1}{\left[1 + \left(\frac{1}{k}\right)^{\frac{an^t - n^t - b}{an^t - b}} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{an^t - n^t - b}{an^t - b}}\right]} \qquad (31)$$

and

$$\frac{\partial Q}{\partial L}\,\frac{L}{Q} = \frac{1}{\left[1 + k^{\frac{an^t - n^t - b}{an^t - b}} \cdot \left(\frac{\alpha}{1-\alpha}\right)^{\frac{an^t - n^t - b}{an^t - b}}\right]} \tag{32}$$

It is interesting to notice that the shares of capital and labour are not constant but for given α vary with the capital/labour ratio and the elasticity of substitution. Nonetheless, the shares sum to unity, as is implied by the assumption of homogenity.

It is not difficult to see that the above production function reduces to a Leontief production function at $t = 0$. We shall demonstrate this in the following way. Consider equation (28) for $t = 0$, recalling that $b = a - \varepsilon$.

$$Q = \Upsilon\left(\frac{1}{2}\right)^{\frac{\varepsilon}{\varepsilon-1}}\left[\left(\frac{K}{1-\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \left(\frac{L}{\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{33}$$

When ε approaches zero the expression in the brackets becomes indeterminate. Therefore we have to consider equation (33) in the limit when $\varepsilon \to 0$.*

$$\lim_{\varepsilon \to 0} Q = \lim_{\varepsilon \to 0} \frac{\Upsilon}{\left[\left(\frac{K}{1-\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \left(\frac{L}{\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{1-\varepsilon}}} \tag{34}$$

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* For the limit analysis of the C.E.S. production function see C.E. Ferguson: "The Neoclassical Theory of Production and Distribution", op.cit., pp. 106-107.

The coefficient does not appear in (34) since $(\frac{1}{2})^{\frac{\varepsilon}{\varepsilon-1}} \to 1$ as $\varepsilon \to 0$. In order to evaluate equation (34) two cases are distinguished. First consider the case when $L < K$.

$$\lim_{\varepsilon \to 0} Q = \lim_{\varepsilon \to 0} \frac{\frac{\Upsilon}{\alpha} L}{\left[(\frac{L}{K})^{\frac{1-\varepsilon}{\varepsilon}} (\frac{\alpha}{1-\alpha})^{\frac{\varepsilon-1}{\varepsilon}} + 1\right]^{\frac{\varepsilon}{1-\varepsilon}}} = \frac{\Upsilon}{\alpha} L \qquad (35)$$

Because the denominator in the above equation has the limit $1^{\frac{\varepsilon}{1-\varepsilon}} = 1$, the limit of Q can be determined.

Now consider the case when $K < L$.

$$\lim_{\varepsilon \to 0} Q = \lim_{\varepsilon \to 0} \frac{\frac{\Upsilon}{1-\alpha} K}{\left[(\frac{\alpha}{1-\alpha})^{\frac{1-\varepsilon}{\varepsilon}} (\frac{K}{L})^{\frac{1-\varepsilon}{\varepsilon}} + 1\right]^{\frac{\varepsilon}{1-\varepsilon}}} = \frac{\Upsilon}{1-\alpha} K \qquad (36)$$

Again $(\frac{K}{L})^{\frac{1-\varepsilon}{\varepsilon}}$ tends to zero as $\varepsilon \to 0$, and the limit of the denominator becomes $1^{\frac{\varepsilon}{1-\varepsilon}} = 1$ as $\varepsilon \to 0$. From equations (35) and (36) we conclude that:

$$\lim_{\varepsilon \to 0} Q = \min[\frac{\Upsilon}{\alpha} L, \frac{\Upsilon}{1-\alpha} K] \qquad (37)$$

Which is the Leontief production function. For a more familiar form we may also write (37) as

$$\lim_{\varepsilon \to 0} Q = \min[\frac{L}{a_L}, \frac{K}{a_K}] \qquad (38)$$

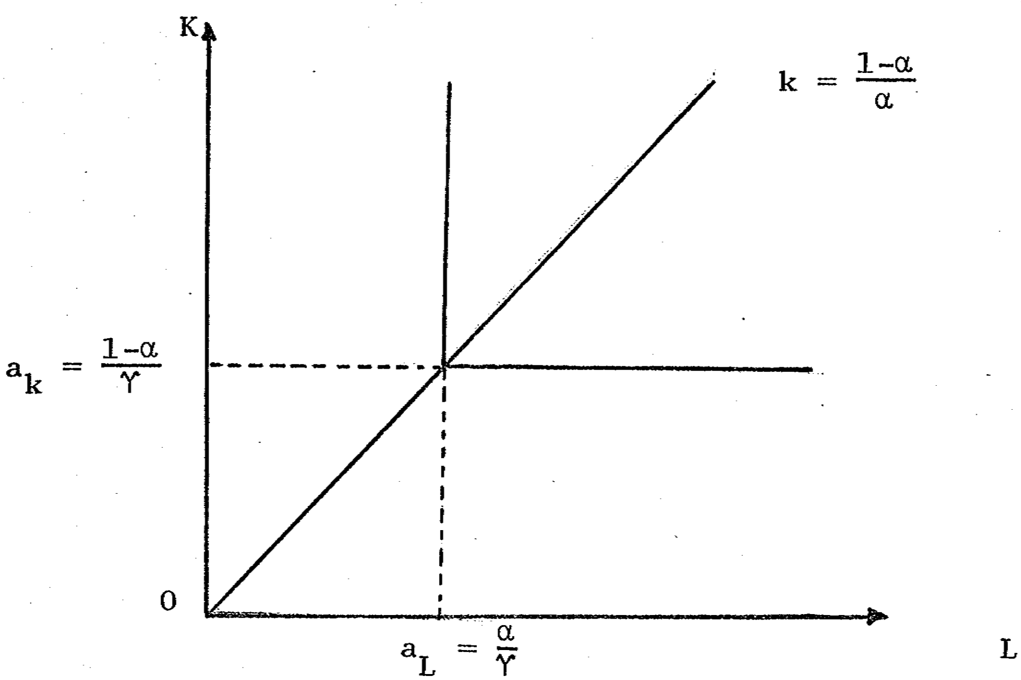
where

$$a_L = \frac{\alpha}{\gamma}$$

and

$$a_K = \frac{1-\alpha}{\gamma} .$$

Figure 3



As illustrated in Figure 3 capital and labour will be used in the proportions $\frac{1-\alpha}{\gamma}$ and $\frac{\alpha}{\gamma}$, respectively.

The production function can be reduced to a Cobb-Douglas production function for a certain value of t. This can be done by applying the similar procedure as above.

The last issue that we would like to take up concerns the long-run strategy of an entrepreneur who has a perfect foresight. Imagine that there is a change in the wage/rental ratio. When will the producer respond to this change if he is faced with a family of isoquants for $t = 0,1,2,\ldots\infty$. If we assume that only one adjustment is possible then it appears obvious that the entrepreneur will remain for some time at his initial position on the unit isoquant, where $\frac{1-\alpha}{\gamma}$ of capital and $\frac{\alpha}{\gamma}$ of labour are used in process of production. He will move to a new isoquant only when enough time has elapsed so as to minimize his unit cost over the entire period stretching from 0 to ∞. It would be possible to set this problem in a more formal way, however it appears obvious that for a given change in the wage/rental ratio, the new equilibrium position will depend on characteristics of the elasticity of substitution function, especially on the speed with which this function approaches its upper bound. Once a new capital/labour ratio is established the sequence is repeated. It always takes time to move to a new position.

Appendix

In the note we have assumed an explicit form of the elasticity of production function. Our result, however, does not depend on that specific form and we can consider a more general case. Suppose we write the elasticity of substitution as:

$$\sigma = \Phi(t) \tag{A.1}$$

where

$$\frac{d}{dt}\{\Phi(t)\} > 0 \tag{A.2}$$

$$\Phi(0) = 0 \tag{A.3}$$

and

$$\lim_{t\to\infty} \Phi(t) = a \tag{A.4}$$

If the production function is homogeneous of degree one it can be written as:

$$q = F(k,1) = f(k) \tag{A.5}$$

It is postulated that

$$-\frac{f'(k)[f(k) - kf'(k)]}{kf(k)f''(k)} = \Phi(t) \tag{A.6}$$

One can similarly derive

$$Q = \Upsilon(\tfrac{1}{2})^{\frac{\Phi(t)}{\Phi(t)-1}} \left[\alpha^{\frac{1-\Phi(t)}{\Phi(t)}} L^{\frac{\Phi(t)-1}{\Phi(t)}} + (1-\alpha)^{\frac{1-\Phi(t)}{\Phi(t)}} K^{\frac{\Phi(t)-1}{\Phi(t)}} \right]^{\frac{\Phi(t)}{\Phi(t)-1}} \quad (A.7)$$

The following can readily be verified:

(i) Marginal product of capital

$$\frac{\partial Q}{\partial K} = \frac{Q}{\left[1 + (\frac{\alpha}{1-\alpha})^{\frac{1-\Phi(t)}{\Phi(t)}} (\frac{1}{k})^{\frac{\Phi(t)-1}{\Phi(t)}}\right] K} \quad (A.8)$$

$$\frac{\partial Q}{\partial K} > 0$$

(ii) Marginal product of labour

$$\frac{\partial Q}{\partial L} = \frac{Q}{\left[1 + (\frac{1-\alpha}{\alpha})^{\frac{1-\Phi(t)}{\Phi(t)}} k^{\frac{\Phi(t)-1}{\Phi(t)}}\right] L} \quad (A.9)$$

$$\frac{\partial Q}{\partial L} > 0$$

At $t = 0$ the production function reduces to a fixed coefficient type:

$$\lim_{t \to 0} Q = \min(\frac{\Upsilon}{\alpha} L, \frac{\Upsilon}{1-\alpha} K) \quad (A.10)$$

Any unit isoquant has to pass through the point given by $L = \frac{\alpha}{\Upsilon}$ and $K = \frac{1-\alpha}{\Upsilon}$. This can be shown in the following way. If $\Phi(t)$ is a single value function of t, then let $\Phi(t) = \eta^*$

for some given t. If we apply $\frac{\alpha}{\gamma}$ of labour and $\frac{1-\alpha}{\gamma}$ of capital we should obtain one unit of output regardless of value of η^*.

Proof.

$$Q = \gamma(\tfrac{1}{2})^{\frac{\eta^*}{\eta^*-1}} \left[\alpha^{\frac{1-\eta^*}{\eta^*}} (\tfrac{\alpha}{\gamma})^{\frac{\eta^*-1}{\eta^*}} + (1-\alpha)^{\frac{1-\eta^*}{\eta^*}} (\tfrac{1-\alpha}{\gamma})^{\frac{\eta^*-1}{\eta^*}}\right]^{\frac{\eta^*}{\eta^*-1}} = \tag{A.11}$$

$$= \gamma(\tfrac{1}{2})^{\frac{\eta^*}{\eta^*-1}} \left[(\tfrac{1}{\gamma})^{\frac{\eta^*-1}{\eta^*}} + (\tfrac{1}{\gamma})^{\frac{\eta^*-1}{\eta^*}}\right]^{\frac{\eta^*}{\eta^*-1}} =$$

$$= \gamma(\tfrac{1}{2})^{\frac{\eta^*}{\eta^*-1}} \, 2^{\frac{\eta^*}{\eta^*-1}} \, \frac{1}{\gamma} = 1 \, .$$

Q.E.D.

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