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Differentiability of the Efficient Frontier when Commitment to Risk Sharing is Limited

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Abstract

This paper shows that the value function describing efficient risk sharing with limited commitment is not necessarily differentiable everywhere. We link differentiability of the value function to history dependence of efficient allocations and provide sufficient conditions for both properties.

Keywords: Risk Sharing; Limited Commitment; Differentiability of Efficient Frontier

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1 Introduction

The literature on limited commitment was very successful in explaining empirical patterns of consumption as the optimal response to commitment frictions. In particular, as pointed out by Kocherlakota (1996), models with commitment frictions seem to be able to explain the positive correlation between consumption and current as well as lagged income. We show here, however, that Kocherlakota's result depends on the efficient frontier of risk sharing being differentiable which need not be the case.[1] As a consequence it can be efficient for current consumption to be *uncorrelated* with past income in the long run even if a commitment problem prevents first-best risk sharing.

There is a sense in which we replicate a finding of Kehoe and Levine (2001) that describe an economy with limited participation where the stochastic steady state equilibrium is history independent. The authors, however, fail to make the connection to the non-differentiability of the efficient frontier. This in turn allows us to go further by partially characterizing when the long-run properties of efficient allocations with limited commitment exhibit history dependence. Interestingly, for our stylized model which features symmetry these conditions are closely linked to differentiability of the value function at the point where all agents are promised the same level of utility.

We proceed as follows. First, we formulate a stylized framework along the lines of Kocherlakota (1996). We then provide a counterexample to the differentiability of the value function and give sufficient conditions for history dependence which are based on differentiability. Finally, we conclude with a short discussion of our findings.

2 Framework

Consider the following stylized environment where people mutually share their endowment risk under limited commitment. Time is discrete and indexed by $t = 0, 1, \ldots$.

[1] In fact, Kocherlakota (1996) falsely claims that the efficient frontier is differentiable.

There are two infinitely lived agents $i = 1, 2$, who receive each period a stochastic endowment of a single good. Let $\theta = \{\theta_1, \theta_2, \dots\}$ be a sequence of independently and identically distributed random variables each having finite support $\Theta = \{1, 2, \dots, S\}$ and denote the probability of θ_t equaling s by $\pi_s > 0$ for all $s \in \Theta$. Define a t-history of θ by $\theta^t = \{\theta_1, \theta_2, \dots, \theta_t\}$ and let Θ^t be the set of all possible t-histories of θ.

The endowment for agent $i = 1, 2$ in period t is determined by the realization of θ_t and denoted by (y_s^1, y_s^2) with aggregate endowment Y_s when $\theta_t = s$ for $t = 0, 1, \dots$. We assume that the joint distribution of the endowment is symmetric; i.e., for every $s \in S$ there exists $s' \in S$ such that $y_s^i = y_{s'}^j$ and $\pi_s = \pi_{s'}$. Preferences for both agents are described over θ^t-measurable consumption processes $c^i \in C = \{\{c_t^i\}_{t=0}^{\infty} | c_t^i : \Theta^t \longrightarrow [0, Y]\}$, $i = 1, 2$, and - given any history θ^t - represented by the utility function

$$E_t \left[\sum_{\tau=0}^{\infty} \beta^\tau u(c_{t+\tau}^i) \Big| \theta^t \right], \tag{1}$$

where $\beta \in (0, 1)$. We assume that u is increasing, strictly concave, twice continuously differentiable and that $\lim_{c \to 0} u'(c) = \infty$.

An *allocation* $(c^1, c^2) \in C^2$ is given by a consumption process for each agent and is *feasible* if

$$c_t^1(\theta^{t-1}, s) + c_t^2(\theta^{t-1}, s) + \leq Y_s \text{ for all } (\theta^{t-1}, s) \text{ and } t \geq 1. \tag{2}$$

Furthermore, we say that an allocation is *incentive feasible* if it is feasible and *incentive compatible* for $i = 1, 2$, i.e.,

$$u(c_t^i(\theta^{t-1}, s)) + E_t \left[\sum_{\tau=1}^{\infty} \beta^\tau u(c_{t+\tau}^i) \Big| (\theta^{t-1}, s) \right] \geq u(y_{t,s}^i) + \beta V_{aut} \tag{3}$$

for all (θ^{t-1}, s) and t, where $V_{aut} = \frac{1}{1-\beta} E[u(y_s)]$ is the ex-ante expected utility of autarky which is equal for both agents.

The concept of incentive feasibility allows us to define optimal allocations. An allocation $(c^1, c^2) \in C^2$ is *optimal* if there exists no other incentive feasible allocation that provides both agents with at least as much expected utility at period 0 and at least one of them with strictly more expected utility at period 0. It is possible to show that optimal allocations are described by the following functional equation:[2]

$$V(u_0) = \max_{\{c_s,u_s\}_{s=1}^S} \sum_{s=1}^{S} \pi_s \left[u(Y_s - c_s) + \beta V(u_s)\right] \tag{4}$$

subject to

$$\sum_{s=1}^{S} \pi_s \left[u(c_s) + \beta u_s\right] = u_0 \tag{5}$$

$$u(Y_s - c_s) + \beta V(u_s) \geq u(y_s^1) + \beta V_{aut} \quad \forall s \tag{6}$$

$$u(c_s) + \beta u_s \geq u(y_s^2) + \beta V_{aut} \quad \forall s \tag{7}$$

$$u_s \in [V_{aut}, V(V_{aut})] \quad \forall s. \tag{8}$$

The state variable u_0 expresses expected utility promised to person 2 while u_s is the level of future expected utility promised when state s is realized. The constraints (6) and (7) are recursive equivalents of the sequential ex-post incentive compatibility constraints for agent 1 and agent 2 respectively. Denote S_1 (S_2) the set of states for which the constraint for agent 1 (agent 2) is binding. We assume throughout the paper that $V(V_{aut}) > V_{aut}$, i.e., there exists some incentive feasible allocation besides autarky.

Notice that $S_1 \cap S_2 = \emptyset$. Furthermore, V is continuous and strictly concave. By Rockafellar (1970), Theorem 25.3 and Theorem 25.5, V is differentiable *almost everywhere* and, since V is a proper concave function, the set $\mathcal{D}$ where V is differentiable is a dense subset of the domain $[V_{aut}, V(V_{aut})]$. However, as we show next and in contrast to the statement in Kocherlakota (1996), V can fail to be differentiable everywhere.

[2]For details on this result and others stated in this section see Kocherlakota (1996).

3 Differentiability and History Dependence

3.1 Counterexample

We proceed now by demonstrating that the value function V is not necessarily differentiable everywhere. Assume for the remainder of the paper that there exists y_s such that

$$\frac{1}{1-\beta}\sum_{s=1}^{S}\pi_s u(Y_s/2) < u(y_s) + \beta V_{aut}. \tag{9}$$

This condition implies that no first-best allocation is incentive feasible. Denote the fixed point of V by $\bar{u}$.

Proposition 3.1. *If $S = 2$, the value function V is not differentiable at $\bar{u}$.*

Proof. Denote the set of states by $S = \{H, L\}$ indicating high and low income for agent 2. Hence, assuming symmetry amounts to $Y_s = Y$ and $\pi_s = \frac{1}{2}$. Guess the following solution to problem (4) for $u_0 \in [V_{aut}, \bar{u}]$: $u_H(u_0) = \bar{u}$; $u_L(u_0) = u_0$; $y_H \geq c_H(u_0) > Y/2$ is the lowest level of consumption that satisfies inequality (7) given $u_H(u_0)$; $c_L(u_0)$ solves equation (5) given $c_H(u_0)$, $u_L(u_0)$ and $u_H(u_0)$. For $u_0 \in [\bar{u}, V(V_{aut})]$ reverse the agents.

Since $Y - c_L(\bar{u}) = c_H(\bar{u})$ and c_L is increasing in u_0, it is straightforward to verify that these policy functions are incentive feasible. Suppose now that V is differentiable on $(V_{aut}, V(V_{aut}))$. Since V is strictly concave, we only have to check the first-order conditions to verify that the guess is a solution. From the Lagrangian we obtain,

$$-V'(u_0) = \frac{u'(Y - c_L)}{u'(c_L)} \tag{10}$$

and

$$V'(\bar{u}) = -\frac{u'(Y - c_H)}{u'(c_H)} \leq V'(u_0). \tag{11}$$

Using the guess, one finds that $V'(u_0) = -\frac{1}{2-\beta}u'(Y - c_L)\frac{\partial c_L}{\partial u_0}$. Since $\frac{\partial c_L}{u_0} = (2-\beta)/u'(c_L)$ and $c_H > c_L$, both conditions hold and the guess is correct. Finally, we check whether

V is differentiable. The solution implies that

$$\lim_{u_0 \nearrow \bar{u}} V'(u_0) = \frac{u'(Y - c_L(\bar{u}))}{u'(c_L(\bar{u}))} \neq \lim_{u_0 \searrow \bar{u}} V'(u_0) = \frac{u'(Y - c_H(\bar{u}))}{u'(c_H(\bar{u}))} \tag{12}$$

which shows that V is not differentiable at $\bar{u}$. □

The solution for the two state case shows that the state variable u_0 converges with probability one to a degenerate distribution of u_0 at $\bar{u}$ independent of initial conditions. Hence, it is optimal for consumption in the long-run to be non-autarkic and iid, i.e., to be completely history independent.

3.2 Sufficient Conditions

The problem of non-differentiability arises from the fact that - at some point in the domain of V - for *every* state s the incentive constraint of either agent 1 or agent 2 is binding, i.e., $S_1 \cup S_2 = S$. In the two state case presented above this is precisely the case at the fixed point of the value function $\bar{u}$ where the role of the agents switches.[3]

Lemma 3.2. *Suppose $S_1 \cup S_2 \neq S$ at u_0. Then V is differentiable at u_0.*

Proof. Let $\hat{u}_0 \in (V_{aut}, V(V_{aut}))$ be given and suppose that $\{1\} \notin S_1 \cup S_2$ at $\hat{u}_0$. Denote the allocation that maximizes the functional equation at $\hat{u}_0$ by $\{c_s^*(\hat{u}_0), u_s^*(\hat{u}_0)\}_{s=1}^S$. Define

$$\gamma(u_0) = u^{-1}\left(\frac{1}{\pi_1}\big(u_0 - \sum_{s=2}^{S} \pi_s \big(u(c_s^*(\hat{u}_0)) + u_s^*(\hat{u}_0)\big)\big)\right) - c_1^*(\hat{u}_0). \tag{13}$$

Define further $c_1(u_0) = c_1^*(\hat{u}_0) + \gamma(u_0)$ and define a new allocation by

$$\{(c_1(u_0), c_2^*(\hat{u}_0), \ldots, c_S^*(\hat{u}_0)), (u_1^*(\hat{u}_0), \ldots, u_S^*(\hat{u}_0))\}.$$

Since $\hat{u}_0 \in (V_{aut}, V(V_{aut}))$, $c_1^*(\hat{u}_0) > 0$ and for state $s = 1$ both incentive constraints are not binding, the new allocation is feasible for a small enough neighborhood around $\hat{u}_0$.

[3]In case some first-best allocation is incentive feasible, one can show that $S_1 \cup S_2 \neq S$ for all u_0 and, hence, that V is differentiable everywhere.

Consider now the function $v(u_0)$ that expresses the value of this new allocation for agent 1. Then, $v(u_0) \leq V(u_0)$ with equality at $\hat{u}_0$. Furthermore, v is differentiable in u_0. Since u is increasing, v is a concave function in u_0. Hence, we have constructed a function v that satisfies the conditions in Lemma 1 of Benveniste and Scheinkman (1979), which proves that V is differentiable. □

This lemma is helpful for establishing a link between differentiability of V and the long-run property of current consumption to depend on lagged income. Denote the Markov process for promised utility associated with the efficient allocation by the sequence of random variables u_t, where $u_t = u_s(u_{t-1})$ with probability π_s for all s.[4]

Proposition 3.3. *If $S_1 \cup S_2 \neq S$ at $\bar{u}$, the Markov process u_t converges weakly to a non-degenerate invariant measure ϕ^*.*

Proof. From the analysis of the first part of Proposition 4.2 in Kocherlakota (1996), it follows that the process u_t converges weakly to a unique invariant measure ϕ^* independent of initial conditions.

Since no first-best allocation is incentive feasible, for all u_0, $S_1 \cap S_2 \neq \emptyset$. Hence, symmetry implies that at $\bar{u}$ there exists $s' \in S_2$. By Lemma 3.2, V is differentiable and the first-order conditions of problem (4) imply that $u_{s'}(\bar{u}) > \bar{u}$. Using the argument of the second part of Proposition 4.2 in Kocherlakota (1996) then proves the result. □

Suppose now, the number of states is odd, i.e., $S/2 \notin I\!N$. Then, it must be the case that $S_1 \cup S_2 \neq S$ at $\bar{u}$. Otherwise, by symmetry, the incentive constraint for both agents is binding in the state where both agents have equal endowment. But then $S_1 \cap S_2 \neq \emptyset$, which cannot be the case. Applying Proposition 3.3 this leads immediately to the following corollary.

Corollary 3.4. *If $S/2 \notin I\!N$, ϕ^* is non-degenerate.*

[4] This result is essentially a correction of Proposition 4.2 in Kocherlakota (1996). I am thankful to Narayana Kocherlakota for discussions on this issue from which conjectures of these results arose.

Whenever the invariant distribution ϕ^* is non-degenerate, consumption is related to lagged income in the long run. What is interesting, however, is that this invariant distribution is non-degenerate precisely when the Pareto frontier V is differentiable at $\bar{u}$, i.e., at the point where both agents receive the same promised utility. Hence, there is a link between differentiability of V and the long-run property of current consumption to depend on lagged income. The results here give then sufficient conditions for history dependence of optimal risk sharing allocations.[5]

[5] In light of the findings on history dependence, our results are then useful for differentiating a static limited commitment model (e.g. Coate and Ravaillon (1993) and Ligon et al. (2002)) - with transfers restricted a priori to be independent of history - from a dynamic limited commitment model.

References

Benveniste, L. and J. Scheinkman, 1979, On the Differentiability of the Value Function in Dynamic Models of Economics, Econometrica 47, 727-732.

Coate, S. and M. Ravallion, 1993, Reciprocity without Commitment: Characterization and Performance of Informal Insurance Arrangements, Journal of Development Economics 40, 1-24.

Kocherlakota, N., 1996, Implications of Efficient Risk Sharing Without Commitment, Review of Economic Studies 63, 595-609.

Kehoe, T. and D. Levine, 2001, Liquidity Constrained Markets versus Debt Constrained Markets, Econometrica 69, 575-598.

Ligon, E., Thomas, J. and T. Worrall, 2002, Informal Insurance Arrangements with Limited Commitment: Theory and Evidence from Village Economies, Review of Economic Studies 69, 209-244.

Rockafellar, T., 1970, Convex Analysis, (Princeton University Press, Princeton).