Queen's Economics Department Working Paper No. 70

KNOWN MOVING-AVERAGE TRANSFORMATIONS AND AUTOREGRESSIVE PROCESSES

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2-1972

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The errors in the linear models which are used so widely by economists may be generated by mixed moving-average autoregressive processes. Aigner [1] has provided a survey of several econometric techniques that have been advocated for use when these processes are suspected to be present. Contributions to this literature fail to take account of the possible availability of prior information on the source of the moving-average component of these processes. This information permits considerable simplification in estimation and its availability has been cited in many areas; notably by Haitovsky [2] for budget studies, Zellner and Montmarquette [3] for money-multiplier models, and Rowley and Wilton [4, 5] for models of wage-determination which take account of the discontinuities due to the presence of bargaining groups in the labour market. A simple procedure for derivation of consistent estimators is illustrated below for a particular situation in which the moving-average component is introduced by a known prior adjustment of data. This procedure combines the well-known autoregressive transformation, suggested by Durbin [6], and the principle of generalized least-squares, due to Aitken [7].

Model

Suppose that the supply of labour to a particular economy can be classified into four groups according to the quarter in which members of each group negotiate and obtain their annual wage-bargains. Then, if each group considers the same collection of variables when negotiating individual contracts, the determination of wage-rates might be described by linear equations of the form

$$(1) \qquad \begin{bmatrix} y_{4t+4} \\ \cdots \\ y_{4t+1} \end{bmatrix} = \sum_{j=1}^{k} \begin{bmatrix} a_{4j} & & 0 \\ & \ddots & \\ 0 & & a_{1j} \end{bmatrix} \begin{bmatrix} x_{4t+4} \\ \cdots \\ x_{4t+1} \end{bmatrix}_j + \begin{bmatrix} e_{4t+4} \\ \cdots \\ e_{4t+1} \end{bmatrix}$$

for $t = 0, 1, 2, \ldots, n_0$ and $n = 4(n_0 + 1)$,

where y_{4t+i} represents the average wage-rate arranged by members of the i-th group in the period indexed by $(4t+i)$, $[x_{4t+i}]_j$ is the level of the j-th explanatory variable considered by the i-th group in obtaining its bargain during period $(4t+i)$. k and n are the number of explanatory variables and observations respectively. The errors $\{e_{4t+i}$ for $i = 1, 2, 3, 4$ and $t = 1, 2, \ldots, n_0\}$ are assumed to be generated by an autoregressive process in the form

$$(2) \qquad \begin{bmatrix} e_{4t+4} \\ \cdots \\ e_{4t+1} \end{bmatrix} = \sum_{i=1}^{4} \begin{bmatrix} b_{4i} & & 0 \\ & \ddots & \\ 0 & & b_{1i} \end{bmatrix} \begin{bmatrix} e_{4t+4-i} \\ \cdots \\ e_{4t+1-i} \end{bmatrix} + \begin{bmatrix} u_{4t+4} \\ \cdots \\ u_{4t+1} \end{bmatrix}$$

for $t = 0, 1, 2, \ldots, n_0$,

where $\{u_{4t+i}\}$ is white noise with constant variance.

These specifications can be written in an alternative and more convenient matrix form. Equations (1) become

$$(3) \qquad y = \sum_{j=1}^{k} [I_n \otimes A_j]\, x_{0j} + e \quad ,$$

where $\{A_j\}$ are diagonal matrices $\{dg(a_{4j}, a_{3j}, a_{2j}, a_{1j})$ for $j = 1, 2, \ldots, k\}$, I_n is an identity matrix of order n, and y, x_{0j} and e are column vectors of order n.

$$y \equiv \begin{bmatrix} y_n \\ .. \\ y_1 \end{bmatrix}, \quad x_{0j} \equiv \begin{bmatrix} x_n \\ .. \\ x_1 \end{bmatrix}_j \quad \text{and} \quad e \equiv \begin{bmatrix} e_n \\ .. \\ e_1 \end{bmatrix}$$

$[I_n \otimes A_j]$ is the right direct product of I_n and A_j (see MacDuffee [8]); namely, a quasidiagonal matrix with $(n_0 + 1)$ diagonal blocks formed by A_j. Similarly, equations (2) can be represented by

$$(4) \qquad e = \sum_{i=1}^{4} [I_n \otimes B_i]\, e_{-i} + u \quad ,$$

where $\{B_i\}$ are diagonal matrices $\{dg(b_{4i}, b_{3i}, b_{2i}, b_{1i})$ for $i = 1, 2, 3, 4\}$,

$$e_{-i} \equiv \begin{bmatrix} e_{n-i} \\ \cdots \\ e_{1-i} \end{bmatrix} \quad \text{and} \quad u \equiv \begin{bmatrix} u_n \\ \cdots \\ u_1 \end{bmatrix}.$$

The autoregressive specification indicates that the error for any particular wage-bargain depends upon the error for the same group in its previous bargain and upon the errors for other groups who have negotiated since this previous bargain. Clearly, this specification is superior to the common one which ignores the presence of annual contracts and embodies a simple first-order autoregressive specification (perhaps because of the general availability of computational programmes for Hildreth-Lu scan procedures for use with this specification).

Aggregative Prior Adjustment

Data for individual wage-bargains are seldom available and economists are compelled to make use of data which have been subjected to prior adjustment by the primary collectors of data. We must distinguish between the "micro-equations", (1) and (2), which would be used if "micro-data" were available and the "macro-equations" which are based upon the aggregative data that are available. Consider the situation in which all variables have been subjected to the same form of prior adjustment. In particular, assume that this adjustment can be represented by the matrix G, where G has rank (n-q), order (n-q) by n, and is given by the following expression.

(5) $$G = \begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_q & & & 0 \\ & d_1 & d_2 & \cdots & d_{q-1} & d_q & & \\ & & \ddots & & & & \ddots & \\ 0 & & & & d_1 & & \cdots & d_q \end{bmatrix}$$

The weights $\{d_1, d_2, \ldots , d_q\}$ are assumed known in this time-invariant choice. More general specifications for G may not introduce any substantive difficulties in practice. They are not discussed here in order to reduce the need for further notation.

With any choice for G, the macro-equations are obtained from (3) and (4).

(6) $$Gy = \sum_{j=1}^{k} G[I_n \otimes A_j]\, x_{0j} + Ge$$

(7) $$Ge = \sum_{i=1}^{4} G[I_n \otimes B_i]\, e_{-i} + Gu$$

In order to write these equations in terms of the "macro-data" $\{Gy\ , Gx_{0j}\ , Ge$ and $(Ge)_{-i}\}$, matrices N_j and M_i must be found such that

(8) $$G[I_n \otimes A_j] \equiv N_j\, G \qquad \text{for } j=1, 2, \ldots , k$$

and

(9) $$G[I_n \otimes B_i] \equiv M_i\, G \qquad \text{for } i=1, 2, 3, 4.$$

Given the particular specification (5) for G, this matrix has full row rank so that its Moore-Penrose inverse, denoted G^- , is $G'(GG')^{-1}$ and GG^- is an identity matrix of order (n-q). Hence, postmultiplication by the Moore-Penrose inverse in (8) and (9) yields explicit expressions for N_j and M_i.

(10) $$N_j = G[I_n \otimes A_j]G^- \qquad \text{for } j=1, 2, \ldots, k$$

(11) $$M_i = G[I_n \otimes B_i]G^- \qquad \text{for } i=1, 2, 3, 4.$$

(12) $$(Gy) = \sum_{j=1}^{k} N_j(Gx_{0j}) + (Ge)$$

(13) $$(Ge) = \sum_{i=1}^{4} M_i(Ge)_{-i} + (Gu)$$

The errors of the macro-equation (12) are generated by a mixed moving-average -autoregressive process for which the weights of the moving-average component are known. This form provides the only basis for estimation when the macro-data are the sole source of information.

Notice that the definitions of N_j and M_i indicate

(14) $$G[I_n \otimes A_j] = G[I_n \otimes A_j]H \qquad \text{for all } j$$

and

(15) $$G[I_n \otimes B_i] = G[I_n \otimes B_i]H \qquad \text{for all } i$$

if H denotes the product G^-G. This product is a symmetric idempotent matrix and has an important role in the estimation procedure which is indicated below. Equations (14) and (15) permit the macro-equation (12) to be arranged in a convenient form after it has been subjected to two more adjustments.

Estimation and the Macro-Equations

Let θ represent the lag operator such that $\theta^s z_t$ is equal to z_{t-s} for an arbitrary vector of observations z_t. Then $L(\theta)$ can be used to denote a matrix function of the lag operator such that the equation for the macro-error (13) can be written as

$$L(\theta) \cdot (Ge) = Gu$$

where
$$L(\theta) \equiv I - \sum_{i=1}^{4} M_i \theta^i .$$

Application of this autoregressive transformation to (12) yields

$$(16) \quad Gy = \sum_{i=1}^{4} M_i (Gy)_{-i} + \sum_{j=1}^{k} N_j (Gx_{0j}) - \sum_{i=1}^{4} \sum_{j=1}^{k} M_i N_j (Gx_{0j})_{-i} + (Gu).$$

Let this equation be pre-multiplied by the Moore-Penrose inverse G^- to obtain an alternative equation which is more convenient to use.

$$(17) \quad Hy = \sum_{i=1}^{4} G^- M_i (Gy)_{-i} + \sum_{j=1}^{k} G^- N_j (Gx_{0j})$$

$$- \sum_{i=1}^{4} \sum_{j=1}^{k} G^- M_i N_j (Gx_{0j})_{-i} + Hu.$$

But
$$G^- M_i (Gy)_{-i} = H[I_n \otimes B_i](Hy)_{-i}$$

$$= [I_n \otimes B_i](Hy)_{-i} \qquad \text{using (15).}$$

$$G^- N_j (Gx_{0j}) = [I_n \otimes A_j](Hx_{0j}) \qquad \text{using (14).}$$

$$G^- M_i N_j (Gx_{0j})_{-i} = H[I_n \otimes B_i] H [I_n \otimes A_j](Hx_{0j})_{-i}$$

$$= [I_n \otimes B_i][I_n \otimes A_j](Hx_{0j})_{-i}$$

$$= [I_n \otimes B_i A_j](Hx_{0j})_{-i}$$

$$(18) \quad (Hy) = \sum_{i=1}^{4} [I_n \otimes B_i](Hy)_{-i} + \sum_{j=1}^{k} [I_n \otimes A_j](Hx_{0j})$$

$$- \sum_{i=1}^{4} \sum_{j=1}^{k} [I_n \otimes B_i A_j](Hx_{0j})_{-i} + (Hu)$$

Further notation is necessary if this equation is to be arranged in a more convenient form. Let a bar associated with an arbitrary column vector z, of order n, represent the operation which takes the elements of the vector and forms a matrix, of order n by 4,

$$\bar{z} \equiv \begin{bmatrix} dg(z_n, z_{n-1}, z_{n-2}, z_{n-3}) \\ \cdots \quad\quad \cdots \\ dg(z_8, z_7, z_6, z_5) \\ dg(z_4, z_3, z_2, z_1) \end{bmatrix} \quad \text{if} \quad z \equiv \begin{bmatrix} z_n \\ \cdots \\ z_2 \\ z_1 \end{bmatrix} .$$

Let

$$b_i \equiv \begin{bmatrix} b_{4i} \\ b_{3i} \\ b_{2i} \\ b_{1i} \end{bmatrix}, \quad a_j \equiv \begin{bmatrix} a_{4j} \\ a_{3j} \\ a_{2j} \\ a_{1j} \end{bmatrix} \quad \text{and} \quad b_i * a_j \equiv \begin{bmatrix} b_{4i}a_{4j} \\ b_{3i}a_{3j} \\ b_{2i}a_{2j} \\ b_{1i}a_{1j} \end{bmatrix} \equiv -c_{ij} ,$$

the Schur product of b_i and a_j, for all i and j.

$$(19) \qquad (Hy) = \sum_{i=1}^{4} \overline{(Hy)}_{-i}\, b_i + \sum_{j=1}^{k} \overline{(Hx_{0j})}\, a_j + \sum_{i=1}^{4} \sum_{j=1}^{k} \overline{(Hx_{0j})}_{-i}\, c_{ij} + (Hu)$$

If the initial explanatory variables (x_{0j}) are non-stochastic and if all redundant variables are omitted from equation (19), then the principle of least-squares can be applied to this equation, even though the final errors (Hu) are not spherical, to derive consistent estimators of b_i, a_j and c_{ij} under fairly general conditions. These estimators will, of course, not take account of the nonlinear constraints on the parametric vectors. An approximation to the principle of generalized least-squares would involve the calculation of a matrix N, of full row rank, such that NHN' is a scalar matrix. Equation (19) would be premultiplied by N and the least-squares technique applied to the new equation. The resultant estimators of the parametric vectors would be identical with those obtained without the use of the further transformation N. This equivalence follows from the equality of H'H and H'N'NH if both N and G have full row rank and NHN' is the identity matrix of order (n-q).

Summary

If errors on an aggregative equation are generated by a mixed moving-average autoregressive process and the weights of the moving-average component of this process are known, then the least-squares procedure can yield consistent estimators of both signal and autoregressive parameters if two adjustments are made to the equation. The autoregressive transformation is combined with pre-multiplication by a Moore-Penrose inverse based on the known weights of the moving-average component.

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