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Endogenous Coalition Formation in Rivalry

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Very Preliminary, Comments Welcome

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Abstract

This paper studies endogenous coalition formation in an environment where continuing conflict exists. A number of players compete for an indivisible prize and the probability of winning for a player depends on his initial resource as well as the distribution of initial resources among the other players. Players can pool their resources together to increase their probabilities of winning through coalition formation. If a coalition wins, the players in the coalition will further compete and possibly form new coalitions. The game continues until one individual winner is left. We determine subgame perfect equilibria for the game of three or four players and provide conditions under which the equilibrium coalition structures involve a balance of power. We also illustrate that there can be no equilibrium coalition structure. Our analysis sheds some lights on problems of temporary cooperation among heterogeneous individuals who are rivals in nature.

I Introduction

An alliance may be considered as a typical device from which countries attain the balance of power. Viewing back to the Warring States period of ancient China (451-221 B.C.), where the nation was dissected into seven parts, forming successful alliance was crucial in safeguarding a party's survival as well as advancing its claim to the future sovereignty. More recent examples include the Triple Alliance in Europe (1880-1914), the North Atlantic Treaty Organization, and the former Soviet Bloc. Timeless as the concept of alliance seems to be, what determines the formation of these alliances and the stability of an alliance system remain to be an unexplored area. The topic is further complicated by taking into account the dynamics of continuing conflicts between the allies. In this paper, we study endogenous alliance (or coalition) formation in an environment where both external effects of alliance formation and continuing conflict within an alliance are present. Stable alliance structures are characterized and shown to exhibit a balance of power.

In the real world, alliances are inevitably made transitory as parties enter into the spiral of continuing conflicts, where synergy comes and goes in an intractable manner. Once an alliance achieves certain objectives, it may face dissolution as its partners engage in new conflict, which then gives rise to new alliances. In order to analyze the externalities from alliance formation and continuing conflict, we provide a simple stylized model: a number of players fight for an indivisible prize and the probability of winning for a player depends on his initial resource and the distribution of initial resources among other players. By forming coalitions, some players increase their probabilities of winning, often at the expense of others' probability of winning. Besides, when players form coalitions, they simply pool their resources together. This indirectly alters the number of effective players in the contest as well as the distribution of initial resources. Moreover, the contest is of a sequential nature. If a coalition loses, the payoffs for all the players within the coalition are zero. If a coalition wins, the allies within the coalition will further form sub-coalitions and fight among themselves. The contest continues until one individual winner is left.

We model coalition formation as a game with multiple stages. In each stage, there is a bargaining process in which the remaining players form coalitions and once coalitions

are formed, there is a fight among the coalitions. The members of the winning coalition will further form coalitions and fight among themselves in the next stage. The bargaining process in each stage is modelled as follows. Partitions of the set of the remaining players are proposed by nature or an outsider in a sequential and exhaustive order. Given a partition, all players vote simultaneously. If all players in a particular subset implied by the partition voted 'yes', the subset becomes a coalition. If a coalition is formed, the current sequence is stopped and a new sequence of partitions is proposed. A subset implied by a partition in the new exhaustive sequence must contain either none of the players from an existing coalition, or all of the players from that coalition. All the players vote again. This process repeats until no new coalition is formed in the most current sequence.

A stable coalition structure is defined as an outcome of a subgame-perfect equilibrium that is independent of the order of partitions proposed. We have characterized the equilibrium coalition structures and obtained the following results. First, in a game consisting of three players, there always exists an equilibrium coalition structure and, under reasonable restrictions on the probability function of winning, the weakest two players form a coalition against the strongest. Here, the strongest is the player who processes the largest amount of initial resource. Therefore, if the strongest player is an aggressor, then the equilibrium implies that weaker players form a counter coalition in an attempt to balance the power of the aggressor.

Second, in a game consisting of four players, however, there may not be any equilibrium coalition structure. To illustrate this fact, we have constructed such an example. If the probability function of winning has reasonable properties, the equilibrium coalition structure exists and takes one of the following two forms: (i) the coalition of three weakest players against the strongest or (ii) the coalition of the weakest and strongest players against the coalition of the other two players. The latter coalition structure results when the players' initial resources or strengths are relatively close, while the former results when their strengths exhibit a wide disparity. Again, the equilibrium coalition structure exhibits a balance of power. Characterizing the equilibrium coalition structures for a game of more than four players is a challenging task. However, since the games of three and four players are subgames of games of more number of players, our results serve to narrow down the set of equilibrium

structures tremendously, and make a general characterization possible.

Contemporary economics literature on alliance formation (in the context of international politics) has mainly focused on the collective action problem within an alliance. In their pioneering study of the budget-sharing problem within an international organization, Olson and Zeckhauser (1966) provide a model in which there is one alliance and each member in the alliance makes a contribution to the collective defense. In other words, collective defense is treated as a collective good and external threats to the alliance are assumed to be constant or not exist.[1] Their analyses have not taken into account two important factors that might affect alliance formation: externalities generated from alliance formation and continuing conflict among the allies within the alliance. Niou and Tan (1995, 1996) analyze the effect of external threats on the formation of alliance, but do not consider the possibility of continuing conflict.

There is a large literature on conflict among several independent states or individuals (see in particular Hirshleifer 1988, 1991, 1995, Skaperdas 1992, 1996, Grossman and Kim 1994, Neary 1996, 1996, and references therein). The main issue analyzed there is how a contestant chooses between productive use (or consumption) of his current resources and fighting to defend his resources or acquire resources from others. In making such effort-allocating decision, the possibility of coalition formation was not considered. Most papers along this line of research, with the exception of Hirshleifer (1988, 1995) and Skaperdas (1996), allow the players to make a one-time investment decision and thus conflict is resolved within one period. Hirshleifer (1988, 1995) addresses the issue of continuing struggle for resources, but does not allow players to form coalitions. In our paper, we analyze the problem of coalition formation in combination with the presence of continuing conflicts. For simplicity, here we do not consider investment decisions made by players. Skaperdas (1996) does not consider the investment decisions either, but takes into account the "strategic endowment" of each player. Focusing on the case of three players, Skaperdas derives conditions under which stable alliances would be formed and provides guidelines on which alliance is most likely to form. We provide a formal model of non-cooperative bargaining on coalition formation and

[1] Further development along this line of research can be found in a survey by Sandler (1993).

extend some of his results to the case of four players. In principle, our model can be used to analyze the case of more players. As we illustrate in the paper, many more complications arise when there are more than three players in the contest.

Our analysis is closely related to several recent papers on endogenous coalition formation. Bloch (1996) studies sequential formation of coalitions in a model where coalitional worths depend on the whole coalition structure and are distributed among the coalition partners according to a fixed sharing rule. Ray and Vohra (1996) provide a general theory of endogenous coalition structure where coalitional worths are endogenously distributed. In both papers the authors explicitly model the formation of coalitions as a non-cooperative sequential bargaining process, and look for stationary subgame perfect equilibria in the spirit of Rubinstein (1982)'s bargaining game. According to Bloch, any core stable coalition structure can be supported by a stationary perfect equilibrium of the game. A general method is proposed for the characterization of equilibria in a class of symmetric games. Ray and Vohra show that there always exists a stationary subgame perfect equilibrium in their bargaining game and further develop an algorithm that generates equilibrium coalition structures for symmetric games.

The main difference between our model and theirs is that, in our model players fight sequentially and thus the total payoff of a winning coalition in any stage of contest is distributed through subsequent stages of contests, and so forth. When players decide to form a coalition, they take into account the consequences of coalition formation in subsequent stages. In this sense, a coalition in our model is a temporary agreement, and that entails a game of multiple stages. There is a sequential bargaining game of coalition formation in each stage, while the number of stages is endogenously determined. The bargaining game in each stage is similar to the one provided by Bloch (1996) and Ray and Vohra (1996). In a related paper, Yi (1996) analyzes a model of stable coalition structures under positive or negative externalities and different rules of coalition formation.

The paper is organized as follows. In Sections II, the formal model is outlined in detail. In Section III, the players' underlying incentives for coalition formation are examined. In Sections IV and V, we characterize equilibrium coalition structures for cases of three and four players. Discussions on the model are provided in Section VI and concluding remarks

follow in Section VII.

II The Model

Consider a model in which there are n players fighting for an indivisible object. The value of the object is the same for all players and is normalized to be equal to 1. Each player is endowed with an absolute strength, or skill, initial resource, and so forth. Let A_i represent player i's strength, where $A_i > 0$ for $i = 1, \ldots, n$, and let $N = \{1, \ldots, n\}$ be the set of all players.

The rule of fighting is the winner-takes-it-all. There is only one final winner who gets the object and all the other players receive zero. The winning probability for a player depends on his strength through a production function $h(A)$, with the properties that $h(A)$ is continuously differentiable, $h(A) > 0$, and $h'(A) > 0$ for all $A > 0$. In particular, when all the players in N fight simultaneously, the probability of winning for player i is assumed to be equal to

$$P_i(N) = \frac{h(A_i)}{\sum_{j=1}^{N} h(A_j)}.$$

This functional form of winning probabilities is commonly employed in the literature on conflict and rent-seeking and is axiomatized by Skaperdas (1996).[2]

The game consists of many stages. In each stage, the only strategy for a surviving player is to form a possible coalition with other players. Once coalitions are formed, they fight against each other. In the initial stage (stage $t = 1$), each subset of N represents a possible coalition. Conversely, any coalition must be represented by a subset of N. A coalition structure of the players can be defined as a partition of N, which is denoted by π. The players in each subset implied by the partition form a coalition. A player who does not

[2]Skaperdas (1996) shows that the crucial property of the functional form is the independence from irrelevant alternatives. That is, if player i only participates in a contest among a subset of players, then his probability of winning is independent of the strengths of the players not included in the subset. This property is useful for our analysis of continuing conflict.

belong to any coalitions is defined as an individual coalition. Given a coalition structure, $\pi = [C_1, C_2, \ldots, C_m]$, where $1 \leq m \leq n$, the probability that coalition C_s wins is assumed to be equal to

$$P_{C_s}(\pi) = \frac{h(\sum_{j \in C_s} A_j)}{\sum_{t=1}^{m} h(\sum_{j \in C_t} A_j)},$$

where $s = 1, ..., m$. There are two effects of forming a coalition. One is that the allied players aggregate their strengths and become stronger. Synergy among the allied players may exist, depending on the property of $h(A)$ function. Another effect is that the number of actively fighting players is essentially reduced. Therefore, coalition formation allows the players to change the fighting game.

The members of any winning coalition will further form coalitions and fight among themselves in the second stage. Coalitions and probabilities of winning are defined analogously for the new set of the players in the winning coalition. The game will continue until there is one final individual winner. Therefore, the number of stages in the game is endogenously determined.

We assume that there is no discounting in the game and that any transfer payment between players is not feasible. The payoff for each player is equal to his probability of being the final individual winner. Each player selects a particular coalition to join in order to maximize his payoff. If we divide player i's payoff by $h(A_i)$, it will not affect the player's optimization problem. We call this divided payoff as player i's *modified payoff*, which are sometimes used in the following analyses.

We now describe the coalition formation process in detail. Let N_t be the set of remaining players in stage t. Notice that $N_1 = N$. In stage t, partitions of the player set N_t are proposed by *nature* in a sequential and exhaustive order. Given a partition, all players vote simultaneously. If and only if all players in a particular subset implied by the partition voted 'yes', the subset becomes a new coalition. If a new coalition is formed, the current sequence is stopped and a new sequence of partitions is proposed. A subset implied by a partition in the new exhaustive sequence must contain either none of the players from an existing coalition, or all of the players from that coalition. All the players vote again. This process

repeats until no new coalition is formed in the most current sequence.

It is important to note that the members of a subset have only the power to decide whether or not their *own* subset will become a coalition. Once a coalition is formed, it is not allowed to break up. Of course, a larger coalition may be formed later, as long as all members in that larger subset of players vote 'yes'. If some of them, who could be from the original coalition or the new members, voted 'no', the larger coalition is not formed. However, the original coalition remains a coalition.

Notice that, in the coalition formation process specified above, once a coalition is formed it cannot be broken up and a new coalition should include all or none of them. We make this restriction primarily because we want the coalition process to converge. Even so, our process is still flexible enough to allow the formation of any particular coalition, as long as all members in this coalition wait for that chance. The delay involved is not significant, since there is no discounting in this game. Of course, a coalition can accept new members, as long as it improves the payoffs of everyone involved and does not break up any coalition already formed.

We shall be looking for subgame-perfect equilibria of the extensive-form game that are independent of the order of partitions proposed. In each stage, the resulting coalition structures in these equilibria should be the same no matter which order that the partitions are proposed. Two equilibria are regarded as equivalent in our model if the resulting coalition structures are the same, even though the formation process may be different. These coalition structures are called *equilibrium coalition structures* for each stage. A collection of the equilibrium coalition structures from different stages is the equilibrium coalition structure for the whole game.

The coalition structure in a subgame-perfect equilibrium in the game may not be independent of the order of partitions proposed. For example, suppose that i wants to form a coalition with j, j wants to form a coalition with k, but k wants to form a coalition with i, and that all players prefer a two-player coalition to an individual coalition. If the relevant partitions are proposed in the following order: $[\{i,j\},\{k\}] \longrightarrow [\{i,k\},\{j\}] \longrightarrow [\{k,j\},\{i\}]$, then $[\{i,j\},\{k\}]$ would be selected in the subgame-perfect equilibrium. This claim can be

proved by using backward induction. If no coalition has been formed when the last above-mentioned partition is proposed, then k and j would form a coalition in the last partition. Given this, in the second partition, i would form a coalition with k, since this is preferred by both players to the last partition. Still given this, i would form a coalition with j when the first partition is proposed, which is the equilibrium outcome. If the relevant partitions are proposed in the following order: $[\{i,k\},\{j\}] \longrightarrow [\{i,j\},\{k\}] \longrightarrow [\{k,j\},\{i\}]$, however, then $[\{i,k\},\{j\}]$ would be selected in the subgame-perfect equilibrium. In this case, k and j would form a coalition when the last partition is proposed. However, i and j would not form a coalition when the second partition is proposed. Given this, i and k would form a coalition when the first partition is proposed, which is the equilibrium outcome. Therefore, in this game the equilibrium outcome depends on the order of partitions proposed.

In what follows, we shall investigate the existence of an equilibrium outcome which is independent of the order of partitions proposed and the characterization of this outcome. The equilibrium coalition structures will depend on the number of players, the distribution of the players' strengths, and the property of $h(A)$ function.

III Synergy of Forming a Coalition

To investigate the players' incentives to form a coalition in the game, we make the following restriction on the function $h(A)$ throughout the paper.

(R1) For any $A > 0$ and $B > 0$, $h(A+B) > h(A) + h(B)$.

Restriction (R1) implies that there exists strictly positive synergy between two players or two coalitions. If two players form a coalition, then their joint winning probability will increase. This is the primary reason for why players form a coalition in our game. (R1) also implies that $h(0) = 0$. This can be seen by letting A go to zero in the above inequality. Examples of $h(A)$ that satisfy (R1) include $h(A) = A^{\alpha}$ for $\alpha > 1$, and $h(A) = e^{\gamma A} - 1$ for $\gamma > 0$. Furthermore, the following lemma shows that (R1) is satisfied if $h(\cdot)$ is convex:

Lemma 1 Suppose $h(0) = 0$, $h(A) > 0$ and $h'(A) > 0$, $\forall A > 0$. If $h(A)$ is also strictly

convex, then (R1) holds.

Proof Notice that $\forall A > 0, \forall B > 0$,

$$\begin{aligned} h(A+B) - h(A) &= \int_0^B h^{'}(x+A)dx \\ &> \int_0^B h^{'}(x)dx \\ &= h(B) - h(0), \end{aligned}$$

where the inequality holds because the convexity of $h(\cdot)$ implies that $h^{'}(\cdot)$ is an increasing function and thus $h'(x+A) > h'(x)$, $\forall A > 0$. The claim then follows from the assumption that $h(0) = 0$. □

Note that (R1) does not necessarily imply that $h(\cdot)$ is convex. For example, $h(A) = A^4 - 20A^3 + 148.5A^2$, which is concave between $A = 4.5$ and $A = 5.5$, and convex everywhere else. It is easy to check that $h(0) = 0$, $h(A) > 0$ and $h'(A) > 0$ for all $A > 0$. Notice that $h(A+B) - h(A) - h(B) = 2ABg(A,B)$, where $g(A,B) = 2A^2 + 3AB + 2B^2 - 30A - 30B + 148.5$. It can be easily verified that $g(A,B)$ reaches its minimum at $A = B = 30/7$, where it is positive. Therefore, $g(A,B)$ is positive everywhere and hence (R1) is satisfied.

The implication of (R1) is presented in the following lemma.

Lemma 2 Suppose that (R1) holds. Then two players fighting individually are dominated by forming a coalition.

Proof Consider a coalition structure $\pi = [C_1, C_2, ..., C_m]$ in which $C_1 = \{i\}$ and $C_2 = \{j\}$. Player i's winning probability is given by

$$P_i = \frac{h(A_i)}{h(A_i) + h(A_j) + \Delta},$$

where $\Delta = \sum_{s=3}^{m} h(\sum_{k \in C_s} A_k)$. If forming a coalition with j, then i's final winning probability is

$$\begin{aligned} \tilde{P}_i &= \frac{h(A_i + A_j)}{h(A_i + A_j) + \Delta} \cdot \frac{h(A_i)}{h(A_i) + h(A_j)} \\ &> \frac{h(A_i) + h(A_j)}{h(A_i) + h(A_j) + \Delta} \cdot \frac{h(A_i)}{h(A_i) + h(A_j)} \\ &= \frac{h(A_i)}{h(A_i) + h(A_j) + \Delta}. \end{aligned}$$

The above inequality holds since $h(A_i + A_j) > h(A_i) + h(A_j)$ and since $x/(x + \Delta)$ is an increasing function of x. Therefore, joining j increases i's probability of winning. Similarly, joining i increases j's probability of winning. □

It should be noted that (R1) does not necessarily imply that any two coalitions have incentives to merge. The reason is that if the merged coalition wins, the players in that coalition might form different new coalitions from the two original coalitions and hence might receive different payoffs.

IV The Case of Three Rivals

To characterize the equilibrium coalition structure, we first consider the case of three players.

From Lemma 2, there will be no two individuals standing alone. The nature of the game implies that the grand coalition is not feasible or given the grand coalition players will further form subcoalitions. Therefore, there are only three possible coalition structures to consider: $[\{1,2\},\{3\}]$, $[\{2,3\},\{1\}]$, and $[\{3,1\},\{2\}]$. The winning probabilities for each player in these coalition structures are presented in Table 1. These probabilities are the players' payoffs in the game. Each player ranks the probabilities of winning across the three coalition structures.

Proposition 1 Suppose (R1) holds and $n = 3$. Then equilibrium coalition structures exist and are in the form of a coalition of two players against one.

	Player 1	Player 2	Player 3
$[\{1,2\},\{3\}]$	$\frac{h(A_1+A_2)}{h(A_1+A_2)+h(A_3)} \cdot \frac{h(A_1)}{h(A_1)+h(A_2)}$	$\frac{h(A_1+A_2)}{h(A_1+A_2)+h(A_3)} \cdot \frac{h(A_2)}{h(A_1)+h(A_2)}$	$\frac{h(A_3)}{h(A_1+A_2)+h(A_3)}$
$[\{2,3\},\{1\}]$	$\frac{h(A_1)}{h(A_1)+h(A_2+A_3)}$	$\frac{h(A_2+A_3)}{h(A_1)+h(A_2+A_3)} \cdot \frac{h(A_2)}{h(A_2)+h(A_3)}$	$\frac{h(A_2+A_3)}{h(A_1)+h(A_2+A_3)} \cdot \frac{h(A_3)}{h(A_2)+h(A_3)}$
$[\{3,1\},\{2\}]$	$\frac{h(A_1+A_3)}{h(A_1+A_3)+h(A_2)} \cdot \frac{h(A_1)}{h(A_1)+h(A_3)}$	$\frac{h(A_2)}{h(A_1+A_3)+h(A_2)}$	$\frac{h(A_1+A_3)}{h(A_1+A_3)+h(A_2)} \cdot \frac{h(A_3)}{h(A_1)+h(A_3)}$

Table 1: Players' Probability of Winning

Proof: Each player prefers a coalition structure that yields him the highest payoff, or equivalently the highest modified payoff. We need to select the highest number among the modified payoffs across coalition structures for each player. Given (R1), this number cannot be the modified payoff when the player stands alone. Furthermore, the modified payoffs for the two members within a coalition are the same. Since there exists at least one highest number of the modified payoffs among the three coalitions, it follows that equilibrium coalition structures always exist and that the two players who obtain the highest modified payoffs form a coalition when the corresponding partition is proposed. □

It should be noted that the coalition structure is not unique in some degenerate cases. Consider the case where $A_1 = A_2 = A_3$. Any two players against the remaining one is an equilibrium coalition structure. If all three players have different strengths, then the equilibrium coalition structure is always unique.

Next, we study how the equilibrium coalition structure depends on the distribution of the players' strengths and the property of $h(A)$ function. In particular, we are interested in when the two weaker players form a coalition against the strongest player. We make the following restrictions on the probability function of winning

(R2) (a) $h(A) = A^{\alpha}$, where $\alpha > 1$; or (b) $h(A) = e^{\gamma A} - 1$, where $\gamma > 0$.

Clearly, (R2) implies (R1). Notice that the power function has the property that winning

probabilities depend only on the ratios of players' strengths and the parameter α measures the decisiveness of strengths. This functional form is widely used in the literature of conflict and rent-seeking, and see Hirshleifer (1995), Skaperdas (1992), and references therein. This literature also uses an exponential functional form which has property that winning probabilities depend only on the differences in strengths, that is, $h(A) = e^{\gamma A}$. However, the exponential function does not satisfy our synergy property. We have therefore made a minor modification by requiring $h(0) = 0$. The following proposition is a direct implication of (R2).

Proposition 2 Suppose (R2) holds and let $A_1 > \max\{A_2, A_3\}$. Then $[\{2,3\},\{1\}]$ is the unique equilibrium coalition structure.

Proof: Given Proposition 1, we need to show that by forming a coalition players 2 and 3 receive the highest modified payoffs, i.e.,

$$\frac{h(A_k + A_j)}{h(A_1) + h(A_k + A_j)} \cdot \frac{1}{h(A_k) + h(A_j)} > \frac{h(A_1 + A_j)}{h(A_1 + A_j) + h(A_k)} \cdot \frac{1}{h(A_1) + h(A_j)}, \tag{1}$$

for $j, k = 2, 3$ and $k \neq j$. We first consider the case in which $h(A) = A^\alpha$, where $\alpha > 1$. Using cross multiplications and the property that $h(AB) = h(A)h(B)$, we can rewrite inequality (1) as

$$\sum_{s=1}^{3} h(x_s) > \sum_{s=1}^{3} h(y_s) \tag{2}$$

where $x_1 = A_1(A_1 + A_j)(A_k + A_j)$, $x_2 = A_1A_k(A_k + A_j)$, $x_3 = A_jA_k(A_k + A_j)$, $y_1 = A_1A_k(A_1 + A_j)$, $y_2 = A_1A_j(A_1 + A_j)$, $y_3 = A_k(A_k + A_j)(A_1 + A_j)$. Notice that

$$\sum_{s=1}^{3} x_s = \sum_{s=1}^{3} y_s = (A_1 + A_j)(A_1 + A_k)(A_k + A_j),$$

and $A_1 > \max\{A_k, A_j\}$ implies that $x_3 < x_2 < x_1$ and $x_3 < y_s < x_1$ for $s = 1, 2, 3$. Consider two uniformly distributed random variables $\tilde{x}$ and $\tilde{y}$ with supports $\{x_3, x_2, x_1\}$ and

$\{y_3, y_2, y_1\}$, respectively. Clearly, $\tilde{x}$ is a mean-preserving spread of $\tilde{y}$. Inequality (2) then follows from the convexity of $h(A)$.

Next, consider the case in which $h(A) = e^{\gamma A} - 1$, where $\gamma > 0$. Using cross multiplications and simple manipulations, we can rewrite inequality (1) as

$$h(A_1)h(A_k)h(A_j)[h(A_j) + 1][h(A_1) - h(A_k)] > 0$$

which holds since $A_1 > A_k$. The claim follows. □

This proposition implies that, given certain restrictions on $h(A)$, the two weaker players will initially form a coalition in an attempt to balance the power of the strongest player. And if they win, they then fight between themselves for the final victory. This result is essentially the same as the one obtained by Skaperdas (1996) (see Proposition 2 and Corollary in his paper). The main difference is that we have provided a non-cooperative justification for the stability concept that he used. This justification proves to be useful when we extend our analysis to the game of more than three players.

It is interesting to note that in equilibrium the strongest player may not have the highest probability of winning. This is because the weak players can increase their probabilities of winning by forming a coalition. For example, let $h(A) = A^2$, $A_1 = 0.6$, $A_2 = 0.5$, and $A_3 = 0.2$. In this case, $[\{1\}, \{2, 3\}]$ is the equilibrium coalition structure. It can be computed that the final probabilities of winning for players 1, 2 and 3 are 0.42, 0.50, and 0.08, respectively. It is player 2 who has the highest probability of winning. This result raises an interesting issue of disarmment. In the example, player 1 has incentives to reduce his strength to a level below A_2 so that he will form a coalition with player 3, provided that such a reduction is credible. For instance, if A_1 is reduced to $A_1^{'} = 0.49$ and A_2 and A_3 are the same as before, then the new equilibrium coalition structure is $[\{1, 3\}, \{2\}]$ in which the final probabilities of winning for players 1, 2 and 3 are 0.56, 0.35, and 0.09, respectively. Clearly, player 1 is better off. Then, player 2 may want to do the same. In order to understand this type of contests, we need a more general model in which players first invest in their strengths or military capabilities and then form coalitions and fight. This is beyond the scope of the

present paper.

One might also conjecture that the strongest player always has a higher probability of winning than the weakest player. However, this conjecture is false when the players have similar strengths. Let $h(A) = A^2$ again, but now $A_1 = 1.2$, $A_2 = 1.1$, and $A_3 = 1.0$. The probabilities of winning for players 1, 2 and 3 are 0.25, 0.41, and 0.34, respectively. The examples clearly illustrate the gains of forming a coalition for the coalition partners.

Proposition 2 provides a set of sufficient conditions on $h(A)$ under which the game exhibits a balance of power through coalition formation. This result may not hold when (R2) is not satisfied. We consider two examples. In the first one, suppose $A_1 = 2A$, $A_2 = A_3 = A$. A balance of power implies that players 2 and 3 should form a coalition, in which each of them has a winning probability 1/4 and player 1 has a winning probability 1/2. Suppose players 1 and 2 form a coalition. Then the winning probability for player 1 is greater than 1/2, for player 3 is less than 1/4, and for player 2 is

$$\frac{h(3A)}{h(3A)+h(A)} \cdot \frac{h(A)}{h(2A)+h(A)}.$$

Suppose $h(A) = A^{1.01} + A^4$ and $A = 0.25$. Then the above probability is equal to 0.250459. Thus, in equilibrium players 1 and 2 or 1 and 3 form a coalition.

In the second example, let

$$h(A) = \begin{cases} \frac{1}{2}A^2, & \text{if } A \leq 1; \\ \frac{1}{100}A^{100} + \frac{49}{100}, & \text{if } A > 1. \end{cases}$$

It is easy to check that $h(\cdot)$ is everywhere increasing and differentiable, $h(0) = 0$, and $h(\cdot)$ is convex. Therefore, by Lemma 1, (R1) is satisfied. Let $A_1 = 0.6$, $A_2 = 0.5$, and $A_3 = 0.45$. Table 2 presents the payoffs for all three players in different coalition structures. Clearly, the unique equilibrium coalition structure is $[\{3,1\},\{2\}]$, which is different from the one predicted by Proposition 2. In this case, the weakest player joins the strongest.

	Player 1	Player 2	Player 3
$[\{1,2\},\{3\}]$	.5897	.4095	.0007
$[\{2,3\},\{1\}]$	.2851	.3949	.3199
$[\{3,1\},\{2\}]$	**.5985**	.0648	**.3367**

Table 2: Players' Probability of Winning

It should also be noted that given the $h(\cdot)$ function in this example, the two strongest players may form a coalition. Take the following values: $A_1 = 0.6$, $A_2 = 0.5$, and $A_3 = 0.43$. It can be computed that the equilibrium requires players 1 and 2 to form a coalition. This implies that coalition formation is not necessarily to balance power, but to increase the probability of winning for the allied partners. In this example, it depends on who makes player 1 happier. If player i makes 1 happier then 1 will join i and i clearly gains. The biggest player can pick whoever he wants in this example.

V The Case of Four Rivals

Suppose now that there are four players in the game. There will be different types of coalition structures in the initial stage of coalition formation. These are $[\{i,j,k\},\{l\}]$ (4 possibilities), $[\{i,j\},\{k,l\}]$ (3 possibilities), $[\{i,j\},\{k\},\{l\}]$ (6 possibilities), and $[\{i\},\{j\},\{k\},\{l\}]$ (1 possibility).

Lemma 2 implies that $[\{i,j\},\{k\},\{l\}]$ dominates $[\{i\},\{j\},\{k\},\{l\}]$ for players i and j, and $[\{i,j\},\{k,l\}]$ dominates $[\{i,j\},\{k\},\{l\}]$ for players k and l. Therefore, the only possible equilibrium coalition structures in the initial stage are of the following two types: $[\{i,j,k\},\{l\}]$ and $[\{i,j\},\{k,l\}]$. Furthermore, when a coalition of two players wins, the two will fight again between themselves. When a coalition of three wins, the three players will further form coalitions and fight among the newly formed coalitions. If $h(A)$ is a power function or an exponential function as in Proposition 2, then the weakest two among the

three will join a coalition. The following lemma is useful for our characterization:

Lemma 3 Suppose (R2) holds and let $A_i > A_j > A_k, A_l$. Then player i prefers $[\{i,j\},\{k,l\}]$ to $[\{i,j,k\},\{l\}]$.

Proof See Appendix. □

Given Proposition 2 and Lemma 3, we are now ready to characterize the equilibrium coalition structures for the case of four players.

Proposition 3 Suppose (R2) holds and $n = 4$, and let $A_1 > A_2 > A_3 > A_4$.

(i) If $A_1 > A_3 + A_4$, then the unique equilibrium coalition structure is $[\{2,3,4\},\{1\}]$ in the initial stage, followed by $[\{2\},\{3,4\}]$ if $\{2,3,4\}$ wins.

(ii) If $A_1 < A_3 + A_4$, then the equilibrium coalition structure is unique and is in the form of $[\{i,j\},\{k,l\}]$.

Proof The proof of part (i) consists of the following three steps. The first step is to show that $[\{2,3,4\},\{1\}]$ is the most favorable coalition structure for players 3 and 4. There are several cases to consider.

Case a): $[\{1,2\},\{3,4\}]$ is dominated by $[\{2,3,4\},\{1\}]$. Consider players 3 and 4 as a group. Since the group has strength $A_3 + A_4$ which is less than A_1, it follows from Proposition 2 that $[\{2,3,4\},\{1\}]$ provides the group with a greater probability of winning than $[\{1,2\},\{3,4\}]$ does. This translates into a greater probability of final victory for both players. The claim follows.

Case b): $[\{2,3\},\{1,4\}]$ is dominated by $[\{2,3,4\},\{1\}]$. By Proposition 2, player 3 prefers $[\{2,3,4\},\{1\}]$ to $[\{2,3,4\},\{1\}]$ followed by $[\{2,3\},\{4\}]$ if $\{2,3,4\}$ wins. On the other hand, notice that

$$\frac{h(A)}{h(A)+h(A_1+A_4)} < \frac{h(A)}{h(A)+h(A_1)+h(A_4)} < \frac{h(A+A_4)}{h(A+A_4)+h(A_1)} \cdot \frac{h(A)}{h(A)+h(A_4)},$$

where $A = A_2 + A_3$, the first inequality holds due to (R1), and the second follows from Lemma 2. That is, player 3 prefers $[\{2,3,4\},\{1\}]$ followed by $[\{2,3\},\{4\}]$ to $[\{2,3\},\{1,4\}]$. Thus, $[\{2,3\},\{1,4\}]$ is dominated by $[\{2,3,4\},\{1\}]$ for player 3. Similarly, for player 4, $[\{2,3\},\{4,1\}]$ is dominated by $[\{2\},\{3,4,1\}]$ followed by $[\{3\},\{4,1\}]$, which is in turn dominated by $[\{2\},\{3,4,1\}]$ followed by $[\{3,4\},\{1\}]$. The latter is again dominated by $[\{2,3,4\},\{1\}]$ followed by $[\{2\},\{3,4\}]$ for player 4. The claim follows.

Case c): $[\{2,4\},\{1,3\}]$ is dominated by $[\{2,3,4\},\{1\}]$. The argument is the same as the one in case b).

Case d): As for $[\{1,2,4\},\{3\}]$ and $[\{1,2,3\},\{4\}]$, the joint separation of $\{2,4\}$ and $\{2,3\}$, respectively, to join the individual player increases the deviators' (as a group) and therefore their individual winning probability. Further increase in winning probability for 3 and 4 occurs as 3 and 4 will form a coalition when $\{2,3,4\}$ wins.

The second step is to show that players 3 and 4 can achieve the maximum probability of winning from $[\{2,3,4\},\{1\}]$ in the following way. They can first join a coalition when the partition implying subset $\{3,4\}$ is proposed. Given this coalition, it follows from Proposition 2 that 2 will join it when the partition implying subset $\{2,3,4\}$ is proposed since $A_1 > A_3 + A_4$. Players 3 and 4 welcome 2, because $A_1 > A_2$, and if the coalition wins, 3 and 4 will form a coalition against 2. Given this, both players 3 and 4 prefer forming the coalition $\{2,3,4\}$. Therefore, $[\{2,3,4\},\{1\}]$ is an equilibrium coalition structure.

The final step is to show the uniqueness. Note that $\{1,2\}$ is the only coalition players can form without the participation of 3 and 4. Suppose that $\{1,2\}$ is first formed. Then, by Lemma 3, player 1 refuses to let either 3 or 4 to join his coalition $\{1,2\}$. Consequently, 3 and 4 will form a coalition $\{3,4\}$. The resulting coalition structure $[\{1,2\},\{3,4\}]$ is not as good as $[\{2,3,4\},\{1\}]$ for player 2. Therefore, 2 will not form a coalition with 1. This means that there is no other equilibrium coalition.

We now prove part (ii). The condition $A_1 < A_3 + A_4$ assures that the combined strength

of any two players must be larger than the strength of any individual player. It follows from Proposition 2 that a coalition of three players against the fourth can never be an equilibrium coalition structure, since the largest among the three allied players would deviate and join the fourth. Hence, the remaining possible equilibrium structures are in the form of 2-against-2.

Look for the highest number in the modified payoffs across all 2-against-2 coalition structures among all players. This payoff must be shared by two players, say i and j. We now show that, after i and j join a coalition, either k does not join $\{i,j\}$, or one of i and j does not welcome k to join their coalition.

Suppose that k join $\{i,j\}$. There are two cases to consider. In the first case, i and j form a coalition against k when $\{i,j,k\}$ wins. Since $A_i + A_j > A_l$ by assumption, Proposition 2 implies that k should join l instead of joining $\{i,j\}$. In the second case, $\{i,k\}$ forms a coalition against j when $\{i,j,k\}$ wins. Again, since $A_i + A_k > A_l$ by assumption, Proposition 2 implies that j prefers coalition structure $[\{i,k\},\{j,l\}]$ to $[\{i,k,j\},\{l\}]$. Since $[\{i,j\},\{k,l\}]$ gives i and j the highest modified payoffs among all 2-and-2 coalitions, j must prefer $[\{i,j\},\{k,l\}]$ to $[\{i,k\},\{j,l\}]$. It follows that j prefers $[\{i,j\},\{k,l\}]$ to $[\{i,k,j\},\{l\}]$. Hence, in this case, j does not welcome k to join his coalition with i.

Finally, $[\{i,j\},\{k,l\}]$ which gives the highest modified payoff across all 2-against-2 coalition structures is the unique equilibrium coalition. If, for instance, $[\{i,k\},\{j,l\}]$ generates the highest modified payoff, say, for player k, then player i gets the same highest number. This violates the fact that player i's highest number is obtained in $[\{i,j\},\{k,l\}]$. Therefore, $[\{i,j\},\{k,l\}]$ must be the equilibrium coalition structure. □

There is no doubt that $[\{2,3,4\},\{1\}]$ gives players 3 and 4 the highest probability of winning, since they collude with as many players as possible in the most favorable way. This coalition structure can be achieved only if $A_3 + A_4 < A_1$, since players 3 and 4 can form an early coalition and induce player 2 to join. If $A_3 + A_4 > A_1$, however, player 2 refuses to join the coalition of 3 and 4. Anticipating this, players 3 and 4 may not want to form a coalition. The following proposition shows that, for some special $h(A)$ function, the unique equilibrium coalition structure is actually $[\{2,3\},\{1,4\}]$.

Proposition 4 Suppose $h(A) = A^2$, $A_1 > A_2 > A_3 > A_4$, and $A_1 < A_3 + A_4$. Then $[\{2,3\},\{1,4\}]$ is the unique equilibrium coalition structure.

Proof First, it follows from the proof of Proposition 3(ii), the 2-against-2 coalition structure which yields the highest modified payoff across all 2-against-2 coalition structures is the unique equilibrium structure. We want to show that $[\{1,4\},\{2,3\}]$ maximizes the modified payoffs for both players 2 and 3. Equivalently, we need to prove the following two inequalities for player 2 and two similar ones for player 3:

$$\frac{(A_2+A_3)^2}{(A_2+A_3)^2+(A_1+A_4)^2}\cdot\frac{1}{A_2^2+A_3^2} > \frac{(A_1+A_2)^2}{(A_1+A_2)^2+(A_3+A_4)^2}\cdot\frac{1}{A_1^2+A_2^2}, \tag{3}$$

and

$$\frac{(A_2+A_3)^2}{(A_2+A_3)^2+(A_1+A_4)^2}\cdot\frac{1}{A_2^2+A_3^2} > \frac{(A_2+A_4)^2}{(A_2+A_4)^2+(A_1+A_3)^2}\cdot\frac{1}{A_2^2+A_4^2}. \tag{4}$$

Inequality (3) holds because

$$\begin{aligned} & (A_2+A_3)^2[(A_1+A_2)^2+(A_3+A_4)^2][A_1^2+A_2^2] \\ & -[(A_2+A_3)^2+(A_1+A_4)^2][A_2^2+A_3^2](A_1+A_2)^2 \\ = \; & 2(A_1-A_3)(A_1A_3-A_2A_4) \\ & \cdot[(A_1+A_2)A_2(A_1+A_4)+A_2(A_2+A_3)^2+A_1(A_2+A_3)(A_2-A_4)] \\ > \; & 0. \end{aligned} \tag{5}$$

Inequality (4) can be obtained by switching subscripts 1 and 4 in (3), and it holds because (5) is also true when we switch subscripts 1 and 4 in it:

$$(A_2+A_3)^2[(A_4+A_2)^2+(A_3+A_1)^2][A_4^2+A_2^2]$$

$$
\begin{aligned}
& -[(A_2+A_3)^2+(A_4+A_1)^2][A_2^2+A_3^2](A_4+A_2)^2 \\
= \; & 2(A_4-A_3)(A_4A_3-A_2A_1) \\
& \cdot[(A_4+A_2)A_2(A_4+A_1)+A_2(A_2+A_3)^2+A_4(A_2+A_3)(A_2-A_1)] \\
= \; & 2(A_3-A_4)(A_1A_2-A_3A_4) \\
& \cdot[A_2A_4(2A_2+A_3+A_4)+A_2(A_2+A_3)^2+A_1(A_2^2-A_3A_4)] \\
> \; & 0. \qquad (6)
\end{aligned}
$$

For player 3, we can just switch 2 and 3 in the above two inequalities. Obviously, (5) continues to hold. Equation (6) holds if $A_3A_4(2A_3+A_2+A_4)+A_3(A_2+A_3)^2+A_1(A_3^2-A_2A_4)>0$, which is implied by assumption $A_1 < A_3+A_4$. Therefore, the proposed structure maximizes players 2 and 3's probabilities of winning and thus is the unique equilibrium coalition structure. □

The equilibrium structures in Propositions 3 and 4 imply a rule of balance. The equilibrium coalition in part (i) is the most evenly balanced one among all 3-and-1 coalition structures. Likewise, the equilibrium coalition in part (ii) in the case of $h(A)=A^2$ is also the most evenly balanced one among all 2-and-2 coalition structures. As a principle, the combined strength of each conflicting side is likely to be close. Of course, a word of caution is in order, since strategic considerations have also to be taken into account. For example, when $A_1=10$, $A_2=9$, $A_3=4$, and $A_4=3$, the equilibrium structure is $[\{1\},\{2,3,4\}]$, instead of the more evenly balanced $[\{1,4\},\{2,3\}]$. In this case, 3 and 4 form a coalition first, and then force 2 to join them.

It should be noted that $[\{2,3\},\{1,4\}]$ may not be the equilibrium coalition for other $h(\cdot)$ functions. For example, for $h(A)=e^A-1$, if A_2, A_3, and A_4 are close, then player 2 would prefer to form a coalition with player 1, not with player 3. By forming a coalition with player 1, player 2 will have a greater probability of winning against the coalition $\{3,4\}$, but a lesser probability of winning against player 1. Given the exponential functional form of $h(A)$, the gain for player 2 by forming a coalition with 1 can outweigh the loss. What will be the equilibrium coalition structure? In the case of $A_1=1$, $A_2=0.52$, $A_3=0.51$, and

$A_4 = 0.50$, $[\{1,2\},\{3,4\}]$ offers the highest modified payoffs for players 1 and 2. Therefore, it is the unique equilibrium coalition structure. Note that this equilibrium structure does not imply any balance of power.

We now discuss the existence issue for a general $h(A)$ function. In the case of three players, Proposition 1 shows that there always exists an equilibrium coalition structure. This may not be the case when there are four players. With four players in the game, there are more possibilities of deviations. We are able to construct such an example in which there is always a deviation.

Suppose $h(A) = A^{1.01} + A^4$, $A_1 = 0.7$, $A_2 = 0.5$, $A_3 = 0.26$, and $A_4 = 0.25$. With four players we need to consider three 2-against-2 coalition structures and four 3-against-1 coalition structures in the initial stage of coalition formation. For each 3-against-1 structure, there are three sub-coalition structures in the next stage. The modified payoffs for all the players under different structures including the sub-coalition structures are presented in Table 3. We first select the equilibrium coalition structures in all the subgames, and use star to indicate these structures in Table 3. Therefore, we only have seven coalition structures to be considered. Notice that, when a coalition with three players wins, a further coalition formation in the next stage may not involve two weakest against the strongest.

Consider $[\{2,3,4\},\{1\}]$ for instance. Player 3 would like to join 1 and 1 will accept 3. Both players prefer $[\{2,3,4\},\{1\}]$ to $[\{1,3\},\{2,4\}]$ and hence there is deviation. However, in $[\{1,3\},\{2,4\}]$, player 4 would like to join the other coalition $\{1,3\}$ and both 1 and 3 will accept 4. Similarly, given $[\{1,3,4\},\{2\}]$, player 1 will join 2 and 2 accept 1 as well. In $[\{1,2\},\{3,4\}]$, player 2 will join the coalition $\{3,4\}$. We are back to the original structure where we started. This implies that there is a cycle of deviation within the four coalition structures. It is easy to check that, if we start with one of the remaining three coalition structures then some players will deviate as well and finally enter the cycle. Therefore, there is no subgame perfect equilibrium in pure strategy that is independent of the order of partitions proposed.

Coalition Structures	Sub-Coalitions	Player 1	Player 2	Player 3	Player 4
[{1,2,3},{4}]	[{1,{2,3}},{4}]*	**0.4730**	**0.6296**	**0.6296**	**0.1597**
	[{2,{1,3}},{4}]	0.6117	0.4054	0.6117	0.1597
	[{3,{1,2}},{4}]	0.5940	0.5940	0.2714	0.1597
[{1,2,4},{3}]	[{1,{2,4}},{3}]*	**0.4782**	**0.6287**	**0.1629**	**0.6287**
	[{2,{1,4}},{3}]	0.6119	0.4121	0.1629	0.6119
	[{4,{1,2}},{3}]	0.5942	0.5942	0.1629	0.2715
[{1,3,4},{2}]	[{1,{3,4}},{2}]*	**0.5669**	**0.2554**	**0.6364**	**0.6364**
	[{3,{1,4}},{2}]	0.6284	0.2554	0.4232	0.6284
	[{4,{1,3}},{2}]	0.6281	0.2554	0.6281	0.4162
[{2,3,4},{1}]	[{2,{3,4}},{1}]	0.3346	0.6055	0.6797	0.6797
	[{3,{2,4}},{1}]*	**0.3346**	**0.6807**	**0.5177**	**0.6807**
	[{4,{2,3}},{1}]	0.3346	0.6805	0.6805	0.5113
[{1,2},{3,4}]		**0.5685**	**0.5685**	**0.2915**	**0.2915**
[{1,3},{2,4}]		**0.5252**	**0.4575**	**0.5252**	**0.4575**
[{1,4},{2,3}]		**0.5199**	**0.4660**	**0.4660**	**0.5199**

Table 3: Modified Payoffs ($A_1 = 0.7$, $A_2 = 0.5$, $A_3 = 0.26$, $A_4 = 0.25$)
(* indicates equilibrium in subgame)

VI Discussions

When there are more than four players in the game, characterizing the equilibrium coalition structures is more difficult. Nevertheless, the results in the previous sections can help us to narrow down the set of possible equilibrium candidates and search for the equilibrium coalition structures in general cases, since the cases of $n = 3$ and $n = 4$ are subgames of the game with players more than four.

We can extend Lemma 2 to the case of n players. Let $\pi = [C_1, C_2, C_3, ..., C_m]$ be a coalition structure in certain stage of coalition formation process. Suppose that the players from C_1 and C_2 form a coalition, denoted by $C_1 \bigcup C_2$ (the union coalition). If $[C_1, C_2]$ is an equilibrium coalition structure in the subsequent stage after $C_1 \bigcup C_2$ wins, then the proof of Lemma 2 implies that all the players in C_1 and C_2 have incentives to form such a coalition. This argument works for any subset of π and therefore, we have the following lemma.

Lemma 4 Suppose (R1) holds. Then any subset of an equilibrium coalition structure cannot be an equilibrium coalition structure of the union of the coalitions in the subset.

Proof: The proof is a minor modification of the proof of Lemma 2. □

When there are more players, the equilibrium coalition structure is much more complicated to analyze, since there are many more possible deviations. In what follows, we consider a special case. If we index the players' strengths from the strongest to the weakest by 1, 2, ..., n, then each player's strength is greater than the strength of all higher indexed players combined. In this case, the equilibrium coalition structure for any subset of the players is the strongest against the rest of the players.

This result is an extension to Propositions 2 and 3:

Proposition 5 Suppose (R2) holds and and $A_i > \sum_{k=i+1}^{n} A_k$ for $i = 1, \ldots, n-2$. Then the equilibrium coalition structure is $[\{1\}, \{2, 3, ..., n\}]$. If $\{t, t+1, \ldots, n\}$ wins, $t \geq 2$, then the equilibrium coalition structure for the next stage is $[\{t\}, \{t+1, \ldots, n\}]$. The game lasts

for at most $n-1$ stages.

Proof We use mathematical induction. By Propositions 2 and 3, the claim holds for $n = 3$ and $n = 4$. The proposed coalition structure gives players $n-1$ and n the highest modified payoffs.

Suppose that the claim holds $n = k$ and that the proposed coalition structure gives players $n-1$ and n the highest modified payoffs. We want to show that it also holds for $n = k+1$.

First, for $n = k+1$ the proposed coalition structure still gives players $n-1$ and n the highest payoffs. Indeed, by induction hypothesis, if $n-1$ and n belong to the same coalition in any other coalition structure, they will be together until they are the only two winners left and we can treat them as one player. If they belong to different coalitions, then they each earn higher payoff in the proposed coalition structure, since player 1 is the biggest player (relative to any coalition without player 1) in this game and two smaller coalitions joining together improve the winning probability of each coalition as a whole. Further increase in winning probability occurs for players $n-1$ and n as the united coalition (when it wins against player 1) restructures in favor of players $n-1$ and n.

Players $n-1$ and n can achieve the proposed equilibrium outcome by simply stick together and form a coalition when it is presented. Players $n-1$ and n certainly do not want to initiate any deviation from the proposed equilibrium structure (in which they get the highest payoffs). Suppose that a coalition that exclude players $n-1$ and n is formed first. Then the strongest player (the player with the lowest index) in that coalition will reject any player with an index higher than the highest index in the coalition (The proof is parallel to that of Proposition 2.). This means that players $n-1$ and n will not be accepted by any existing coalition. Hence, in the subgame after a deviation coalition is formed, both players $n-1$ and n will belong to some new coalition (by induction). This means that players $n-1$ and n will stick together until they are the final winners (again by induction). Given that $n-1$ and n will form a coalition no matter what happen, they can be treated as one individual player. By induction, the unique equilibrium coalition structure is the one proposed by the proposition.

From the above arguments, we conclude that the proposition is true for any n. □

The above equilibrium coalition structure exhibits the rule of balance of power. In each stage of the game, the weaker players form a coalition against the strongest. This is the most balanced structure since the strongest is stronger than the strength of the rest of the players combined. Note that in the above structure, there are always exactly two coalitions in each stage. We conjecture that this is always the case in general settings. That is, there are always two coalitions fighting against each other in any stage of the game given any number of players. Unfortunately, we are still unable to prove it.

VII Conclusions

This paper analyzes a model of endogenous coalition formation in a situation where players can pool their resources together to compete for a final prize. We look for coalition structures that do not depend on the order of proposals for such structures. Therefore, the resulting coalition structures are stable.

We show that, given the synergy assumption, two individual players always have the incentive to cooperate in fighting against other players. When there are three players, the equilibrium coalition must be in the form of two players against one. For many probability functions of winning, the weakest two players form a coalition and fight against the strongest. Similar results are obtained when there are four players. The weakest three players form a coalition and fight against the strongest, as long as the strongest is significantly stronger than the remaining three players. When the players' strengths are comparable, however, the equilibrium coalition structure is always in the form of two players against two players, where the weakest and strongest form one coalition. We have also illustrated how these results can help us search for the equilibrium coalition structures when the number of players is large. However, we have constructed an example in a game of four players in which the equilibrium coalition structure does not exist. This also implies the difficulties of characterizing the equilibrium outcomes for a game of more than four players.

The analysis is intended to shed some lights on the problem of temporary cooperation between players who are rivals in nature. Examples include war among different nations, political party members fighting for leadership, and firms competing for monopoly powers. The model provided in the paper can also be used to analyze long-term cooperation, as it gives a means for how the surplus generated from cooperation is shared when players are asymmetric.

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Appendix

Proof of Lemma 3: First, by the assumption and Proposition 2, j and k will form a coalition when $\{i,j,k\}$ wins in $[\{i,j,k\},\{l\}]$. Thus, we need to show that the following inequality holds:

$$\frac{h(A_i+A_j)}{h(A_i+A_j)+h(A_k+A_l)}\cdot\frac{1}{h(A_i)+h(A_j)}$$

$$>\frac{h(A_i+A_j+A_k)}{h(A_i+A_j+A_k)+h(A_l)}\cdot\frac{1}{h(A_i)+h(A_j+A_k)}. \tag{7}$$

Using cross multiplications, we can rewrite (7) as

$$h(A_i+A_j)[h(A_l)h(A_i)+h(A_l)h(A_j+A_k)+h(A_i+A_j+A_k)h(A_j+A_k)]$$

$$>h(A_i+A_j+A_k)[h(A_k+A_l)h(A_j)+h(A_k+A_l)h(A_i)+h(A_i+A_j)h(A_j)]. \tag{8}$$

When $h(A)=A^\alpha$, $h(AB)=h(A)h(B)$. It follows that (8) can be equivalently rewritten as

$$\sum_{s=1}^{3}h(x_s)>\sum_{s=1}^{3}h(y_s) \tag{9}$$

where $x_1=(A_i+A_j)A_lA_i$, $x_2=(A_i+A_j)A_l(A_j+A_k)$, $x_3=(A_i+A_j)(A_i+A_j+A_k)(A_j+A_k)$, $y_1=(A_i+A_j+A_k)(A_k+A_l)A_j$, $y_2=(A_i+A_j+A_k)(A_k+A_l)A_i$, $y_3=(A_i+A_j+A_k)(A_i+A_j)A_j$. Clearly,

$$\sum_{s=1}^{3}x_s=\sum_{s=1}^{3}y_s=(A_i+A_j)(A_i+A_k)(A_k+A_j),$$

and $y_1<y_2<y_3<x_3$. If $A_i\le A_j+A_k$, then $x_1\le x_2$ and $x_1<y_1$. If $A_i>A_j+A_k$, then $x_2<x_1$ and $x_2<y_1$. Thus, $\mathrm{Min}_s\{x_s\}<y_s<\mathrm{Max}_s\{x_s\}$. Consider two uniformly distributed

random variables $\tilde{x}$ and $\tilde{y}$ with supports $\{x_3, x_2, x_1\}$ and $\{y_3, y_2, y_1\}$, respectively. Clearly, $\tilde{x}$ is a mean-preserving spread of $\tilde{y}$. Inequality (9) then follows from the convexity of $h(A)$.

Next, consider the case $h(A) = e^{\gamma A} - 1$, $\gamma > 0$. Notice that, in this case,

$$h(A + B) = h(B) + h(A)[h(B) + 1].$$

We can rewrite (8) as

$$h(A_i + A_j + A_k)\Gamma + h(A_i + A_j)h(A_l)[h(A_i) + h(A_j + A_k)] > 0, \tag{10}$$

where

$$\begin{aligned}
\Gamma &= h(A_i + A_j)h(A_k)[h(A_j) + 1] - h(A_k + A_l)[h(A_i) + h(A_j)] \\
&= h(A_i)h(A_k)[h(A_j) - h(A_l)] + h(A_j)h(A_k)[h(A_i + A_j) - h(A_l)] \\
&\quad -h(A_l)[h(A_i) + h(A_j)] \\
&= h(A_k)[h(A_i) + h(A_j)][h(A_j) - h(A_l)] + h(A_j)h(A_k)h(A_i)[h(A_j) + 1] \\
&\quad -h(A_l)[h(A_i) + h(A_j)]
\end{aligned}$$

where the second equality follows from the facts that $h(A_i + A_j) = h(A_i) + h(A_j)[h(A_i) + 1]$ and that $h(A_k + A_l) = h(A_l) + h(A_k)[h(A_l) + 1]$. Thus, inequality (10) is equivalent to

$$\begin{gathered}
h(A_i + A_j + A_k)h(A_k)[h(A_i) + h(A_j)][h(A_j) - h(A_l)] \\
+h(A_i + A_j + A_k)h(A_j)h(A_k)h(A_i)[h(A_j) + 1] \\
-h(A_i + A_j + A_k)h(A_l)[h(A_i) + h(A_j)]
\end{gathered}$$

$$+h(A_i + A_j)h(A_l)[h(A_i) + h(A_j + A_k)] > 0. \tag{11}$$

Since $h(A_i + A_j + A_k) = h(A_i + A_j) + h(A_k)[h(A_i + A_j) + 1]$, the last two terms of the left hand side of (11) can be written as

$$-h(A_i + A_j)h(A_l)[h(A_i) + h(A_j)] - h(A_k)h(A_l)[h(A_i) + h(A_j)][h(A_i + A_j) + 1]$$
$$+h(A_i + A_j)h(A_l)[h(A_i) + h(A_j + A_k)]$$

$$\begin{aligned} &= h(A_i + A_j)h(A_l)h(A_k)[h(A_j) + 1] \\ &\quad -h(A_k)h(A_l)[h(A_i) + h(A_j)][h(A_i + A_j) + 1] \\ &= -h(A_k)h(A_l)h(A_i)^2[h(A_j) + 1], \end{aligned} \tag{12}$$

where the first equality follows from $h(A_j + A_k) = h(A_j) + h(A_k)[h(A_j) + 1]$ and the second equality follows from $h(A_i + A_j) = h(A_i) + h(A_j)[h(A_i) + 1]$. Therefore, inequality (11) is equivalent to

$$h(A_i + A_j + A_k)h(A_k)[h(A_i) + h(A_j)][h(A_j) - h(A_l)]$$
$$+h(A_i)h(A_k)[h(A_j) + 1][h(A_i + A_j + A_k)h(A_j) - h(A_i)h(A_l)] > 0.$$

which holds since $A_j > A_l$. The claim follows. □