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A GENERAL EQUILIBRIUM MODEL OF REGIONAL PUBLIC GOODS AND OPTIMIZING SUBSIDIES

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1. Introduction

There has in recent years been a considerable amount of writing on behavioural theories of government. This body of literature has been concerned with the specification of an objective function for representative governments in order to gain insights into public sector activity. The growing role of government in "private enterprise" economies increases the importance of developing a realistic behavioural theory. The work of Anthony Downs (1957) and of Buchanan and Tullock (1962) typifies the work that has been done at the behavioural level. However, a great deal of public sector analysis has been carried out within the traditional framework of the theory of the profit-maximizing firm or the theory of consumer behaviour. The most general type of objective function found in this work is a social welfare (or utility) function whose arguments consist of some set of private and public goods.

The present paper follows the neoclassical tradition of private goods general equilibrium theory and integrates public goods into a simple equilibrium model. In so doing no attempt is made to construct a behavioural theory of government to explain the articulation of individual demand for public goods. In what follows it is assumed that private and public goods are provided by a set of governments and private firms whose market behaviour replicates the long-run equilibrium behaviour of perfectly competitive industries.

The paper will use a general equilibrium model to examine the effects of a federal subsidy to a provincial government, which produces a single public good for a province whose population is endogenously determined.

The general equilibrium model to be used is the standard two-good, two-factor model that is almost ubiquitous in the international trade literature. Factor intensities will be assumed different in each of the two goods. Our concern will differ from that of the trade literature in that we are interested in one private and one public good, rather than with two private goods. In addition, the endowment of one of our two factors will be endogenous while that of the other factor will be fixed.

Some important theorems have been derived in the standard trade model and our interest here is in extending these to the public goods model. In particular, the aim of this paper is to find the effects of a subsidy to producers of public goods. The trade theorems to which we refer are typically derived from the production side of the standard general equilibrium model, at given product prices. However, R.W. Jones (1965) has generalized these theorems to a model in which demand is endogenous. The Rybczynski theorem (Rybczynski, 1955) states that at unchanged commodity prices an expansion in one factor results in an increase in the output of the commodity intensive in the use of that factor, and that the proportional change in that output will exceed the proportional change in the intensive factor. However, when demand is endogenous, and commodity prices adjust, then output may not change as much as factor endowments. The dual of the Rybczynski theorem for two private goods states that an increase in the price of a good will raise the factor return of the factor used intensively in producing that good by a greater relative amount. This leads to the issue of subsidies. The theorem indicates that, at fixed commodity prices a subsidy will be shifted backward to the intensive factor in a magnified way. This magnification effect may be dampened when demand is endogenously introduced.

The discussion of international trade theorems has been presented for a purpose. Let it be assumed that a federal government is making conditional grants to a provincial government and let us not for the present be concerned with the purpose for these grants or subsidies but note that the subsidy is "conditional" in the sense that it relates to a particular public good. It is very likely that the subsidy program will have some distributional effects. In our model however, to be presented below, the effects will go beyond the standard trade results to which reference has been made. Since provincial population is a variable the subsidy may change factor endowments. When provincial population changes, the optimizing subsidy may also be changed. The analysis which follows is designed to trace the general equilibrium incidence effects.

In Section II the basic model will be presented. Section III will integrate population mobility into the analysis. In Section IV the model will be closed by making demand endogenous. Section V will conclude the paper.

II. The Model

The model that will be used here is an adaptation of the general equilibrium model developed by R.W. Jones (1965). This model employs an activity analysis approach for an economy with two final goods and two primary factors of production. For our purposes it will be convenient to concentrate our attention upon a province which is a member of a federation of provinces and which is small enough for us to ignore any repercussion effects on other provinces. We assume that our province is so small that changes in its public goods provision do not appreciably affect public goods output, taxation or factor prices in the rest of the federation.

In the provincial economy that is of interest to us the two primary factors of production are land (fixed in amount T_0) and labour (N units). The supply of land is fixed while the supply of labour is variable, because of the possibility of induced migration. Both the population and the labour force are given by N and each worker supplies a fixed amount of homogeneous labour. There are few landlords in relation to workers and they may be ignored or assumed to be included in N. The number of landlords is assumed to be fixed so that workers alone account for the variability of N. There are two final goods, public goods (measured as R units) and private goods (X). The production functions for both of these goods exhibit constant returns to scale. Wages (w) and rentals (r) are the returns earned by workers and landlords for the use of factor services. The market (unit) prices of public and private goods are denoted by P_R and P_X respectively. The notation S_R is used to denote one plus the ad valorem rate of subsidy so that $S_R P_R$ represents the price received by producers. An ad valorem rate of 50% then represents cost sharing of one-third.

A requirement of the model is that both factors be fully employed and this is given by (1) and (2), in which a_{ij} refers to the quantity of factor i required to produce a unit of commodity j.

(1) $a_{NR}R + a_{NX}X = N$, and

(2) $a_{TR}R + a_{TX}X = T_0$.

Long-run equilibrium in perfect competition requires zero profits so that unit costs must be equal to the price received by producers. The zero profit conditions are given by (3) and (4).

(3) $a_{NR}w + a_{TR}r = P_R S_R \quad = P$, and

(4) $a_{NX}w + a_{TX}r = P_X$.

Following Jones, an asterisk will be used to denote the relative change in a variable or parameter. If we let

$$\lambda_{NR} = a_{NR}\frac{R}{N}, \quad \lambda_{NX} = a_{NX}\frac{X}{N},$$

$$\theta_{NR} = a_{NR}\frac{w}{P}, \quad \theta_{NX} = a_{NX}\frac{w}{P_X},$$

and so forth, then the four equations in the rates of change, from (1) to (4), are given by (5) to (8).

(5) $\lambda_{NR}R^* + \lambda_{NX}X^* = N^* - [\lambda_{NR}\overset{*}{a}_{NR} + \lambda_{NX}\overset{*}{a}_{NX}]$,

(6) $\lambda_{TR}R^* + \lambda_{TX}X^* = T^* - [\lambda_{TR}\overset{*}{a}_{TR} + \lambda_{TX}\overset{*}{a}_{TX}]$,

(7) $\theta_{NR}w^* + \theta_{TR}r^* = P^* - [\theta_{NR}\overset{*}{a}_{NR} + \theta_{TR}\overset{*}{a}_{TR}]$,

and

(8) $\theta_{NX}w^* + \theta_{TX}r^* = P_X^* - [\theta_{NX}\overset{*}{a}_{NX} + \theta_{TX}\overset{*}{a}_{TX}]$.

In what follows it will be assumed that the public goods sector is labour intensive. Let λ and θ denote the matrices of coefficients given in (5,6) and (7,8):

(9) $\lambda = \begin{pmatrix} \lambda_{NR} & \lambda_{NX} \\ \lambda_{TR} & \lambda_{TX} \end{pmatrix}$, and

$$(10) \quad \theta = \begin{pmatrix} \theta_{NR} & \theta_{TR} \\ \theta_{NX} & \theta_{TX} \end{pmatrix} .$$

By the factor intensity assumption both $|\lambda|$ and $|\theta|$, in (11) and (12) are positive.

$$(11) \quad |\lambda| = \lambda_{NR}-\lambda_{TR} = \lambda_{TX}-\lambda_{NX}, \quad \text{and}$$

$$(12) \quad |\theta| = \theta_{NR}-\theta_{NX} = \theta_{TX}-\theta_{TR} .$$

In generalizing the model to variable coefficients Jones (1965, p. 560) has shown that cost minimization leads to equations (13) and (14), which simplify the ensuing derivations.

$$(13) \quad \theta_{NR}\overset{*}{a}_{NR} + \theta_{TR}\overset{*}{a}_{TR} = 0, \quad \text{and}$$

$$(14) \quad \theta_{NX}\overset{*}{a}_{NX} + \theta_{TX}\overset{*}{a}_{TX} = 0.$$

To supplement the assumption of variable coefficients and to provide more information about the production functions the elasticities of substitution in production are introduced in (15) and (16).

$$(15) \quad \sigma_R = \frac{\overset{*}{a}_{TR} - \overset{*}{a}_{NR}}{w^* - r^*} , \quad \text{and}$$

$$(16) \quad \sigma_X = \frac{\overset{*}{a}_{TX} - \overset{*}{a}_{NX}}{w^* - r^*}$$

Equations (13,15) and (14,16) can be solved for the technical coefficients, to yield equations (17) to (20).

$$(17)\quad a^*_{NR} = -\theta_{TR}\sigma_R(w^*-r^*),$$

$$(18)\quad a^*_{TR} = \theta_{NR}\sigma_R(w^*-r^*),$$

$$(19)\quad a^*_{NX} = -\theta_{TX}\sigma_X(w^*-r^*), \text{ and}$$

$$(20)\quad a^*_{TX} = \theta_{NX}\sigma_X(w^*-r^*).$$

These solutions can be substituted into equations (5) to (8) to give:

$$(21)\quad \lambda_{NR}R^* + \lambda_{NX}X^* = N^* + \zeta_N(w^*-r^*),$$

$$(22)\quad \lambda_{TR}R^* + \lambda_{TX}X^* = T^* - \zeta_T(w^*-r^*),$$

$$(23)\quad \theta_{NR}w^* + \theta_{TR}r^* = P^* , \text{ and}$$

$$(24)\quad \theta_{NX}{}^*w^* + \theta_{TX}r^* = P_X{}^* ,$$

where

$$(25)\quad \zeta_N = \lambda_{NR}\theta_{TR}\sigma_R + \lambda_{NX}\theta_{TX}\sigma_X, \text{ and}$$

$$(26)\quad \zeta_T = \lambda_{TR}\theta_{NR}\sigma_R + \lambda_{TX}\theta_{NX}\sigma_X.$$

"In general, ζ_N is the aggregate percentage saving in labor inputs at unchanged outputs associated with a 1 per cent rise in the relative wage rate, the saving resulting from the adjustment to less labor-intensive techniques in both industries as relative wages rise." (Jones, 1965, p. 561)

If we subtract (22) from (21) we get

$$(27)\quad R^*-X^* = \frac{N^*-T^*}{|\lambda|} + \frac{\varsigma_N+\varsigma_T}{|\lambda|}(w^*-r^*).$$

This equation shows how the change in the ratio of outputs produced depends upon the growth in factor supplies and the change in the ratio of factor prices. The change in the factor price ratio can be derived from (23) and (24).

$$(28)\quad w^*-r^* = \frac{P^*-P_X^*}{|\theta|} = \frac{P_R^*+S_R^*-P_X^*}{|\theta|} .$$

If (28) is substituted into (27) then (29) is obtained.

$$(29)\quad R^*-X^* = \frac{N^*-T^*}{|\lambda|} + \sigma_S(P^*-P_X^*),$$

where σ_S is the elasticity of substitution between commodities on the supply side. From (29) it is clear that if $N^* > 0$ and $T^* = 0$ then $(R^*-X^*) > 0$ at constant supply prices (i.e., for $P^*-P_X^* = 0$). Furthermore, from (28) it is clear that if $(P^*-P_X^*) > 0$ then $(w^*-r^*) > 0$.

Let us now prove the Rybczynski theorem. Substitute (28) into (21) and (22) and solve for R^*.

$$(30)\quad R^* = \frac{1}{|\lambda|}[\lambda_{TX}N^*-\lambda_{NX}T^* + \frac{1}{|\theta|}(\lambda_{TX}\varsigma_N+\lambda_{NX}\varsigma_T)(P^*-P_X^*)]$$

At constant supply prices and for $N^* > 0$ and $T^* = 0$ it is clear that $\frac{R^*}{N^*} = \frac{\lambda_{TX}}{|\lambda|} = \frac{|\lambda|}{|\lambda|} + \frac{\lambda_{NX}}{|\lambda|} > 1$,

from equation (11). Again, using (28), (21) and (22) we may solve for X^*:

$$(31) \quad X^* = \frac{1}{|\lambda|}[\lambda_{NR}T^* - \lambda_{TR}N^* - \frac{1}{|\theta|}(\lambda_{TR}\varsigma_N + \lambda_{NR}\varsigma_T)(P^* - P_X^*)].$$

In this case we find that for constant supply prices and for $N^* > 0$ and $I^* = 0$ then

$$\frac{X^*}{N^*} = -\frac{\lambda_{TR}}{|\lambda|} = -\left(\frac{\lambda_{NR}}{|\lambda|} - \frac{|\lambda|}{|\lambda|}\right) < 0 = T^* .$$

This is the Rybczynski theorem.

Next it will be proved that if the supply price of R (i.e., P) grows more rapidly than P_X then $w^* > P^* > P_X^* > r^*$. Let us solve (23) and (24) for each of w^* and r^*:

$$(32) \quad w^* = \frac{P^*\theta_{TX} - P_X^*\theta_{TR}}{|\theta|} = P^* + \frac{\theta_{TR}}{|\theta|}(P^* - P_X^*)$$

$$(33) \quad r^* = \frac{P_X^*\theta_{NR} - P^*\theta_{NX}}{|\theta|} = P_X^* - \frac{\theta_{NX}}{|\theta|}(P^* - P_X^*).$$

If $(P^* - P_X^*) > 0$ then (32) and (33) show clearly that $w^* > P^* > P_X^* > r^*$. Note that up to this point in our analysis nothing new has been added to the trade theory model by assuming that R is a regional public good. We turn now to the integration of population mobility into the analysis.

III. The Mobility Function

Equation (29) represents the relationship between growth in factor supplies and the change in the ratio of outputs produced. It will be assumed here that $T^*=0$ but that the supply of workers (i.e., citizens) is variable. The population of the region is assumed to be variable and to depend upon the wage rate net of the individual tax price of the public good. It is assumed that the cost of public goods provision (i.e., $P_R R$, assuming no subsidies) is divided equally among all citizens. Resident population will also depend upon the output level of the regional public good, which is equally available to all residents. The mobility function is given by

(34) $N = \tilde{N}(\tilde{w}, R)$, where

$\tilde{w}$ is the wage rate deflated by an index[1] of prices $\tilde{P}$:

(35) $\tilde{w} = \frac{w}{\tilde{P}}$,

(36) $\tilde{P} = P_X Q_X + \pi Q_R$, where

(37) $\pi = \frac{P_R}{N}$.

The first partial derivatives, $\tilde{N}_1$ and $\tilde{N}_2$ of equation (34), are both positive by assumption. Equation (36) is a price index which is derived by invoking a set of weights, Q_X and Q_R, and by making the tax price of public goods equal to $\frac{P_R}{N}$. From equations (35) and (36) it is possible to write:

$$(38)\quad \widetilde{w}^* = w^* - \widetilde{P}^*$$
$$= w^* - (P_X Q_X + \pi Q_R)^*$$
$$= w^* - Q_X \frac{dP_X}{\widetilde{P}} - Q_R \frac{d\pi}{\widetilde{P}}$$
$$= w^* - \frac{P_X Q_X}{\widetilde{P}} \frac{dP_X}{P_X} - \pi \frac{Q_R}{\widetilde{P}} \frac{d\pi}{\pi} .$$

From (37) we have:

$$(39)\quad \frac{d\pi}{\pi} = \left(\frac{N dP_R - P_R dN}{N^2}\right) \frac{N}{P_R} ,$$

so that (38) may be rewritten:

$$(40)\quad \widetilde{w}^* = w^* - m_X P_X^* - m_R P_R^* + m_R N^* ,$$

where m_X and m_R are the proportions of income spent on X and R, respectively, in the index.

The mobility function of equation (34) will play an important role in what follows.[2] From (34) we may write:

$$(41)\quad N^* = e_W \widetilde{w}^* + e_R R^*,$$

where e_W and e_R are the elasticities of N with respect to

$\tilde{w}$ and R, respectively. From (40), equation (41) becomes (42):

$$(42)\quad N^* = e_W(w^* - m_X P_X^* - m_R P_R^* + m_R N^*) + e_R R^* \ .$$

If we substitute (32) into (42) then we obtain (43).

$$(43)\quad N^* = e_W\left[P_R^*\left(\frac{\theta_{TX}}{|\theta|} - m_R\right) + S_R^* \frac{\theta_{TX}}{|\theta|} - P_X^*\left(\frac{\theta_{TR}}{|\theta|} + m_X\right) + m_R N^*\right] + e_R R^*$$

$$= e_W\left[P_R^*\left(\frac{\theta_{TX}}{|\theta|} - m_R\right) - P_X^*\left(\frac{\theta_{TX}}{|\theta|} - m_R\right) + S_R^* \frac{\theta_{TX}}{|\theta|} + m_R N^*\right] + e_R R^* \ .$$

$$= e_W\left[(P_R^* - P_X^*)\left(\frac{\theta_{TX}}{|\theta|} - m_R\right) + S_R^* \frac{\theta_{TX}}{|\theta|} + m_R N^*\right] + e_R R^* \ .$$

Equation (43) expresses the rate of change in population in terms of supply prices, subsidies and the output level of the regional public good.

IV. Integration of Demand Into the Model

The next step in the construction of our general equilibrium model is the introduction of demand. Following Jones (1965, p. 563) we assume that taste patterns are homothetic so that the quantities of R and X consumed depends only upon the relative (individual) price ratio, as in equation (44).

$$(44)\quad \frac{R}{X} = f\left(\frac{\pi}{P_X}\right) \ .$$

If we substitute (37) into (44) then we may write (45), where σ_D is the elasticity of substitution between the two goods.

$$(45) \quad R^*-X^* = -\sigma_D(P_R^*-N^*-P_X^*)$$

This gives us the demand side of our analysis and if we invoke the concept of market equilibrium then prices will adjust to equate demand and supply. The supply side is given by (29) and if we equate (R^*-X^*) in (29) with R^*-X^* in (45) then we obtain (46).

$$(46) \quad -\sigma_D(P_R^*-N^*-P_X^*) = \frac{N^*}{|\lambda|} + \sigma_S(P_R^*+S_R^*-P_X^*),$$

which may be rewritten as

$$(47) \quad P_R^*-P_X^* = \frac{N^*}{\sigma_D+\sigma_S}\left(\sigma_D - \frac{1}{|\lambda|}\right) - \frac{\sigma_S S_R^*}{\sigma_D+\sigma_S}.$$

The equilibrium set of prices will depend upon population growth and subsidies.

In what follows, two equations of the form $R^* = g(N^*,S_R^*)$ and $N^* = h(R^*,S_R^*)$ will be derived and these will be employed to solve for R^* in terms of S_R^*. First, rewrite (30) as

$$(48) \quad R^* = \frac{\lambda_{TX}}{|\lambda|}N^* + E(P^*-P_X^*),$$

where $E = \dfrac{\lambda_{TX}\zeta_N + \lambda_{NX}\zeta_T}{|\lambda|\,|\theta|}$

Next substitute (47) into (48) to yield (49).

$$(49)\quad R^* = \frac{\lambda_{TX}}{|\lambda|}N^* + E\left[S_R^* + \frac{N^*}{\sigma_D+\sigma_S}\left(\sigma_D - \frac{1}{|\lambda|}\right) - \frac{\sigma_S S_R^*}{\sigma_D+\sigma_S}\right]$$

$$= \frac{\lambda_{TX}}{|\lambda|}N^* + E\left[S_R^*\frac{\sigma_D}{\sigma_D+\sigma_S} + \frac{N}{\sigma_D+\sigma_S}\left(\sigma_D - \frac{1}{|\lambda|}\right)\right]$$

$$= N^*\left[\frac{\lambda_{TX}}{|\lambda|} + E\,\frac{\left(\sigma_D - \frac{1}{|\lambda|}\right)}{\sigma_D+\sigma_S}\right] + ES_R^*\,\frac{\sigma_D}{\sigma_D+\sigma_S}\ .$$

If (49) is compared with (30) then it may be noted that some exogenous change in N may have an amplified effect on R but that this will depend upon the sign of $(\sigma_D - \frac{1}{|\lambda|})$. Note that (49) is an equation of the form $R^* = g(N^*, S_R^*)$.

To derive $N^* = h(R^*, S_R^*)$ we substitute (47) into (43) to yield (50).

$$(50)\quad N^* = e_W\left(\frac{\theta_{TX}}{|\theta|} - m_R\right)\left[\frac{N^*}{\sigma_D+\sigma_S}\left(\sigma_D - \frac{1}{|\lambda|}\right) - \sigma_S\frac{S_R^*}{\sigma_D+\sigma_S}\right]$$

$$+ e_W S_R^*\,\frac{\theta_{TX}}{|\theta|} + e_W m_R N^* + e_R R^*\ .$$

Solving this equation for N* we obtain

$$(51)\quad N^*\left[1 - e_W\frac{\theta_{TX}}{|\theta|}\,\frac{\bar{\sigma}_D}{\sigma_D+\sigma_S} + e_W m_R\frac{\bar{\sigma}_D}{\sigma_D+\sigma_S} - e_W m_R\right]$$

$$= e_W S_R^*\left[\frac{\theta_{TX}}{|\theta|} - \frac{\sigma_S}{\sigma_D+\sigma_S}\left(\frac{\theta_{TX}}{|\theta|} - m_R\right)\right] + e_R R^*\ ,\quad \text{or}$$

$$(52)\quad N^* = \frac{e_W S_R^* \left[\frac{\theta_{TX}}{|\theta|} - \frac{\sigma_S}{\sigma_D+\sigma_S}\left(\frac{\theta_{TX}}{|\theta|} - m_R\right)\right] + e_R R^*}{\left[1 - \frac{e_W}{\sigma_D+\sigma_S}\left(\frac{\theta_{TX}}{|\theta|}\tilde{\sigma}_D + m_R\tilde{\sigma}_S\right)\right]},$$

where $\tilde{\sigma}_S = \sigma_S + \frac{1}{|\lambda|}$,

$\tilde{\sigma}_D = \sigma_D - \frac{1}{|\lambda|}$, and $\frac{\tilde{\sigma}_D+\tilde{\sigma}_S}{\sigma_D+\sigma_S} = 1$

To get R^* in terms of S_R^* we substitute the rather unwieldy expression given in (52) into (49). In so doing let

$$Q^{-1} = \left[1 - \frac{e_W}{\sigma_D+\sigma_S}\left(\frac{\theta_{TX}}{|\theta|}\tilde{\sigma}_D + m_R\tilde{\sigma}_S\right)\right]^{-1}.$$

$$(53)\quad R^* = e_W\frac{S_R^*}{Q}\left[\left(\frac{\lambda_{TX}}{|\lambda|} + E\frac{\tilde{\sigma}_D}{\sigma_D+\sigma_S}\right)\left\{\frac{\theta_{TX}}{|\theta|}\left(1 - \frac{\sigma_S}{\sigma_D+\sigma_S}\right) + m_R\frac{\sigma_S}{\sigma_D+\sigma_S}\right\}\right.$$

$$\left. + e_R\frac{R^*}{Q}\left[\frac{\lambda_{TX}}{|\lambda|} + E\frac{\tilde{\sigma}_D}{\sigma_D+\sigma_S}\right]\right] + ES_R^*\frac{\sigma_D}{\sigma_D+\sigma_S}, \text{ or}$$

$$(54)\quad R^*\left[1 - \frac{e_R}{Q}\left(\frac{\lambda_{TX}}{|\lambda|} + E\frac{\tilde{\sigma}_D}{\sigma_D+\sigma_S}\right)\right]$$

$$= S_R^*\left[e_W Q^{-1}\left(\frac{\lambda_{TX}}{|\lambda|} + E\frac{\tilde{\sigma}_D}{\sigma_D+\sigma_S}\right)\left(\frac{\sigma_D}{\sigma_D+\sigma_S}\frac{\theta_{TX}}{|\theta|} + \frac{\sigma_S m_R}{\sigma_D+\sigma_S}\right) + E\frac{\sigma_D}{\sigma_D+\sigma_S}\right].$$

Let us now attempt to simplify the expressions that have been derived. Define the following:

$$W_D = \frac{\sigma_D}{\sigma_D+\sigma_S}, \quad W_S = \frac{\sigma_S}{\sigma_D+\sigma_S},$$

$$\widetilde{W}_D = \frac{\sigma_D - 1/|\lambda|}{\sigma_D + \sigma_S} \quad \text{and} \quad \widetilde{W}_S = \frac{\sigma_S + 1/|\lambda|}{\sigma_D + \sigma_S} .$$

The denominator in (52) may now be written

$$(55) \quad Q = \left[1 - e_W\left(\frac{\theta_{TX}}{|\theta|}\widetilde{W}_D + m_R\widetilde{W}_S\right)\right] .$$

We know that $\frac{\theta_{TX}}{|\theta|}$ exceeds unity, while $m_R < 1$ and $\widetilde{W}_D + \widetilde{W}_S = 1$. Assume for simplicity that $(\sigma_D - \frac{1}{|\lambda|}) > 0$ so that $\widetilde{W}_D > 0$ although $\widetilde{W}_D < W_D$. Assume also that $Q > 0$ so that a small increase in e_W will reduce Q, but increase Q^{-1}. Returning to (54) define

$$Z = \left[1 - \frac{e_R}{Q}\left(\frac{\lambda_{TX}}{|\lambda|} + \widetilde{W}_D E\right)\right] .$$

The expression $\left(\frac{\lambda_{TX}}{|\lambda|} + \widetilde{W}_D E\right)$ is bound to exceed unity by our assumptions. For simplicity in interpretation let it be assumed that Z is positive. A small increase in e_W will reduce Z but increase Z^{-1}. A small increase in e_R will reduce Z but also increase Z^{-1}. Rewrite (54) as (56).

$$(56) \quad R^* = \frac{S_R^*}{Z}\left[\frac{e_W}{Q}\left(\frac{\lambda_{TX}}{|\lambda|} + \widetilde{W}_D E\right)\left(W_D\frac{\theta_{TX}}{|\theta|} + W_S m_R\right) + EW_D\right]$$

This is the expression that indicates the effects of population endogeneity upon the output of the public good, when federal subsidies are paid. If population were constant then (56) would simply be $R^* = S_R^* EW_D$. But, by our assumptions, $R^* > S_R^* EW_D$ so that the subsidy effects are amplified.

The solution for w* in (32) will be bypassed and we shall now derive the solution for w*-r*. From (28) and (47) we have

$$(57)\quad w^*-r^* = \frac{S_R^*}{|\theta|} + \frac{P_R^*-P_X^*}{|\theta|}$$

$$= \frac{S_R^*}{|\theta|}\left(1-\frac{\sigma_S}{\sigma_D+\sigma_S}\right) + \frac{N^*}{\sigma_D+\sigma_S}\left(\sigma_D-\frac{1}{|\lambda|}\right)\frac{1}{|\theta|}$$

$$= \frac{1}{|\theta|(\sigma_D+\sigma_S)}\left[\sigma_D S_R^* + N^*\tilde{\sigma}_D\right].$$

Next substitute R* from (56) into (52) to derive N*

$$(58)\quad N^* = e_W\frac{S_R^*}{Q}\left[\frac{\theta_{TX}}{|\theta|} - W_S\left(\frac{\theta_{TX}}{|\theta|} - m_R\right)\right]$$

$$+ \frac{e_R}{Q}\frac{S_R^*}{Z}\left[\frac{e_W}{Q}\left(\frac{\lambda_{TX}}{|\lambda|} + \tilde{W}_D E\right)\left(W_D\frac{\theta_{TX}}{|\theta|} + W_S m_R\right) + EW_D\right]$$

From (57) and (58) we have

$$(59)\quad w^*-r^* = \frac{W_D}{|\theta|}S_R^* + \frac{\tilde{W}_D}{|\theta|}\frac{e_W}{Q}S_R^*\left[\frac{\theta_{TX}}{|\theta|} - W_S\left(\frac{\theta_{TX}}{|\theta|} - m_R\right)\right]$$

$$+ \frac{e_R}{Q}\frac{\tilde{W}_D}{|\theta|}\frac{S_R^*}{Z}\left[\frac{e_W}{Q}\left(\frac{\lambda_{TX}}{|\lambda|} + \tilde{W}_D E\right)\left(W_D\frac{\theta_{TX}}{|\theta|} + W_S m_R\right) + EW_D\right].$$

Equation (59) indicates that the effect of the subsidy upon the ratio of factor prices when population is constant is given by $w^*-r^* = \frac{W_D}{|\theta|} S_R^*$. If Q, Z and $(\sigma_D - \frac{1}{|\lambda|})$ are positive then the introduction of population mobility will increase the backward shifting of the subsidy to the factor intensive in the subsidized industry. If, however, the factor intensity expression $(|\lambda|)$ is small so that $1/|\lambda|$ is large in relation to σ_D then the extent of backward shifting is reduced.

V. Conclusion

Federal subsidies to a province have been examined in a general equilibrium setting. General equilibrium analysis has been used to determine some of the incidence effects on the assumption that both private and public goods industries behave as if they are competitive. For a single small province it has been shown that when population is endogenous the effects of a subsidy may be amplified. The model is set out in a way that makes in-migration a substitute for federal subsidies since new citizens participate in an ongoing benefit- and cost-sharing arrangement. Since migration is endogenous the model allows federal subsidies to the provincial public good to generate a population influx.

In the previous section specific assumptions were made concerning the parameters of the model. The possibility of amplified effects upon public good output and relative factor prices was demonstrated. Some caveats are now in order. The small province assumption is of particular importance to our results. Also, federal subsidies were given only to the province of interest. Hence, the model is not fully general equilibrium in that it must be heavily qualified for subsidy systems that apply generally to all provinces.

The mobility function, $N=\widetilde{N}(\widetilde{w},R)$, has been central to our analysis. The net real wage rate, $\frac{w-\pi}{\widetilde{P}}$, depends upon the marginal product of labour, relative product prices, and the size of the population. However, changes in population are likely to affect relative prices and marginal products, in a general equilibrium system. Let us look in particular at the wage rate and the marginal product of labour.

On the production side of the model the Rybczynski theorem states that at unchanged commodity prices an exogenous increase in the labour supply will lead to an expansion in the output of the labour-intensive public good and a contraction in the output of the land intensive private good. The adjustment involved in moving from the initial position to the final equilibrium requires a change in factor prices but factor prices return to their original level. The factor intensities for the two linear homogeneous production functions will also return to their original level. This follows from the well-known theorem for linear homogenous production functions that factor prices depend upon the ratio of labour to land. This means that a Rybczynski-type displacement will not change factor prices and, in particular, that an increase in the supply of fully-employed workers will not reduce the wage rate. When demand is introduced and prices are allowed to adjust then factor intensities are likely to change and diminishing returns to labour are possible. The relevance of diminishing returns to our model is clear since diminishing returns tend to choke off in-migration. Note that when factor intensities are the same in both industries then no factor saving is possible through inter-industry shifting of factors and diminishing returns to labour appears whenever the supply of labour increases. It has been shown above that it is possible for population mobility to increase the effect of federal subsidies upon the wage-rental ratio, when factor intensities differ. But when factor intensities are the same in both industries then it can be shown, from (59), that subsidies to the provincial public good will result in a fall in the wage-rental ratio. If factor intensities are the same in both industries (at all wage-rental ratios) then relative prices will not change (i.e., $P^*-P_X^*=0$) but an increase in the subsidy will reduce P_R and π. However, if the reduction in π induces an influx of people then diminishing returns to labour will reduce the wage-rental rate. The size of this reduction will depend upon the elasticity of substitution in demand and upon the proportion of income spent upon the public good. It will also depend upon the elasticities of population with respect to the net wage and with respect to the output of the public good. This is shown in equation (60).

$$(60) \quad w^*-r^* = -\frac{S_R^*}{Z}[e_W m_R + e_R \lambda_{TX} \sigma_D],$$

$$\text{where } Z = [1 + e_W(\theta_{TR} - m_R) - e_R \lambda_{TX} \sigma_D]$$

A final comment may be made with respect to the special case where fixed supply prices result from equal factor intensities. From equation (56) it has been shown that, when factor intensities differ between the two industries, the subsidy may have an amplified effect on R^* because of induced changes in population. This possibility remains when factor intensities are the same since equation (56) then becomes

$$(61) \quad R^* = \frac{S_R^* \sigma_D \lambda_{TX}}{Z}(1 + e_W \theta_{TX}).$$

FOOTNOTES

1. The price deflation technique applied here is adapted from that used by Acheson (1971).

2. For a more complete treatment of the role of population mobility in the theory of public goods see Tiebout (1956) and Vardy (1971a, pp. 42-52, and 1971b).

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