The asymptotic distribution of impact multipliers for a non-linear structural econometric model,

Bianchi, Carlo; Calzolari, Giorgio and Corsi , Paolo
IBM Scientific Center, Pisa, Italy.

1977

C. Bianchi - G. Calzolari - P. Corsi

THE ASYMPTOTIC DISTRIBUTION OF IMPACT MULTIPLIERS FOR A NON-LINEAR STRUCTURAL ECONOMETRIC MODEL

ABSTRACT

The problem of deriving asymptotic statistical properties of impact multipliers from a consistent estimate of a structural non-linear econometric model is discussed. The theoretical aspects, which generalize the results derived by Goldberger, Nagar and Odeh [9] for linear models, are analyzed in detail, as well as the numerical (computational) aspects. Numerical results are finally displayed for an econometric model well known in the literature.

Lawrence R. Klein and Harry Eisenpress are kindly acknowledged for encouragement and comments on a first draft. Morris Norman and Arthur M. Havenner are also acknowledged for useful suggestions on the use of their data and programs. Of course, the final form and contents of this paper, including any errors which might remain, are the joint responsibility of the authors.

1. INTRODUCTION

In 1961 Goldberger, Nagar and Odeh [9] proposed a method for deriving the statistical properties of the reduced form coefficients of a structural linear econometric model in terms of asymptotic distribution theory.

Even if with substantial modifications in the approach, Dhrymes in 1973 [5] confirmed the validity of those results, still for linear models.

The main reason for deriving asymptotic statistical properties was that, also when the model is linear, the computation of reduced form coefficients involves non-linear transformations of the structural coefficients, thus making in general impossible the computation of finite sample moments, while it is well known that an asymptotic normal distribution can be maintained also through non-linear transformations provided they are continuous and differentiable [15, p. 374]. As these conditions are, usually, largely satisfied by the functions involved also in non-linear econometric models, it should be possible to extend the method of analysis to a quite general class of non-linear models. This is the subject of the paper.

A general structural econometric model(1), linear or non-linear in the variables as well as in the coefficients, can be represented as:

$$(1.1) \qquad F(Y_t, X_t, A, U_t) = 0$$

where:

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}$$

is a vector of functional operators

$$Y_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{m,t} \end{bmatrix}; \quad X_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{bmatrix}$$

are the vectors of observed endogenous and predetermined variables at time $t = 1, 2, .., T$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{bmatrix}$$

is the vector of all the structural stochastic coefficients of the model (all the other coefficients being excluded from this vector and included in the functional operators)

$$U_t = \begin{bmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{m,t} \end{bmatrix}$$

is the vector of structural stochastic disturbances at time t, assumed to be with zero means, independent from all the predetermined variables and with finite covariance matrix.

Following Dhrymes [4] and Goldberger [7] the generic impact multiplier is defined as the partial derivative of the conditional expectation of a current endogenous variable with respect to an exogenous variable, with all other variables held constant, after all contemporaneous feedbacks due to simultaneity of the system have been allowed

---

(1) As impact multipliers analysis deals with instantaneous response of a system [4, p. 518], exogenous and lagged endogenous do not need, in this context, to be distinguished and can be simply treated as predetermined variables. The locution impact multiplier is here preferred to that of reduced form coefficient only because it seems to be of more current use when dealing with non-linear models; the validity of the results is, however, also for the lagged endogenous variables.

for.

In linear stochastic models, the conditional expectation can be easily derived through the reduced form. As the reduced form involves only linear transformations of the structural disturbances, the conditional expectation can be derived setting these random terms a-priori to their expected value ($U_t = 0$). Therefore, the impact multipliers, in this case, can be practically defined without involving the concept of conditional expectation, but simply as the coefficients of the current exogenous variables in the reduced form equations.

Also when dealing with non-linear stochastic models, the derivation of the impact multipliers moves, at least in principle, from the reduced from equations. The reduced form, however, involves in this case non-linear transformations of the structural disturbances, so that they should not be set a-priori to zero for a correct derivation of the conditional expectation of the current endogenous variables (see, for example, the analysis in [11], even if there the approach is dynamic).

In spite of the correct theoretical definition, however, empirical multiplier analysis is very often carried out without taking into account the structural disturbances also when the model is non-linear. This consideration has suggested to develop two different theoretical approaches to the problem of the statistical properties of impact multipliers in the general framework of non-linear models.

The first should be properly called the derivation of the asymptotic distribution of impact multipliers when the reduced form is drawn disregarding the existence of the disturbances in the structural form. Once the theory has been developed, the simplest way to derive in practice numerical results is simulation, so that we shall simply speak of impact multipliers in non-stochastic simulation (using the terminology of [11]; of course, one-step or static simulation). This first approach will be dealt with in chapter (2.).

The second approach should be properly called the derivation of the asymptotic distribution of impact multipliers when the reduced form takes into account the structural disturbances. Once the theory has been developed, simulation is, in this case, the only practical method to get numerical results, so that we shall speak, still using the terminology in [11], of impact multipliers in stochastic simulation. This approach will be described in chapter (3.).

In chapter (4.) numerical results are drawn for the non-linear Klein-Goldberger revised model.

2. ASYMPTOTIC DISTRIBUTION OF IMPACT MULTIPLIERS IN NON-STOCHASTIC SIMULATION

2.1. Statement of the problem and major assumptions

The definition of impact multipliers in the theoretical framework that disregards the presence of the structural disturbances is well known in the econometric literature, as well known are the methods of empirical computation via non-stochastic simulation [6], [8].

Subject of this section will be the study of the statistical asymptotic properties of the impact multipliers, when they are computed via non-stochastic simulation on the basis of estimated coefficients. The conditions under which the procedures that will be described can be applied are very wide and general; nevertheless, for methodological correctness, they are listed hereunder. Reference is made to the system (1.1).

a) *Assumption*

Setting U_t to its expected value ($U_t = 0$) the system can be solved at time t (non-stochastic solution); let Y_t^o be the solution vector (or one of the solution vectors, the one in which we are going to compute the multipliers).

b) *Assumption*

Setting U_t to its expected value ($U_t = 0$) $F_1, F_2, \dots, F_m$ are continuous and differentiable functions, with respect to the elements of Y_t, X_t and A, in a domain of the $(m + n + s)$ dimensional space, including the point (Y_t^o, X_t, A), with continuous derivatives of the first order.

c) Assumption

The Jacobian matrix of $F_1, F_2, \dots, F_m$ is non-singular in some neighborhood of the solution point (Y°_t, X_t, A) $(m + n + s$ dimensional space$)$.

d) Assumption

$F_1, F_2, \dots, F_m$ have continuous second order mixed derivatives, with respect to the elements of A and Y_t or A and X_t, in some neighborhood of the point (Y°_t, X_t, A).

Under these assumptions the well known implicit functions theorem (see, for example, [12, p. 389]) can be applied, showing that Y_t can be made explicit as a vector of m functions of X_t and A in some neighborhood of the solution point (Y°_t, X_t, A) of the $(m + n + s)$ dimensional space; the m functions $f_1, f_2, \dots, f_m$(2) are, in this case, the reduced form equations.

The f_i functions are defined in some neighborhood of the point (X_t, A) of the $(n + s)$ dimensional space, where they are also continuous and have continuous partial detivatives of the first order with respect to all the elements of X_t and A.

The same theorem states that each of these derivatives is equal to the ratio of two Jacobian determinants (the denominator being different from zero, for assumption (c)). In particular

$$(2.1.1) \qquad \frac{\partial f_i(X, A)}{\partial x_j} = - \frac{\dfrac{\partial(F_1, F_2, \dots, F_m)}{\partial(y_1, y_2, \dots, y_{i-1}, x_j, y_{i+1}, \dots, y_m)}}{\dfrac{\partial(F_1, F_2, \dots, F_m)}{\partial(y_1, y_2, \dots, y_m)}}$$

$$\begin{cases} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \end{cases}$$

---

(2) These functions are such that $y^\circ_{i,t} = f_i(X_t, A)$; we prefer this notation to the more used $y^\circ_{i,t} = y_i(X_t, A)$, to avoid confusion in the use of the symbol y.

in some neighborhood of the $(n + s)$ space point (X_t, A).

In this context, the i,j-th. impact multiplier is the value assumed by the function $\partial f_i / \partial x_j$ at the point (X_t, A); it will be called, hereunder, $\pi^\circ_{i,j,t}$, where the superscript "°" recalls that it disregards the presence of the disturbances U_t in the model. In formula:

$$(2.1.2) \qquad \pi^\circ_{i,j,t} = \left(\frac{\partial f_i(X, A)}{\partial x_j} \right)_{(X_t, A)}$$

where the domain of the function f_i is, for the implicit functions theorem, an $(n + s)$ dimensional interval containing the point (X_t, A).

Observing now that, for assumption (*d*), every partial derivative in equation (2.1.1) in the two Jacobian determinants is again continuous and differentiable with respect to the elements of A, with continuous derivatives, it is clear that also $\partial f_i / \partial x_j$ is continuous and differentiable with respect to the elements of A, with continuous derivatives.

All the impact multipliers at time t can be arranged into a (mn) column vector Π°_t. The order within the vector is not really important; for example, we could arrange the multipliers by reduced form equations as in [9], so that the subscripts i,j for the elements should be abandoned and replaced by a single subscript h, where

$$h = n(i - 1) + j.$$

When passing to the structural model with estimated coefficients, the following further assumptions are required.

e) Assumption

The model's coefficients can be estimated by means of a consistent estimation method; let $\hat{A}$ be the vector of all the structural coefficients estimated using a sample of length T. The corresponding solution vector of system (1.1) at time t is indicated as $\hat{Y}^\circ_t$.

f) Assumption

No matter of what are the "small sample" properties of $\hat{A}$, let $\sqrt{T}(\hat{A} - A)$ converge in distribution to the multivariate normal $N(0, \Sigma)$

(for 2SLS, 3SLS, FIML and LIML see, for example, Dhrymes [4, pp. 191, 216, 323 and 351]); let $\hat{\Sigma}$ be a consistent estimator of Σ, covariance matrix of the asymptotic distribution of the structural stochastic coefficients.

The next subsections (from 2.2. to 2.10.) will deal first of all with some asymptotic properties of the estimated non-stochastic solution vector $\hat{Y}^{\circ}_{t}$; then the definition of estimated impact multipliers will be introduced, and the related asymptotic distribution analyzed; finally the computational procedures will be displayed.

2.2. Lemma

(2.2.1) $\quad \operatorname*{plim}_{T\to\infty} \hat{Y}^{\circ}_{t} = Y^{\circ}_{t}$

Proof:

We have observed in section (2.1.) that, for every i, $y^{\circ}_{i,t}$ is the value of the continuous function f_i at the point (X_t, A), being this function defined in some neighborhood of that point. As soon as $\hat{A}$ falls within the domain of f_i, $\hat{y}^{\circ}_{i,t}$ can be regarded as the value of the function f_i at the point $(X_t, \hat{A})$.

As $\operatorname*{plim}_{T\to\infty} \hat{A} = A$, and the functions f_i are continuous, the desired result follows immediately from Slutsky theorem.(3).

2.3. Definition

Observing that $\hat{y}^{\circ}_{i,t}$ is the value of the function f_i at the point $(X_t, \hat{A})$, it is clear that the estimated i,j-th impact multiplier at time t is the value of the function $(\partial f_i/\partial x_j)$ at the point $(X_t, \hat{A})$. In

---

(3) "If plim x = x* and g(x) is a continuous function, then plim g(x) = g(x*)" [7, p. 118]; for a proof see Wilks [16, p. 102].

formula:

(2.3.1) $\quad \hat{\pi}^{\circ}_{i,j,t} = \left(\dfrac{\partial f_i(X, A)}{\partial x_j}\right)_{(X_t,\hat{A})}$

$\hat{\Pi}^{\circ}_{t}$ is the (mn) column vector of the estimated impact multipliers.

2.4. Theorem

(2.4.1) $\quad \operatorname*{plim}_{T\to\infty} \hat{\Pi}^{\circ}_{t} = \Pi^{\circ}_{t}$

Proof:

As we have proved in section (2.1.) the continuity of the derivatives of the functions $f_1, f_2, \ldots, f_m$, the above result follows from a straightforward application of Slutsky theorem.

Theorem (2.4.) states that, in a non-stochastic simulation framework, the impact multipliers derived from the consistently estimated structural model are consistent estimates of Π°_{t}.

2.5. Lemma

(Note that the symbols in this lemma should not be identified with the same symbols elsewhere in this paper).

"Let T be a k-dimensional statistic $(T_{1,n}, \ldots, T_{k,n})$ such that the asymptotic distribution of $\sqrt{n}(T_{1,n} - \theta_1), \ldots, \sqrt{n}(T_{k,n} - \theta_k)$ is k-variate normal with mean zero and dispersion matrix Σ". "Let $g_1, \ldots, g_q$ be q functions of k variables and each g_i is totally differentiable. Then the asymptotic distribution of

$$\sqrt{n}[g_i(T_{1,n}, \ldots, T_{k,n}) - g_i(\theta_1, \ldots, \theta_k)] \quad i = 1, 2, \ldots, q$$

is q-variate normal with zero means and dispersion matrix $G\Sigma G'$, where $G = [\partial g_i/\partial\theta_j]$. The rank of the distribution is equal to $R(G\Sigma G')$".

The complete proof can be found in [14, p. 322] (see also [15, p. 383]).

2.6. Theorem

The asymptotic distribution of

$$(2.6.1) \qquad \sqrt{T}(\hat{\Pi}^\circ_t - \Pi^\circ_t) \qquad \text{(as } T \to +\infty)$$

is mn-variate normal with zero means and covariance matrix

$$(2.6.2) \qquad \Omega^\circ_t = G^\circ_t \Sigma G^{\circ\prime}_t$$

where Σ is the covariance matrix of the asymptotic distribution of $\sqrt{T}(\hat{A} - A)$, and G°_t is the $(mn \times s)$ matrix whose general element is

$$(2.6.3) \qquad g^\circ_{h,k,t} = \left(\frac{\partial^2 f_i(X, A)}{\partial x_j \, \partial a_k} \right)_{(X_t,\, A)} \qquad \left\{ \begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \\ h = n(i-1) + j \\ k = 1, 2, \ldots, s \end{array} \right.$$

Proof:

The proof immediately follows from lemma (2.5.) and from the consideration that the mn functions $(\partial f_i / \partial x_j)$, computed for X_t fixed at its historical value, are continuous and differentiable functions with respect to the elements of A.

This theorem states the existence and the properties of the asymptotic distribution of the derived estimated impact multipliers for non-linear econometric models in a non-stochastic simulation approach.

2.7. Remarks

The rank of the above distribution, as observed in lemma (2.5.), is that of the matrix $G^\circ_t \Sigma G^{\circ\prime}_t$. The distribution will be, therefore, generally singular (as soon as mn is greater than s), thus making it impossible to determine explicitly the density function. This,however does not create any difficulty in theory, as well pointed out by Rao [14, pp. 445 - 446].

2.8. Corollary

As proved in section (2.1.), the second order mixed derivative in equation (2.6.3) is still a continuous function of the structural coefficients. If we define $\hat{g}^\circ_{h,k,t}$ as the value of that derivative at the point $(X_t, \hat{A})$,

$$(2.8.1) \qquad \hat{g}^\circ_{h,k,t} = \left(\frac{\partial^2 f_i(X, A)}{\partial x_j \, \partial a_k} \right)_{(X_t,\, \hat{A})}$$

Slutsky theorem can be applied once again, leading to:

$$(2.8.2) \qquad \underset{T \to \infty}{\text{plim}} \; \hat{g}^\circ_{h,k,t} = g^\circ_{h,k,t}$$

and, in matrix notation:

$$(2.8.3) \qquad \underset{T \to \infty}{\text{plim}} \; \hat{G}^\circ_t = G^\circ_t$$

Furthermore, being $\hat{\Sigma}$ a consistent estimator of the covariance matrix of the asymptotic distribution of the coefficients (assumption (6), section (2.1.)):

$$(2.8.4) \qquad \underset{T \to \infty}{\text{plim}} \; \hat{\Sigma} = \Sigma$$

Defining now $\hat{\Omega}^\circ_t$ as:

$$(2.8.5) \qquad \hat{\Omega}^\circ_t = \hat{G}^\circ_t \hat{\Sigma} \hat{G}^{\circ\prime}_t$$

the application of Slutsky theorem to this last equation leads, finally, to the desired result:

$$(2.8.6) \qquad \operatorname*{plim}_{T\to\infty} \hat{\Omega}^\circ_t = \Omega^\circ_t$$

that is: $\hat{\Omega}^\circ_t$ is a consistent estimate of Ω°_t.

2.9. Computational procedures

The problem to be empirically solved is that of developing a simple and reliable technique to compute the matrix $\hat{\Omega}^\circ_t$ above defined, the computation of $\hat{\Pi}^\circ_t$ being no more a problem for econometricians [6].

2.9.1. The analytical approach

In the case of linear models, where Π°_t is costant over time, the expression of the elements of G°_t can be further developed, leading to the formula [9, eq. (4.6)] (in the approach by Goldberger, Nagar and Odeh, however, the dimensions of the matrices Σ and G are much larger than in this approach, because the vector of the coefficients is composed not only of the structural estimated ones, but of all the coefficients of the model, including zeroes).

In the case of non-linear models, one should proceed as follows.

1) Find a solution vector $\hat{Y}^\circ_t$ for a given year by means of a numerical simulation technique.

2) For any i and j, compute the analytical expression of the function $(\partial f_i/\partial x_j)$ (in its domain), as the ratio of two Jacobian determinants (implicit functions theorem), without making explicit the elements of Y.

3) Compute the partial derivatives of the function $(\partial f_i/\partial x_j)$ with respect to a_k, for any k; in this case, only the numerical value in the solution point $(\hat{Y}^\circ_t, X_t, \hat{A})$, and not the full analytical expression, is required.

4) With the obtained values, build the matrix $\hat{G}^\circ_t$ and compute $\hat{\Omega}^\circ_t = \hat{G}^\circ_t \Sigma \hat{G}^{\circ\prime}_t$.

The above method, though possible, is dreadfully complicated, even for small models.

An alternative and much simpler numerical method is suggested.

2.9.2. The numerical approach

The numerical method is a direct application of equations (2.8.1) and (2.8.5). It is simply based on numerical computation of the second order mixed derivatives (eq. 2.8.1), obtained via simulation as:

$$(2.9.2.1) \qquad \hat{g}^\circ_{h,k,t} = \frac{\Delta\left(\dfrac{\Delta\hat{y}^\circ_{i,t}}{\Delta x_{j,t}}\right)}{\Delta\hat{a}_k}$$

with carefully assigned values of $\Delta x_{j,t}$ and $\Delta\hat{a}_k$. It must be recalled that the values of $\Delta x_{j,t}$ do not influence the results in linear models but they do in the case of non-linear models, so that in these cases they must be chosen as small as possible, being the definition of derivative a limit as $\Delta \to 0$. The values of $\Delta\hat{a}_k$, on the contrary, do influence the results both in linear and in non-linear models (the solution, in fact, always involves non-linear transformations of the coefficients, such as inversion of matrix of coefficients): for the same reason they must be, therefore, chosen always as small as possible.

2.10. Remarks

Numerical differencing methods, especially of second and higher order, are notoriously poor substitutes for analytic differentiation. To overcome, in practice, this difficulty, it is recommended to repeat the computation several times, using different values of $\Delta x_{j,t}$ and $\Delta\hat{a}_k$, for example reducing them more and more, until results do not change any more up to a convenient number of significant digits.

3.THE ASYMPTOTIC DISTRIBUTION OF IMPACT MULTIPLIERS IN STOCHASTIC SIMULATION

Subject of this section will be first of all the definition of impact multipliers in the framework of models with stochastic structural disturbances; then a careful definition of impact multipliers estimated on the basis of the estimated structural coefficients will be introduced, and the related statistical asymptotic properties analyzed; finally the computational procedures will be described.

3.1 Further assumptions

As we have to take into account the presence of the disturbances U in the model (1.1), the basic assumptions described in section (2.1) must be modified, becoming slightly more restrictive (but still remaining very wide and general to allow empirical application).More precisely, all the assumptions, from (*a*) to (*f*), are maintained, with the difference that the structural form functions $F_1, F_2, \dots, F_m$ must be considered defined in the $(m + n + s + m)$ dimensional space (for the inclusion of the m-dimensional U within the arguments).

Furthermore, we assume that:

g) Assumption

Once the system (1.1) has been solved for Y_t, according to the implicit functions theorem, making explicit the functions $f_1, f_2, \dots, f_m$ of the arguments X_t, A and U_t (reduced form equations), the m-dimensional projection of the domain of these functions, into the subspace of U_t, must include any possible value of U_t (in other words,the reduced form equations must be defined for the whole set of values of U_t for which the structural form is defined). This assumption, though strong, is implicitly adopted by econometricians when dealing with the reduced form of non-linear models including disturbances [11].

h) Assumption

The distribution function of U is known, and is such that the conditional expectation of Y_t, given X_t and A, exists and is finite. The necessity of this condition, not only here,but in the more general context of stochastic properties of non-linear models, should be clear when considering that non-linear transformations of random variables (for example normally distributed) can lead to variables whose expectation does not exist (see, for example, [15, pp. 376-377]). The distribution of the structural disturbances,therefore, should be regarded with great care. This is not the case here; what we need is the normality of the asymptotic distribution of the coefficients and this, generally,does not depend on a particular form of the distribution of U [4, p. 191]. Without any loss of generality we can, therefore, simply suppose structural disturbances to have a distribution that satisfies assumption (*h*).

3.2. Definitions

Making use of the reduced form equations,with the properties assumed in section (3.1.), we can now put in formula the definition of the i,j-th impact multiplier at time t already anticipated in the introduction:

$$\pi_{i,j,t} = \frac{\partial E[f_i(X, A, U)|X_t, A]}{\partial x_j} \tag{3.2.1}$$

that is:

$$\pi_{i,j,t} = \left[\frac{\partial}{\partial x_j}\int f_i(X, A, U)\varphi(U)\, dU\right]_{(X_t, A)} \tag{3.2.2}$$

being $\varphi(U)$ the joint density function of the disturbances and being the integration performed over the m-dimensional domain of $\varphi(U)$.Recalling that the disturbances are supposed to be uncorrelated with the predetermined variables and that integration is performed with respect to elements of U, while derivation is with respect to x_j [12, p. 291], [14, p. 81], then the integral is a continuous and differentiable function of X and A with continuous second order mixed derivatives in the (n + s) dimensional projection of the domain of f_i, that is an interval around (X_t, A). Two consequences originate from the previous considerations: first the partial derivative can be moved within the integral, obtaining:

$$(3.2.3)\qquad \pi_{i,j,t} = \left[\int \frac{\partial f_i(X, A, U)}{\partial x_j}\,\varphi(U)\,dU\right]_{(X_t,\,A)} = E\left[\frac{\partial f_i(X, A, U)}{\partial x_j}\,\Big|\,X_t, A\right]$$

that is the partial derivative, with respect to x_j, of the conditional expectation is equal to the conditional expectation of the partial derivative.

The second consequence is that if the set of estimated structural coefficients $\hat{A}$ is used instead of A, for large sample size (see the remarks in section (2.3.)) the "estimated" impact multiplier at time t (in words the partial derivative of the conditional expectation, given X_t and A) can be simply defined as the value assumed by the function "partial derivative of the integral" in equation (3.2.2) (or the function in equation (3.2.3)) at point $(X_t, \hat{A})$. That is:

$$(3.2.4)\qquad \hat{\pi}_{i,j,t} = \left[\frac{\partial}{\partial x_j}\int f_i(X, A, U)\,\varphi(U)\,dU\right]_{(X_t,\,\hat{A})}$$

Also here the partial derivative can be moved within the integral, obtaining:

$$(3.2.5)\qquad \hat{\pi}_{i,j,t} = \left[\int \frac{\partial f_i(X, A, U)}{\partial x_j}\,\varphi(U)\,dU\right]_{(X_t,\,\hat{A})}$$

All the $\pi_{i,j,t}$ and $\hat{\pi}_{i,j,t}$, as in sections (2.1.) and (2.3.), will be arranged in two column vectors of mn elements Π_t and $\hat{\Pi}_t$.

3.3. The asymptotic distribution of $\sqrt{T}(\hat{\Pi}_t - \Pi_t)$

The function "partial derivative of the integral" in equation (3.2.2) (or that in equation (3.2.3)) is continuous and differentiable with respect to the elements of A, with continuous partial derivatives. The theorems already mentioned in sections (2.2.) (Slutsky) and (2.5.) can be applied in the same way leading to:

$$(3.3.1)\qquad \operatorname*{plim}_{T\to\infty} \hat{\Pi}_t = \Pi_t$$

$$(3.3.2)\qquad \sqrt{T}(\hat{\Pi}_t - \Pi_t) \xrightarrow[T\to\infty]{\text{in distribution}} N(0,\ \Omega_t = G_t \Sigma G_t')$$

$$(3.3.3)\qquad \hat{\Omega}_t = \hat{G}_t \hat{\Sigma} \hat{G}_t'$$

$$(3.3.4)\qquad \operatorname*{plim}_{T\to\infty} \hat{\Omega}_t = \Omega_t$$

Some comments are necessary for the matrices G_t and $\hat{G}_t$; the general element of G_t is:

$$(3.3.5)\qquad g_{h,k,t} = \left[\frac{\partial}{\partial a_k}\int \frac{\partial f_i(X, A, U)}{\partial x_j}\,\varphi(U)\,dU\right]_{(X_t,\,A)}$$

As in section (3.2.) it is possible to move the derivative under the integral, obtaining:

$$(3.3.6)\qquad g_{h,k,t} = \left[\int \frac{\partial^2 f_i(X, A, U)}{\partial x_j\,\partial a_k}\,\varphi(U)\,dU\right]_{(X_t,\,A)} = E\left[\frac{\partial^2 f_i(X, A, U)}{\partial x_j\,\partial a_k}\,\Big|\,X_t, A\right]$$

The general element of $\hat{G}_t$ is:

$$(3.3.7) \qquad \hat{g}_{h,k,t} = \left[\int \frac{\partial^2 f_i(X, A, U)}{\partial x_j \, \partial a_k} \, \varphi(U) \, dU \right]_{(X_t, \hat{A})}$$

in perfect analogy with equations (2.6.3) and (2.8.1), with the difference that the conditional expectation replaces the computation of the derivative at the point (X_t, A) or $(X_t, \hat{A})$.

3.4. Computational procedures

In the case of linear models, as already mentioned in the introduction, it is well known [11] that the conditional expectation of Y_t is equal to the non-stochastic solution of the model (given the exact coefficients A) that has been called Y_t° in section (2.1.). An immediate consequence is that, for linear models, also $\Pi^\circ = \Pi$, $G^\circ = G$, $\hat{\Pi}^\circ = \hat{\Pi}$, $\hat{G}^\circ = \hat{G}$ (the subscript t can be dropped, as they are constant over time); the proof follows from equations (3.2.3), (3.2.5), (3.3.6) and (3.3.7) as soon as f_i is replaced by its analytical expression$^{(4)}$.

If the model is non-linear, the Monte Carlo method (stochastic simulation [11]) seems to be the only approach, at the same time correct and practical, to get the values for $\hat{\Pi}_t$ and $\hat{G}_t$.

Equations (3.2.5) and (3.3.7) supply the elements of $\hat{\Pi}_t$ and $\hat{G}_t$ as conditional expectations of partial derivatives, given X_t and $\hat{A}$; therefore, as usual in Monte Carlo methods, they can be computed by means of replicated simulations; in each simulation, first of all a pseudo-random vector of disturbances W, with the same statistical properties of U, is introduced; then $(\Delta\hat{y}_{i,t}/\Delta x_{j,t})$ and $\Delta(\Delta\hat{y}_{i,t}/\Delta x_{j,t})/ /\Delta\hat{a}_k$ are computed at the solution point, at time t; the sample means of these statistics are finally computed. As the distribution function of U (assumption *(h)*, section (3.1.)) is supposed to be known a-priori and a "perfect" random numbers generator is supposed to be available, the computed sample means converge to the elements of $\hat{\Pi}_t$ and $\hat{G}_t$ when the number of replicated solutions increases.

3.5. Remarks

If the parameters of the distribution of U, instead of known as hypothesized in *(h)*, were simply estimated, other problems would arise. Roughly speaking, the approach to the problem should take into account the uncertainty deriving from the estimate of these parameters, besides that related to $\hat{A}$. The development of the theory in this case is beyond the purposes of this paper. Nevertheless, the authors are aware of the fact that, in most cases, the parameters of the distribution of U are estimated, so that the theoretical approach and the empirical applications here developed should be considered, in some sense, conditional on these estimates.

4. APPLICATION

When passing to empirical applications, it is worth recalling that the covariance matrix of the structural coefficients estimated by any consistent estimation method is not the $\hat{\Sigma}$ introduced in section (2.1.), assumption *(f)*: in fact, as well pointed out by Christ [3, p. 379] and Theil [15, p. 497], what is of practical interest is not the covariance matrix of the asymptotic distribution of $\sqrt{T}(\hat{A} - A)$, but an estimated approximate covariance matrix of the distribution of $\hat{A}$ for a finite sample of length T; and this is $\hat{\Sigma}/T$. To be more precise, again following Christ, $\hat{\Sigma}/T$ "is an estimate of the exact variance [covariance matrix, in our case] of a distribution that approximates the distribution of " $\hat{A}$, while the exact distribution of $\hat{A}$ may not have, in general, finite moments [4, p. 193].

Using $\hat{\Sigma}/T$ in equations (3.3.3) (or 2.8.5), the resulting matrix

---

(4) A linear model, using the notation of [7, p. 279], can be represented in its structural form as: $Y\Gamma + XB + U = 0$; the reduced form is: $Y = X\Pi + V$ where $\Pi = -B\Gamma^{-1}$ and $V = -U\Gamma^{-1}$. U disappears after derivation, and the integral of $\varphi(U)$ is equal to 1.

$\hat{\Omega}_t/T$ (or $\hat{\Omega}^\circ_t/T$) is the estimated approximate covariance matrix of the impact multipliers $\hat{\Pi}_t$ (or $\hat{\Pi}^\circ_t$), that is the result of practical interest.

4.1. Application to the Klein-Goldberger revised model

The correctness of the methods described in this paper has been preliminarily tested on a linear model, the Klein-I model. The numerical results were equal, up to 5 significant decimal digits, to those obtained, using the analytical procedure described in [9],by means of Hävenner's program [10], thus convincing the authors of the correctness of the adopted procedures; they were, however,substantially different from the numerical results presented in [9] and this led to a revision of the numerical results of the literature relating to linear models [2].

An application was then undertaken on a larger non-linear model, the Klein-Goldberger revised model [13], estimated by means of Two Stage Least Squares with 4 Principal Components. The model's structure, the meaning of the symbols and the results of the structural estimate can be found in [13]. The only difference in the re-estimation performed by the authors concerns the asymptotic standard errors of the structural coefficients and of the structural equations,where no correction has been done for the degrees of freedom according to [15, p. 451, (5.3)]. Theil's formula [15, p. 500] has been used to compute $\hat{\Sigma}/T$, asymptotic covariance matrix of the structural coefficients.

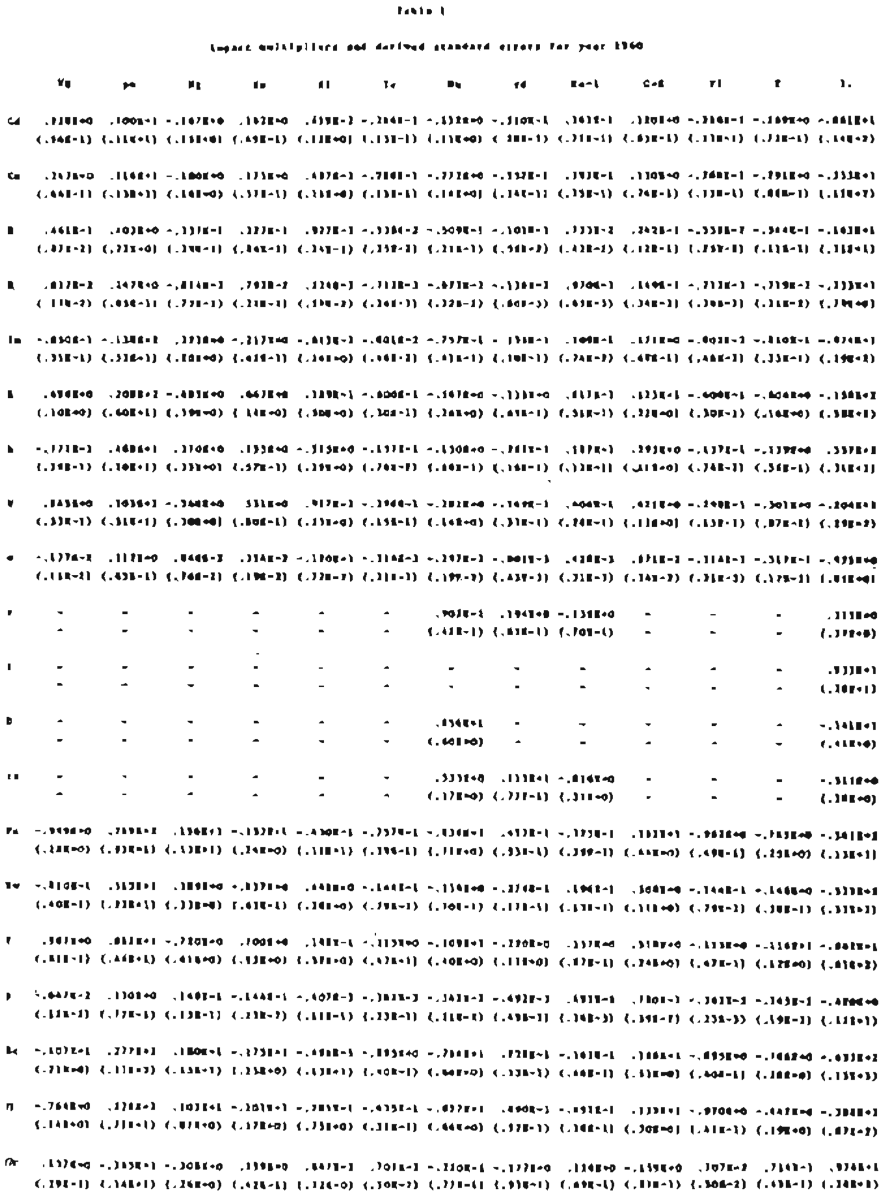
Being the model non-linear in the endogenous variables, the impact multipliers change (slightly, in this model) over time. Table 1 presents, as an example, the impact multipliers and derived standard errors in one year of the sample period (1960).

The computations have been performed as described in section (3.4.) by means of special features introduced into the package described in [1]; the displayed results have been obtained as sample means of 200 replications, each of which requiring 30 seconds of CPU time on a computer IBM/370 model 168. The same computation has

been carried out also by means of non-stochastic simulation, as in section (2.9.2). These results, even if theoretically biased [11], have been found practically equal, up to 3 (sometimes 4) significant decimal digits, to those displayed in the table. In other words, table 1 can be regarded as summarizing the numerical results relating to chapter (2.) as well. Of course, this result must be considered strictly related to the structure of this particular model, which has been recognized to be "nonlinear, but mildly so" [13, p. 188].

Table 1

Impact multipliers and derived standard errors for year 1960



REFERENCES

[1] BLANCHI C., G. CALZOLARI and P. CORSI, "A Program for Stochastic Simulation of Econometric Models", *Econometrica*, 46 (1978), 235-236.

[2] BLANCHI C., G. CALZOLARI and P. CORSI, "A Note on the Numerical Results by Goldberger, Nagar and Odeh", *Econometrica*, 47 (1979), 505-506

[3] CHRIST C. F., *Econometric Models and Methods*. John Wiley, New York, (1967).

[4] DHRYMES P. J., *Econometrics: Statistical Foundations and Applications*. Harper & Row, New York, (1970).

[5] DHRYMES P. J., "Restricted and Unrestricted Reduced Forms: Asymptotic Distribution and Relative Efficiency", *Econometrica*, 41 (1973), 119-134.

[6] EVANS M. K. and L. R. KLEIN, *The Wharton Econometric Forecasting Model*,.Economics Research Unit,University of Pennsylvania,(1968).

[7] GOLDBERGER A.S., *Econometric Theory*,. John Wiley, New York, (1964).

[8] GOLDBERGER A. S., *Impact Multipliers and Dynamic Properties of the Klein-Goldberger Model*, North Holland, Amsterdam, (1970).

[9] GOLDBERGER A. S., A. L. NAGAR and H.S. ODEH, "The Covariance Matrices of Reduced-Form Coefficients and of Forecasts for a Structural Econometric Model", *Econometrica*, 29 (1961), 556-573.

[10] HAVENNER A. M., "Computer Algorithm: Derived Reduced Form Coefficient Covariances", *Econometrica*, 44 (1976), 837.

[11] HOWREY E. P. and H.H. KELEJIAN, "Simulation Versus Analytical Solutions: The Case of Econometric Models", in *Computer Simulation Experiments with Models of Economic Systems*. ed. by T. H. Naylor, John Wiley, New York, (1971), 299-319.

[12] JAMES R.C., *Advanced Calculus*, Wadsworth, Belmont, (1966).

[13] KLEIN L. R., "Estimation of Interdependent Systems in Macroeconometrics", *Econometrica*, 37 (1969), 171-192.

[14] RAO C. R., *Linear Statistical Inference and its Applications*, John Wiley, New York, (1965).

[15] THEIL H., *Principles of Econometrics*, North Holland, Amsterdam, (1971).

[16] WILKS S.S., *Mathematical Statistics*, John Wiley, New York, (1962).