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the conditional autoregressive wishart model for multivariate stock market volatility

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The Conditional Autoregressive Wishart Model for Multivariate Stock Market Volatility

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Abstract

We propose a Conditional Autoregressive Wishart (CAW) model for the analysis of realized covariance matrices of asset returns. Our model assumes a generalized linear autoregressive moving average structure for the scale matrix of the Wishart distribution allowing to accommodate for complex dynamic interdependence between the variances and covariances of assets. In addition, it accounts for symmetry and positive definiteness of covariance matrices without imposing parametric restrictions, and can easily be estimated by Maximum Likelihood. We also propose extensions of the CAW model obtained by including a Mixed Data Sampling (MIDAS) component and Heterogeneous Autoregressive (HAR) dynamics for long-run fluctuations. The CAW models are applied to time series of daily realized variances and covariances for five New York Stock Exchange (NYSE) stocks.

Keywords: Component volatility models, Covariance matrix, Mixed data sampling, Observation-driven models, Realized volatility;

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1. Introduction

Multivariate modeling and forecasting the variances and covariances of asset returns play a prominent role in many practical situations, ranging from portfolio allocation and asset pricing to risk assessment. In practice, the covariance matrix of asset returns is not directly observable and most existing models treat it either as measurable given past observations, such as multivariate GARCH models introduced by Bollerslev et al. (1988), or as an inherently latent quantity, such as multivariate stochastic volatility (SV) models introduced by Harvey et al. (1994). Excellent overviews on multivariate GARCH and SV models can be found in Bauwens et al. (2006) and Asai et al. (2006), respectively. An alternative approach of covariance estimation and modeling, which has attracted substantial interest in recent years, uses high-frequency returns data to construct the realized variances and covariances as precise estimates for the variances and covariances of low-frequency returns (see e.g., Andersen et al. 2003 and Barndorff-Nielsen and Shephard, 2004). As such, the observed realized variances and covariances can be modeled directly as advocated, for example, by Andersen et al. (2003). Multivariate models for the realized covariance matrix should satisfy two important requirements, namely that the predicted covariance matrices remain positive definite, and, second, that the specification is parsimoniously parameterized yet empirically realistic with the ability to account for the strong serial dependence typically observed for realized variances and covariances.

Pioneering multivariate approaches to model the dynamics in the realized covariance matrix are found in Gourieroux et al. (2009), Jin und Maheu (2009), Chiriac and Voev (2010), and Bauer and Vorkink (2010). The specification proposed by Gourieroux et al. (2009) extends the Wishart distribution of the sample covariance for i.i.d. multivariate Gaussian random variables by allowing the multivariate Gaussian random variables to be serially correlated. Under the resulting Wishart Autoregressive (WAR) process the realized covariance has a transition distribution which is noncentral Wishart with a non-centrality parameter depending on lagged covariances and a fixed scale matrix. As such the WAR model naturally accommodates the positive definiteness of predicted covariance matrices without any parametric restrictions. The approach followed by Jin und Maheu (2009) also relies on a Wishart transition distribution, but assumes a central rather than a noncentral Wishart distribution, and decomposes its scale matrix into multiplicative components, which are driven by sample averages of lagged realized covariance matrices. In order to account for the positive definiteness, the approaches of Chiriac and Voev (2010) and Bauer and Vorkink (2010) use appropriate transformations of the covariance matrix. The former approach is based upon a Cholesky decomposition of the covariance matrix and assumes fractionally integrated processes for the individual elements of the Cholesky factor. The latter approach transforms the covariance matrix by using the matrix logarithm function and specifies the individual elements of the transformation as functions of latent factors driven by lagged volatilities and lagged returns.

In the present paper, we adopt a conditional autoregressive Wishart (CAW) approach and propose a new flexible dynamic model for the realized covariance matrix of asset returns. Its baseline specification assumes a simple generalized linear autoregressive structure for the scale matrix of the Wishart distribution allowing to account for nontrivial serial dependencies in the variances and covariances. In particular, under our model the predicted covariance matrix depends on lagged covariance matrices as well as on their lagged predictions. As such it presents a dynamic generalization of the models proposed by Gourieroux et al. (2009) and Jin und Maheu (2009), where the predicted covariance matrix is specified as a function of lagged covariances only. Our model also accounts for symmetry and positive definiteness of the predicted covariance matrices without imposing parametric restrictions and can easily be estimated by Maximum Likelihood (ML). In addition, it allows us to derive in a straightforward manner conditions for stationarity and other important time series properties. A further advantage of our approach is that its baseline specification can easily be generalized. The extensions of the baseline CAW model we explore, are specifically designed to capture the long-run fluctuations in the variances and covariances. For this purpose, we combine the CAW specification with the mixed data sampling (MIDAS) approach of Engle et al. (2008) and, alternatively, with an heterogeneous autoregressive (HAR) component as used by Corsi (2009) and Bonato et al. (2009).

The rest of the paper is organized as follows. Section 2 introduces the baseline CAW model and discusses its stochastic properties. Extensions of the baseline model are proposed in Section 3. The empirical application to NYSE data is presented in Section 4. Section 5 concludes. The proofs are provided in the Appendix.

2. Conditional Autoregressive Wishart (CAW) Model

2.1. CAW(p,q) Model

Consider the stochastic, symmetric positive definite matrix $R_t = (r_{ij,t})$ of realized covariances with dimension $n \times n$ recorded at time t $(t = 1, ..., T)$. The matrix R_t given the past history $\mathcal{F}_{t-1} = \{R_{t-1}, R_{t-2}, ...\}$ is assumed to follow a central Wishart distribution

$$R_t|\mathcal{F}_{t-1} \sim \mathcal{W}_n(\nu, S_t/\nu), \tag{1}$$

where $\nu > 0$ is the scalar degree of freedom and S_t/ν is the $n \times n$ symmetric, positive definite scale matrix with $S_t = (s_{ij,t})$, such that the conditional mean and covariances are (see Muirhead, 1982)

$$E(R_t|\mathcal{F}_{t-1}) = S_t, \qquad \mathrm{Cov}(r_{ij,t}, r_{lm,t}|\mathcal{F}_{t-1}) = \frac{1}{\nu}(s_{il,t} \cdot s_{jm,t} + s_{im,t} \cdot s_{jl,t}). \tag{2}$$

The density function for $R_t|\mathcal{F}_{t-1}$ has the form

$$f(R_t|\mathcal{F}_{t-1}) = \frac{|S_t/\nu|^{-\nu/2}|R_t|^{(\nu-n-1)/2}}{2^{\nu n/2}\pi^{n(n-1)/4}\prod_{i=1}^{n}\Gamma([\nu+1-i]/2)} \exp\{-\frac{1}{2}\mathrm{tr}(\nu S_t^{-1}R_t)\}, \tag{3}$$

where $\Gamma(\cdot)$ denotes the Gamma function. In order to account for serial- and cross-correlation across the elements in R_t we assume that the matrix-variate process S_t follows the linear recursion of order (p,q)

$$S_t = CC' + \sum_{i=1}^{p} B_i S_{t-i} B_i' + \sum_{j=1}^{q} A_j R_{t-j} A_j', \tag{4}$$

where C is a $n \times n$ lower-triangular matrix and A_j, B_i are $n \times n$ parameter matrices. This recursion of order (p, q) resembles the BEKK-GARCH(p, q) specification of Engle and Kroner (1995) for the conditional covariance in models for multivariate returns, and has the appealing property to guarantee the symmetry and positive-definiteness of the conditional mean S_t essentially without imposing parametric restrictions on (C, A_j, B_i).

The CAW(p, q) model defined by Equations (1) and (4) can be interpreted as a state-space model with S_t as a state variable measured by the observable matrix R_t and with measurement density given by Equation (3). The corresponding measurement equation obtains as (see Muirhead, 1982, p. 95, Theorem 3.2.11)

$$R_t = \frac{1}{\nu} S_t^{1/2} U_t (S_t^{1/2})', \qquad U_t \sim \mathcal{W}_n(I_n, \nu), \tag{5}$$

where $S_t^{1/2}$ denotes the lower-triangular Cholesky factor of S_t such that $S_t = S_t^{1/2}(S_t^{1/2})'$ and U_t represents the measurement error following a standardized Wishart distribution with ν degree of freedom and a scale matrix given by the identity matrix I_n. This allows us to interpret S_t as the 'true' integrated covariance for a broad class of multivariate continuous-time stochastic volatility models which is, under fairly general conditions, consistently estimated by the realized covariance R_t (see Barndorff-Nielsen and Shephard, 2004). Within this context, the matrix U_t in Equation (5) plays the role of the corresponding estimation error.

The CAW(p, q) model as specified is unidentified. Sufficient conditions for identification are that the main diagonal elements of C, denoted by c_{ll}, and the first diagonal element for each of the matrices A_j, B_i denoted by $a_{11,j}$ and $b_{11,i}$ are restricted to be positive (see Engle and Kroner, 1995).

The CAW(p, q) model is designed to capture nontrivial dynamic interactions across $n(n+1)/2$ elements of the realized covariance matrix for the returns of n assets. It involves $n(n+1)/2 + (p + q)n^2 + 1$ parameters. For the multivariate GARCH class of models with such a highly parameterized covariance process, the estimation can be computationally very demanding when the number of assets increases. However, note that the CAW model is directly fitted to $n(n+1)/2$ realized (co-)variances, while the corresponding GARCH models are estimated based on the returns of n assets only. Hence, the number of observations per parameter is significantly larger for the CAW model than that for the corresponding GARCH specification, such that the curse-of-dimensionality problem appears to be less acute for the CAW model. Furthermore, the number of CAW-parameters can be reduced by imposing restrictions on the matrices (A_j, B_i). A natural restriction is to impose a diagonal structure on the dynamics of S_t by assuming that A_j and B_j are diagonal matrices. This reduces the number of parameters to $n(n+1)/2 + (p+q)n + 1$.

The CAW model is related to the Wishart autoregressive (WAR) model introduced by Gourieroux et al. (2009), which is based upon a conditional non-central Wishart distribution for R_t. Under the WAR model, it is the matrix of non-centrality parameters of the Wishart distribution which is assumed to depend on lagged R_ts, rather than the scale matrix as under the CAW model. In particular, the WAR(p) process is characterized by ν degrees of freedom, a fixed scale matrix S, and a matrix of non-centrality parameters given by $S^{-1}(\sum_{i=1}^{p} A_i R_{t-i} A_i')$ such that

$$E(R_t|\mathcal{F}_{t-1}) = \nu \cdot S + \sum_{i=1}^{p} A_i R_{t-i} A_i'. \tag{6}$$

Hence, the WAR(p) and CAW$(0,q)$ model with $q = p$ have conditional expectations for the covariance matrix of the same form and with the same number of parameters. This allows us to interpret the CAW(p,q) model as a dynamic generalization of the WAR(p) specification. Note, however, that the two model specifications are nonnested, except for the trivial cases, that the WAR(0) obtains as a restricted CAW(p,q) and the CAW$(0,0)$ represents a restricted WAR(p) model.

2.2. Stochastic Properties of the CAW(p,q) Model

For the discussion of the stochastic properties of the CAW model, it proves convenient to use its VARMA representation which obtains from the recursion (4).

Let vech$(\cdot)$ denote the operator that stacks the lower triangular portion, including the diagonal of a matrix into a vector, and let vec$(\cdot)$ denote the operator that stacks all columns of a matrix into a vector. Then defining $r_t = \text{vech}(R_t)$, $s_t = \text{vech}(S_t)$ and $c = \text{vech}(CC')$, the vector representation of recursion (4) is

$$s_t = c + \sum_{i=1}^{p} \mathcal{B}_i s_{t-i} + \sum_{j=1}^{q} \mathcal{A}_j r_{t-j}, \tag{7}$$

where ($\mathcal{A}_j$, $\mathcal{B}_i$) are $k \times k$ matrices with $k = n(n+1)/2$. They obtain as

$$\mathcal{A}_j = L_n(A_j \otimes A_j)D_n, \qquad \mathcal{B}_i = L_n(B_i \otimes B_i)D_n, \tag{8}$$

where L_n and D_n denote the elimination and duplication matrix, respectively, defined so that $\text{vec}(X) = D_n\text{vech}(X)$ and $\text{vech}(X) = L_n\text{vec}(X)$ for any symmetric $n \times n$ matrix X (see Lütkepohl, 1996, p. 9-10).

Notice further that r_t can be written as

$$r_t = \mathrm{E}(r_t|\mathcal{F}_{t-1}) + v_t = s_t + v_t, \quad \text{with} \quad \mathrm{E}(v_t) = 0, \quad \mathrm{E}(v_t v_s') = 0 \quad \forall s \neq t, \tag{9}$$

where v_t is a martingale difference. By plugging $s_{t-i} = r_{t-i} - v_{t-i}$ $(i = 1, ..., p)$ into Equation (7), the CAW(p, q) can be represented as a VARMA$(\max(p, q), p)$ model:

$$r_t = c + \sum_{i=1}^{\max(p,q)} (\mathcal{B}_i + \mathcal{A}_i) r_{t-i} - \sum_{j=1}^{p} \mathcal{B}_j v_{t-j} + v_t, \tag{10}$$

with $\mathcal{A}_{q+1} = \cdots = \mathcal{A}_p = 0$ if $q < p$ and $\mathcal{B}_{p+1} = \cdots = \mathcal{B}_q = 0$ if $p < q$. From the VARMA representation (10) we immediately obtain the conditions for the existence of the unconditional mean for the CAW(p, q) model, which are given in the following proposition.

Proposition 1. *The unconditional mean of the CAW(p, q) model (1) - (4) is finite iff all eigenvalues of the matrix* $\Psi_1 = \sum_{i=1}^{\max(p,q)} (\mathcal{B}_i + \mathcal{A}_i)$ *are less than 1 in modulus. In that case the unconditional mean is given by*

$$E(r_t) = \bar{r} = \left(I_k - \sum_{i=1}^{\max(p,q)} (\mathcal{B}_i + \mathcal{A}_i) \right)^{-1} c. \tag{11}$$

The following discussion of the second moments of r_t, which represent the fourth moments of the asset returns, is based on the VMA(∞) representation of the CAW(p, q) model and resembles that of Hafner (2003) who derives the existence conditions and the analytic expressions for the fourth moments of multivariate GARCH processes.

The VMA(∞) representation which obtains from the VARMA$(\max(p, q), p)$ specification (10) is given by (see Lütkepohl, 2005, p. 424)

$$r_t = \bar{r} + \sum_{i=0}^{\infty} \Phi_i v_{t-i}, \quad \text{with} \quad \Phi_i = -\mathcal{B}_i + \sum_{j=1}^{i} (\mathcal{A}_j + \mathcal{B}_j) \Phi_{i-j}, \quad i = 1, 2, ..., \quad \Phi_0 = I_k. \tag{12}$$

Then the autocovariance and variance of r_t, provided that they exist, have the form

$$\Gamma(\tau) = \mathrm{E}[(r_t - \bar{r})(r_{t-\tau} - \bar{r})'] = \sum_{i=0}^{\infty} \Phi_{\tau+i} \mathrm{E}(v_t v_t') \Phi_i', \qquad \tau = 1, 2, \ldots, \tag{13}$$

and

$$\Gamma(0) = \mathrm{E}(r_t r_t') - \bar{r}\bar{r}' = \sum_{i=0}^{\infty} \Phi_i \mathrm{E}(v_t v_t') \Phi_i'. \tag{14}$$

The following lemma establishes the particular relationship between the second moment of r_t and the second moment of the state process s_t obtained under the conditional Wishart distribution in Equation (1).

Lemma 1. *Under the CAW(p,q) model (1) - (4) and the assumption that $E(r_t r_t')$ exists,*

$$vec[E(r_t r_t')] = (\Omega + I_{k^2}) vec[E(s_t s_t')], \tag{15}$$

with

$$\Omega = \frac{1}{\nu}(L_n \otimes L_n)[I_{n^2} \otimes (I_{n^2} + K_{nn})](I_n \otimes K_{nn} \otimes I_n)(D_n \otimes D_n), \tag{16}$$

where K_{nn} denotes the commutation matrix (as given in Lütkepohl, 1996 p. 115).

Based upon this result, we can derive necessary and sufficient conditions for the existence of $\mathrm{E}(r_t r_t')$ and its explicit form, which are given in proposition 2.

Proposition 2. *The unconditional second moment for the CAW(p,q) model (1) - (4) is finite iff all eigenvalues of the matrix $\Psi_2 = \sum_{i=1}^{\infty}(\Phi_i \otimes \Phi_i)\Omega$ are less than 1 in modulus. In that case the second moment is given by*

$$vec[E(r_t r_t')] = (\Omega + I_{k^2}) \left(I_{k^2} - \sum_{i=1}^{\infty} (\Phi_i \otimes \Phi_i)\Omega \right)^{-1} vec(\bar{r}\bar{r}'). \tag{17}$$

Proposition 2 implies that under the CAW(p,q) model the process $\{R_t\}$ is covariance stationary if and only if the eigenvalues of the matrix $\sum_{i=1}^{\infty}(\Phi_i \otimes \Phi_i)\Omega$ are less than 1 in modulus. The explicit form of the unconditional variance obtains by inserting $\bar{r}$ and $\mathrm{E}(r_t r_t')$ as given by Equations (11) and (17), respectively, into $\Gamma(0) = \mathrm{E}(r_t r_t') - \bar{r}\bar{r}'$. The unconditional variance of the martingale difference v_t, which is required for the computation of the autocovariance for r_t in Equation (13), is given by

$$\mathrm{vec}[\mathrm{E}(v_t v_t')] = \left(I_{k^2} - (\Omega + I_{k^2})^{-1}\right) \mathrm{vec}[\mathrm{E}(r_t r_t')], \tag{18}$$

which follows from Equation (15) and the fact that $\mathrm{E}(v_t v_t') = \mathrm{E}(r_t r_t') - \mathrm{E}(s_t s_t')$ (see Equation 9).

If the order of the CAW(p,q) model is small, it is possible to obtain a more convenient expression for the second moment than that in Equation (17). For the CAW$(1,1)$ specification, for example, we derive the following results:

Corollary 1. *The unconditional second moment for the CAW(*$1,1$*) model is finite iff all eigenvalues of the matrix*

$$\Delta = (\mathcal{A}_1 \otimes \mathcal{A}_1)(\Omega + I_{k^2}) + (\mathcal{B}_1 \otimes \mathcal{A}_1) + (\mathcal{A}_1 \otimes \mathcal{B}_1) + (\mathcal{B}_1 \otimes \mathcal{B}_1) \tag{19}$$

are less than 1 in modulus. In that case the mean and the second moment are given by

$$E(r_t) = \bar{r} = (I_k - (\mathcal{A}_1 + \mathcal{B}_1))^{-1} c, \tag{20}$$

$$vec[E(r_t r_t')] = (\Omega + I_{k^2})\,(I_{k^2} - \Delta)^{-1}\, vec\left(cc' + c\bar{r}'(\mathcal{A}_1 + \mathcal{B}_1)' + (\mathcal{A}_1 + \mathcal{B}_1)\bar{r}c'\right). \tag{21}$$

2.3. Estimation of the CAW(p,q) Model and Diagnostics

Estimation of the parameters $\psi = (\nu, \mathrm{vech}(C),' \mathrm{vec}(B_1)', ..., \mathrm{vec}(B_p)', \mathrm{vec}(A_1)', ..., \mathrm{vec}(A_q)')'$ of the CAW(p,q) model can be carried out by maximizing the log-likelihood function using numerical techniques routinely available in standard software packages. The log-likelihood function obtains as

$$\begin{aligned}\mathcal{L}(\psi) = \sum_{t=1}^{T} \Bigg\{ & -\frac{\nu n}{2}\ln(2) - \frac{n(n-1)}{4}\ln(\pi) - \sum_{i=1}^{n} \ln\Gamma\left(\frac{\nu+1-i}{2}\right) \\ & -\frac{\nu}{2}\ln|\frac{S_t}{\nu}| + \left(\frac{\nu-n-1}{2}\right)\ln|R_t| - \frac{1}{2}\mathrm{tr}(\nu S_t^{-1} R_t)\Bigg\}.\end{aligned} \tag{22}$$

The ML-estimates presented below are obtained by using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) optimization procedure. Positivity of the diagonal elements c_{ll}, $a_{11,i}$, and $b_{11,j}$ are enforced by estimating $\sqrt{c}_{ll}$, $\sqrt{a}_{11,i}$, and $\sqrt{b}_{11,j}$. The estimation of an unrestricted CAW$(2,2)$ for the covariance of 5 assets with 116 parameters, e.g., requires approximately 150 BFGS iterations and takes of the order of 10 minutes on a CORE 2 Duo Intel 2.7 GHz processor using GAUSS on Windows XP.

Diagnostic tests for the fitted models are conducted from the vector of standardized residuals

$$e_t^* = \mathrm{Var}(r_t|\mathcal{F}_{t-1})^{-1/2}\left[r_t - \mathrm{E}(r_t|\mathcal{F}_{t-1})\right], \tag{23}$$

where $\mathrm{Var}(r_t|\mathcal{F}_{t-1})^{-1/2}$ denotes the inverse Cholesky factor of $\mathrm{Var}(r_t|\mathcal{F}_{t-1})$. For a correctly specified model, the standardized residuals $e_{ij,t}^*$ in the vector e_t^* are serially uncorrelated and not predictable based on past residuals $e_{t-\tau}^*$. In order to test this implication, we regress each series $e_{ij,t}^*$ on a constant and $\{e_{t-\tau}^*\}_{\tau\geq 1}$ and test the hypotheses that all regression coefficients other than the constant are equal to zero by using the F-statistic.

3. Extensions of the Baseline CAW Model

Due to the nonlinear dynamics and the number $k = n(n+1)/2$ of different elements in R_t a low order CAW(p, q) model can be expected to accommodate a large variety of dynamic patterns in the variances and covariances of asset returns, including a long-memory type of persistence. The ability to accommodate long-memory type dependence patterns can be expected due to the well-known fact that low-order multivariate VARMA models typically imply univariate ARMA specifications of a very high order (see, e.g., Cubadda et al., 2009). Nevertheless, it might be useful to consider extensions of the basic CAW(p, q) model, introduced in Section 2.1, which are specifically designed to capture the long-run movements of volatilities and co-volatilities. In the following, we introduce two of such extensions, where we combine the CAW model with the MIDAS (mixed data sampling) approach of Engle et al. (2008) and Colacito et al. (2009), and with an HAR (heterogenous autoregressive) component as used by Corsi (2009) and Bonato et al. (2009).

3.1 MIDAS-CAW Model

Component GARCH models with short- and long-run components have been proven to be useful representations of complex dependence structures in the volatility (see, e.g., Engle and Lee, 1999). Under the GARCH-MIDAS component model, recently proposed by Engle et al. (2008), the short-run component is specified as a mean-reverting GARCH process based on daily returns that moves around a long-run component driven by realized volatilities computed over a monthly, quarterly or semi-annual basis. Following this idea, we decompose the scale matrix S_t for the daily covariance matrix R_t in Equations (1)-(3) into a secular component M_t and a mean-reverting short-run component S_t^*:

$$E(R_t|\mathcal{F}_{t-1}) = S_t = C_t S_t^* C_t', \quad \text{with} \quad M_t = C_t C_t', \tag{24}$$

where C_t is the lower-triangular Cholesky factor of the secular component M_t. The short-run component S_t^* is assumed to follow a covariance-stationary CAW(p,q) process with $\mathrm{E}(S_t^*) = I_n$, namely,

$$S_t^* = \left(I_n - \sum_{j=1}^{q} A_j A_j' - \sum_{i=1}^{p} B_i B_i' \right) + \sum_{i=1}^{p} B_i S_{t-i}^* B_i' + \sum_{j=1}^{q} A_j \left[C_{t-j}^{-1} R_{t-j} (C_{t-j}')^{-1} \right] A_j'. \tag{25}$$

The long-run component M_t is specified as a parsimonious multivariate extension of the univariate MIDAS polynomial proposed by Engle et al. (2008). This multivariate extension is applied to realized covariance matrices ($\bar{R}_t^{(m)}$) computed over a horizon of m trading days using rolling samples that change from day to day. In particular, our MIDAS component is given by the following weighted sum of L lags of m-period realized covariances:

$$M_t = \bar{C}\bar{C}' + \theta \cdot \sum_{\ell=1}^{L} \varphi_\ell(\omega) \cdot \bar{R}_{t,\ell}^{(m)}, \tag{26}$$

$$\bar{R}_{t,\ell}^{(m)} = \sum_{\tau=t-m\cdot\ell}^{t-m\cdot(\ell-1)-1} R_\tau, \qquad \ell = 1, ..., L, \tag{27}$$

where $\bar{C}$ denotes a lower triangular matrix, θ is a slope parameter restricted to be non-negative, and $\varphi_\ell(\cdot)$ represents a scalar-valued function of weights. Following Engle et al. (2008), we specify the function $\varphi_\ell(\cdot)$ using so-called Beta weights, defined as

$$\varphi_\ell(\omega) = \frac{\left(1 - \frac{\ell}{L}\right)^{\omega-1}}{\sum_{j=1}^{L} \left(1 - \frac{j}{L}\right)^{\omega-1}}, \tag{28}$$

where the parameter ω controls the weights' decay pattern. In the empirical application discussed below, we use for the m-period realized covariance in Equation (27) a window length m of one month

(20 trading days) and take a lag order L of 12 such that the MIDAS filter aggregates daily covariances of one year.

Note that the MIDAS filter in Equations (26)-(28) with the same weighting scheme and the same slope across all series imposes a common pattern in the long-run dynamics for all elements in the covariance matrix, thereby preserving parsimony for the specification of the secular component. This restriction is justified by the finding that the long-run movements of the individual realized (co)variances (see Figure 1) appear to be very similar.

An interesting alternative to the parametric MIDAS filter for the secular component would be to use for M_t a non-parametric function which smoothes realized covariances in the spirit of the multivariate component GARCH approach recently proposed by Hafner and Linton (2010). Note, however, that this approach does not directly allow to forecast covariances since such a non-parametric smoother involves future covariances.

3.2 HAR-CAW Model

A related alternative to the MIDAS component specification of Engle et al. (2008) for highly persistent volatility processes is the HAR model proposed by Corsi (2009). It accommodates the long-memory type of dependence patterns in daily volatility by an hierarchical autoregressive specification including lagged daily as well as weekly and monthly volatilities. Bonato et al. (2009) extend this univariate approach by combing the multivariate WAR process with HAR dynamics. Following this idea, we consider the following specification for the scale matrix S_t in Equations (1)-(3):

$$S_t = CC' + AR_{t-1}A' + A^{(w)}\bar{R}_{t-1}^{(w)}A^{(w)\prime} + A^{(bw)}\bar{R}_{t-1}^{(bw)}A^{(bw)\prime} + A^{(m)}\bar{R}_{t-1}^{(m)}A^{(m)\prime}, \tag{29}$$

with $\bar{R}_{t-1}^{(x)}$ denoting the realized covariance computed over a time window $x = \{w, bw, m\}$, where w stands for the weekly (5 days), bw for the biweekly (10 days), and m for the monthly (20 days) horizon. A and $A^{(x)}$ are $n \times n$ parameter matrices.

Using the vector representation of specification (29), it can be written as a restricted CAW(0,20) model:

$$
\begin{aligned}
s_t \quad = \quad & c + [\mathcal{A} + \mathcal{A}^{(w)} + \mathcal{A}^{(bw)} + \mathcal{A}^{(m)}] \cdot r_{t-1} + \cdots + [\mathcal{A}^{(w)} + \mathcal{A}^{(bw)} + \mathcal{A}^{(m)}] \cdot r_{t-5} \\
& + [\mathcal{A}^{(bw)} + \mathcal{A}^{(m)}] \cdot r_{t-6} + \cdots + [\mathcal{A}^{(bw)} + \mathcal{A}^{(m)}] \cdot r_{t-10} \\
& + [\mathcal{A}^{(m)}] \cdot r_{t-11} + \cdots + [\mathcal{A}^{(m)}] \cdot r_{t-20},
\end{aligned} \tag{30}
$$

where the matrices $\mathcal{A}$, $\mathcal{A}^{(x)}$ are obtained as described in Equation (8). Note that this representation of the HAR-CAW model allows to use directly the results discussed in Section 2.2 for the derivation of its stochastic properties.

4. Empirical Application

4.1 Data

The CAW model introduced in Sections 2 and 3 is used to analyze the dynamics of daily realized covariance matrices for five stocks traded at the New York Stock Exchange: American Express (AXP), Citigroup (C), General Electric (GE), Home Depot (HD), and International Business Machines (IBM). The data set is the same as that used by Chiriac and Voev (2010). The daily realized covariance matrix can be computed as $R_t = \sum_{j=1}^{M} y_{t,j} y_{t,j}'$, where $y_{t,j}$ is the vector of returns for the $n = 5$ stocks computed for the jth 5-minute interval of trading day t between 9:30 a.m. and 4:00 p.m. This realized covariance measure is further refined, as in Chiriac and Voev, by averaging over 30 subsampling subgrids per day in order to exploit the data richness more efficiently and to cope with market microstructure noise. The sample period starts at January 1, 2000, and ends on July 30, 2008, covering 2156 trading days. The first 240 covariance matrices are reserved as starting values for the initialization of the MIDAS filter discussed in Section 3.1. This leaves a sample of $T = 1916$ observations. Figure 1 shows time series plots of the realized variances and covariances. Descriptive statistics are provided in Table 1. The empirical distribution of the variances and covariances is heavily skewed to the right and is highly leptokurtic. The autocorrelation function of the variances and covariances plotted in Figure 2 dies out at a very slow rate indicating very strong serial correlation.

4.2 Estimation Results

4.2.1 Baseline CAW Model

Our first attempt to describe the data uses the baseline CAW(p,q) model as given by Equations (1) and (4), with lag orders (p,q) ranging from (0,1) to (2,2). The upper panel of Table 2 reports the values of the maximized log-likelihood function, Schwarz's (1978) information criterion, the largest eigenvalues of the estimated matrices $\Psi_1 = \sum_{i=1}^{\max(p,q)}(\mathcal{B}_i + \mathcal{A}_i)$ and $\Psi_2 = \sum_{i=1}^{\infty}(\Phi_i \otimes \Phi_i)\Omega$, and the results of diagnostics checks on the standardized residuals e_t^*, obtained for the fitted unrestricted CAW specifications.

Of the unrestricted models, the Schwarz-preferred specification is the CAW(2,2) with 116 parameters. The largest eigenvalues of the estimated Ψ_1 and Ψ_2 matrices imply that the unrestricted CAW(2,2) is the only specification which is covariance stationary with finite unconditional first- and second-order moments. However, the largest eigenvalue of Ψ_1 is very close to unity indicating an extremely high persistence in the conditional mean. The result of the F-test for predictability of the standardized residuals $e_{ij,t}^*$ using 50 lags reveals that the unrestricted CAW(2,2) model successfully accounts for the dynamics in the variance of four stocks and in six covariances, but points towards residual predictability of the variance for one stock and four covariances. Hence, the model seems to capture the dynamic interdependence in most, though not all, elements of the realized covariance matrix. This is corroborated by the autocorrelation function of the standardized residuals (see Figure 3), indicating that the CAW(2,2) model dramatically reduces the serial correlation for the raw data in Figure 2. The ML parameter estimates for the unrestricted CAW(2,2) model are provided in the upper panel of Table 3. They indicate that, apart from $a_{11,2}$ and $b_{11,2}$, all diagonal elements of the autoregressive matrices B_i and A_j are statistically significant at the 1% level, while many of the off-diagonal entries are not significantly different from zero. This seems to suggest the use of a diagonal CAW specification obtained by restricting the matrices B_i and A_j in Equation (4) to be diagonal.

The estimation results for the diagonal CAW models are reported in the lower panels of Tables

2 and 3. In fact, the Schwarz criterion typically favors the parsimonious diagonal CAW(p,q) specifications over their unrestricted counterparts and selects the diagonal CAW(2,1) with 31 parameters as the best CAW formulation. However, this specification is clearly dominated by the unrestricted CAW(2,2) model in terms of accounting for serial correlation in the covariance matrix. In particular, under the diagonal CAW(2,1) only 7 out of 15 elements of the residual vector e_t^* pass the F-test for predictability using the 1% significance level, while under the unrestricted CAW(2,2) 10 elements pass this test. Finally, the largest eigenvalues of the estimated matrices characterizing the unconditional first- and second-order moments indicate that the diagonal CAW(2,1) is not covariance stationary due to an explosive behavior in the second-order moments.

4.2.2 MIDAS-CAW and HAR-CAW Model

In order to explicitly account for long-run fluctuations we generalize the baseline CAW models by including a MIDAS component as described in Equations (24)-(28). The upper panel of Table 4 summarizes goodness-of-fit measures for those CAW extensions. For each lag order (p, q), the inclusion of the MIDAS component significantly increases the value of the maximized log-likelihood function, indicative for a much better fit. The Schwarz-preferred MIDAS specification is the diagonal MIDAS-CAW(2,2)-model. The largest eigenvalues of Ψ_1 and Ψ_2, obtained for all MIDAS specifications, are all less than one and are noticeably smaller than under the baseline CAW models. This suggests that the inclusion of the MIDAS component has significant effects on the dynamic structure of the model. The same finding is reported by Engle et al. (2008) for a univariate GARCH model, where the inclusion of a MIDAS filter substantially reduces the persistence in the GARCH component. The results of the F-test for predictability of the standardized residuals show that the MIDAS specifications better account for the dynamics in the realized covariance matrix than their baseline counterparts in Table 2. However, none of the fitted MIDAS-CAW specifications can completely capture the serial and cross correlation in all the variances and covariances. The parameter estimates of the unrestricted and diagonal MIDAS-CAW(2,2) model provided in Table 5 show that under both specifications the slope parameter for the MIDAS component θ is significantly larger than zero, which underscores the importance of allowing for a long-run component. Figure 4 shows plots of the predicted variance, covariance and correlation for the AXP and C stock together with their respective long-run MIDAS components obtained under the MIDAS-CAW(2,2) model. As expected, the MIDAS component explains a significant part of the variation in the predicted conditional (co)variances and correlations.

As an alternative to the MIDAS approach, we explore the HAR-CAW model as specified by Equations (1) and (29). The goodness-of-fit measures provided in the lower part of Table 4 reveal that the Schwarz criterion strongly favors the unrestricted and diagonal HAR-CAW models over the baseline CAW formulations in Table 2. However, the Schwarz criterion for both HAR specifications remains substantially larger than that for the best MIDAS alternative, the diagonal MIDAS-CAW(2,2) model. Furthermore, the F-test for residual predictability shows that the HAR specifications also have difficulties capturing the full dynamics in the realized covariance matrix.

All in all, the Schwarz criterion together with the F-test for residual predictability indicate that the diagonal MIDAS-CAW(2,2) model represents the preferred CAW specification for our data. In order to more completely capture the dynamics in the realized covariance than it is possible with this diagonal MIDAS-CAW model one could further increase the lag order. However, neither the Schwarz criterion nor the F-statistic obtained for diagonal MIDAS-CAW models with lag orders larger than $(p, q) = (2, 2)$ (not presented here) indicate a significant improvement of the in-sample fit.

4.3 Forecasting Results

We now compare the out-of-sample-forecast performance of the CAW-specifications and alternative forecasting models, focusing on forecast horizons of $h = \{1, 5, 10\}$ days. The forecast of the h-period-ahead realized covariance, denoted by $\hat{R}_{t+h} = \mathrm{E}(R_{t+h}|\mathcal{F}_t)$, can be compared with the ex-post realization of the realized covariance R_{t+h}. For the forecast experiment we use the last 240 trading days in our sample as the out-of-sample window. Every model is re-estimated daily and new forecasts are generated based upon the updated parameter estimates.

For the baseline CAW and the HAR-CAW models, given the parameter estimates, the h-step-ahead forecasts are easily obtained by recursion. In particular, it can be shown for the baseline

CAW(p,q) model that

$$\mathrm{E}(r_{t+h}|\mathcal{F}_t) = \mathrm{E}(s_{t+h}|\mathcal{F}_t) = c + \mathcal{B}_1\mathrm{E}(s_{t+h-1}|\mathcal{F}_t) + \cdots + \mathcal{B}_p\mathrm{E}(s_{t+h-p}|\mathcal{F}_t) + \mathcal{A}_1\mathrm{E}(r_{t+h-1}|\mathcal{F}_t) + \cdots + \mathcal{A}_q\mathrm{E}(r_{t+h-q}|\mathcal{F}_t), \tag{31}$$

where

$$\mathrm{E}(s_{t+h-\tau}|\mathcal{F}_t) = \begin{cases} s_{t+h-\tau}, & \text{if} \quad \tau \geq h-1 \\ \mathrm{E}(r_{t+h-\tau}|\mathcal{F}_t), & \text{if} \quad \tau < h-1. \end{cases} \tag{32}$$

The forecasts under the HAR-CAW model are obtained by exploiting its restricted CAW(0,20) representation as given by Equation (30).

Under the MIDAS-CAW model the functional relationship between $\mathrm{E}(r_{t+1}|\mathcal{F}_t) = s_{t+1}$ and $\mathcal{F}_t = \{r_t, r_{t-1}, \ldots\}$ as specified in Equations (24)-(28) is non-linear. This implies that the forecast $\mathrm{E}(r_{t+h}|\mathcal{F}_t)$ for $h > 1$ depends upon the entire h-step-ahead forecast distribution $f(r_{t+h}|\mathcal{F}_t)$, which is not available in a closed form. Hence, we approximate this distribution by Monte-Carlo (MC) simulation from the convolution of the h one-step-ahead forecast distributions $\{f(r_{t+\tau}|\mathcal{F}_{t+\tau-1})\}_{\tau=1}^{h}$ as specified by the model and evaluate the forecasts $\mathrm{E}(r_{t+h}|\mathcal{F}_t)$ by MC integration. The MC integration is implemented using 10,000 artificial trajectories simulated from the convolution of the one-step-ahead forecast distributions.

As alternative forecasting models for the daily realized covariance, we consider a simple Exponentially Weighted Moving Average (EWMA) approach applied to the realized covariance matrices and a BEKK-GARCH(p,q) model fitted to the daily stock returns. The EWMA model, which is often used in risk management systems like RiskMetrics (see J.P. Morgan, 1996) to forecast variances and covariances, is given by

$$\mathrm{E}(r_t|\mathcal{F}_{t-1}) = (1-\lambda)r_{t-1} + \lambda\mathrm{E}(r_{t-1}|\mathcal{F}_{t-2}), \tag{33}$$

where we set λ to its typical value given by 0.94. For this approach the h-step-ahead forecast obtains as $\mathrm{E}(r_{t+h}|\mathcal{F}_t) = \mathrm{E}(r_{t+h-1}|\mathcal{F}_t)$.

The BEKK-GARCH(p,q) model for a vector of n stock returns, denoted by y_t, has the form

$y_t = \Sigma_t^{1/2} \upsilon_t$, where $\upsilon_t \sim \mathcal{N}(0, I_n)$. The conditional covariance matrix Σ_t is specified as

$$\Sigma_t = D_0 D_0' + \sum_{i=1}^{p} D_i \Sigma_{t-i} D_i' + \sum_{j=1}^{q} G_j [y_{t-j} y_{t-j}'] G_j', \tag{34}$$

where D_0 is a lower triangular $n \times n$ matrix and D_i, G_j are $n \times n$ matrices. The h-step-ahead forecasts $\hat{R}_{t+h} = \mathrm{E}(\Sigma_{t+h} | y_t, y_{t-1}, ...)$ are obtained by a recursion, which is analogous to that used to generate forecasts of the baseline CAW model (see Equations 31 and 32).

In order to assess the predictive accuracy for a given model we follow Ledoit et al. (2003) and use the root-mean-square error (RMSE) based on the Frobenius norm of the forecast error, given by

$$FN_h = \frac{1}{T_h} \sum_t ||R_{t+h} - \hat{R}_{t+h}|| = \frac{1}{T_h} \sum_t \left[\sum_{i,j} (r_{ij,t+h} - \hat{r}_{ij,t+h})^2 \right]^{1/2}, \tag{35}$$

where T_h is number of forecast periods.

Table 6 contains the results on the forecasting accuracy of the different models measured by the Frobenius norm. It appears that the diagonal MIDAS-CAW(2,2) model outperforms the other CAW specifications as well as the EWMA and the BEKK-GARCH models at the short horizons ($h = 1, 5$), whereas at the longer horizon ($h = 10$) the diagonal MIDAS-CAW(2,1) formulation yields the most accurate forecasts. However note, that the differences in forecasting accuracy between the the best CAW specification, the EWMA and the best BEKK-GARCH model become smaller if we move from the short to the longer forecast horizon. This might reflect the fact that it is extremely hard to accurately predict variances and covariances at longer horizons during a highly volatile regime with many 'outliers' as we face it in our forecasting period covering the last 240 trading days of our sample (see Figure 1). Overall, the out-of-sample performance of our CAW approach appears to be favorable in relation to the alternative forecasting models.

5. Conclusions

In this paper, we propose a Conditional Autoregressive Wishart (CAW) model for the realized covariance of asset returns. The model is designed to represent nontrivial temporal interdependencies across variances and covariances and is based upon a generalized linear autoregressive moving average (GLARMA) structure. Since our model ensures positive-definite covariance matrices without imposing parameter constraints and can easily be estimated by maximum likelihood, it represents a convenient framework for the analysis of high-dimensional variance-covariance processes. A further advantage offered by our CAW approach is that its baseline specification is easily generalizable. The extensions of the baseline CAW model we explored include a mixed data sampling (MIDAS) component and heterogeneous autoregressive (HAR) dynamics for long-run fluctuations.

The empirical application to daily realized covariance matrices for the returns of five stocks shows that the MIDAS-CAW specification dominates the baseline CAW model and the HAR-CAW alternative in terms of accounting for the observed dynamic behavior of the realized covariance as well as in terms of out-of-sample covariance predictions. Furthermore, we find that the MIDAS-CAW model is able to remove most, though not all, of the observed serial dependence in the variances and covariances. This indicates that in order to more completely capture the highly complex dynamics in the realized covariance matrix it might be useful to consider further alternative extensions of the baseline CAW model. Extensions being currently explored include CAW specifications with Markov switching regimes allowing the parameters of the GLARMA specification (4) to differ across two regimes. A further extension we currently investigate allows for asymmetric effects of positive and negative news on the covariance matrix using a specification which is in line with that of Cappiello et al. (2006) for a multivariate GARCH model.

Appendix: Proofs

Proof for Lemma 1. First, we derive the functional form of the conditional variance of $r_t|\mathcal{F}_{t-1}$ obtained under the conditional Wishart distribution. Since $r_t = \text{vech}(R_t) = L_n \text{ vec}(R_t)$, we can write

$$\text{Var}\left(r_t \mid \mathcal{F}_{t-1}\right) = \text{var}\left(\text{vech}(R_t) \mid \mathcal{F}_{t-1}\right) = \text{Var}\left(L_n \text{vec}(R_t) \mid \mathcal{F}_{t-1}\right) = L_n \text{Var}\left(\text{vec}(R_t) \mid \mathcal{F}_{t-1}\right) L_n'. \quad (36)$$

Under the Wishart distribution in Equation (1) the conditional variance $\text{Var}\left(\text{vec}(R_t) \mid \mathcal{F}_{t-1}\right)$ is (see Muirhead 1982, p. 90)

$$\text{Var}\left(\text{vec}(R_t) \mid \mathcal{F}_{t-1}\right) = \frac{1}{\nu}\left(I_{n^2} + K_{nn}\right)\left(S_t \otimes S_t\right), \quad (37)$$

where K_{nn} is the commutation matrix defined so that $K_{mn}\text{vec}(X) = \text{vec}(X')$ for any $m \times n$ matrix X. Due to $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$ we obtain from Equations (36) and (37)

$$\text{vec}[\text{Var}\left(r_t \mid \mathcal{F}_{t-1}\right)] = \frac{1}{\nu}\left(L_n \otimes L_n\right) \text{vec}\left[\left(I_{n^2} + K_{nn}\right)\left(S_t \otimes S_t\right)\right]. \quad (38)$$

Since $\text{vec}(AB) = (I_p \otimes A)\text{vec}(B)$ for A $(m \times n)$, B $(n \times p)$ and $\text{vec}(A \otimes B) = (I_n \otimes K_{sm} \otimes I_r)[\text{vec}(A) \otimes \text{vec}(B)]$ for A $(m \times n)$, B $(r \times s)$ (see Lütkepohl 1996, p. 97), we can write

$$\begin{aligned} \text{vec}\left[\text{Var}\left(r_t \mid \mathcal{F}_{t-1}\right)\right] &= \frac{1}{\nu}\left(L_n \otimes L_n\right) \cdot \left[I_{n^2} \otimes \left(I_{n^2} + K_{nn}\right)\right] \text{vec}\left(S_t \otimes S_t\right) \quad (39) \\ &= \frac{1}{\nu}\left(L_n \otimes L_n\right) \cdot \left[I_{n^2} \otimes \left(I_{n^2} + K_{nn}\right)\right]\left(I_n \otimes K_{nn} \otimes I_n\right)\left[\text{vec}(S_t) \otimes \text{vec}(S_t)\right], \end{aligned}$$

where $\text{vec}(S_t) \otimes \text{vec}(S_t) = (D_n \otimes D_n)\text{vec}(s_t s_t')$. Thus

$$\begin{aligned} \text{vec}[\text{Var}\left(r_t \mid \mathcal{F}_{t-1}\right)] &= \Omega \, \text{vec}\left(s_t s_t'\right), \quad \text{with} \quad (40) \\ \Omega &= \frac{1}{\nu}\left(L_n \otimes L_n\right)\left[I_{n^2} \otimes \left(I_{n^2} + K_{nn}\right)\right]\left(I_n \otimes K_{nn} \otimes I_n\right)\left(D_n \otimes D_n\right). \quad (41) \end{aligned}$$

The law of iterated expectations applied to $\mathrm{Var}(r_t \mid \mathcal{F}_{t-1}) = \mathrm{E}(r_t r_t' \mid \mathcal{F}_{t-1}) - s_t s_t'$ leads to

$$\mathrm{E}\left[\mathrm{var}\left(r_t \mid \mathcal{F}_{t-1}\right)\right] = \mathrm{E}\left(r_t r_t'\right) - \mathrm{E}\left(s_t s_t'\right), \tag{42}$$

such that $\mathrm{E}\left(r_t r_t'\right) = \mathrm{E}\left[\mathrm{var}\left(r_t \mid \mathcal{F}_{t-1}\right)\right] + \mathrm{E}\left(s_t s_t'\right)$. Taking vecs and accounting for Equation (40) we obtain

$$\begin{aligned} \mathrm{vec}[\mathrm{E}\left(r_t r_t'\right)] &= \mathrm{vec}\left(\mathrm{E}\left[\mathrm{Var}\left(r_t \mid \mathcal{F}_{t-1}\right)\right]\right) + \mathrm{vec}\left[\mathrm{E}\left(s_t s_t'\right)\right] = \mathrm{E}\left(\mathrm{vec}\left[\mathrm{Var}\left(r_t \mid \mathcal{F}_{t-1}\right)\right]\right) + \mathrm{vec}\left[\mathrm{E}\left(s_t s_t'\right)\right] \\ &= \Omega\,\mathrm{vec}\left[\mathrm{E}\left(s_t s_t'\right)\right] + \mathrm{vec}\left[\mathrm{E}\left(s_t s_t'\right)\right] = \left(\Omega + I_{k^2}\right)\mathrm{vec}\left[\mathrm{E}\left(s_t s_t'\right)\right], \end{aligned}$$

which completes the proof. □

Proof for Proposition 1. The VARMA representation in Equation (10) allows us to write $\mathrm{E}\left(r_t\right) = c + \sum_{i=1}^{\max(p,q)}\left(\mathcal{B}_i + \mathcal{A}_i\right)\mathrm{E}\left(r_t\right)$, which can be solved for $\mathrm{E}\left(r_t\right) = \bar{r}$ to obtain Equation (11) iff all eigenvalues of the matrix $\sum_{i=1}^{\max(p,q)}(\mathcal{B}_i + \mathcal{A}_i)$ are less than 1 in modulus. □

Proof for Proposition 2. Since $\mathrm{Var}(r_t) = \Gamma(0) = \mathrm{E}(r_t r_t') - \bar{r}\bar{r}'$ and $\mathrm{E}(v_t v_t') = \mathrm{E}(r_t r_t') - \mathrm{E}(s_t s_t')$ (see Equation 9), we obtain from covariance Equation (14)

$$\mathrm{E}\left(r_t r_t'\right) = \sum_{i=0}^{\infty}\Phi_i\left[\mathrm{E}\left(r_t r_t'\right) - \mathrm{E}\left(s_t s_t'\right)\right]\Phi_i' + \bar{r}\bar{r}', \tag{43}$$

$$\mathrm{vec}[\mathrm{E}\left(r_t r_t'\right)] = \sum_{i=0}^{\infty}\left(\Phi_i \otimes \Phi_i\right)\mathrm{vec}\left[\mathrm{E}\left(r_t r_t'\right) - \mathrm{E}\left(s_t s_t'\right)\right] + \mathrm{vec}(\bar{r}\bar{r}'). \tag{44}$$

Applying the result of Lemma 1 that $\mathrm{vec}[\mathrm{E}\left(r_t r_t'\right)] = \left(\Omega + I_{k^2}\right)\mathrm{vec}\left[\mathrm{E}\left(s_t s_t'\right)\right]$, we obtain

$$\mathrm{vec}[\mathrm{E}\left(r_t r_t'\right)] = \sum_{i=0}^{\infty}\left(\Phi_i \otimes \Phi_i\right)\Omega\,\mathrm{vec}[\mathrm{E}\left(s_t s_t'\right)] + \mathrm{vec}(\bar{r}\bar{r}'). \tag{45}$$

Since $\Phi_0 = I_k$, Equation (45) can be rewritten as

$$\mathrm{vec}(\bar{r}\bar{r}') = \left(I_{k^2} - \sum_{i=1}^{\infty}\left(\Phi_i \otimes \Phi_i\right)\Omega\right)\mathrm{vec}[\mathrm{E}\left(s_t s_t'\right)]. \tag{46}$$

Iff all eigenvalues of the matrix $\Psi_2 = \sum_{i=1}^{\infty} (\Phi_i \otimes \Phi_i)\, \Omega$ are less than 1 in modulus, Equation (46) can be solved for $\text{vec}[\text{E}\,(s_t s_t')]$ to obtain

$$\text{vec}[\text{E}\left(s_t s_t'\right)] \;=\; \left(I_{k^2} - \sum_{i=1}^{\infty} (\Phi_i \otimes \Phi_i)\, \Omega\right)^{-1} \text{vec}(\bar{r}\bar{r}'). \tag{47}$$

Inserting Equation (47) into $\text{vec}[\text{E}(r_t r_t')] = (\Omega + I_{k^2})\ \text{vec}[\text{E}(s_t s_t')]$ completes the proof. $\square$

Proof for Corollary 1. The mean $\text{E}(r_t)$ is obtained directly from Proposition 1. Furthermore, note that the CAW(1,1) model can be written as a VARMA(1,1) with a VMA(∞) representation characterized by the parameters (see Equation 12)

$$\Phi_0 = I_k, \quad \Phi_1 = \mathcal{A}_1, \quad \Phi_2 = (\mathcal{A}_1 + \mathcal{B}_1)\mathcal{A}_1,\ \ldots,\ \Phi_i = (\mathcal{A}_1 + \mathcal{B}_1)^{i-1}\mathcal{A}_1.$$

Then using the result that $AC \otimes BD = (A \otimes B)(C \otimes D)$ (see Lütkepohl 1996, p. 19), we can write under the assumption that the second moment exists

$$\begin{aligned}
\sum_{i=1}^{\infty} (\Phi_i \otimes \Phi_i) &= \sum_{i=1}^{\infty} [(\mathcal{A}_1 + \mathcal{B}_1)^{i-1}\mathcal{A}_1] \otimes [(\mathcal{A}_1 + \mathcal{B}_1)^{i-1}\mathcal{A}_1] \\
&= \sum_{i=0}^{\infty} \left[(\mathcal{A}_1 + \mathcal{B}_1) \otimes (\mathcal{A}_1 + \mathcal{B}_1)\right]^i (\mathcal{A}_1 \otimes \mathcal{A}_1) \\
&= [I_{k^2} - (\mathcal{A}_1 + \mathcal{B}_1) \otimes (\mathcal{A}_1 + \mathcal{B}_1)]^{-1} (\mathcal{A}_1 \otimes \mathcal{A}_1) \\
&= Q^{-1}(\mathcal{A}_1 \otimes \mathcal{A}_1) \quad \text{(say)}.
\end{aligned} \tag{48}$$

Plugging Equation (48) into Equation (17) we obtain

$$\begin{aligned}
\text{vec}[\text{E}(r_t r_t')] &= (\Omega + I_{k^2})\, (I_{k^2} - [\mathcal{A}_1 + \mathcal{B}_1] \otimes [\mathcal{A}_1 + \mathcal{B}_1] - (\mathcal{A}_1 \otimes \mathcal{A}_1)\, \Omega)^{-1}\, Q\, \text{vec}(\bar{r}\bar{r}') \qquad (49) \\
&= (\Omega + I_{k^2})\, (I_{k^2} - \Delta)^{-1}\, Q\, \text{vec}(\bar{r}\bar{r}'),
\end{aligned}$$

where $Q\,\text{vec}(\bar{r}\bar{r}') = \text{vec}\left(cc' + c\bar{r}'(\mathcal{A}_1 + \mathcal{B}_1)' + (\mathcal{A}_1 + \mathcal{B}_1)\bar{r}c'\right)$, which completes the proof. $\square$

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Table 1. Descriptive Statistics for the Realized Variances and Covariances.

Stock	Mean	Max.	Min.	Std. dev.	Skewness	Kurtosis
			Realized Variance			
AXP (r_{11})	3.44	57.58	.07	4.68	4.23	32.78
C (r_{22})	3.61	119.86	.11	5.91	7.65	108.49
GE (r_{33})	2.43	51.40	.10	3.17	4.90	46.97
HD (r_{44})	3.46	51.38	.16	3.97	3.92	28.01
IBM (r_{55})	2.26	56.91	.12	3.05	5.68	67.60
			Realized Covariance			
C-AXP (r_{21})	1.59	37.66	−0.55	2.78	5.32	46.13
GE-AXP (r_{31})	1.11	26.32	−1.47	1.85	5.90	58.08
HD-AXP (r_{41})	1.16	27.66	−2.46	1.97	5.33	47.60
IBM-AXP (r_{51})	.92	23.43	−.79	1.46	5.65	55.89
GE-C (r_{32})	1.24	41.69	−.58	2.12	7.02	91.59
HD-C (r_{42})	1.27	27.34	−.93	2.17	5.02	39.51
IBM-C (r_{52})	1.03	36.73	−3.27	1.74	5.33	109.96
HD-GE (r_{43})	1.04	26.85	−1.14	1.70	5.90	59.20
IBM-GE (r_{53})	.90	24.05	−.33	1.44	5.76	57.77
IBM-HD (r_{54})	.87	18.32	−1.20	1.34	5.21	44.18

Note: The number of observations for each (co)variance series is $T = 1916$.

Table 2. Optimized Log-likelihood and Diagnostics for the Baseline CAW Model.

						p-value for the F-test for predictability of the residuals e^*_{ijt}														
(p,q)	dim(ψ)	Log-lik.	Schwarz	meig1	meig2	e^*_{11}	e^*_{21}	e^*_{31}	e^*_{41}	e^*_{51}	e^*_{22}	e^*_{32}	e^*_{42}	e^*_{52}	e^*_{33}	e^*_{43}	e^*_{53}	e^*_{44}	e^*_{54}	e^*_{55}
						Unrestricted CAW(p,q)														
(0,1)	41	-10681.2	21672.4	1.022		**.00**	.03	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**
(1,1)	66	-7794.4	16087.7	1.001		.80	.42	.42	.05	.07	.03	.25	**.00**	**.00**	**.00**	**.00**	**.00**	.18	**.00**	.01
(1,2)	91	-7567.3	15822.5	.997	1.635	.90	.59	.67	.04	.08	.04	.24	.02	**.00**	**.00**	**.00**	**.00**	.47	**.00**	.04
(2,1)	91	-7493.7	15675.3	.996	1.086	.89	.61	.65	.07	.07	.02	.16	.02	**.00**	**.00**	**.00**	**.00**	.37	**.00**	.04
(2,2)	116	-7391.7	15660.2	.995	0.817	.87	.57	.71	.02	.13	.03	.32	**.01**	.02	**.00**	**.00**	**.00**	.41	**.00**	.11
						Diagonal CAW(p,q)														
(0,1)	21	-10715.0	21588.8	1.021		**.00**	.01	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**	**.00**
(1,1)	26	-7839.9	15876.3	.999	9.968	.83	.56	.45	.03	.17	.04	.27	**.01**	**.00**	**.00**	**.00**	**.00**	.21	**.00**	.01
(1,2)	31	-7827.2	15888.6	.997	2.159	.78	.10	.03	**.00**	.05	**.01**	.02	**.01**	**.00**	**.00**	**.00**	**.00**	.24	**.00**	.01
(2,1)	31	-7711.8	15657.9	.998	3.397	.81	.24	.19	**.00**	.02	.05	.10	**.01**	**.00**	**.00**	**.00**	**.00**	.30	**.00**	**.01**
(2,2)	36	-7704.6	15681.3	.996	1.621	.80	.20	.19	**.00**	**.01**	.04	.10	.01	**.00**	**.00**	**.00**	**.00**	.27	**.00**	**.01**

Note: dim(ψ) is the number of parameters; meig1 and meig2 are the maximum eigenvalues of the matrices $\Psi_1 = \sum_{i=1}^{\max(p,q)}(\mathcal{B}_i + \mathcal{A}_i)$ and $\Psi_2 = \sum_{i=1}^{\infty}(\Phi_i \otimes \Phi_i)\Omega$, respectively, see Equations (11) and (17); bold p-values indicate significance at the 1% level.

Table 3. ML-Parameter Estimates for the Baseline CAW(2,2) model.

Unrestricted CAW(2,2)

Param.	Estimate					Param.	Estimate				
A_1	.774*	.060*	.014	−.018	.010	A_2	.003	.191*	−.057	.052	−.251*
	.033	.649*	−.004	−.008	−.037		−.072*	−.096*	.198*	−.012	.053*
	.019	.049*	.583*	−.021	−.013		−.043	−.019	.234*	−.036	−.079*
	.030	.067*	−.035	.559*	−.022		.050	−.016	.061	.131*	−.107*
	.016	.041*	−.032*	−.029*	.597*		.115*	−.087*	.036	.059*	.115*
B_1	.860*	−.034	.001	.014	−.092*	B_2	.002	−.163*	−.037	.043*	−.154*
	.008	.659*	−.175*	.070*	−.010		−.005	−.306*	−.132*	.010	−.077*
	.012	−.026	.506*	.040*	.120*		.068*	.022	−.571*	.069*	.002
	−.030	−.099*	.057*	.617*	.227*		−.149*	−.272*	−.021	.360*	.042
	.022	−.007	.067*	.017	.408*		.106*	−.058*	.009	.096*	−.625*
ν	22.51*										

Diagonal CAW(2,2)

Param.	Estimate					Param.	Estimate				
A_1	.790*	.644*	.609*	.591*	.593*	A_2	.000	.073	.018	.101	−.093*
B_1	.806*	.577*	.609*	.582*	.587*	B_2	.656*	.493*	.494*	.525*	.518*
ν	22.10*										

Note: The estimated model is described in Equations (1) and (4). The first diagonal element for A_1, A_2, B_1, B_2 represents the estimates of $\sqrt{a}_{11,1}$, $\sqrt{a}_{11,2}$, $\sqrt{b}_{11,1}$, $\sqrt{b}_{11,2}$.

* Significant at the 1% level.

Table 4. Optimized Log-likelihood and Diagnostics for the MIDAS-CAW and HAR-CAW Model.

						p-value for the F-test for predictability of the residuals e^*_{ijt}														
(p,q)	dim(ψ)	Log-lik.	Schwarz	meig1	meig2	e^*_{11}	e^*_{21}	e^*_{31}	e^*_{41}	e^*_{51}	e^*_{22}	e^*_{32}	e^*_{42}	e^*_{52}	e^*_{33}	e^*_{43}	e^*_{53}	e^*_{44}	e^*_{54}	e^*_{55}
						Unrestricted MIDAS-CAW(p,q)														
(0,1)	43	-8104.8	16534.5	.519	.034	.10	.21	**.01**	**.01**	.02	**.00**	.01	**.00**	**.00**	**.00**	**.00**	**.00**	.21	**.00**	**.00**
(1,1)	68	-7447.8	15409.4	.845	.037	.88	.35	.26	**.01**	**.01**	.06	.05	**.01**	**.00**	.02	**.00**	**.00**	.50	**.00**	.08
(1,2)	93	-7252.8	15208.3	.866	.041	.83	.63	.49	**.01**	.14	.05	.21	**.01**	.01	.04	**.00**	**.00**	.61	**.00**	.19
(2,1)	93	-7297.1	15297.2	.881	.050	.83	.57	.29	.04	.08	.01	.24	.01	**.00**	.06	**.00**	**.00**	.57	**.00**	.10
(2,2)	118	-7157.7	15207.2	.880	.040	.84	.66	.61	.05	.25	.04	.21	.01	**.01**	.03	**.00**	**.00**	.70	**.00**	.22
						Diagonal MIDAS-CAW(p,q)														
(0,1)	23	-8117.3	16408.4	.516	.034	.11	.27	**.01**	**.01**	.02	**.00**	.01	**.00**	**.00**	**.00**	**.00**	**.00**	.18	**.00**	**.00**
(1,1)	28	-7505.9	15223.6	.881	.056	.65	.43	.35	.04	.17	.02	.16	**.01**	**.01**	.02	**.00**	**.00**	.38	**.00**	.04
(1,2)	33	-7465.1	15179.7	.886	.059	.69	.57	.46	.03	.32	.02	.20	.02	**.00**	.02	**.00**	**.00**	.37	**.01**	.04
(2,1)	33	-7456.6	15162.5	.889	.060	.69	.56	.42	.08	.25	.02	.18	.01	**.00**	.02	**.00**	**.00**	.40	**.00**	.03
(2,2)	38	-7412.5	15112.2	.882	.065	.63	.57	.35	.05	.25	.03	.19	.04	**.01**	.02	**.00**	**.00**	.37	**.00**	.04
						Unrestricted HAR-CAW														
	116	-7337.8	15552.3	.986	.592	.73	.73	.30	.02	.16	.05	.05	.01	**.00**	.02	**.00**	**.00**	.56	**.00**	.03
						Diagonal HAR-CAW														
	36	-7606.8	15485.7	.988	1.005	.45	.80	.08	.02	.05	.02	.13	**.01**	**.00**	**.01**	**.00**	**.00**	.32	**.00**	.01

Note: dim(ψ) is the number of parameters; meig1 and meig2 are the maximum eigenvalues of the matrices $\Psi_1 = \sum_{i=1}^{\max(p,q)}(\mathcal{B}_i + \mathcal{A}_i)$ and $\Psi_2 = \sum_{i=1}^{\infty}(\Phi_i \otimes \Phi_i)\Omega$, respectively, see Equations (11) and (17); bold p-values indicate significance at the 1% level.

Table 5. ML-Parameter Estimates for the MIDAS-CAW(2,2) model.

Unrestricted MIDAS-CAW(2,2)

Param.	Estimate					Param.	Estimate				
A_1	.767*	.064*	−.004	−.015	.003	A_2	.000	−.002	.152*	−.121*	.276*
	.014	.604*	−.034	−.004	−.049*		.049	.094*	−.080*	.075*	−.180*
	.016	.013	.554*	−.013	−.013		.184*	−.037	−.193*	.141*	.058
	.003	.003	−.007	.569*	.004		−.063*	−.196*	−.005	−.080	−.054
	−.005	.005	−.010	−.019	.575*		−.171*	.153*	−.074*	−.090*	−.097*
B_1	.745*	−.021	.209*	.093	−.060	B_2	.017	−.219*	.090	.106*	.206*
	.027	.603*	−.007	.049	.076		.233*	−.125	.194*	−.050	−.102
	.048	.022	.420*	.104	.250*		−.173*	.165*	.016	−.015	−.234*
	.048	.077*	.007	.471*	.075		−.041	.022	−.191*	−.405*	.023
	−.036	−.001	.061	.046	.539*		−.003	.083*	.016	.091	.347*
$\sqrt{\theta}$	.927*					ω	8.413*				
ν	22.819*										

Diagonal MIDAS-CAW(2,2)

Param.	Estimate					Param.	Estimate				
A_1	.788*	.628*	.583*	.588*	.575*	A_2	−.000	.231*	−.169	−.292*	−.393*
B_1	.829*	.522*	.586*	.371*	.066	B_2	−.261	.401*	.277*	.474*	.574*
$\sqrt{\theta}$	.920*					ω	7.271*				
ν	22.481*										

Note: The estimated model is described in Equations (1) and (24)-(28). The first diagonal element for A_1, A_2, B_1, B_2 represents the estimates of $\sqrt{a}_{11,1}$, $\sqrt{a}_{11,2}$, $\sqrt{b}_{11,1}$, $\sqrt{b}_{11,2}$.

* Significant at the 1% level.

Table 6. Evaluation of Forecasting Accuracy in Terms of RMSE.

Model	(p,q)	Frobenius norm of forecast error		
		$h=1$	$h=5$	$h=10$
Unrestricted CAW	(0,1)	8.369	12.135	16.743
	(1,1)	7.405	9.516	10.962
	(1,2)	7.379	9.431	10.766
	(2,1)	7.338	9.518	10.945
	(2,2)	7.256	9.368	10.744
Diagonal CAW	(0,1)	8.043	10.010	11.612
	(1,1)	7.304	9.313	10.612
	(1,2)	7.276	9.239	10.508
	(2,1)	7.239	9.283	10.569
	(2,2)	7.212	9.223	10.487
Unrestricted MIDAS-CAW	(0,1)	7.418	9.734	10.850
	(1,1)	7.367	9.529	10.607
	(1,2)	7.325	9.478	10.601
	(2,1)	7.297	9.516	10.678
	(2,2)	7.283	9.427	10.623
Diagonal MIDAS-CAW	(0,1)	7.376	9.757	10.846
	(1,1)	7.192	9.176	10.373
	(1,2)	7.192	9.153	10.396
	(2,1)	7.164	9.105	**10.324**
	(2,2)	**7.161**	**9.102**	10.337
Unrestricted HAR-CAW		7.357	9.718	11.146
Diagonal HAR-CAW		7.184	9.311	10.600
EWMA		8.749	9.842	10.865
Unrestricted BEKK-GARCH	(0,1)	10.165	11.546	11.749
	(1,1)	10.232	10.989	11.685
	(1,2)	9.251	9.987	10.855
	(2,1)	10.005	10.641	11.288
	(2,2)	9.442	10.141	10.856
Diagonal BEKK-GARCH	(0,1)	10.774	11.570	11.765
	(1,1)	9.964	10.776	11.355
	(1,2)	9.476	10.319	10.945
	(2,1)	9.555	10.412	11.004
	(2,2)	9.153	10.074	10.753

Note: Reported are the average Frobenius norm of the forecast error as given by Equation (35). Bold numbers indicate smallest value of the average Frobenius norm.

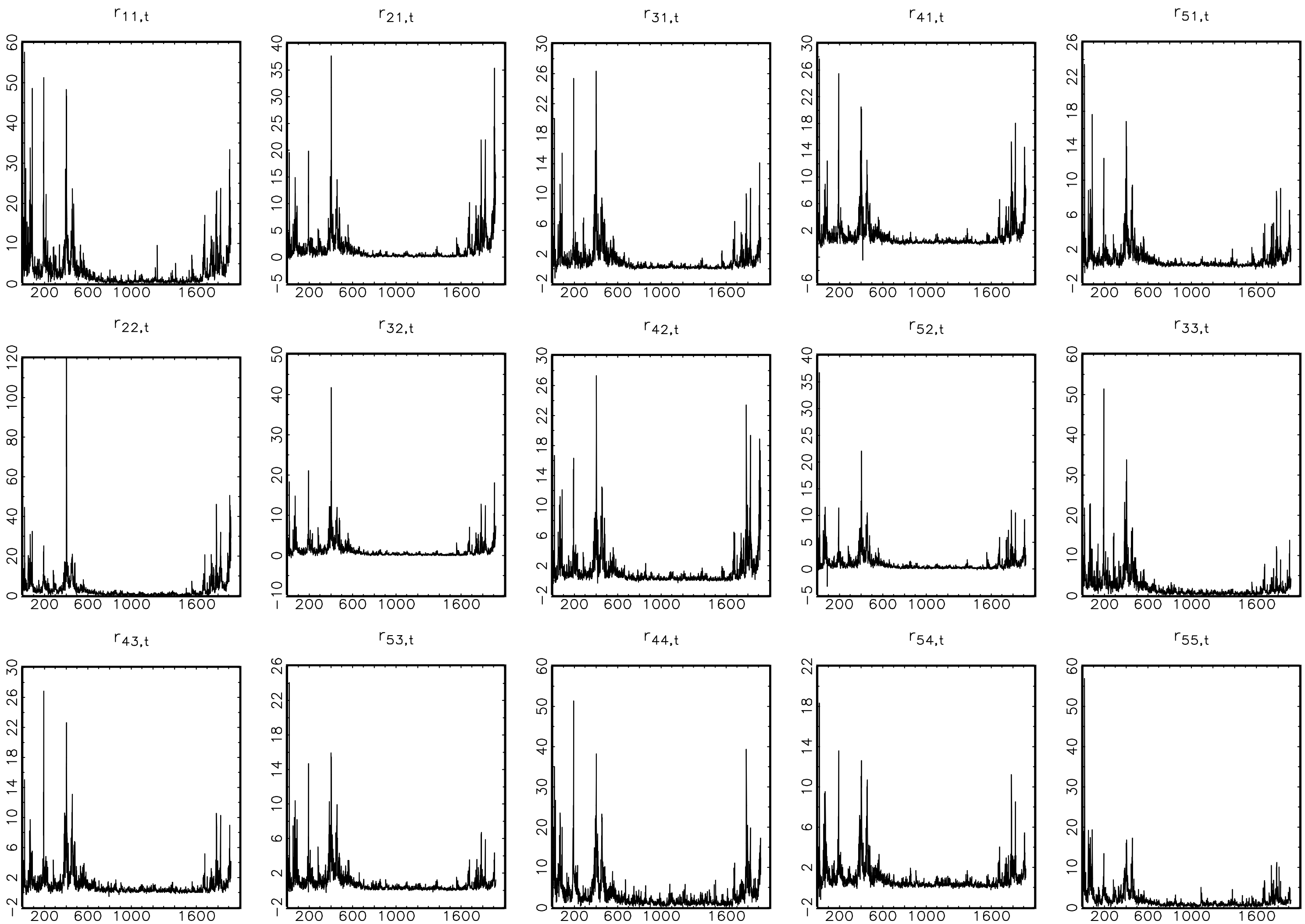

Figure 1. Time series of daily realized variances and covariances $r_{ij,t}$ for AXP ($i = 1$), C ($i = 2$), GE ($i = 3$), HD (i=4), and IBM (i=5) stock.

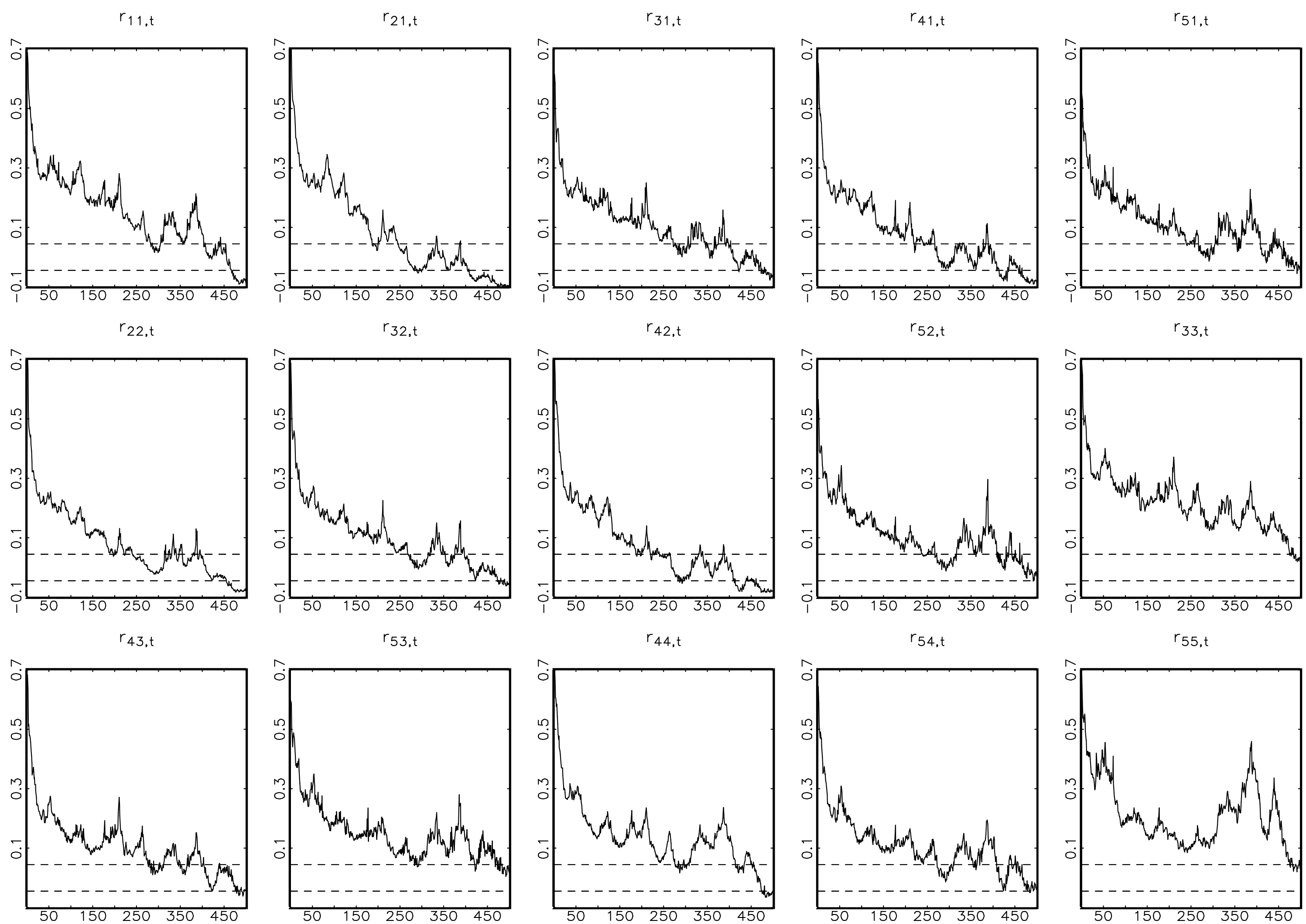

Figure 2. Sample autocorrelation function of daily realized variances and covariances $r_{ij,t}$ for AXP ($i=1$), C ($i=2$), GE ($i=3$), HD (i=4), and IBM (i=5) stock; the dashed lines indicate the 95% Bartlett confidence bands.

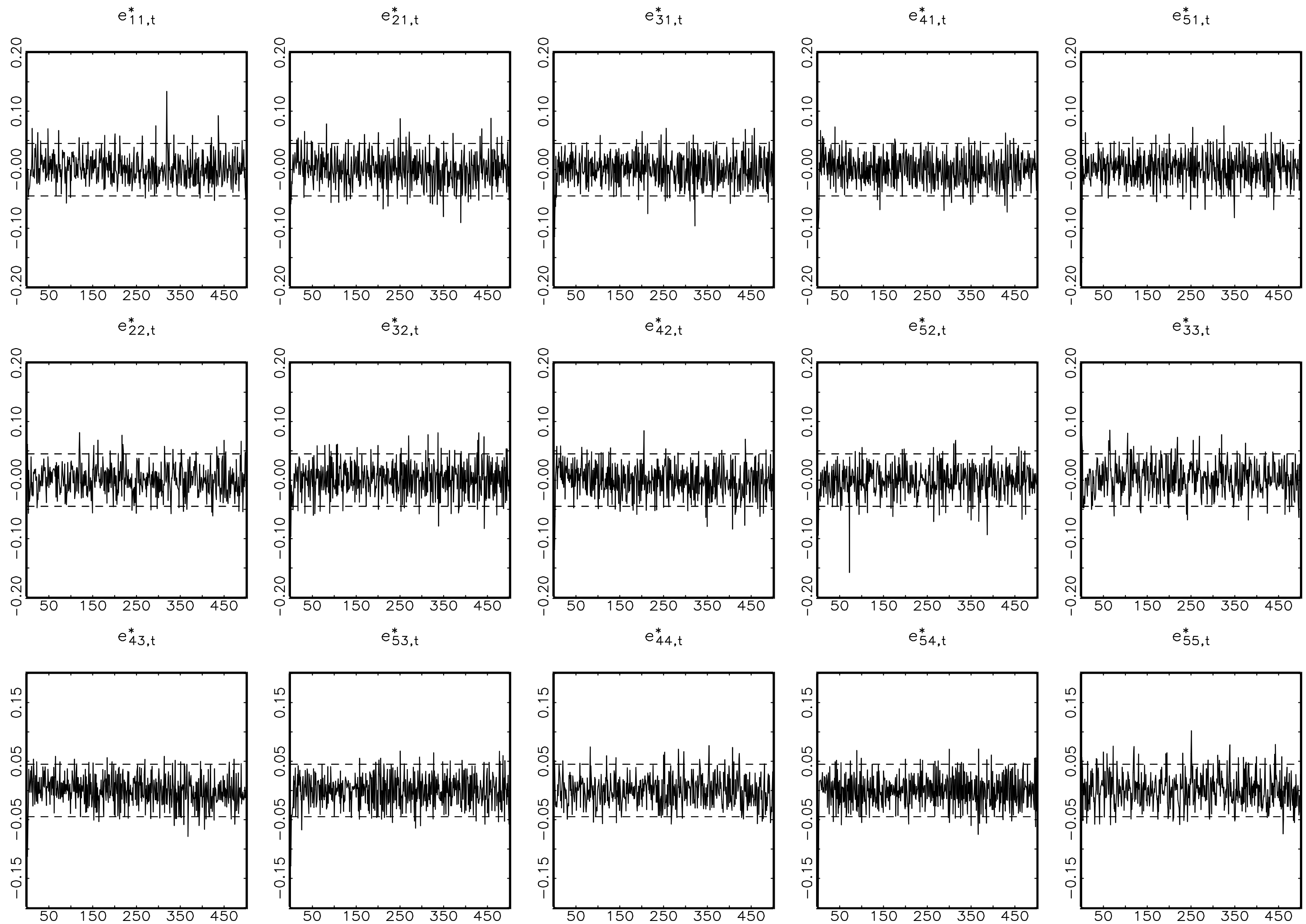

Figure 3. Sample autocorrelation function of the standardized residuals $e^{*}_{ij,t}$ from the baseline CAW(2,2) model; the dashed lines indicate the 95% Bartlett confidence bands for no serial dependence.

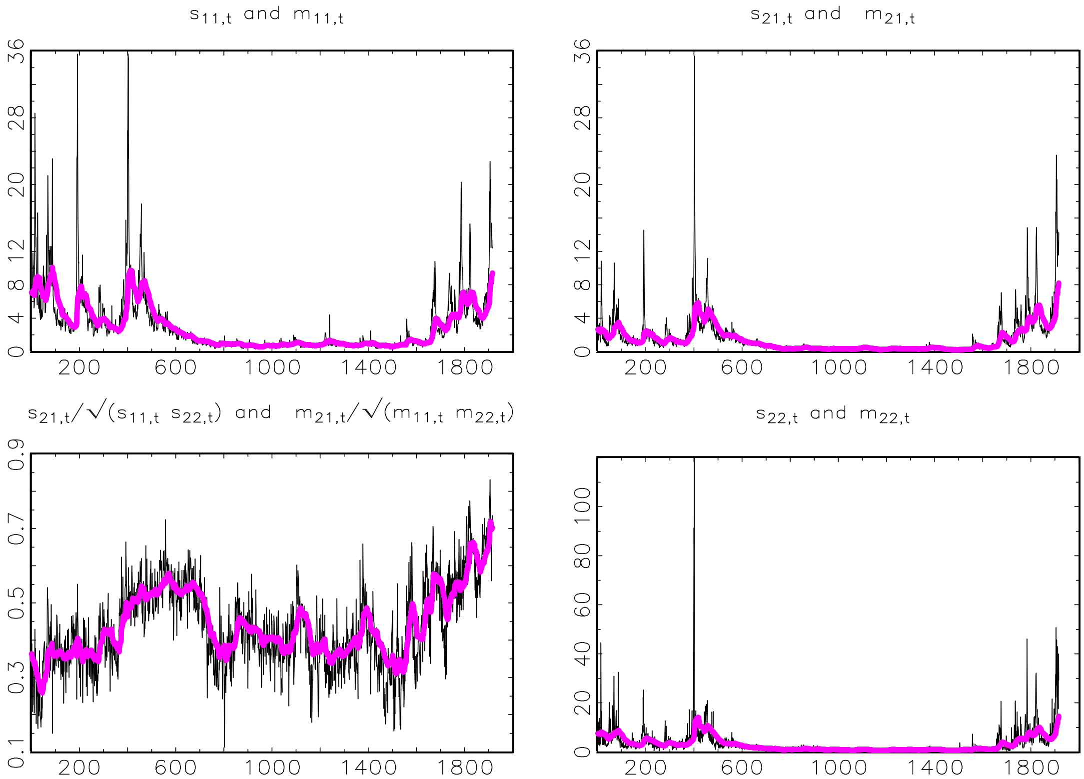

Figure 4. Predicted (co)variances $s_{ij,t}$ and correlations $s_{ij,t}/\sqrt{s_{ii,t}s_{jj,t}}$ for AXP ($i=1$) and C ($j=2$) stock, together with their predicted secular components $m_{ij,t}$ and $m_{ij,t}/\sqrt{m_{ii,t}m_{jj,t}}$ obtained under the MIDAS-CAW(2,2) model. The bold lines represent the secular components.